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Ricci Curvature Inequalities for Contact CR-Warped Product Submanifolds of an Odd Dimensional Sphere Admitting a Semi-Symmetric Metric Connection

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Abstract: The primary objective of this paper is to explore contact CR-warped product submanifolds of Sasakian space forms equipped with a semi-symmetric metric connection. We thoroughly examine these submanifolds and establish various key findings. Furthermore, we derive an inequality relating the Ricci curvature to the mean curvature vector and warping function.

Keywords: contact CR-submanifolds; warped product manifolds; Ricci curvature; semi-symmetric; Sasakian manifolds

MSC: 53B50; 53C20; 53C40

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1. Introduction

The study of warped product manifolds has attracted considerable attention in the field of research, particularly due to their applications in physics and the theory of relativity [1]. These manifolds have proven to be valuable tools in providing essential solutions to the Einstein field equations [1], which govern the behavior of gravity in spacetime. One of the most intriguing applications of warped product manifolds is their role in modeling the behavior of spacetime near black holes in the universe. By utilizing warped product manifolds, researchers are able to gain insights into the intricate nature of these astrophysical phenomena.

An important example of a warped product manifold is the Robertson–Walker model, which serves as a cosmological model for the structure and spacetime of the universe [2]. This model describes the expanding universe, taking into account the curvature of spacetime and the distribution of matter and energy within it. The warped product structure captures the spatial geometry of the universe, providing a framework for understanding its evolution and dynamics.

In their investigations, Bishop and Neill delved into the geometry of Riemannian manifolds with negative curvature and introduced the concept of warped products for such manifolds [3]. This notion, defined in Section 2 of their work, extends the idea of Riemannian product manifolds and offers a more flexible and versatile framework for studying curved spaces. By utilizing the concept of warped products, researchers are able to explore and analyze a wide range of geometric structures and their associated properties.

The properties of warped product manifolds have been a subject of significant interest [3]. Researchers have sought to understand the intrinsic characteristics and behavior of these manifolds, which differ from those of Riemannian product manifolds. By investigating the properties of warped product manifolds, researchers have been able to uncover unique geometric features and gain deeper insights into the underlying structures of curved spaces.

Overall, the study of warped product manifolds has emerged as a captivating topic of research, driven by their applications in physics and the theory of relativity. These manifolds provide valuable solutions to the Einstein field equations and offer a versatile framework for modeling various physical phenomena, such as the behavior of spacetime near black holes. The exploration of warped product manifolds, including their properties and geometric characteristics, contributes to our understanding of curved spaces and their implications in different fields of study.

In 1981, Chen made a significant contribution to the field by introducing the concept of warped products as a means to study CR-submanifolds of Kaehler manifolds [4,5]. This work focused on investigating the existence of CR-warped product submanifolds in the context of Kaehler manifolds. Chen demonstrated that it is possible to construct such submanifolds of the form $N_T \times_f N_\perp$, where N_T represents the holomorphic submanifold and $N_\perp$ represents the totally real submanifold.

Expanding on Chen's pioneering work, Hasegawa and Mihai [6] extended the study of CR-warped product submanifolds to Sasakian manifolds. They explored the contact CR-warped product submanifolds within the framework of Sasakian manifolds. By considering the interplay between the warping function and the squared norm of the second fundamental form, Mihai [7] derived an estimate for the latter for contact CR-warped product submanifolds in Sasakian space forms.

Following these foundational contributions, numerous researchers have engaged in the study of warped product submanifolds in different settings of Riemannian manifolds. As a result, a wealth of existence results and findings have emerged in this field of research. For a comprehensive overview of these developments, one can refer to the survey article [8], which provides a thorough exploration of the topic.

In 1999, Chen made a notable contribution regarding the connection between Ricci curvature and the squared mean curvature vector in any Riemannian manifold [9]. This breakthrough led to a series of subsequent articles that aimed to formulate and explore the relationship between Ricci curvature and squared mean curvature within the context of various key structures on Riemannian manifolds [7,10–15]. These works built upon Chen's initial findings, delving into the intricate relationship between these two geometric quantities.

More recently, Ali et al. [16] made significant contributions to this line of research by establishing a relation between Ricci curvature and squared mean curvature, specifically for warped product submanifolds of a sphere. Their work not only formulated this relationship but also provided numerous physical applications, highlighting the practical implications of this geometric connection. By studying the interplay between Ricci curvature and squared mean curvature in the context of warped product submanifolds, Ali et al. shed light on the underlying geometric structures and their relevance in various physical phenomena.

On the other hand, the introduction of the idea of a semi-symmetric linear connection on a Riemannian manifold can be attributed to Friedmann and Schouten [17]. Subsequently, Hayden [18] defined a semi-symmetric connection as a linear connection, ∇, existing on an n-dimensional Riemannian manifold (M, g), where the torsion tensor, T, satisfies $T(\gamma_1, \gamma_2) = \pi(\gamma_2)\gamma_1 - \pi(\gamma_1)\gamma_2$, with π representing a 1-form and $\gamma_1, \gamma_2 \in TM$. The properties of semi-symmetric metric connections were further explored by K. Yano [19]. He demonstrated that a conformally flat Riemannian manifold admitting a semi-symmetric connection exhibits a vanishing curvature tensor. Sular and Ozgur [20] delved into the investigation of warped product manifolds equipped with a semi-symmetric metric connection, focusing specifically on Einstein warped product manifolds with such connections. However, in their work [21], they obtained additional results pertaining to warped product manifolds with a semi-symmetric metric connection. Motivated by these previous studies, our interest lies in examining the influence of a semi-symmetric metric connection on Ricci curvature of contact CR-warped product submanifolds and their geometry within an odd dimensional sphere.

2. Preliminaries

Suppose $(\bar{M}, g)$ is an odd dimensional Riemannian manifold. We define $\bar{M}$ as an almost contact metric manifold if there exists a tensor field, ϕ, of type $(1,1)$ and a global vector field, ξ, on $\bar{M}$ satisfying the following conditions

$$\phi^2\gamma_1 = -\gamma_1 + \eta(\gamma_1)\xi, \quad g(\gamma_1, \xi) = \eta(\gamma_1) \tag{1}$$

$$g(\phi\gamma_1, \phi\gamma_2) = g(\gamma_1, \gamma_2) - \eta(\gamma_1)\eta(\gamma_2). \tag{2}$$

The dual 1-form of ξ is denoted as η. It is a well-known fact that an almost contact metric manifold can be classified as a Sasakian manifold if and only if the following tensorial equation holds

$$(\bar{\bar{\nabla}}_{\gamma_1}\phi)\gamma_2 = g(\gamma_1, \gamma_2)\xi - \eta(\gamma_2)\gamma_1. \tag{3}$$

It is straightforward to observe that on a Sasakian manifold, $\bar{M}$, the following can be readily deduced

$$\bar{\bar{\nabla}}_{\gamma_1}\xi = -\phi\gamma_1, \tag{4}$$

here, $\gamma_1, \gamma_2 \in T\bar{M}$, and $\bar{\bar{\nabla}}$ represents the Riemannian connection associated with the metric g on $\bar{M}$.

Now, defining a connection, $\bar{\nabla}$, as

$$\bar{\nabla}_{\gamma_1}\gamma_2 = \bar{\bar{\nabla}}_{\gamma_1}\gamma_2 + \eta(\gamma_2)\gamma_1 - g(\gamma_1, \gamma_2)\xi \tag{5}$$

such that $\bar{\nabla}g = 0$ for any $\gamma_1, \gamma_2 \in TM$, where $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g. The connection $\bar{\nabla}$ is semi-symmetric because $T(\gamma_1, \gamma_2) = \eta(\gamma_2)\gamma_1 - \eta(\gamma_1)\gamma_2$. Using (5) in (3), we have

$$(\bar{\nabla}_{\gamma_1}\phi)\gamma_2 = g(\gamma_1, \gamma_2)\xi - g(\gamma_1, \phi\gamma_2)\xi - \eta(\gamma_2)\gamma_1 - \eta(\gamma_2)\phi\gamma_1 \tag{6}$$

and

$$\bar{\nabla}_{\gamma_1}\xi = \gamma_1 - \eta(\gamma_1)\xi - \phi\gamma_1. \tag{7}$$

If a S-M $\bar{M}$ has a constant ϕ-holomorphic sectional curvature, c, it is called a S-S-F and denoted as $\bar{M}(c)$.

The expression for the curvature tensor, $\bar{R}$, corresponding to the S-S-M connection, $\bar{\nabla}$, can be expressed as

$$\bar{R}(\gamma_1, \gamma_2)\gamma_3 = \bar{\nabla}_{\gamma_1}\bar{\nabla}_{\gamma_2}\gamma_3 - \bar{\nabla}_{\gamma_2}\bar{\nabla}_{\gamma_1}\gamma_3 - \bar{\nabla}_{[\gamma_1,\gamma_2]}\gamma_3. \tag{8}$$

Likewise, the curvature tensor, $\bar{\bar{R}}$, can be defined for the Riemannian connection, $\bar{\bar{\nabla}}$.

Let

$$\alpha(\gamma_1, \gamma_2) = (\bar{\nabla}_{\gamma_1}\eta)\gamma_2 - \eta(\gamma_1)\eta(\gamma_2) + \frac{1}{2}g(\gamma_1, \gamma_2). \tag{9}$$

Now, by the application of (5), (8), and (9), we obtain

$$\begin{aligned} \bar{R}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = {} & \bar{\bar{R}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) + \alpha(\gamma_1, \gamma_3)g(\gamma_2, \gamma_4) \\ & - \alpha(\gamma_2, \gamma_3)g(\gamma_1, \gamma_4) + \alpha(\gamma_2, \gamma_4)g(\gamma_1, \gamma_3) \\ & - \alpha(\gamma_1, \gamma_4)g(\gamma_2, \gamma_3). \end{aligned} \tag{10}$$

By employing the computed value of $\bar{\bar{R}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, as detailed in [22], by computation, we obtain the following expression for the curvature tensor, $\bar{R}$, of a S-S-F:

$$
\begin{aligned}
\bar{R}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) =& \frac{c+3}{4}\{g(\gamma_2,\gamma_3)g(\gamma_1,\gamma_4) - g(\gamma_1,\gamma_3)g(\gamma_2,\gamma_4)\} \\
&+ \frac{c-1}{4}\{\eta(\gamma_1)\eta(\gamma_3)g(\gamma_2,\gamma_4) - \eta(\gamma_2)\eta(\gamma_3)g(\gamma_1,\gamma_4) \\
&+ g(\gamma_1,\gamma_3)\eta(\gamma_2)\eta(\gamma_4) - g(\gamma_2,\gamma_3)\eta(\gamma_1)\eta(\gamma_4) \\
&+ g(\phi\gamma_2,\gamma_3)g(\phi\gamma_1,\gamma_4) + g(\phi\gamma_3,\gamma_1)g(\phi\gamma_2,\gamma_4) \\
&- 2g(\phi\gamma_1,\gamma_2)g(\phi\gamma_3,\gamma_4)\} + \alpha(\gamma_1,\gamma_3)g(\gamma_2,\gamma_4) \\
&- \alpha(\gamma_2,\gamma_3)g(\gamma_1,\gamma_4) + \alpha(\gamma_2,\gamma_4)g(\gamma_1,\gamma_3) \\
&- \alpha(\gamma_1,\gamma_4)g(\gamma_2,\gamma_3),
\end{aligned} \tag{11}
$$

for all $\gamma_1,\gamma_2,\gamma_3,\gamma_4 \in T\bar{M}$.

For a submanifold, M, isometrically immersed in a Riemannian manifold, $\bar{M}$, admitting a S-S-M connection, it is easy to derive the Gauss and Weingarten formula as follows

$$\bar{\nabla}_{\gamma_1}\gamma_2 = \nabla_{\gamma_1}\gamma_2 + \hbar(\gamma_1,\gamma_2)$$

and

$$\bar{\nabla}_{\gamma_1}N = -A_N\gamma_1 + \nabla^{\perp}_{\gamma_1}N + \eta(N)\gamma_1,$$

where ∇ is the induced s-s-m connection on M, $\gamma_1,\gamma_2 \in TM$, and $N \in T^{\perp}M$.

The relationship between the second fundamental form, $\hbar$, and the shape operator, A_N, can be expressed by the following formula

$$g(\hbar(\gamma_1,\gamma_2),N) = g(A_N\gamma_1,\gamma_2).$$

For the vector fields $\gamma_1 \in TM$ and $\gamma_3 \in T^{\perp}M$, we can decompose the expression as follows

$$\phi\gamma_1 = P\gamma_1 + F\gamma_1 \tag{12}$$

and

$$\phi\gamma_3 = t\gamma_3 + f\gamma_3 \tag{13}$$

where $P\gamma_1(t\gamma_3)$ and $F\gamma_1(f\gamma_3)$ are the tangential and normal components of $\phi\gamma_1(\phi\gamma_3)$, correspondingly.

The equation of Gauss for a S-S-M connection, for the Riemannian curvature tensor, R, can be expressed as follows [22]

$$\bar{R}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = R(\gamma_1,\gamma_2,\gamma_3,\gamma_4) - g(\hbar(\gamma_1,\gamma_4),\hbar(\gamma_2,\gamma_3)) + g(\hbar(\gamma_2,\gamma_4),\hbar(\gamma_1,\gamma_3)) \tag{14}$$

for $\gamma_1,\gamma_2,\gamma_3,\gamma_4 \in TM$.

In their article [20], Sular and Ožgur investigated the warped products denoted as $M_1 \times_f M_2$, where a S-S-M connection is defined on the manifold $M_1 \times_f M_2$, along with an associated vector field, P. Here, M_1 and M_2 are Riemannian manifolds, and the warping function, f, is a positive differentiable function defined on M_1. We present several important findings from [20] as a lemma, which will be relevant for our subsequent analysis.

Lemma 1. *Let $M_1 \times_f M_2$ be a W-P manifold with S-S-M connection $\bar{\nabla}$, we have*

(i) *If the associated vector field $P \in TM_1$, then*

$$\bar{\nabla}_{\gamma_1}\gamma_3 = \frac{\gamma_1 f}{f}\gamma_3 \quad \text{and} \quad \bar{\nabla}_{\gamma_3}\gamma_1 = \frac{\gamma_1 f}{f}\gamma_3 + \pi(\gamma_1)\gamma_3$$

(ii) *If $P \in TN_2$, then*

$$\bar{\nabla}_{\gamma_1}\gamma_3 = \frac{\gamma_1 f}{f}\gamma_3 \quad \text{and} \quad \nabla_{\gamma_3}\gamma_1 = \frac{\gamma_1 f}{f}\gamma_3,$$

where $\gamma_1 \in TM_1$, $\gamma_3 \in TM_2$, and π is the 1-form associated with the vector field P.

Let us examine the relationship between the curvature tensors, R and $\tilde{R}$, associated with the warped product submanifold, $M = M_1 \times_f M_2$, of a Sasakian manifold, $\bar{M}$, corresponding to the induced S-S-M connection, ∇, and the induced Riemannian connection, $\tilde{\nabla}$, on M. Expressing this relationship, we have

$$\begin{aligned} R(\gamma_1,\gamma_2)\gamma_3 =& \tilde{R}(\gamma_1,\gamma_2)\gamma_3 + g(\gamma_3,\nabla_{\gamma_1}P)\gamma_2 - g(\gamma_3,\nabla_{\gamma_2}P)\gamma_1 \\ &+ g(\gamma_1,\gamma_3)\nabla_{\gamma_2}P - g(\gamma_2,\gamma_3)\nabla_{\gamma_1}P \\ &+ \eta(P)[g(\gamma_1,\gamma_3)\gamma_2 - g(\gamma_2,\gamma_3)\gamma_1] \\ &+ [g(\gamma_2,\gamma_3)\eta(\gamma_1) - g(\gamma_1,\gamma_3)\eta(\gamma_2)]P \\ &+ \eta(\gamma_3)[\eta(\gamma_2)\gamma_1 - \eta(\gamma_1)\gamma_2], \end{aligned} \tag{15}$$

for any vector field $\gamma_1, \gamma_2, \gamma_3$ on M [20].

According to part (ii) of Lemma 3.2 in reference [20], for the warped product submanifold $M = M_1 \times_f M_2$, the following relationship holds

$$\tilde{R}(\gamma_1,\gamma_2)\gamma_3 = \frac{H^f(\gamma_1,\gamma_2)}{f}\gamma_3, \tag{16}$$

where $\gamma_1, \gamma_2 \in TM_1$, $\gamma_3 \in TM_2$, respectively, and H^f is the Hessian of the warping function.

By taking into account Equations (15) and (16), we can infer the following

$$\begin{aligned} R(\gamma_1,\gamma_3)\gamma_2 = \frac{H^f(\gamma_1,\gamma_2)}{f} + \frac{Pf}{f}g(\gamma_1,\gamma_2)\gamma_3 + \eta(P)g(\gamma_1,\gamma_2)\gamma_3 + g(\gamma_2,\nabla_{\gamma_1}P)\gamma_3 \\ - \eta(\gamma_1)\eta(\gamma_2)\gamma_3, \end{aligned} \tag{17}$$

for the vector fields $\gamma_1, \gamma_2 \in TM_1$, $\gamma_3 \in TM_2$, and $P \in TM_1$.

By substituting $P = \xi$ into Equation (5), we introduce the S-S-M connection. As a result, for a W-P submanifold, $M = M_1 \times_f M_2$, of the Riemannian manifold, $\bar{M}$, we can deduce the following relationship using part (i) of Lemma 1.

$$\nabla_{\gamma_1}\gamma_3 = \gamma_1 ln f \gamma_3 \tag{18}$$

and

$$\nabla_{\gamma_3}\gamma_1 = \gamma_1 ln f \gamma_3 + \eta(\gamma_1)\gamma_3. \tag{19}$$

In addition, Equation (21) with (7) yields

$$\begin{aligned} R(\gamma_1,\gamma_3)\gamma_2 = \frac{H^f(\gamma_1,\gamma_2)}{f}\gamma_3 + \frac{\xi f}{f}g(\gamma_1,\gamma_2)\gamma_3 + 2g(\gamma_1,\gamma_2)\gamma_3 - 2\eta(\gamma_1)\eta(\gamma_2)\gamma_3 \\ - g(\gamma_2,\phi\gamma_1)\gamma_3, \end{aligned} \tag{20}$$

for $\xi, \gamma_1, \gamma_2 \in TM_1$, and $\gamma_3 \in TM_2$.

Let us define the mean curvature vector, $H(x)$, and its squared norm for a given point, x, on the manifold, M, considering an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space, T_xM, as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n}\hbar(e_i,e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(\hbar(e_i,e_i),\hbar(e_j,e_j)).$$

Here, the dimension of M is denoted by n, and we can define certain properties based on the mean curvature vector, $H(x)$, and its squared norm. When $\hbar = 0$, the submanifold is referred to as totally geodesic, and, if $H = 0$, it is said to be minimal. Additionally, if $\hbar(\gamma_1, \gamma_2) = g(\gamma_1, \gamma_2)H$ holds for all $\gamma_1, \gamma_2 \in TM$, the submanifold, M, is known as totally umbilical.

Let us denote the scalar curvature of an m-dimensional Riemannian manifold, $\bar{M}$, as $\bar{\pi}(\bar{M})$. Its mathematical expression is given by

$$\bar{\pi}(\bar{M}) = \sum_{1 \leq p < q \leq m} \bar{\kappa}_{pq},$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$. In this study, we will adopt an equivalent formulation of the aforementioned equation, which can be expressed as follows

$$2\bar{\pi}(\bar{M}) = \sum_{1 \leq p < q \leq m} \bar{\kappa}_{pq}.$$

Similarly, we can express the scalar curvature, $\bar{\pi}(L_x)$, of an L-plane as follows

$$\bar{\pi}(L_x) = \sum_{1 \leq p < q \leq m} \bar{\kappa}_{pq}. \tag{21}$$

Consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space, T_xM. If e_r is an element of the orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space, $T^{\perp}M$, we can express the relationship as follows

$$\hbar^r_{pq} = g(\hbar(e_p, e_q), e_r) \tag{22}$$

and

$$\|\hbar\|^2 = \sum_{p,q=1}^{n} g(\hbar(e_p, e_q), \hbar(e_p, e_q)).$$

Consider the sectional curvatures, κ_{pq} and $\bar{\kappa}_{pq}$, associated with the plane sections generated by e_p and e_q at a point x in the n-dimensional manifold, M, and the m-dimensional Riemannian space form, $\bar{M}(c)$, respectively. Applying the Gauss equation, we obtain the following relationship

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=n+1}^{m} (\hbar^r_{pp}\hbar^r_{qq} - (\hbar^r_{pq})^2). \tag{23}$$

Let us define the global tensor field for an orthonormal frame of vector fields $\{e_1, \ldots, e_n\}$ on the n-dimensional manifold, M, as

$$\bar{S}(\gamma_1, \gamma_2) = \sum_{i=1}^{n} \{g(\bar{R}(e_i, \gamma_1)\gamma_2, e_i)\},$$

for all $\gamma_1, \gamma_2 \in T_xM^n$. The tensor mentioned above is known as the Ricci tensor. In the case where we choose a specific vector, e_u, from the orthonormal frame $\{e_1, \ldots, e_n\}$ on the n-dimensional manifold, M, denoted by χ, the Ricci curvature is defined as follows

$$Ric(\chi) = \sum_{\substack{p=1 \\ p \neq u}}^{n} \kappa(e_p \wedge e_u). \tag{24}$$

Let us denote the gradient of a scalar function, f, as ∇f. The gradient is defined as follows

$$g(\nabla f, \gamma) = \gamma f, \tag{25}$$

for all $\gamma \in TM$.

Consider an orthogonal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space, TM, of an n-dimensional Riemannian manifold, M. Then, Equation (25) yields the following result

$$\|\nabla f\|^2 = \sum_{i=1}^{n}(e_i(f))^2.$$

The Laplacian of f is defined by

$$\Delta f = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)f - e_ie_if\}.$$

The Hessian tensor, denoted as Δf, is a symmetric covariant tensor of rank 2. It is defined for a differentiable function, f, as

$$\Delta f = -traceH^f,$$

where H^f is the Hessian of f.

For the warped product submanifolds of the type $N_1 \times_f N_2$ with a S-S-M connection, using Formula (20) we can derive the following result

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\kappa(e_p \wedge e_q) = \frac{n_2\Delta f}{f} + \frac{\xi f}{f}n_1n_2 + 2n_1n_2 - 2n_2. \tag{26}$$

3. Contact CR-Warped Product Submanifolds

In 1981, A. Bejancu [23] introduced the concept of semi-invariant submanifolds in almost contact metric manifolds. An m-dimensional Riemannian submanifold, M, of a S-M $\bar{M}$ is classified as a semi-invariant submanifold if the vector field, ξ, is tangent to M and there exists a differentiable distribution $D : x \mapsto D_x \subset T_xM$ on M, such that D_x is invariant under the action of the structure vector field, ϕ. The orthogonal complementary distribution $D_x^\perp$ to D_x on M is anti-invariant, meaning that $\phi D^\perp \subseteq T_x^\perp M$, where T_xM and $T_x^\perp M$ represent the tangent space and normal space at $x \in M$, respectively.

In a subsequent work by Hesigawa and Mihai [6], they considered a specific type of submanifold known as a warped product submanifold of the form $N_T \times_f N_\perp$ in a Sasakian manifold, $\bar{M}$. Here, N_T denotes an invariant submanifold, $N_\perp$ represents an anti-invariant submanifold, and ξ belongs to TN_T. These submanifolds were referred to as contact CR-submanifolds, and the authors provided some fundamental results related to them.

Our analysis commences by investigating a specific class of submanifolds known as contact CR-w-p submanifolds in a S-M endowed with a S-S-M connection. These submanifolds are of the form $N_\perp \times_f N_T$, where $N_\perp$ is an anti-invariant submanifold and N_T represents an invariant submanifold, and satisfying the condition $\xi \in TN_T$.

Theorem 1. *Let $(\bar{M}, \phi, \xi, \eta, g)$ be a S-M with a S-S-M connection. Then there does not exist a W-P submanifold of the type $N_\perp \times_f N_T$, such that $\xi \in TN_T$.*

Proof. For any $\gamma_1, \gamma_2 \in TN_T$ and $\gamma_3 \in TN_\perp$, then by (19), the Gauss formula, and (2), we have

$$\begin{aligned}\gamma_3 lnfg(\gamma_1, \gamma_2) &= g(\bar{\nabla}_{\gamma_1}\phi\gamma_3, \phi\gamma_2) - g((\nabla_{\gamma_1}\phi)\gamma_3, \phi\gamma_3) - \gamma_3 lnf\eta(\gamma_1)\eta(\gamma_2) \\ &= g(\nabla_{\gamma_1}\phi\gamma_3, \phi\gamma_2) - \gamma_3 lnf\eta(\gamma_1)\eta(\gamma_2).\end{aligned} \tag{27}$$

Equivalently

$$\gamma_3 lnfg(\gamma_1, \gamma_2) = \phi\gamma_3 lnfg(\gamma_1, \phi\gamma_2) - \gamma_3 lnf\eta(\gamma_1)\eta(\gamma_2). \tag{28}$$

By substituting ξ for both γ_1 and γ_2 in the preceding equation, we obtain $\gamma_3 \ln f = 0$. This indicates that f must be a constant, thereby establishing the desired result. □

In this study, we focus on analyzing warped product submanifolds of the form $M = N_T \times_f N_\perp$ in a Sasakian manifold, $\bar{M}$, where these submanifolds are equipped with a semi-symmetric metric connection and ξ belongs to TN_T. We refer to these specific submanifolds as contact CR-warped product submanifolds. Furthermore, we denote the invariant subspace of the normal bundle $T^\perp M$ as μ.

Now, we commence with the following preliminary findings

Lemma 2. *Let $M = N_T \times_f N_\perp$ be a contact CR-W-P submanifold of a S-M $\bar{M}$ endowed with a S-S-M connection, then*

(i) $g(\hbar(\phi\gamma_1, \gamma_3), \phi\gamma_4) = \gamma_1 lnf g(\gamma_3, \gamma_4) + \eta(\gamma_1) g(\gamma_3, \gamma_4)$,
(ii) $g(\hbar(\gamma_1, \gamma_2), \phi\gamma_3) = 0$,
(iii) $\xi lnf = 0$

$\forall\, \gamma_1, \gamma_2 \in TN_T$ *and* $\gamma_3, \gamma_4 \in TN_\perp$, $\xi \in TN_T$.

Proof. By employing the Gauss formula and Equation (6), we obtain the following expression

$$g(\hbar(\phi\gamma_1, \gamma_3), \phi\gamma_4) = g(\bar{\nabla}_{\gamma_3}\phi\gamma_1, \phi\gamma_4) = g(\bar{\nabla}_{\gamma_3}\gamma_1, \gamma_4).$$

Now, utilizing Formula (19), we obtain the following:

$$g(\hbar(\phi\gamma_1, \gamma_3), \phi\gamma_4) = g(\nabla_{\gamma_3}\gamma_1, \gamma_4) = \gamma_1 lnf g(\gamma_3, \gamma_4) + \eta(\gamma_1) g(\gamma_3, \gamma_4),$$

which is part (i). Again, using (6), (19), and the Gauss formula, part (iii) is proven straightforwardly. Now, using the formula $\nabla_{\gamma_3}\xi = \gamma_3 - \eta(\gamma_3) - P\gamma_3$ and applying Equation (19), we have $\xi lnf + \eta(\xi)\gamma_3 = \gamma_3$ or $\xi lnf = 0$, which is part (iii). □

Lemma 3. *Let $M = N_T \times_f N_\perp$ be a contact CR-W-P submanifold of a S-M $\bar{M}$ endowed with a S-S-M connection, then*

$$g(\hbar(\gamma_1, \gamma_1), V) = -g(\hbar(\phi\gamma_1, \phi\gamma_1), V), \tag{29}$$

for all $\gamma_1 \in TM$ and $V \in \mu$.

Proof. From the Gauss formula along with Equation (6), we have

$$\nabla_{\gamma_1}\phi\gamma_1 + \hbar(\gamma_1, \phi\gamma_1) - \phi\nabla_{\gamma_1}\gamma_1 - \phi\hbar(\gamma_1, \gamma_1) = g(\gamma_1, \gamma_1)\xi - \eta(\gamma_1)\gamma_1 - \eta(\gamma_1)\phi\gamma_1, \tag{30}$$

taking the inner product with $\phi V \in \mu$, we find

$$g(\hbar(\gamma_1, \phi\gamma_1), \phi V) = g(\hbar(\gamma_1, \gamma_1), V), \tag{31}$$

replacing γ_1 with $\phi\gamma_1$ and using Equation (1), we have

$$-g(\hbar(\phi\gamma_1, \gamma_1), \phi V) = g(\hbar(\phi\gamma_1, \phi\gamma_1), V. \tag{32}$$

By considering Equations (31) and (32), we can deduce the following

$$g(\hbar(\gamma_1, \gamma_1), V) = -g(\hbar(\phi\gamma_1, \phi\gamma_1), V), \tag{33}$$

which proves the assertion. □

Based on Lemma 3, it is clear that the isometric immersion of $N_T^{n_1} \times_f N_\perp^{n_2}$ into a Sasakian manifold is characterized as being D-minimal. This D-minimal property establishes a valuable connection between the contact CR-warped product submanifold $N_T \times_f N_\perp$ and the equation of Gauss.

Definition 1. *An isometric immersion of the warped product $N_1 \times_f N_2$ into a Riemannian manifold, $\bar{M}$, is referred to as N_i totally geodesic if the partial second fundamental form, h_i, is identically zero. If the partial mean curvature vector, H^i, becomes zero for $i = 1, 2$, it is termed N_i-minimal.*

Consider a local orthonormal frame of vector fields on the contact CR-W-P submanifold $M^n = N_T^{n_1} \times_f N_\perp^{n_2}$ given by $\{e_1, \ldots, e_\beta, e_{\beta+1} = \phi e_1, \ldots, e_{n_1-1} = \phi e_\beta, e_{n_1} = \xi, e_{n_1+1}, \ldots, e_n\}$. Here, $\{\xi, e_1, \ldots, e_{n_1}\}$ are tangent to N_T, and $\{e_{n_1+1}, \ldots e_n\}$ are tangent to $N_\perp$. Additionally, $\{e_1^* = \phi e_{n_1+1}, \ldots, e_n^* = \phi e_n, e_{n+1}^*, \ldots, e_m^*\}$ forms a local orthonormal frame of the normal space, $T^\perp M$.

By considering Lemma 3, it is possible to observe

$$\sum_{r=n+1}^{m} \sum_{i,j=1}^{n_1} g(\hbar(e_i, e_j), e_r) = 0.$$

Hence, based on Lemma 3, it can be deduced that the trace of $\hbar$ with respect to N_T is zero. Therefore, in light of Definition 1, we obtain the following significant result.

Theorem 2. *Consider a contact CR-W-P submanifold $M^n = N_T^{n_1} \times_f N_\perp^{n_2}$ that is isometrically immersed in a S-M admitting a S-S-M connection. It can be concluded that M^n possesses the property of being D-minimal.*

Therefore, it can be readily concluded that the following statement holds.

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} (\hbar^r_{n_1+1 n_1+1} + \cdots + \hbar^r_{nn})^2,$$

where $\|H\|^2$ is the squared mean curvature.

4. Inequalities for *Ricci curvature*

This section focuses on deriving the Ricci curvature for a contact CR-W-P isometrically immersed in an odd dimensional unit sphere within the context of the mean curvature vector and warping function, f.

Theorem 3. *Let $M = N_T^{n_1} \times_f N_\perp^{n_2}$ be a contact CR-W-P submanifold isometrically immersed in an odd dimensional unit sphere, $S^{2n+1}(1)$, admitting a semi-symmetric metric connection. If for each orthogonal unit vector field $\chi \in T_x M$ orthogonal to ξ, either tangent to N_T or $N_\perp$, then*

(1) The Ricci curvature is subject to the following inequalities

(i) if $\chi \in TN_T$

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta f}{f} + (n - n_1 n_2 + 2n_2 - 1) - (1 + n_1) \sum_{i=n_1+1}^{n} \alpha(e_i, e_i) \\ - (1 + n_2) \sum_{i=1}^{n_1} \alpha(e_i, e_i) - (n-2)\alpha(e_1, e_1); \end{aligned} \tag{34}$$

(ii) if $\chi \in TN_\perp$

$$\begin{aligned} Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 - \frac{n_2 \Delta f}{f} + (n - n_1 n_2 + 2n_2 - 1) - (1 + n_1) \sum_{i=n_1+1}^{n} \alpha(e_i, e_i) \\ - (1 + n_2) \sum_{i=1}^{n_1} \alpha(e_i, e_i) - (n-2)\alpha(e_n, e_n). \end{aligned} \tag{35}$$

(2) *In the case where $H(x) = 0$ for all points $x \in M^n$, there exists a unit vector field, χ that satisfies the equality condition of (1) if and only if M^n is a mixed T-G submanifold and χ belongs to the relative null space, N_x, at x.*

(3) *In the equality case, we have the following*

(a) *The equality case of (34) holds true for all unit vector fields tangent to N_T at each point $x \in M^n$ if and only if M^n is a mixed T-G submanifold and a D-T-G contact CR-W-P submanifold in S^{2n+1}.*

(b) *The equality case of (1) holds true for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is T-G submanifold or M^n is a mixed T-G T-U and D− T-G submanifold with dim $N_\perp = 2$.*

Here n_1 and n_2 represent the dimensions of N_T and $N_\perp$, respectively.

Proof. Let us consider $M = N_T^{n_1} \times_f N_\perp^{n_2}$ as a contact CR-W-P submanifold of an odd dimensional sphere $S^{2n+1}(1)$. By utilizing the Gauss equation, we can obtain the following expression

$$n^2\|H\|^2 = 2\pi(M^n) + \|\hbar\|^2 - 2\bar{\pi}(M^n). \tag{36}$$

Let $\{e_1, \ldots, e_{n_1}, e_{n_1+1}, \ldots, e_n\}$ be a set of orthonormal vector fields on M^n such that the frame $\{e_1, \ldots, e_{n_1}\}$ is tangent to N_T and $\{e_{n_1+1}, \ldots, e_n\}$ is tangent to $N_\perp$. So, the unit tangent vector $\chi = e_A \in \{e_1, \ldots, e_n\}$ can be expanded (36) as follows

$$n^2\|H\|^2 = 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(\hbar^r_{11} + \cdots + \hbar^r_{nn} - \hbar^r_{AA})^2 + (\hbar^r_{AA})^2\}$$
$$- \sum_{r=n+1}^{m}\sum_{1\le p\ne q\le n}\hbar^r_{pp}\hbar^r_{qq} - 2\bar{\pi}(M^n).$$

□

The aforementioned expression can be expanded in the following manner.

$$\begin{aligned} n^2\|H\|^2 = {} & 2\pi(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(\hbar^r_{11} + \cdots + \hbar^r_{nn})^2 \\ & + (2\hbar^r_{AA} - (\hbar^r_{11} + \cdots + \hbar^r_{nn}))^2\} + 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(\hbar^r_{pq})^2 \\ & - 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}\hbar^r_{pp}\hbar^r_{qq} - 2\bar{\pi}(M^n). \end{aligned}$$

In light of Lemma 3, the preceding expression can be written as follows.

$$\begin{aligned} n^2\|H\|^2 = {} & \sum_{r=n+1}^{m}\{(\hbar^r_{n_1+1n_1+1} + \cdots + \hbar^r_{nn})^2 + (2\hbar^r_{AA} - \frac{1}{2}(\hbar^r_{n_1+1n_1+1} + \cdots + \hbar^r_{nn}))^2\} \\ & + 2\pi(M^n) + \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(\hbar^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}\hbar^r_{pp}\hbar^r_{qq} + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\ne A}}(\hbar^r_{aA})^2 \\ & + \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}(\hbar^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}\hbar^r_{pp}\hbar^r_{qq} - 2\bar{\pi}(M^n). \end{aligned} \tag{37}$$

According to Equation (23), we obtain the following

$$\begin{aligned}\sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}(\hbar^r_{pq})^2-\sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\hbar^r_{pp}\hbar^r_{qq}\\ =\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\bar{\kappa}_{pq}-\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\kappa_{pq}\end{aligned}\tag{38}$$

By substituting the values from Equation (38) into Equation (37), we uncover the following

$$\begin{aligned}\frac{1}{2}n^2\|H\|^2 =&2\pi(M^n)+\frac{1}{2}\sum_{r=n+1}^{m}(2\hbar^r_{AA}-(\hbar^r_{n_1+1n_1+1}+\cdots+\hbar^r_{nn}))^2\\ &+\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(\hbar^r_{pq})^2-\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}\hbar^r_{pp}\hbar^r_{qq}-2\bar{\pi}(M^n)\\ &+\sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(\hbar^r_{aA})^2+\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\bar{\kappa}_{pq}-\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\kappa_{pq}.\end{aligned}\tag{39}$$

For the submanifold $M^n = N_T^{n_1}\times_f N_\perp^{n_2}$, we can define the scalar curvature of M^n based on Equation (21) as follows:

$$\begin{aligned}\pi(M^n)=&\sum_{1\le p<q\le n}\kappa(e_p\wedge e_q)\\ =&\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\kappa(e_i\wedge e_j)+\sum_{1\le i<k\le n_1}\kappa(e_i\wedge e_k)+\sum_{n_1+1\le l<o\le n}\kappa(e_l\wedge e_o)\end{aligned}\tag{40}$$

By utilizing Equations (17), (21) and (26), we can derive the following expression:

$$\pi(M^n)=\frac{n_2\Delta f}{f}+2n_1n_2-2n_2+\pi(N_T^{n_1})+\pi(N_\perp^{n_2})\tag{41}$$

By combining Equation (41) with Equation (5) in Equation (39), we obtain the following

$$\begin{aligned}\frac{1}{2}n^2\|H\|^2 =&\frac{n_2\Delta f}{f}+2n_1n_2-2n_2+\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\bar{\kappa}_{pq}+\bar{\pi}(N_T^{n_1})+\bar{\pi}(N_\perp^{n_2})\\ &+\sum_{r=n+1}^{m}\{\sum_{1\le p<q\le n}(\hbar^r_{pq})^2-\sum_{\substack{1\le p<q\le n\\ p,q\neq A}}\hbar^r_{pp}\hbar^r_{qq}\}\\ &+\sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(\hbar^r_{aA})^2+\sum_{r=n+1}^{m}\sum_{1\le i\neq j\le n_1}(\hbar^r_{ii}\hbar^r_{jj}-(\hbar^r_{ij})^2)\\ &+\sum_{r=n+1}^{m}\sum_{n_1+1\le s\neq t\le n}(\hbar^r_{ss}\hbar^r_{tt}-(\hbar^r_{st})^2)\\ &+\frac{1}{2}\sum_{r=n+1}^{m}(2\hbar^r_{AA}-(\hbar^r_{n_1+1n_1+1}+\cdots+\hbar^r_{nn}))^2\\ &-\{n(n-1)-2(n-1)trace\alpha\}.\end{aligned}\tag{42}$$

When considering $\chi = e_a$, we have two possible scenarios: either χ is tangent to the submanifold $N_T^{n_1}$ or it is tangent to the fiber $N_\perp^{n_2}$.

Case 1: Assuming that e_a is tangent to $N_T^{n_1}$, let us consider a unit tangent vector from the set $\{e_1, \ldots, e_{n_1}\}$ and suppose $\chi = e_a = e_1$. By using Equations (42) and (24), we can determine the following

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta f}{f} - 2n_1n_2 + 2n_2 - \frac{1}{2}\sum_{r=n+1}^{m}(2\hbar_{11}^r - (\hbar_{n_1+1n_1+1}^r + \ldots \hbar_{nn}^r))^2 \\ &- \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(\hbar_{pq}^r)^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(\hbar_{ij}^r)^2 - \sum_{1\leq i<j\leq n_1}\hbar_{ii}^r\hbar_{jj}^r] \\ &+ \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(\hbar_{st}^r)^2 + \sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(\hbar_{ij}^r)^2 - \sum_{n_1+1\leq s<t\leq n}\hbar_{ss}^r\hbar_{tt}^r] \\ &+ \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}\hbar_{pp}^r\hbar_{qq}^r + n(n-1) - 2(n-1)trace\alpha \\ &- \sum_{2\leq p<q\leq n}\bar{\kappa}_{pq} - \bar{\pi}(N_T^{n_1}) - \bar{\pi}(N_\perp^{n_2}). \end{aligned} \tag{43}$$

By combining Equations (5), (21) and (22), we obtain the following expression

$$\sum_{2\leq p<q\leq n}\bar{\kappa}_{pq} = \frac{1}{2}(n-1)(n-2) - (n-2)\sum_{i=2}^{n}\alpha(e_i,e_i), \tag{44}$$

$$\bar{\pi}(N_T^{n_1}) = \frac{1}{2}n_1(n_1-1) - (n_1-1)\sum_{i=1}^{n_1}\alpha(e_i,e_i), \tag{45}$$

$$\bar{\pi}(N_\perp^{n_1}) = \frac{1}{2}n_2(n_2-1) - (n_2-1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i). \tag{46}$$

Substituting into Equation (43), we obtain

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta f}{f} + n - n_1n_2 + 2n_2 - 1 \\ &- \frac{1}{2}\sum_{r=n+1}^{m}(2\hbar_{11}^r - (\hbar_{n_1+1n_1+1}^r + \cdots + \hbar_{nn}^r))^2 \\ &- \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(\hbar_{pq}^r)^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(\hbar_{ij}^r)^2 + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(\hbar_{st}^r)^2] \\ &- \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}\hbar_{ii}^r\hbar_{jj}^r + \sum_{n_1+1\leq s<t\leq n}\hbar_{ss}^r\hbar_{tt}^r] \\ &+ \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}\hbar_{pp}^r\hbar_{qq}^r - (1+n_1)\sum_{i=1}^{n_2}\alpha(e_i,e_i) - (1+n_2)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i) \\ &- (n-2)\alpha(e_1,e_1). \end{aligned} \tag{47}$$

Moreover, we can express the seventh and eighth terms on the right-hand side of Equation (47) as follows

$$\begin{aligned} \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(\hbar_{ij}^r)^2 + \sum_{n_1+1\leq s<t\leq n}(\hbar_{st}^r)^2] - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(\hbar_{pq}^r)^2 \\ = -\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar_{pq}^r)^2. \end{aligned}$$

Similarly, we obtain

$$\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}\hbar_{ii}^r\hbar_{jj}^r+\sum_{n_1+1\leq s\neq t\leq n}\hbar_{ss}^r\hbar_{tt}^r-\sum_{2\leq p<q\leq n}\hbar_{pp}^r\hbar_{qq}^r]$$
$$=\sum_{r=n+1}^{m}[\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n}\hbar_{pp}^r\hbar_{qq}^r-\sum_{j=2}^{n_1}\hbar_{11}^r\hbar_{jj}^r].$$

By substituting the above two values into Equation (47), we obtain

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}+(n-n_1n_2+2n_2-1)\\&-\frac{1}{2}\sum_{r=n+1}^{m}(2\hbar_{11}^r-(\hbar_{n_1+1n_1+1}^r+\dots\hbar_{nn}^r))^2\\&-\sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar_{pq}^r)^2+\sum_{b=2}^{n_1}\hbar_{11}^r\hbar_{bb}^r-\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n}\hbar_{pp}^r\hbar_{qq}^r].\\&-(1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i)-(1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i)-(n-2)\alpha(e_1,e_1)\end{aligned}\tag{48}$$

Since $M^n=N_T^{n_1}\times_f N_\perp^{n_2}$ is $N_T^{n_1}$-minimal then we can observe the following

$$\sum_{r=n+1}^{m}\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n}\hbar_{pp}^r\hbar_{qq}^r=-\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}\hbar_{11}^r\hbar_{qq}^r\tag{49}$$

and

$$\sum_{r=n+1}^{m}\sum_{b=2}^{n_1}\hbar_{11}^r\hbar_{bb}^r=-\sum_{r=n+1}^{m}(\hbar_{11}^r)^2.\tag{50}$$

Furthermore, we can reach the following conclusion

$$\begin{aligned}\frac{1}{2}\sum_{r=n+1}^{m}(2\hbar_{11}^r-(\hbar_{n_1+1n_1+1}^r+\cdots+\hbar_{nn}^r))^2+\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}\hbar_{11}^r\hbar_{qq}^r\\=2\sum_{r=n+1}^{m}(\hbar_{11}^r)^2+\frac{1}{2}n^2\|H\|^2.\end{aligned}\tag{51}$$

By substituting Equations (49) and (50) into Equation (48), and evaluating Equation (51), we obtain the final expression:

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}+(n+n_1n_2+2n_2-1)\\&-(1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i)-(1+n_2)\sum_{i=1}^{n_2}\alpha(e_i,e_i)-(n-2)\alpha(e_1,e_1)\\&-\frac{1}{4}\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}(\hbar_{qq}^r)^2-\sum_{r=n+1}^{m}\{(\hbar_{11}^r)^2-\sum_{q=n_1+1}^{n}\hbar_{11}^r\hbar_{qq}^r\\&+\frac{1}{4}(\hbar_{n_1+1n_1+1}^r+\cdots+\hbar_{nn}^r)^2\}.\end{aligned}$$

Moreover, by utilizing the fact that $\sum_{r=n+1}^{m}(\hbar^{r}_{n_1+1n_1+1}+\cdots+\hbar^{r}_{nn})=n^2\|H\|^2$, we obtain

$$\begin{aligned}Ric(\chi)\leq\ &\frac{1}{4}n^2\|H\|^2-\frac{n_2\Delta f}{f}+(n+n_1n_2+2n_2-1)\\&-(1+n_1)\sum_{i=1}^{n_2}\alpha(e_i,e_i)-(1+n_2)\sum_{i=1}^{n_2}\alpha(e_i,e_i)-(n-2)\alpha(e_1,e_1)\\&-\frac{1}{4}\sum_{r=n+1}^{m}(2h^{r}_{11}-\sum_{q=n_1+1}^{n}\hbar^{r}_{qq})^2.\end{aligned}$$

From the inequality mentioned earlier, we can deduce the inequality stated in Equation (34).
Case 2: Assuming that e_a is tangent to $N^{n_2}_{\perp}$, we choose the unit vector from the set $\{e_{n_1+1},\ldots,e_n\}$ and suppose it is e_n, i.e., $\chi=e_n$. Then, using Equations (5), (21) and (22), we obtain the following expressions

$$\sum_{1\leq p<q\leq n-1}\bar{\kappa}_{pq}=\frac{1}{2}((n-1)(n-2))-(n-2)\sum_{i=1}^{n-1}\alpha(e_i,e_i). \tag{52}$$

$$\bar{\pi}(N^{n_1}_{T})=\frac{1}{2}(n_1(n_1-1))-(n_1-1)\sum_{i=1}^{n_1}\alpha(e_i,e_i). \tag{53}$$

$$\bar{\pi}(N^{n_2}_{\perp})=\frac{1}{2}(n_2(n_2-1))-(n_2-1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i). \tag{54}$$

Now, following a similar approach as in case 1, and utilizing Equation (52), we obtain

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}-\frac{1}{2}\sum_{r=n+1}^{m}((\hbar^{r}_{n_1+1n_1+1}+\cdots\hbar^{r}_{nn})-2\hbar^{r}_{nn})^2\\&-\sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(\hbar^{r}_{pq})^2+\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(\hbar^{r}_{ij})^2-\sum_{1\leq i<j\leq n_1}\hbar^{r}_{ii}\hbar^{r}_{jj}]\\&+\sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(\hbar^{r}_{st})^2+\sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(\hbar^{r}_{ij})^2-\sum_{n_1+1\leq s<t\leq n}\hbar^{r}_{ss}\hbar^{r}_{tt}]\\&+\sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n-1}\hbar^{r}_{pp}\hbar^{r}_{qq}+n-n_1n_2+2n_2-1\\&-(1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i)-(1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i)-(n-2)\alpha(e_n,e_n).\end{aligned} \tag{55}$$

By following similar steps as in case i, the above inequality can be expressed as

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}+n-n_1n_2+2n_2-1\\&-(1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i)-(1+n_1)\sum_{i=n_1+1}^{n_2}\alpha(e_i,e_i)-(n-2)\alpha(e_n,e_n)\\&-\frac{1}{2}\sum_{r=n+1}^{m}((\hbar^{r}_{n_1+1n_1+1}+\cdots\hbar^{r}_{nn})-2\hbar^{r}_{nn})^2\\&-\sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar^{r}_{pq})^2+\sum_{b=n_1+1}^{n-1}\hbar^{r}_{nn}\hbar^{r}_{bb}-\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}\hbar^{r}_{pp}\hbar^{r}_{qq}]\end{aligned} \tag{56}$$

Observing Lemma 3, it can be seen that

$$\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}\hbar^r_{pp}\hbar^r_{qq}=0. \tag{57}$$

By utilizing this result in Equation (56), we obtain

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}+n-n_1n_2+2n_2-1\\&-\frac{1}{2}\sum_{r=n+1}^{m}((\hbar^r_{n_1+1n_1+1}+\ldots\hbar^r_{nn})-2\hbar^r_{nn})^2\\&-\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar^r_{pq})^2-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}\hbar^r_{nn}\hbar^r_{bb}\\&-(1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i)-(1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i)-(n-2)\alpha(e_n,e_n).\end{aligned}\tag{58}$$

The final term in the aforementioned inequality can be expressed as

$$-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}\hbar^r_{nn}\hbar^r_{bb}=-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}\hbar^r_{nn}\hbar^r_{bb}+\sum_{r=n+1}^{m}(\hbar^r_{nn})^2$$

Additionally, we can expand the fifth term on the right-hand side of Equation (58) as follows

$$\begin{aligned}-\frac{1}{2}\sum_{r=n+1}^{m}((\hbar^r_{n_1+1n_1+1}+\cdots+\hbar^r_{nn})-2\hbar^r_{nn})^2=&\\-\frac{1}{2}\sum_{r=n+1}^{m}(\hbar^r_{n_1+1n_1+1}+\cdots+\hbar^r_{nn})^2&\\-2\sum_{r=n+1}^{m}(\hbar^r_{nn})^2+\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}\hbar^r_{nn}\hbar^r_{jj}.&\end{aligned}$$

By substituting the last two values into Equation (58), we have

$$\begin{aligned}Ric(\chi)\leq&\frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta f}{f}+n-n_1n_2+2n_2-1\\&-\frac{1}{2}\sum_{r=n+1}^{m}(\hbar^r_{n_1+1n_1+1}+\cdots\hbar^r_{nn})^2-2\sum_{r=n+1}^{m}(\hbar^r_{nn})^2\\&+2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}\hbar^r_{nn}\hbar^r_{jj}-\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar^r_{pq})^2\\&-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}\hbar^r_{nn}\hbar^r_{bb}+\sum_{r=n+1}^{m}(\hbar^r_{nn})^2\\&-(1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i)-(1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i)-(n-2)\alpha(e_n,e_n),\end{aligned}$$

or equivalently

$$\begin{aligned}
Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta f}{f} + n - n_1 n_2 + 2n_2 - 1 \\
& - (1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i) - (1+n_1)\sum_{i=n_1+1}^{n_2}\alpha(e_i,e_i) - (n-2)\alpha(e_n,e_n) \\
& - \frac{1}{2}\sum_{r=n+1}^{m}(\hbar^r_{n_1+1n_1+1} + \cdots \hbar^r_{nn})^2 - \sum_{r=n+1}^{m}(\hbar^r_{nn})^2 \\
& + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}\hbar^r_{nn}\hbar^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(\hbar^r_{pq})^2
\end{aligned}$$

Applying similar techniques as in the proof of case 1, we reach the following expression

$$\begin{aligned}
Ric(\chi) \leq & \frac{1}{4}n^2\|H\|^2 - \frac{n_2\Delta f}{f} + n - n_1 n_2 + 2n_2 - 1 \\
& - (1+n_2)\sum_{i=1}^{n_1}\alpha(e_i,e_i) - (1+n_1)\sum_{i=n_1+1}^{n}\alpha(e_i,e_i) - (n-2)\alpha(e_n,e_n) \\
& - \frac{1}{4}\sum_{r=n+1}^{m}(\hbar^r_{nn} - (\hbar^r_{n_1+1n_1+1} + \cdots + \hbar^r_{nn}))^2,
\end{aligned}$$

which gives the inequality (36).

Let us now investigate the equality cases of the inequality (34). Firstly, we redefine the concept of the relative null space, N_x, of the submanifold, M^n, within the odd dimensional sphere, S^{2n+1}, at any given point, $x \in M^n$. The notion of the relative null space was originally introduced by B. Y. Chen [9] and can be defined as follows

$$N_x = \{X \in T_xM^n : h(X,Y) = 0, \forall Y \in T_xM^n\}.$$

For a unit vector field, e_A, tangent to M^n at point x, the equality in (34) holds true if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}\hbar^r_{pq} = 0 \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n}\hbar^r_{bA} = 0 \ (iii)\ 2\hbar^r_{AA} = \sum_{q=n_1+1}^{n}\hbar^r_{qq}.$$

When $r \in \{n+1,\ldots,m\}$ and condition (i) is satisfied, it indicates that M^n is a mixed T-G contact CR-W-P submanifold. By combining statements (ii) and (iii) with the fact that M^n is a contact CR-W-P submanifold, we can conclude that the unit vector field $\chi = e_A$ belongs to the relative null space, N_x. The converse of this statement is straightforward, thus proving statement (2).

For a contact CR-W-P submanifold, the equality satisfies in (34) for all unit tangent vectors belonging to N_T at x if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}\hbar^r_{pq} = 0 \ \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n_1}\hbar^r_{bA} = 0 \ \ (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n}\hbar^r_{qq}, \tag{59}$$

where $p \in \{1,\ldots,n_1\}$ and $r \in \{n+1,\ldots,m\}$. Considering that M^n is a contact CR-W-P submanifold, the third condition implies that $\hbar^r_{pp} = 0$ for $p \in \{1,\ldots,n_1\}$. By incorporating this information into condition (ii), we can assert that M^n is a D-T-G contact CR-W-P submanifold in $S^{2n+1}(1)$, and its mixed T-G nature arises from condition (i). This establishes statement (a) in (3).

The equality sign in (34) holds identically for all unit tangent vector fields tangent to $N_\perp$ at x in the case of a contact CR-W-P submanifold if and only if

$$(i) \sum_{p=1}^{n_1} \sum_{q=n_1+1}^{n} \hbar^r_{pq} = 0 \ (ii) \sum_{b=1}^{n} \sum_{\substack{A=n_1+1 \\ b\neq A}}^{n} \hbar^r_{bA} = 0 \ (iii) \ 2h^r_{KK} = \sum_{q=n_1+1}^{n} \hbar^r_{qq}, \tag{60}$$

such that $K \in \{n_1+1,\ldots,n\}$ and $r \in \{n+1,\ldots,m\}$. Two cases arise from condition (iii), namely

$$\hbar^r_{KK} = 0, \ \ \forall K \in \{n_1+1,\ldots,n\} \ \text{ and } \ r \in \{n+1,\ldots,m\} \ \text{ or } \ dim\ N_\perp = 2.$$

If the first case of Equation (60) is satisfied, it can be easily concluded, based on condition (ii), that M^n is a $D^\perp$-T-G contact CR-W-P submanifold in $S^{2n+1}(1)$. This corresponds to the first case of part (b) in statement (3).

Alternatively, if M^n is not a $D^\perp$-T-G contact CR-warped product submanifold and $\dim(N^\perp) = 2$, condition (ii) of Equation (60) implies that M^n is a $D^\perp$-T-U contact CR-W-P submanifold in $S^{2n+1}(1)$. This corresponds to the second case in part (b) of statement (3). Thus, part (b) of statement (3) is confirmed.

To establish (c) using parts (a) and (b) of (3), we combine Equations (59) and (60). In the first case of this part, let us assume that $\dim(N_\perp) \neq 2$. Based on parts (a) and (b) of (3), we can conclude that M^n is both a D-T-G and $D^\perp$-T-G submanifold in $\bar{S}^{2n+1}(1)$. Consequently, M^n is a T-G submanifold in $S^{2n+1}(1)$.

In the case where the first case does not satisfy, we can conclude from parts (a) and (b) that M^n is a mixed T-G and D-T-G submanifold of $S^{2n+1}(1)$ with $\dim(N_\perp) = 2$. Based on condition (b), it is evident that M^n is a $D^\perp$-T-U contact CR-W-P submanifold, and from part (a), it is a D-T-G. This satisfies part (c) of the theorem. Thus, the theorem is proven.

5. Conclusions

This paper has delved into the study of contact CR-warped product submanifolds within the framework of Sasakian space forms endowed with a semi-symmetric metric connection. Through our comprehensive investigation, we have uncovered several important results and made significant contributions to the understanding of these submanifolds.

One of the main achievements of this paper is the establishment of various key findings regarding contact CR-warped product submanifolds. We have explored their geometric properties, such as the characterization of their induced metric, and the determination of the necessary and sufficient conditions for a submanifold to be contact CR-warped. Additionally, we have investigated the behavior of the mean curvature vector and the warping function on these submanifolds.

Moreover, we have derived an inequality that relates the Ricci curvature to the mean curvature vector and the warping function. This inequality provides a valuable geometric constraint on contact CR-warped product submanifolds in Sasakian space forms with a semi-symmetric metric connection. It deepens our understanding of the interplay between the intrinsic curvature of the submanifold and its extrinsic mean curvature vector.

The findings presented in this paper have implications for various areas of differential geometry and mathematical physics. They contribute to the broader field of Riemannian geometry, particularly in the study of Sasakian space forms and submanifold theory. Furthermore, these results can potentially be applied in other mathematical and physical contexts where contact CR-warped product submanifolds arise.

In conclusion, this paper expands our knowledge of contact CR-warped product submanifolds in Sasakian space forms equipped with a semi-symmetric metric connection. The insights gained from this research, along with the derived inequality, provide a solid foundation for further investigations and applications in related fields. In future research, we will explore how our results can be applied with soliton theory, submanifold theory, and related fields presented in the papers [1,6,7,14,24–32] to obtain new results.

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Abbreviations

The following abbreviations are used in this manuscript:

S-S-M	Semi-symmetric metric
S-M	Sasakian manifold
S-S-F	Sasakian space form
T-G	Totally geodesic
T-U	Totally umbilical

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