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Jacobi Stability for T-System

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Abstract: In this paper will be considered a three-dimensional autonomous quadratic polynomial system of first-order differential equations with three real parameters, the so-called T-system. This system is symmetric relative to the *Oz*-axis and represents a special type of the generalized Lorenz system. The approach of this work will consist of the study of the nonlinear dynamics of this system through the Kosambi–Cartan–Chern (KCC) geometric theory. More exactly, we will focus on the associated system of second-order differential equations (SODE) from the point of view of Jacobi stability by determining the five invariants of the KCC theory. These invariants determine the internal geometrical characteristics of the system, and particularly, the deviation curvature tensor is decisive for Jacobi stability. Furthermore, we will look for necessary and sufficient conditions that the system parameters must satisfy in order to have Jacobi stability for every equilibrium point.

Keywords: T-system; the deviation curvature tensor; Jacobi stability; KCC geometric theory

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1. Introduction

In the present work, the Jacobi stability of T-system through the use of the Kosambi-Cartan–Chern (KCC) geometric theory will be investigated. In order to reach the Jacobi stability requirements, we will compute the five invariants of KCC theory that express the intrinsic geometric properties of the T-system, especially the tensor of deviation curvature, which characterizes the Jacobi stability of the T-system at each equilibrium point.

The T-system was introduced by G. Tigan in [1–3], and although it looks like the Lorenz system, this system is not topologically equivalent to the Lorenz system, nor to other Lorenz-type system. For this reason alone, this system and its nonlinear dynamics are of interest, and it deserves to be addressed by any methods and by any tools. From the linear stability (classical or Lyapunov) point of view, the nonlinear dynamics and the local and global behaviour of this new chaotic system and its very interesting properties have been recently studied, as follows. Local and global stability analysis was performed in [4], the existence of heteroclinic orbits and the horseshoe chaos by the heteroclinic Shilnikov method was investigated in [3,5,6], and bifurcations with delayed feedback were investigated in [7]. Moreover, the coexistence of chaotic butterfly attractors and unstable limit cycles for T-system was deeply studied in [8], and pitchfork and Hopf bifurcations of this system were approached in [9]. Also, many properties of the global dynamics of this system were obtained in [8], by using techniques with invariant algebraic surfaces and first integrals [10]. Moreover, a fractional-order version of a T-system was introduced and studied in [11].

Although the T-system has a simple analytical form, its dynamics is very complex, even having a chaotic behavior, including the coexistence of isolated unstable periodic orbits with a chaotic attractor [8]. Chaotic behaviors, with the presence of a strange attractor, were first discovered for the following famous dynamical systems: Lorentz [12], Chua [13], Lü and Chen [14], Chen and Ueta [15]. Another chaotic modified Lorenz system close to the T-system with a completely different dynamics was studied in [16,17]. All these systems have an analytical form very close to the T-system, but with a different complex dynamics. A geometric approach to the Jacobi stability of these systems was recently studied in [18–21].

In the present work, by using the geometric tools of Kosambi–Cartan–Chern theory, the Jacobi stability of the T-system and its geometric associated objects are investigated for the first time.

The study of this type of chaotic dynamical system is of essential significance in practical applications, because approaches to chaos have not only focused on finding new chaotic systems or investigating chaos control and chaos synchronization, but also on analyzing local and global behavior of these dynamical systems, behavior that is crucial to understand what chaos means. Generally speaking, in a dynamical system mathematical approach, the catastrophe theory is closely related to the bifurcations theory, because any changes in a chaotic regime or any transition from (or to) a chaotic regime is made by some bifurcation of the dynamical system. So, the study of the nonlinear dynamics and the Jacobi stability of the T-system through the Kosambi–Cartan–Chern (KCC) geometric theory represents a step forward in the investigation of the complex dynamics of this system, with possible applications in catastrophe theory and other research fields. Moreover, in the whole region with Jacobi stability, we have no bifurcations, and therefore no catastrophic events [22,23].

By the transformation of the system with three first-order differential equations into a system with two second-order differential equations, we will study the local nonlinear dynamics of the T-system from the Jacobi stability point of view, by using the KCC geometric theory. The own geometric characteristics of the T-system will be determined by finding the corresponding geometric objects, i.e., the tensor of zero-connection curvature Z^i_j, the nonlinear connection N^i_j, the Berwald connection G^i_{jl}, and the five KCC geometric invariants: the external force ε^i—the first invariant; the tensor of deviation curvature P^i_j—the second invariant; the tensor of torsion P^i_{jk}—the third invariant; the tensor of Riemann–Christoffel curvature P^i_{jkl}—the fourth invariant; the tensor of Douglas D^i_{jkl}—the fifth invariant. For the purpose of obtaining the necessary and sufficient conditions for the Jacobi stability near every equilibrium point, the tensor of deviation curvature will be found at every equilibrium point. In addition, a comparison of classical (or linear) Lyapunov stability with Jacobi stability will be made, including diagrams related to system parameter values.

This approach of Jacobi stability appears as a prolongation of the geometric stability approach of the geodesic flow, from a Riemann or Finsler manifold to a differentiable manifold without any metric [24–29]. More precisely, the concept of Jacobi stability plays the role of proof of the resilience of a dynamical system defined by a system of second-order differential equations (semi-spray or SODE), where this resilience reflects the adaptability and the preservation of the system's basic behavior to changes in internal parameters and to influences from outside circumstances. Through the Kosambi–Cartan–Chern theory, i.e., from the perspective of the concept of Jacobi stability, the dynamics of various dynamical systems has recently been approached in [18–21,25,26,30–37]. Therefore, the local behavior of the system is revealed through the use of geometric objects corresponding to the system of second-order differential equations (SODE), because this SODE is derived from the system of first-order differential equations [38–40].

The principal target of the KCC theory is to investigate the deviation of the neighboring trajectories, and this will provide us the opportunity to measure the allowed perturbation close to the equilibrium points of the SODE. Initially, this approach consisted of the investigation of the variation equations (or Jacobi field equations) associated with the differential manifold structure. In particular, P. L. Antonelli, R. Ingarden, and M. Matsumoto started the investigation of the geometric stability of the geodesics given through a Riemannian or a Finslerian metric by deviating the geodesics and by the use of the KCC covariant derivative associated to the differential system in variations [24–26]. So, the second Kosambi–Cartan–Chern (KCC) invariant resulted from the covariant form of the differential system in variations. This geometric invariant is called the deviation curvature tensor, and is essential to the approach of Jacobi stability, both for geodesics and for the trajectories associated to a system of second-order differential equations (SODE). In differential geometry theory, a second-order differential equations system (SODE) is also called semi-spray. Once we have given a semi-spray, we can associate a nonlinear connection on the differential manifold, and conversely, by giving a nonlinear connection, we can associate a semi-spray. In conclusion, starting with any SODE (or semi-spray), we can construct a geometry on the base manifold by using the geometric associated objects [27,41–43]. Moreover, all these geometrical objects are invariant with respect to any local coordinates change, i.e., they are tensors that can fulfilled the symmetry conditions or skew-symmetry conditions, or not, according to the type of the second-order differential equations system.

It is known that the origins of the KCC theory are in the works of D. D. Kosambi [38], E. Cartan [39], and S. S. Chern [40]. So, the acronym KCC (Kosambi–Cartan–Chern) is very clear, and this consistent geometric theory can be applied with success in a lot of research fields, e.g., biology, chemistry, physics and engineering [18–21,30,44]. Furthermore, recent and valuable approaches to KCC theory in cosmology and gravity can be found in [45,46]. More exactly, in [32], C.G. Boehmer, T. Harko, and S.V. Sabau performed a comprehensive analysis of Jacobi stability and its relationship with the linear stability of dynamical systems representing phenomena from astrophysics and gravity. For the present study, the novelty is the use of the geometric instruments of KCC theory in order to obtain new information about the local dynamics of T-system. More precisely, in this work, the Jacobi stability around to an equilibrium point was obtained for some parameters values. Bearing in mind that Jacobi stability near an equilibrium point involves this equilibrium point being a stable or unstable focus, it results in Hopf bifurcations, and even isolated periodic orbits, being able to occur. Moreover, if the conditions for Jacobi stability are fulfilled at an equilibrium point, then any chaotic behavior for the T-system near this equilibrium point is not possible.

Following an introductory section, Section 2 deals with a short description of the T-system, together with the basic known facts about the local and global stability of the T-system. In Section 3, the T-system will be reformulated as an equivalent system of second-order differential equations (SODE), and the five geometrical invariants of the system will be determined. The main results related to the Jacobi stability of the T-system will be presented in Section 4. More precisely, the necessary and sufficient conditions to meet the Jacobi stability of the system near every equilibrium point will be obtained. Moreover, in order to understand the connection between Jacobi stability and classical (linear or Lyapunov) stability, we will present a diagram which illustrate the dependence of the Jacobi stability for the T-system relative to the parameters of the system. Moreover, at the ending of Section 4, the deviation equations around every equilibrium point and the deviation vector curvature will be presented. In the last section, conclusions and possible next research will be pointed out.

At the end, in Appendix A, an enlightening description of the basic definitions and principal instruments of the Kosambi–Cartan–Chern geometric theory will be presented, with an accent on the five invariants of this theory and the definition of Jacobi stability. The sum over repeated cross-indexes will be used.

2. Preliminary Results

The T-system is a three-dimensional autonomous system of first-order differential equations with three real parameters a, b, c, defined by

$$\begin{cases} \dot{x} &=& a(y-x), \\ \dot{y} &=& (c-a)x - axz, \\ \dot{z} &=& -bz + xy. \end{cases} \tag{1}$$

If $a = 0$, the system becomes linear, and we have no interest from the dynamical behaviour perspective. Also, for $b = 0$, this system has an infinite number of non-hyperbolic equilibrium points, i.e., the entire Oz-axis [1–3,9]. Therefore, for the next considerations, we will assume that $a \neq 0$ and $b \neq 0$.

If $\dfrac{b(c-a)}{a} > 0$, then the T-system (1) has three equilibrium points: $O(0,0,0)$, $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$ and $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, -\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$.

Otherwise, for $\dfrac{b(c-a)}{a} \leq 0$, the T-system has only one equilibrium point, the trivial equilibrium, the origin $O(0,0,0)$.

Due to the fact that it is not certain that we can analytically solve this first-order differential system, it remains only to find information about the dynamics of this system by means of the qualitative theory of dynamical systems.

Let us remark that the T-system is invariant relative to the transformation $(x,y,z) \mapsto (-x-y,z)$, i.e., the integral curves of the system are symmetrical with respect to Oz-axis. So, if the system has the integral curve $\gamma_1(t) = (x(t), y(t), z(t))$, then it has too the integral curve $\gamma_2(t) = (-x(t), -y(t), z(t))$ and the two integral curves $\gamma_1(t)$ and $\gamma_2(t)$ are symmetric with respect to Oz-axis. Moreover, the equilibrium points E_1 and E_2 are symmetric relative to Oz-axis and the dynamics around the E_1 and E_2 have the same characteristics [9].

The Jacobi matrix at the point (x,y,z) is

$$A = \begin{pmatrix} -a & a & 0 \\ c-a-az & 0 & -ax \\ y & x & -b \end{pmatrix}.$$

For the trivial equilibrium $O(0,0,0)$ we obtain

$$A = \begin{pmatrix} -a & a & 0 \\ c-a & 0 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

with eigenvalues $\lambda_1 = -b$, $\lambda_2 + \lambda_3 = -a$, $\lambda_2\lambda_3 = -a(c-a)$.

Consequently, if $a > 0$, $b > 0$ and $c \leq a$, then $O(0,0,0)$ is an attractor or an asymptotically stable equilibrium point. Otherwise, if $b < 0$ or $a < 0$ or ($a > 0$ and $c > a$), then the origin $O(0,0,0)$ is a saddle point, i.e., an unstable equilibrium point. For $a = c$, the origin is a non-hyperbolic equilibrium point because there is an eigenvalue equal to zero.

If $\dfrac{b(c-a)}{a} > 0$, then the Jacobi matrix at $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$ is:

$$A = \begin{pmatrix} -a & a & 0 \\ c-a-az_o & 0 & -ax_0 \\ y_0 & x_0 & -b \end{pmatrix},$$

where $x_0 = y_0 = \sqrt{\frac{b(c-a)}{a}}$, $z_0 = \frac{c-a}{a}$. The characteristic polynomial at E_1 is

$$P(\lambda) = \lambda^3 + (a+b)\lambda^2 + bc\lambda + 2ab(c-a).$$

Of course, for the equilibrium point $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, -\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$, similar results appear.

If we remember the Routh–Hurwitz criterion, then we will be able to state that the characteristic polynomial $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ has all roots in the open left half plane (which means $\lambda_i < 0$ or Re $\lambda_i < 0$, for all i) if and only if $a_2 > 0$, $a_0 > 0$ and $a_2a_1 > a_0$. Therefore, the equilibrium points E_1 and E_2 are asymptotically stable if and only if $a + b > 0$, $ab(c-a) > 0$, and $b(2a^2 + bc - ac) > 0$. Because E_1 and E_2 exist if and only if $ab(c-a) > 0$, it results in the equilibrium points E_1 and E_2 being asymptotically stable if and only if $a + b > 0$ and $b(2a^2 + bc - ac) > 0$.

Therefore, we can conclude that it is very hard to determine the behavior of the T-system around the equilibrium points, due to the fact that there are a lot of parameters that appear in calculations. For more details and comprehensive studies, see [1–3,5,7–9].

Next, we collected the results obtained for local classical stability in the Table 1.

Table 1. The equilibrium points for T-system.

Case	Conditions	Equilibrium Point Type
1	$\frac{b(c-a)}{a} \leq 0$	O is the only one equilibrium point O is asymptotically stable iff $a > 0, b > 0$ and $c \leq a$ or O is unstable iff $b < 0$ or $a < 0$ or $(a > 0$ and $c > a)$.
2	$\frac{b(c-a)}{a} > 0$	O, E_1, E_2 are three equilibrium points, O is asymptotically stable iff $a > 0, b > 0$ and $c < a$ or O is unstable iff $b < 0$ or $a < 0$ or $(a > 0$ and $c > a)$ and $E_{1,2}$ is asymptotically stable iff $a + b > 0$ and $b(2a^2 + bc - ac) > 0$, otherwise $E_{1,2}$ is unstable.

Next, we are interested in investigating the Jacobi stability of T-system by the use of the geometrical methods of the Kosambi–Cartan–Chern (KCC) theory. So, to clarify the behavior of the T-system, we will concentrate on the investigation of Jacobi stability, and we will present the characteristics of the associated geometric objects and their relationships with the local dynamics of the T-system.

3. SODE Formulation of the T-System

Let us consider the T-system (1) with three real parameters, $a, b, c, a \neq 0$:

$$\begin{cases} \dot{x} &= a(y-x), \\ \dot{y} &= (c-a)x - axz, \\ \dot{z} &= -bz + xy. \end{cases}$$

By substituting

$$y = \frac{1}{a}\dot{x} + x$$

from the first equation to the second equation, we obtain

$$\ddot{x} + a\dot{x} + a^2 xz - a(c-a)x = 0\,.$$

Further, by replacing $y = \frac{1}{a}\dot{x} + x$ in the third equation and taking the derivative with respect to time t, we have

$$\ddot{z} + b\dot{z} - \frac{1}{a}(\dot{x})^2 - x\dot{x} - (c-a)x^2 + ax^2 z = 0\,.$$

Using the repeated crossed indices rule from differential geometry, the next notations for the variables will be adopted:

$$x = x^1, \dot{x} = y^1, z = x^2, \dot{z} = y^2.$$

Consequently, we can write the previous two second-order differential equations as the following second-order system of differential equations (SODE):

$$\begin{cases} \ddot{x}^1 + ay^1 + a^2 x^1 x^2 - a(c-a)x^1 &= 0, \\ \ddot{x}^2 + by^2 - \frac{1}{a}(y^1)^2 - x^1 y^1 - (c-a)(x^1)^2 + a(x^1)^2 x^2 &= 0, \end{cases} \tag{2}$$

where $y = \frac{1}{a}y^1 + x^1$. The last system can be written as

$$\begin{cases} \frac{d^2x^1}{dt^2} + ay^1 + a^2x^1x^2 - a(c-a)x^1 & = & 0, \\ \frac{d^2x^2}{dt^2} + by^2 - \frac{1}{a}(y^1)^2 - x^1y^1 - (c-a)(x^1)^2 + a(x^1)^2x^2 & = & 0, \end{cases} \tag{3}$$

where $\frac{dx^i}{dt} = y^i$, $i = 1, 2$.

By using the formalism of Kosambi–Cartan–Chern (KCC) theory (see Appendix A), the above system of second-order differential Equation (3) can be viewed as a SODE (or semi-spray), i.e.,

$$\begin{cases} \frac{d^2x^1}{dt^2} + 2G^1(x^1, x^2, y^1, y^2) & = & 0, \\ \frac{d^2x^2}{dt^2} + 2G^2(x^1, x^2, y^1, y^2) & = & 0, \end{cases} \tag{4}$$

where $\frac{dx^i}{dt} = y^i$, $i = 1, 2$ and

$$\begin{aligned} G^1(x^i, y^i) & = \tfrac{1}{2}\left[ay^1 + a^2x^1x^2 - a(c-a)x^1\right], \\ G^2(x^i, y^i) & = \tfrac{1}{2}\left[by^2 - \tfrac{1}{a}(y^1)^2 - x^1y^1 - (c-a)(x^1)^2 + a(x^1)^2x^2\right]. \end{aligned}$$

The coefficients of the zero-connection curvature tensor $Z^i_j = 2\frac{\partial G^i}{\partial x^j}$ are: $Z^1_1 = a^2x^2 - a(c-a)$, $Z^1_2 = a^2x^1$, $Z^2_1 = -y^1 + 2(ax^2 - c + a)x^1$, $Z^2_2 = a\left(x^1\right)^2$.

Because $N^1_1 = \frac{\partial G^1}{\partial y^1} = \frac{1}{2}a$, $N^1_2 = \frac{\partial G^1}{\partial y^2} = 0$, $N^2_1 = \frac{\partial G^2}{\partial y^1} = -\frac{1}{a}y^1 - \frac{1}{2}x^1$, $N^2_2 = \frac{\partial G^2}{\partial y^2} = \frac{1}{2}b$, the coefficients N^i_j of the nonlinear connection is given in the next matrix

$$N = \begin{pmatrix} \frac{1}{2}a & 0 \\ -\frac{1}{a}y^1 - \frac{1}{2}x^1 & \frac{1}{2}b \end{pmatrix}.$$

Then the Berwald connection has the coefficients $G^i_{jk} = \frac{\partial N^i_j}{\partial y^k}$ and all these coefficients are equal to zero, with one exception, $G^2_{11} = \frac{\partial N^2_1}{\partial y^1} = -\frac{1}{a}$.

The coefficients of the first invariant of KCC theory are

$$\varepsilon^i = -\left(N^i_j y^j - 2G^i\right),$$

or, more exactly,

$$\begin{aligned} \varepsilon^1 & = \tfrac{1}{2}ay^1 + a^2x^1x^2 - a(c-a)x^1, \\ \varepsilon^2 & = \tfrac{1}{2}by^2 - \tfrac{1}{2}x^1y^1 + \left(ax^2 - c + a\right)\left(x^1\right)^2. \end{aligned} \tag{5}$$

Let us note that there is not any real scalar λ such that $\varepsilon^i = \lambda G^i$ for $i = 1, 2$, or $\frac{\partial G^i}{\partial y^j}y^j = (2-\lambda)G^i$ for $i = 1, 2$, i.e., there is no homogeneity property relative to y^i for the functions G^i.

Since

$$G^i_{jl} = \begin{cases} -\frac{1}{a} & , & \text{if } j = 1,\ l = 1,\ i = 2 \\ 0 & , & \text{in rest} \end{cases}$$

and by using (A10),

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j,$$

we obtain the coefficients of the second invariant of KCC geometric theory, the deviation curvature tensor of the T-system (3):

$$\begin{aligned} P_1^1 &= -a^2x^2 + a(c-a) + \tfrac{1}{4}a^2, \\ P_2^1 &= -a^2x^1, \\ P_1^2 &= \left(1 - \tfrac{b}{2a}\right)y^1 - ax^1x^2 + \frac{4c-5a-b}{4}x^1, \\ P_2^2 &= -a\left(x^1\right)^2 + \tfrac{1}{4}b^2. \end{aligned} \tag{6}$$

Taking into account that the trace and the determinant of the deviation curvature matrix

$$P = \begin{pmatrix} P_1^1 & P_2^1 \\ P_1^2 & P_2^2 \end{pmatrix}$$

are trace $(P) = P_1^1 + P_2^2$ and det $(P) = P_1^1 P_2^2 - P_1^2 P_2^1$, and by following Appendix A, we obtain the basic result:

Theorem 1. *All roots of the characteristic polynomial of P are negative or have negative real parts (which means the Jacobi stability occurs) if and only if*

$$P_1^1 + P_2^2 < 0 \text{ and } P_1^1 P_2^2 - P_1^2 P_2^1 > 0\,.$$

According to Appendix A, $P_{jk}^i = \dfrac{1}{3}\left(\dfrac{\partial P_j^i}{\partial y^k} - \dfrac{\partial P_k^i}{\partial y^j}\right)$, $P_{jkl}^i = \dfrac{\partial P_{jk}^i}{\partial y^l}$, $D_{jkl}^i = \dfrac{\partial G_{jk}^i}{\partial y^l}$, and then, we can express the coefficients of the third invariant, fourth invariant, and fifth invariant of the T-system (3):

Theorem 2. *The eight coefficients of the third invariant of KCC theory, i.e., the torsion tensor P_{jk}^i, are all equal to zero,*

$$P_{jk}^i = 0\,, \forall\, i, j, k. \tag{7}$$

The sixteen coefficients of the fourth invariant of KCC theory, i.e., the Riemann–Christoffel curvature tensor P_{jkl}^i, are all equal to zero,

$$P_{jkl}^i = 0\,, \forall\, i, j, k, l. \tag{8}$$

The sixteen coefficients of the fifth invariant of KCC theory, i.e., the Douglas tensor D_{jkl}^i, are all equal to zero,

$$D_{jkl}^i = 0\,, \forall\, i, j, k, l. \tag{9}$$

4. Jacobi Stability Analysis of the T-System

In this section, for every equilibrium point the first invariant and the second invariant of the first-order differential T-system (1) will be calculated. So, the Jacobi stability conditions of the T-system near every equilibrium point will be obtained.

If $\dfrac{b(c-a)}{a} > 0$, then for equilibrium points $O(0,0,0)$, $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$ and $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, -\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}\right)$ of the initial T-system (1), we obtain the associated equilibrium points $O(0,0,0,0)$, $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ and $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ for the second-order differential system (SODE or semi-spray) (3).

For $O(0,0,0,0)$, we have that the first invariant of the theory has all coefficients null, i.e., $\varepsilon^1 = \varepsilon^2 = 0$. Then, the matrix of the coefficients of the second invariant is as follows:

$$P = \begin{pmatrix} a(c-a) + \frac{1}{4}a^2 & 0 \\ 0 & \frac{1}{4}b^2 \end{pmatrix}.$$

We have the following result:

Theorem 3. *The trivial equilibrium point O is always Jacobi-unstable.*

Proof. Since $\operatorname{tr} P = a(c-a) + \frac{1}{4}a^2 + \frac{1}{4}b^2$ and $\det P = \frac{1}{4}b^2\left(a(c-a) + \frac{1}{4}a^2\right)$, it results in $\operatorname{tr} P > 0$ for the case $\frac{c-a}{a} \geq 0$. If $\frac{c-a}{a} < 0$, then it is impossible to fulfill simultaneously $\operatorname{tr} P < 0$ and $\det P > 0$. Therefore, the conclusion is true by using Theorem 1. $\square$

For $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ the first invariant of the KCC geometric theory ε^i has the components $\varepsilon^1 = 0$, $\varepsilon^2 = 0$, and the coefficients of the second invariant are as follows:

$$\begin{aligned} P_1^1 &= \tfrac{1}{4}a^2, \\ P_2^1 &= -a^2\sqrt{\tfrac{b(c-a)}{a}}, \\ P_1^2 &= -\left(\tfrac{a+b}{4}\right)\sqrt{\tfrac{b(c-a)}{a}}, \\ P_2^2 &= -b(c-a) + \tfrac{1}{4}b^2. \end{aligned}$$

Then, we have:

Theorem 4. *E_1 is Jacobi-stable if and only if $P_1^1 + P_2^2 < 0$ and $P_1^1P_2^2 - P_1^2P_2^1 > 0$, i.e.,*

$$a^2 + b^2 - 4b(c-a) < 0 \textit{ and } a^2b^2 - 4ab(c-a)(2a+b) > 0\,.$$

For $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ the first invariant of the KCC geometric theory ε^i has the components $\varepsilon^1 = 0$, $\varepsilon^2 = 0$, and the components of the second invariant are as follows:

$$\begin{aligned} P_1^1 &= \tfrac{1}{4}a^2, \\ P_2^1 &= a^2\sqrt{\tfrac{b(c-a)}{a}}, \\ P_1^2 &= \left(\tfrac{a+b}{4}\right)\sqrt{\tfrac{b(c-a)}{a}}, \\ P_2^2 &= -b(c-a) + \tfrac{1}{4}b^2. \end{aligned}$$

Then, we have the same result as for E_1:

Theorem 5. *E_2 is Jacobi-stable if and only if $P_1^1 + P_2^2 < 0$ and $P_1^1P_2^2 - P_1^2P_2^1 > 0$, i.e.,*

$$a^2 + b^2 - 4b(c-a) < 0 \textit{ and } a^2b^2 - 4ab(c-a)(2a+b) > 0\,.$$

Let us emphasize that E_2 is Jacobi-stable if and only if E_1 is Jacobi-stable, due to the fact that for both equilibrium points, we have

$$\begin{aligned} \operatorname{tr} P &= P_1^1 + P_2^2 = \tfrac{1}{4}\left[a^2 + b^2 - b(c-a)\right], \\ \det P &= P_1^1P_2^2 - P_1^2P_2^1 = \tfrac{1}{16}\left[a^2b^2 - 4ab(c-a)(2a+b)\right]. \end{aligned}$$

Remark 1. *If $a < 0$, then it result that $b(c-a) < 0$ and then $\operatorname{tr} P > 0$. Therefore, $E_{1,2}$ are Jacobi-unstable whenever $a < 0$, and then it remains to study the Jacobi stability of the T-system only for the case $a > 0$.*

Hence, we have the following result:

Theorem 6. *If they exist, the equilibrium points E_1 and E_2 are Jacobi-stable if and only if the following two conditions are simultaneously fulfilled:*

$$a^2 + b^2 - 4b(c-a) < 0 \textit{ and } ab^2 - 4b(c-a)(2a+b) > 0\,.$$

If $a > 0$, to simplify the approach, next, we will denote by $m = \dfrac{c-a}{a}$, $n = \dfrac{b}{a}$. Then, the equilibrium points $E_{1,2}$ exist if and only if $mn > 0$. In this case, for $E_{1,2}$, we have

$$\operatorname{tr} P = \frac{a^2}{4}\left(1 + n^2 - 4mn\right)$$

and

$$\det\, P = \frac{a^4}{16}\left(n^2 - 4mn(n+2)\right),$$

and then by using Theorem 1, we obtain the following result:

Theorem 7. *If $mn > 0$, then the equilibrium points $E_{1,2}$ are Jacobi-stable if and only if*

$$1 + n^2 < 4mn \text{ and } n^2 > 4mn^2 + 8mn\,.$$

According to [8,9], if $mn > 0$, then we have that the equilibrium points $E_{1,2}$ are stable foci if $m(n-1) + n + 1 > 0$. Otherwise, $E_{1,2}$ are saddle foci if $m(n-1) + n + 1 < 0$. To clear up the relation between the Jacobi stability and classical linear (Lyapunov) stability at the equilibrium points $E_{1,2}$, and to highlight the dependence of Jacobi stability on the parameters of the T-system, we will use the next diagram relative to m and n in Figure 1:

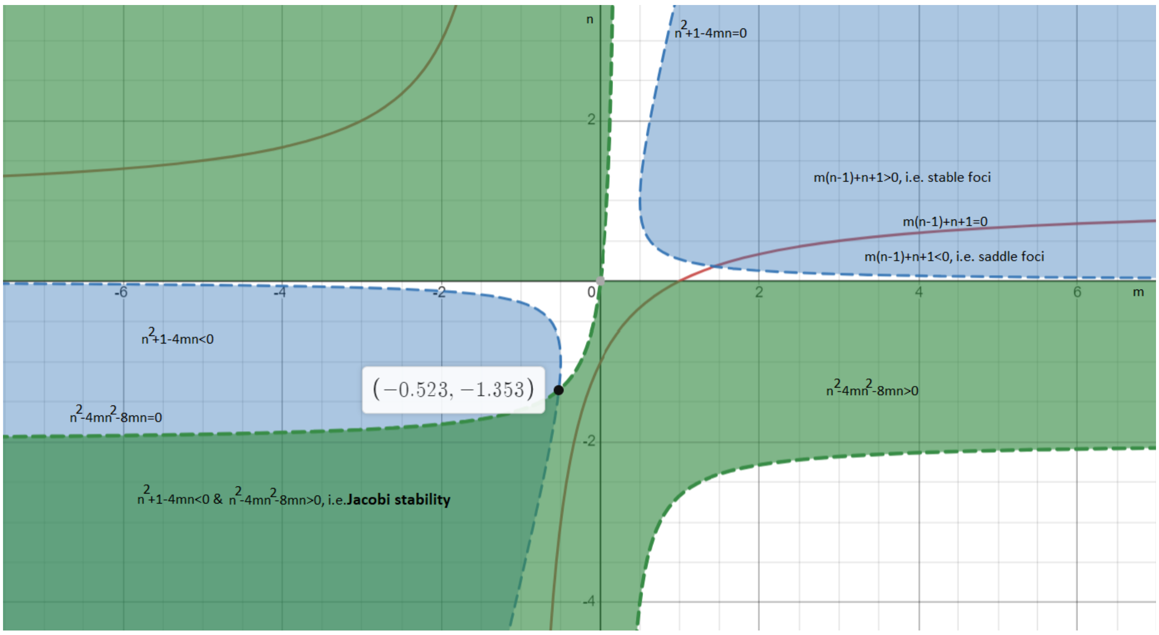

Figure 1. The Jacobi stability for T-system.

Remark 2. *Let us remark that the Jacobi stability for the equilibrium points $E_{1,2}$ it only happens in the third quadrant, which means for $m < 0$ and $n < 0$. Moreover if $E_{1,2}$ are Jacobi-stable, then $E_{1,2}$ are stable foci equilibrium points. So, whenever E_1 (and E_2) exists and satisfies the Jacobi stability conditions, any chaotic behaviour of the T-system in a sufficiently small vicinity of E_1 (and E_2) is not possible.*

Remark 3. *Let us point out that any chaotic behaviour of the T-system is known only for some values of parameters $m > 0$ and $n > 0$. More exactly, near to the Hopf bifurcation curve, $m(n-1) + n + 1 = 0$, with $0 < n < 1$, the T-system could have an attractor of butterfly type or even a butterfly attractor coexisting with two unstable limit cycles [8,9]. From this perspective, the Jacobi stability obtained for a whole region of the third quadrant (which means for $n^2 + 1 - 4mn < 0$ and $n^2 - 4mn^2 - 8mn > 0$) is a very important result for the investigation of the dynamics of*

T-system, because in this region, due to Jacobi stability, we do not have any chaotic behaviour around the equilibrium points $E_{1,2}$.

If $\dfrac{b(c-a)}{a} \leq 0$, then the T-system (1) has only the trivial equilibrium point $O(0,0,0)$, and the associated equilibrium point $O(0,0,0,0)$ for SODE (3) is obviously Jacobi-unstable.

Dynamics of the Deviation Vector for the T-System

Due to the fact that the deviation vector ξ^i, $i = 1,2$ shows us the time evolution of the integral curves of the associated dynamical system near any equilibrium point, it results in the study of the behavior of the deviation vector being very important. This time evolution is given by the deviation equations system (A8), which are named Jacobi equations. Also, this system can be written in the covariant form (A9).

For the T-system, the system of deviation Equation (A8) is given by

$$\begin{cases} \frac{d^2\xi^1}{dt^2} + a\frac{d\xi^1}{dt} + \left(a^2x^2 - a(c-a)\right)\xi^1 + a^2x^1\xi^2 & = 0, \\ \frac{d^2\xi^2}{dt^2} - \left(\frac{2}{a}y^1 + x^1\right)\frac{d\xi^1}{dt} + b\frac{d\xi^2}{dt} + \left(-y^1 + 2(ax^2 - c + a)x^1\right)\xi^1 + a\left(x^1\right)^2\xi^2 & = 0. \end{cases} \tag{10}$$

The deviation vector $\xi(t) = \left(\xi^1(t), \xi^2(t)\right)$ has the length defined by

$$\|\xi(t)\| = \sqrt{\left(\xi^1(t)\right)^2 + \left(\xi^2(t)\right)^2}.$$

Further, the system of deviation equations near every equilibrium point for the T-system will be presented. So, the evolution in time of the deviation vector near to the trivial equilibrium point $O(0,0,0,0)$ is carried out by the next SODE:

$$\begin{cases} \frac{d^2\xi^1}{dt^2} + a\frac{d\xi^1}{dt} - a(c-a)\xi^1 & = 0, \\ \frac{d^2\xi^2}{dt^2} + b\frac{d\xi^2}{dt} & = 0. \end{cases} \tag{11}$$

The evolution in time of the deviation vector near to the equilibrium point $E_1\left(\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ is carried out by the next SODE:

$$\begin{cases} \frac{d^2\xi^1}{dt^2} + a\frac{d\xi^1}{dt} + a^2\sqrt{\frac{b(c-a)}{a}}\xi^2 & = 0, \\ \frac{d^2\xi^2}{dt^2} - \sqrt{\frac{b(c-a)}{a}}\frac{d\xi^1}{dt} + b\frac{d\xi^2}{dt} + b(c-a)\xi^2 & = 0. \end{cases} \tag{12}$$

The evolution in time of the deviation vector near to the equilibrium points $E_2\left(-\sqrt{\frac{b(c-a)}{a}}, \frac{c-a}{a}, 0, 0\right)$ is carried out by the next SODE:

$$\begin{cases} \frac{d^2\xi^1}{dt^2} + a\frac{d\xi^1}{dt} - a^2\sqrt{\frac{b(c-a)}{a}}\xi^2 & = 0, \\ \frac{d^2\xi^2}{dt^2} + \sqrt{\frac{b(c-a)}{a}}\frac{d\xi^1}{dt} + b\frac{d\xi^2}{dt} + b(c-a)\xi^2 & = 0. \end{cases} \tag{13}$$

According to the theory of differential geometry of plane curves [18], the curvature $\kappa(t)$ of the trajectory $\xi(t) = \left(\xi^1(t), \xi^2(t)\right)$ associated to the system of deviation Equation (10) represents a quantitative indicator of the dynamics that is carried out by the deviation vector ξ^i, and it is defined by the formula

$$\kappa(t) = \frac{\dot{\xi}^1(t)\ddot{\xi}^2(t) - \ddot{\xi}^1(t)\dot{\xi}^2(t)}{\left[\left(\dot{\xi}^1(t)\right)^2 + \left(\dot{\xi}^2(t)\right)^2\right]^{3/2}}, \tag{14}$$

where $\dot{\xi}^i(t) = \dfrac{d\xi^i}{dt}$, $\ddot{\xi}^i(t) = \dfrac{d^2\xi^i}{dt^2}$, $i = 1,2$.

5. Conclusions

The basic target of this work is to investigate the dynamics of the second-order differential equations system (SODE or semi-spray) associated with the T-system from the Jacobi stability perspective through the instruments of the geometric Kosambi–Cartan–Chern (KCC) theory. Firstly, we recalled briefly the Lyapunov, linear, or classical local dynamics near each equilibrium point, and secondly, we reformulated the system of first-order nonlinear differential equations as an equivalent system of second-order differential equations (SODE). We calculated the first invariant and the second invariant of the Kosambi–Cartan–Chern (KCC) geometric theory, and for the third invariant, the fourth invariant, and the fifth invariant, we obtained that they have all coefficients equal to zero. Moreover, all coefficients of the Berwald connection are equal to zero, with a single exception. Also, near every equilibrium point, the coefficients of the tensor of zero-connection curvature and the coefficients of the nonlinear connection defined by the semi-spray (SODE) were computed. Moreover, the tensor of the deviation curvature was determined in order to obtain the Jacobi stability conditions at each equilibrium point.

The Jacobi stability of the T-system by the Kosambi–Cartan–Chern geometric theory for a whole region with negative values of the parameters m and n, i.e., a big region from the third quadrant, is very important in order to know more information about the local dynamics of this new system with chaotic behavior. These new results are very precious because the Jacobi stability near to an equilibrium point excludes a chaotic behavior of the system in a neighborhood of this point. Moreover, for the purpose of comparing the two kinds of stability, a comparative analysis of the Jacobi stability and the classical (linear or Lyapunov) stability near the nontrivial equilibrium points was performed. Also, the system of deviation equations near each equilibrium point of the T-system was determined. A possible continuation of this work could consist of performing a numerical study on the variation in time of the deviation vector and of its curvature in order to deduce new data and information about the behaviour of the T-system close to each equilibrium point. Also, in a future paper, we will look for possible applications of this new chaotic system to the catastrophe theory by using mathematical models inspired by sociology, ecology, epidemiology, and others.

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Appendix A. Kosambi–Cartan–Chern Geometric Theory and Jacobi Stability

The main aim of this appendix is to show briefly the basic of the Kosambi–Cartan–Chern geometric theory, because all these notions and results are necessary to understand the obtained results about the Jacobi stability of the T-system [19,20,25,26,30,31,35–40].

If M is a real n-dimensional C^∞–manifold and TM denotes the tangent bundle of M, then $u = (x, y)$ will be denoted as a point from TM, where $x = \left(x^1, \dots, x^n\right)$, $y = \left(y^1, \dots, y^n\right)$, and $y^i = \frac{dx^i}{dt}$, $i = 1, \dots, n$. Most often, M is $\mathbf{R}^n$ or an open subset of $\mathbf{R}^n$. Let the next system of second-order differential equations (in brief, SODE), in the normalized form [24] be

$$\left\{ \frac{d^2x^i}{dt^2} + 2G^i(x, y) = 0, \ i = 1, \dots, n. \right. \tag{A1}$$

where $G^i(x,y)$ are C^∞–functions given on a domain of a local system of coordinates on TM, i.e., an open neighborhood for some initial conditions (x_0, y_0). The system (A1) can be interpreted as a system of Euler–Lagrange equations from classical mechanics [24,41]:

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} &= F^i \\ y^i &= \frac{dx^i}{dt} \end{cases}, \quad i = 1, \ldots, n. \tag{A2}$$

where $L(x,y)$ is a regular Lagrangian on TM, and F^i are the coefficients of the external force.

The SODE (A1) has "a geometrical meaning" if and only if "the accelerations" $\frac{d^2x^i}{dt^2}$ and "the forces" $G^i(x^j, y^j)$ are tensors of type $(0,1)$ relative to the change in local coordinates:

$$\begin{cases} \tilde{x}^i &= \tilde{x}^i(x^1, \ldots, x^n) \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j \end{cases}, \quad i = 1, \ldots, n. \tag{A3}$$

More clearly, the SODE (A1) has a geometrical meaning (and then this system is called semi-spray) if and only if the changing of coefficients $G^i(x^j, y^j)$ relative to the change in local coordinates (A3) is going following the next relations [24,41]:

$$2\tilde{G}^i = 2G^j \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{y}^i}{\partial x^j} y^j \,. \tag{A4}$$

The geometric thought of the Kosambi–Cartan–Chern (KCC) theory is to obtain from the system of second-order differential Equation (A1) an equivalent system (i.e., with the same solutions), while preserving a geometric sense. Next, for the second-order differential equations system (SODE or semi-spray), we will introduce five tensor fields, named the geometric (or differential) invariants of the theory [25,26]. Surely, they do not change, i.e., they are invariant relative to the local change in coordinates (A3). Further, we will use the KCC covariant derivative of a vector field $\xi = \xi^i \frac{\partial}{\partial x^i}$ on an open domain of TM (sometimes, even on $TM = \mathbf{R}^n \times \mathbf{R}^n$) [25,38–40]:

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N^i_j \xi^j \,, \tag{A5}$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}$ are the coefficients of a nonlinear connection N on the tangent bundle TM corresponding to the semi-spray (A1).

For $\xi^i = y^i$,

$$\frac{Dy^i}{dt} = -2G^i + N^i_j y^j = -\varepsilon^i \,. \tag{A6}$$

and the contravariant vector field $\varepsilon^i = -\left(N^i_j y^j - 2G^i\right)$ is called the first invariant of the theory. This invariant represents an external force, and its coefficients ε^i have a geometrical sense, because with respect to a change in local coordinates (A3), we have [25]:

$$\tilde{\varepsilon}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \varepsilon^j \,.$$

If the coefficients G^i of the semi-spray (A1) are homogeneous functions of degree 2 with respect to y^i (i.e., $\frac{\partial G^i}{\partial y^j} y^j = 2G^i$, for all i), then the system (A1) is called a spray. Therefore, the first invariant is null ($\varepsilon^i = 0$ for all $i = 1$) if and only if the semi-spray is a spray. More that, this is available for the geodesic spray associated to a Riemannian or Finslerian metric [24,41].

Among the main objectives of the Kosambi–Cartan–Chern theory, we have to investigate the orbits that deviate slightly from an orbit of (A1). More exactly, the behavior of the system in variations will be investigated, and thus, the orbits $x^i(t)$ of (A1) will be changed into close ones, as given by the relations

$$\tilde{x}^i(t) = x^i(t) + \eta\xi^i(t), \tag{A7}$$

where $|\eta|$ is a is an enough small parameter and $\xi^i(t)$ are the coefficients of a contravariant vector field on the orbits $x^i(t)$, and named the deviation vector. Next, by using (A7) into (A1) and by following the limit $\eta \to 0$, we obtain the variational equations system [24–26]:

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0. \tag{A8}$$

If we use the formula of the KCC covariant derivative from (A5), then the system (A8) can be written in the equivalent covariant form [24–26]:

$$\frac{D^2\xi^i}{dt^2} = P^i_j \xi^j, \tag{A9}$$

where, on the right side, we have the $(1,1)$- type tensor P^i_j, with the following coefficients:

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j. \tag{A10}$$

According to [24,41], the coefficients

$$G^i_{jl} = \frac{\partial N^i_j}{\partial y^l} \tag{A11}$$

represent the Berwald connection corresponding to the nonlinear connection N of the semi-spray (A1).

The coefficients P^i_j represent the so-called deviation curvature tensor or the second invariant of the Kosambi–Cartan–Chern (KCC) geometric theory. If all coefficients of the nonlinear connection and all coefficients of the Berwald connection are null, then the deviation curvature tensor from (A10) has the coefficients $P^i_j = -2\frac{\partial G^i}{\partial x^j}$. So, it very useful to define the so-called zero-connection curvature tensor Z with the coefficients [44]:

$$Z^i_j = 2\frac{\partial G^i}{\partial x^j}. \tag{A12}$$

The system of second-order differential Equation (A8) represents the deviation equations (or Jacobi equations), and the invariant Equation (A9) also represents the Jacobi equations. In Riemann geometry or Finsler geometry, when a system of second-order equations represents the geodesic curve, Equations (A8) (or (A9)) are exactly the Jacobi field equations for the manifold geometry.

Next, we can introduce the third, the fourth, and the fifth invariants of the Kosambi–Cartan–Chern theory for the system of second-order differential equations (semi-spray or SODE) (A1). The coefficients of these invariants are given by:

$$P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right), \; P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l}, \; D^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}. \tag{A13}$$

From the geometric point of view, the third invariant P^i_{jk} is called the torsion tensor, and the fourth and the fifth invariants P^i_{jkl} and D^i_{jkl} are called the Riemann–Christoffel curvature tensor and the Douglas tensor. Let us remark that all these tensors always exist [24–26,31,41].

According to [24,39,41], all these five geometric objects are the main invariants that determine the geometrical properties of the system and give us the geometrical characteristics of the system of second-order differential Equation (A1). Next, we present a basic theorem of the KCC geometric theory, which belongs to P.L. Antonelli [25]:

Theorem A1. *Two systems of second-order differential equations (SODE) of type (A1), e.g.,*

$$\frac{d^2x^i}{dt^2} + 2G^i(x^j, y^j) = 0,\ y^j = \frac{dx^j}{dt}$$

and

$$\frac{d^2\tilde{x}^i}{dt^2} + 2\tilde{G}^i(\tilde{x}^j, \tilde{y}^j) = 0,\ \tilde{y}^j = \frac{d\tilde{x}^j}{dt},$$

can be locally changed, from one into another, by local coordinate changing (A3) if and only if the five geometrical invariants ε^i, P^i_j, P^i_{jk}, P^i_{jkl}, and D^i_{jkl} are equivalent tensors of $\tilde{\varepsilon}^i$, $\tilde{P}^i_j$, $\tilde{P}^i_{jk}$, $\tilde{P}^i_{jkl}$, and $\tilde{D}^i_{jkl}$, respectively.

Moreover, there exists a local coordinates chart $(U; x^1, \ldots, x^n)$ on the basic manifold M, for which $G^i = 0$ on U, for all i, if and only if all five invariant tensors have all coefficients equal to zero. In this case, the orbits of the dynamical system are straight lines.

The name "Jacobi stability" from the Kosambi–Cartan–Chern theory cames from Riemann geometry or Finsler geometry, when the system (A1) is the system of second-order differential equations that define the geodesics associated with the manifold's metric. Then, the system (A9) represents the Jacobi field equations for the geodesic deviation. More generally, the Jacobi Equation (A9) of a Finsler manifold (M, F) can be rewritten in the scalar form [28]:

$$\frac{d^2v}{ds^2} + K \cdot vs. = 0, \tag{A14}$$

where $\xi^i = v(s)\eta^i$ is the Jacobi tensor field along the geodesic $\gamma : x^i = x^i(s)$, η^i is the unit normal vector field on γ, and K is the flag curvature associated with the Finslerian function F.

Additionally, related to the sign of the flag curvature K of the Finsler manifold, we can determine the following [29]:

- If $K > 0$, then the geodesics "add up together" (i.e., Jacobi stability of the geodesics occurs);
- If $K < 0$, then the geodesics "disperse" (i.e., no Jacobi stability of the geodesics occurs).

Due to the equivalence of Equations (A9) and (A14), we can conclude that the flag curvature K is positive if and only if the eigenvalues of the curvature deviation tensor P^i_j are negative, and the flag curvature K is negative if and only if the eigenvalues of P^i_j are positive [29,32].

Now, we can present the main theorem of the KCC geometric theory [29,32]:

Theorem A2. *The trajectories of the second-order differential system (A1) are Jacobi-stable if and only if the real parts of the eigenvalues of the deviation curvature tensor P^i_j are strictly negative everywhere. Otherwise, the trajectories are Jacobi-unstable.*

In conclusion, the geometric stability in the Jacobi sense of a system of second-order differential equations (SODE) of type (A1) means the stability in the Lyapunov sense (linear or classical) of the variations system (A9). Therefore, the investigation of Jacobi stability is based on the study of the Lyapunov stability of all trajectories in a region, but without considering speed. Moreover, although the local analysis is concentrated at an equilibrium point, this kind of approach provides us details of the behavior of the trajectories of the system in a vicinity of this equilibrium point.

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