symmetry

MDPI

Article

# Fixed Point Results via Orthogonal $(\alpha - \mathfrak{y} - \mathbb{G})$-Contraction in Orthogonal Complete Metric Space

**Xiaolan Liu [1,2,3], Gunasekaran Nallaselli [4], Absar Ul Haq [5] and Arul Joseph Gnanaprakasam [4,*] and Imran Abbas Baloch [6,7,*]**

1 College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong 643000, China; xiaolanliu@suse.edu.cn
2 Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong 643000, China
3 South Sichuan Center for Applied Mathematics, Zigong 643000, China
4 Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, India; gn4255@srmist.edu.in
5 Department of Natural Sciences and Humanities, University of Engineering and Technology (Narowal Campus), Lahore 54000, Pakistan; absarulhaq@uet.edu.pk
6 Abdus Salam School of Mathematical Sciences, GC University, Lahore 54600, Pakistan
7 Higher Education Department, Government Graduate College for Boys Gulberg, Lahore 54600, Pakistan
* Correspondence: aruljoseph.alex@gmail.com (A.J.G.); iabbasbaloch@sms.edu.pk (I.A.B.)

**Abstract:** In this publication, we establish a suitable symmetry structure for orthogonal $(\alpha - \mathfrak{y} - \mathbb{G})$-contractive mappings and prove fixed point results for an orthogonal $(\alpha - \mathfrak{y} - \mathbb{G})$-contractive via orthogonal metric spaces. We give an application to strengthen our main results from the existing literature to prove the existence of a unique analytical solution to the differential equation by converting it into an integral equation through fixed point analysis.

**Keywords:** fixed point; orthogonal metric space; orthogonal $\alpha$-admissible mapping; orthogonal $(\alpha - \mathfrak{y} - \mathbb{G})$-contraction

check for updates

**Citation:** Liu, X.; Nallaselli, G.; Haq, A.U.; Gnanaprakasam, A.J.; Baloch, I.A. Fixed Point Results via Orthogonal $(\alpha - \mathfrak{y} - \mathbb{G})$-Contraction in Orthogonal Complete Metric Space. *Symmetry* **2023**, *15*, 1762. https://doi.org/10.3390/sym15091762

Academic Editors: Oluwatosin Mewomo, Qiaoli Dong and Alexander Zaslavski

Received: 22 July 2023
Revised: 29 August 2023
Accepted: 6 September 2023
Published: 14 September 2023



## 1. Introduction

Fixed point theory is a fascinating subject with an enormous number of symmetry applications in various fields of mathematics. The Banach fixed point theorem is the most significant test for the solutions of some problems in mathematics and engineering. In 1922, Stefan Banach [1] introduced contraction principle theorems. It has shown symmetry in the existing problems in various fields of mathematical analysis and a simple structure. In 1985, Droz et al. [2] presented as an abstract formulation of Picard's method of successive approximations. In 2010, Emmanouil [3] established an extension of the Banach fixed point theorem. Very recently, in 2012, Samet et al. [4] introduced the concepts of $\alpha - \psi$-contractive and $\alpha$-admissible maps, as well as proved some fixed point theorems for such functions stated on complete metric spaces (complete metric space). Following that, in 2013, Salimi, Latif, and Hussain [5] modified $\alpha - \psi$-contractive maps. In 2014, Hussain, Kutbi, and Salimi [6] updated the version of $\alpha - \psi$-contractive and also described fixed point theories that are appropriate generalizations in $\alpha$-admissible via complete metric spaces with an application of the recent results in symmetry manner. In 2014, Jleli and Samet [7] created a novel sort of $\mathbb{G}$-contraction and produced a unique fixed point for like contraction in the notion of nonspecific metric spaces. In 2017, Zheng et al. [8] introduced a new concept of $\theta - \phi$-contractions and established some fixed point results for such mappings in complete metric spaces. On the other side, Gordji, Rameani, De La Sen, and Cho [9] presented the idea of an orthogonal set, sometimes known as a $\mathscr{O}$-set, as well as certain examples and properties of these sets. In 2017, Hussain et al. [10] improved and expanded certain fixed point theorems for generalized $\mathbb{G}$-contractive axioms in complete metric space occupation. For further details, see [11–20].

This paper is written as follows. In the first part, we give the required background about an orthogonal $\alpha - \mathfrak{y} - \mathbb{G}$-complete metric space and an orthogonal $\alpha - \mathfrak{y}$-continuous function. In the next section, we state and prove the main results of an orthogonal $\alpha - \mathfrak{y} - \mathbb{G}$-complete metric space and an orthogonal $\alpha - \mathfrak{y}$-continuous. Finally, we give an application of the differential equation of fixed point theorem to an orthogonal $\alpha - \mathfrak{y} - \mathbb{G}$-complete metric space and an orthogonal $\alpha - \mathfrak{y}$-continuous.

## 2. Preliminaries

**Definition 1.** *[10] Let $\mathbb{G} : \mathfrak{R}^+ \to (1, \infty)$ be a function satisfying*

$(\mathbb{G}_1)$ *$\mathbb{G}$ is nondecreasing,*
$(\mathbb{G}_2)$ *for each sequence $\{\alpha_n\} \subset \mathfrak{R}^+$, $\lim_{n\to\infty} \mathbb{G}(\alpha_n) = 1$ if $\lim_{n\to\infty} (\alpha_n) = 0$,*
$(\mathbb{G}_3)$ *there exist $0 < \theta < 1$ and $\kappa \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\mathbb{G}(\alpha)-1}{\alpha^\theta} = \kappa$.*

**Definition 2.** *[10] A self-map $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ is known to be $\mathbb{G}$-contraction if there exists such a function $\mathbb{G}$ satisfying $(\mathbb{G}_1) - (\mathbb{G}_3)$ and a constant $\theta \in \mathfrak{R}^+$ such that for all $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$,*

$$\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) \neq 0 \textit{ implies } \mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^\theta.$$

**Theorem 1.** *[7] Let $(\mathfrak{U}, \mathtt{d})$ be a complete metric space and $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a $\mathbb{G}$-contraction, then $\mathbb{M}$ has a unique fixed point.*

Jleli et al. [7], denoted by the $\Psi$ set of all functions $\mathbb{G} : \mathfrak{R}^+ \to (\infty, \infty)$ satisfying the above axioms $(\mathbb{G}_1) - (\mathbb{G}_3)$.

**Theorem 2.** *[10] Let $(\mathfrak{U}, \mathtt{d})$ be a complete metric space and a self-map $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$. If there exist $\mathbb{G} \in \Psi$ and real numbers $\alpha > 0, \mathfrak{l} > 0, \mathbb{M} > 0, \mathtt{x} > 0$ with $0 \leq \alpha + \mathfrak{l} + \mathtt{m} + 2\mathtt{k} < 1$ such that*

$$\begin{aligned} \mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq {} & [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^\alpha \cdot [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathbb{M}\mathtt{x}))]^{\mathfrak{l}} \cdot \\ & [\mathbb{G}(\mathtt{d}(\mathtt{y}, \mathbb{M}\mathtt{y}))]^{\mathtt{m}} \cdot [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathbb{M}\mathtt{y}) + \mathtt{d}(\mathtt{y}, \mathbb{M}\mathtt{x}))]^{\mathtt{k}} \end{aligned}$$

*for all $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$, then $\mathbb{M}$ has a unique fixed point.*

Ahmad, Al-Mazrooei, Cho, and YangAhmad [11] used by $\Omega$ the family of all maps fulfilling the axioms $(\mathbb{G}_1), (\mathbb{G}_2)$ and also utilized the weaker axiom
$(\mathbb{G}_3')$ $\mathbb{G}$ is continuous on $\mathfrak{R}^+$ instead of the axiom $(\mathbb{G}_3)$.

**Example 1.** *[11] Let $\mathbb{G}_1(\delta) = \mathfrak{e}^{\sqrt{\delta}}, \mathbb{G}_2(\delta) = \mathfrak{e}^{\sqrt{\delta \mathfrak{e}^\delta}}, \mathbb{G}_3(\delta) = \mathfrak{e}^\delta, \mathbb{G}_4(\delta) = cosh\delta, \mathbb{G}_5(\delta) = 1 + ln$ $(1 + \delta)$ and $\mathbb{G}_6(\delta) = \mathfrak{e}^{\delta \mathfrak{e}^\delta}$ for all $\delta > 0$. Then $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3, \mathbb{G}_4, \mathbb{G}_5, \mathbb{G}_6 \in \Omega$.*

**Example 2.** *[11] Note that the axioms $\mathbb{G}_3$ and $\mathbb{G}_3'$ are independent. Indeed, for $\eta > 1, \mathbb{G}(\delta) = \mathfrak{e}^{\delta^\eta}$ fulfills the axioms ($\mathbb{G}_1$) and ($\mathbb{G}_2$), but it does not fulfill ($\mathbb{G}_3$), while it fulfills the axiom ($\mathbb{G}_3'$). Hence, $\Omega \nsubseteq \Psi$. For $\eta > 1, \varrho \in (0, \frac{1}{\eta})$ and $\mathbb{G}(\delta) = 1 + \delta^\varrho(1 + [\delta])$ where $[\delta]$ denotes the integral part of $\delta$, fulfills the axioms ($\mathbb{G}_1$) and ($\mathbb{G}_2$), but it does not fulfill ($\mathbb{G}_3'$), while it assures the axiom ($\mathbb{G}_3$) for any $\theta \in (\frac{1}{\eta}, 1)$. Therefore, $\Psi \nsupseteq \Omega$. Also, if we let $\mathbb{G}(\delta) = \mathfrak{e}^{\eta^\delta}$, then $\mathbb{G} \in \Psi$ and $\mathbb{G} \in \Omega$. Hence, $\Psi \cap \Omega \neq \emptyset$.*

**Definition 3.** *[9] Let $\mathfrak{U} \neq \phi$ and a binary relation $\perp \subseteq \mathfrak{U} \times \mathfrak{U}$. If $\perp$ satisfies the following axiom,*

$$\textit{there exists } \mathtt{x}_0 \in \mathfrak{U}\, [(\textit{for all } \mathtt{y} \in \mathfrak{U}, \mathtt{y} \perp \mathtt{x}_0) \quad \textit{or} \quad (\textit{for all } \mathtt{y} \in \mathfrak{U}, \mathtt{x}_0 \perp \mathtt{y})],$$

*then, it is known as $\mathcal{O}$-set and $\mathtt{x}_0$ is an orthogonal element. We will denote this $\mathcal{O}$-set by $(\mathfrak{U}, \perp)$.*

**Example 3.** *[9] Let $(\mathfrak{U}, \mathtt{d})$ be a metric space and $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a Picard operator, that is, $\mathbb{M}$ has a unique fixed point $\mathtt{x}^* \in \mathfrak{U}$ and $\lim_{\mathtt{n}\to\infty} \mathbb{M}^{\mathtt{n}}(\mathtt{y}) = \mathtt{x}^*$ for all $\mathtt{y} \in \mathfrak{U}$. Define $\perp$ on $\mathfrak{U}$ by $\mathtt{y} \perp \mathtt{x}$ if*

$$\lim_{\mathtt{n}\to\infty} \mathtt{d}(\mathtt{x}, \mathbb{M}^{\mathtt{n}}(\mathtt{y})) = 0.$$

*Then, $(\mathfrak{U}, \perp)$ is an $\mathcal{O}$-set.*

**Example 4.** *[9] Let $\mathfrak{U}$ be an inner product space with the inner product $< \cdot, \cdot >$. Define the binary relation $\perp$ on $\mathfrak{U}$ by $\mathtt{x} \perp \mathtt{y}$ if $< \mathtt{x}, \mathtt{y} >= 0$. Easily seen that $0 \perp \mathtt{x}$ for all $\mathtt{x} \in \mathfrak{U}$. Therefore, $(\mathfrak{U}, \perp)$ is an $\mathcal{O}$-set.*

**Definition 4.** *[9] A sequence $\{\mathtt{x}_{\mathtt{n}}\}_{\mathtt{n}=1}^{\infty}$ of an $\mathcal{O}$-set is known as an orthogonal sequence (shortly, $\mathcal{O}$-sequence) if*

$$(\forall \mathtt{n} \in \mathcal{N}, \mathtt{x}_{\mathtt{n}} \perp \mathtt{x}_{\mathtt{n}+1}) \quad \text{or} \quad (\forall \mathtt{n} \in \mathcal{N}, \mathtt{x}_{\mathtt{n}+1} \perp \mathtt{x}_{\mathtt{n}}).$$

**Definition 5.** *[12] The triple $(\mathfrak{U}, \perp, \mathtt{d})$ is known as orthogonal metric space (shortly, OMS) if $(\mathfrak{U}, \perp)$ is an $\mathcal{O}$-set and $(\mathfrak{U}, \mathtt{d})$ is a metric space.*

**Definition 6.** *[9] A self-map $\mathbb{M}$ defined on OMS $\mathfrak{U}$ is known as orthogonal continuous (or $\perp$-continuous) in $\mathtt{x} \in \mathfrak{U}$ if there exists an $\mathcal{O}$-sequence $\{\mathtt{x}_{\mathtt{n}}\}_{\mathtt{n}=1}^{\infty}$ in $\mathfrak{U}$ which implies $\mathtt{x}_{\mathtt{n}} \to \mathtt{x}$ as $\mathtt{n} \to \infty$, that is, $\mathbb{M}(\mathtt{x}_{\mathtt{n}}) \to \mathbb{M}(\mathtt{x})$ as $\mathtt{n} \to \infty$. Also, $\mathbb{M}$ is known as $\perp$-continuous on $\mathfrak{U}$ if $\mathbb{M}$ is $\perp$-continuous at $\mathtt{x} \in \mathfrak{U}$.*

**Definition 7.** *[9] Let $(\mathfrak{U}, \perp, \mathtt{d})$ be an OMS. If every $\mathcal{O}$-Cauchy sequence in $\mathfrak{U}$ is convergent, then, $\mathfrak{U}$ is said to be orthogonal complete (shortly, $\mathcal{O}$-complete).*

**Definition 8.** *[9] A self-map $\mathbb{M}$ defined on $\mathcal{O}$-set $\mathfrak{U}$ is known as $\perp$-preserving for each $\mathtt{x} \perp \mathtt{y}$ if $\mathbb{M}\mathtt{x} \perp \mathbb{M}\mathtt{y}$. Also, $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ is known to be weakly $\perp$-preserving forever $\mathtt{x} \perp \mathtt{y}$ if $\mathbb{M}(\mathtt{x}) \perp \mathbb{M}(\mathtt{y})$ or $\mathbb{M}(\mathtt{y}) \perp \mathbb{M}(\mathtt{x})$.*

**Definition 9.** *[15] Let $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a self-map and let $\alpha : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ be a function. Then, $\mathbb{M}$ is called an orthogonally $\alpha$-admissible whenever $\mathtt{x} \perp \mathtt{y}, \alpha(\mathtt{x}, \mathtt{y}) \geq 1 \implies \alpha(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) \geq 1$.*

**Definition 10.** *[5] Let $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a self-map and let $\alpha, \eta : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ be two functions. Then, $\mathbb{M}$ is called an orthogonally $\alpha$-admissible with respect to $\eta$ if $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}, \alpha(\mathtt{x}, \mathtt{y}) \geq \mathfrak{y}(\mathtt{x}, \mathtt{y}) \implies \alpha(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) \geq \mathfrak{y}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})$.*

**Definition 11.** *[6] Let $\alpha, \mathfrak{y} : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ be two functions. Then, $\mathbb{M}$ is known to be an orthogonally $\alpha - \mathfrak{y}$-continuous (shortly, $O - (\alpha - \mathfrak{y})$-continuous) on $(\mathfrak{U}, \perp, \mathtt{d})$, if for $\mathtt{x} \in \mathfrak{U}$, an $\mathcal{O}$-sequence $\{\mathtt{x}_{\mathtt{n}}\}_{\mathtt{n}=1}^{\infty}$, which implies $\mathtt{x}_{\mathtt{n}} \to \mathtt{x}$ as $\mathtt{n} \to \infty, \alpha(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1}) \geq \mathfrak{y}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1})$ for all $\mathtt{n} \in \mathcal{N}$ implies $\mathbb{M}\mathtt{x}_{\mathtt{n}} \to \mathbb{M}\mathtt{x}$.*

**Definition 12.** *[6] A map $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ is known as orbitally $\perp$-continuous at $\eta \in \mathfrak{U}$ if $\lim_{\mathtt{n}\to\infty} \mathbb{M}^{\mathtt{n}}\mathtt{x} = \eta \implies \lim_{\mathtt{n}\to\infty} \mathbb{M}\mathbb{M}^{\mathtt{n}}\mathtt{x} = \mathbb{M}\eta$. The map $\mathbb{M}$ is orbitally $\perp$-continuous on $\mathfrak{U}$ if $\mathbb{M}$ is orbitally $\perp$-continuous at $\eta \in \mathfrak{U}$.*

**Remark 1.** *[6] Consider a self-map $\mathbb{M}$ on orbitally $\mathbb{M}$-complete metric space $\mathfrak{U}$. Define $\alpha, \mathfrak{y} : \mathfrak{U} \times \mathfrak{U} \to [0, +\infty)$ by*

$$\alpha(\mathtt{x}, \mathtt{y}) = \begin{cases} 5, & \text{if } \mathtt{x}, \mathtt{y} \in \mathcal{O}(\vartheta), \\ 0, & \text{otherwise}, \end{cases} \text{and } \mathfrak{y}(\mathtt{x}, \mathtt{y}) = 1,$$

*where $\mathcal{O}(\vartheta)$ is an orbit of a point $\vartheta \in \mathfrak{U}$. If $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ is an orbitally $\perp$-continuous on $(\mathfrak{U}, \perp, \mathtt{d})$, then, $\mathbb{M}$ is an $O - (\alpha - \mathfrak{y})$-continuous on $(\mathfrak{U}, \perp, \mathtt{d})$.*

In this section, we define an $O - \alpha - \mathfrak{y} - \mathbb{G}$-contraction and prove some fixed point theorems, inspired by Hussain and Gordji and also utilized the axiom $(\mathbb{G}_3')$ in the proof of fixed point theorems in the notion of an $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contraction map via complete metric space.

## 3. Absolute Results

First, we define an $O - \alpha - \mathfrak{y} - \mathbb{G}$-contractions. We also prove fixed point theorems for an $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contractive map via orthogonal complete metric space.

**Definition 13.** *Let $(\mathfrak{U}, \perp, \mathtt{d})$ be an OMS and $\mathbb{M}$ be a self-map on $\mathfrak{U}$. Also, we assume two functions $\alpha, \mathfrak{y} : \mathfrak{U} \times \mathfrak{U} \to [0, +\infty)$. $\mathbb{M}$ is said to be an $O - \alpha - \mathfrak{y} - \mathbb{G}$-contraction (shortly, $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contraction) if for $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$ and $\mathfrak{y}(\mathtt{x}, \mathbb{M}\mathtt{x}) \leq \alpha(\mathtt{x}, \mathtt{y})$ and $\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) > 0$, we have*

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^{\theta},$$

*where $\mathbb{G} \in \Omega$ and $\theta \in (0, 1)$.*

**Theorem 3.** *Let $(\mathfrak{U}, \perp, \mathtt{d})$ be an orthogonal complete metric space. Let $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a self-map satisfying the axioms:*

*(i) $\mathbb{M}$ is $\alpha$-admissible map with respect to $\mathfrak{y}$,*
*(ii) $\mathbb{M}$ is an $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contraction,*
*(iii) there exists $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) \geq \mathfrak{y}(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0)$,*
*(iv) $\mathbb{M}$ is an $O - (\alpha - \mathfrak{y})$-continuous.*

*Then, $\mathbb{M}$ has a fixed point. Furthermore, $\mathbb{M}$ has a unique fixed point whenever $\alpha(\mathtt{x}, \mathtt{y}) \geq \mathfrak{y}(\mathtt{x}, \mathtt{x})$ for all $\mathtt{x}, \mathtt{y} \in Fix(\mathbb{M})$.*

**Proof.** From orthogonality, it follows that

$$\mathtt{x}_0 \perp \mathbb{M}(\mathtt{x}_0) \text{ or } \mathbb{M}(\mathtt{x}_0) \perp \mathtt{x}_0.$$

Let $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a mapping with $Fix(\mathbb{M}) \neq \emptyset$. For a given $\mathtt{x}_0 \in \mathfrak{U}$, the fixed point iteration method generates a sequence $\{\mathtt{x}_\mathtt{n}\}$ in $\mathfrak{U}$ as follows:

$$\mathtt{x}_1 = \mathbb{M}(\mathtt{x}_0),\ \mathtt{x}_2 = \mathbb{M}(\mathtt{x}_1) = \mathbb{M}^2\mathtt{x}_0,\ \ldots,\ \mathtt{x}_\mathtt{n} = \mathbb{M}\mathtt{x}_{\mathtt{n}-1} = \mathbb{M}^\mathtt{n}\mathtt{x}_0$$

for every $\mathtt{n} \in \mathscr{N} \cup \{0\}$. If $\mathtt{x}_{\mathtt{n}^*} = \mathtt{x}_{\mathtt{n}^*+1}$ for each $\mathtt{n}^* \in \mathscr{N} \cup \{0\}$, then, $\mathtt{x}_{\mathtt{n}^*}$ is a fixed point of $\mathbb{M}$ and so, the proof is obvious. Suppose it is not true, then $\mathtt{x}_\mathtt{n} \neq \mathtt{x}_{\mathtt{n}+1}$ for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. Thus, we have $\mathtt{d}(\mathbb{M}\mathtt{x}_{\mathtt{n}-1}, \mathbb{M}\mathtt{x}_\mathtt{n}) > 0$ for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. From $\mathbb{M}$ is $\perp$-preserving, we have

$$\mathtt{x}_\mathtt{n} \perp \mathtt{x}_{\mathtt{n}+1} \text{ or } \mathtt{x}_{\mathtt{n}+1} \perp \mathtt{x}_\mathtt{n} \tag{1}$$

for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. It provides that $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-sequence. Let $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) > \mathfrak{y}(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0)$. Now, since $\mathbb{M}$ is an $\alpha$-admissible map with respect to $\mathfrak{y}$, then, $\alpha(\mathtt{x}_0, \mathtt{x}_1) = \alpha(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) > \mathfrak{y}(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) = \mathfrak{y}(\mathtt{x}_0, \mathtt{x}_1)$. By continuing in this way, we have

$$\mathfrak{y}(\mathtt{x}_{\mathtt{n}-1}, \mathbb{M}\mathtt{x}_{\mathtt{n}-1}) = \mathfrak{y}(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_\mathtt{n}) \leq \alpha(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_\mathtt{n}) \text{ for all } \mathtt{n} \in \mathscr{N}.$$

Since $\mathbb{M}$ is an $O-(\alpha-\mathfrak{y}-\mathbb{G})$-contractive map, we have

$$\begin{aligned}1 &< \mathbb{G}(\mathtt{d}(\mathtt{x_n},\mathtt{x_{n+1}}))\\ &= \mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x_{n-1}},\mathbb{M}\mathtt{x_n}))\\ &\le [\mathbb{G}(\mathtt{d}(\mathtt{x_{n-1}},\mathtt{x_n}))]^{\theta}\\ &= [\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x_{n-2}},\mathbb{M}\mathtt{x_{n-1}}))]^{\theta}\\ &\le [\mathbb{G}(\mathtt{d}(\mathtt{x_{n-2}},\mathtt{x_{n-1}}))]^{\theta^2}\\ &\dots\\ &\le [\mathbb{G}(\mathtt{d}(\mathtt{x_0},\mathtt{x_1}))]^{\theta^{\mathtt{n}}}\end{aligned}$$

for all $\mathtt{n}\in\mathscr{N}$. Since $\mathbb{G}\in\Omega$, letting $\lim\limits_{\mathtt{n}\to\infty}$ in the above inequality, we obtain

$$\lim_{\mathtt{n}\to\infty}\mathbb{G}(\mathtt{d}(\mathtt{x_n},\mathtt{x_{n+1}}))=1.$$

By $(\mathbb{G}_2)$, we have

$$\lim_{\mathtt{n}\to\infty}\mathtt{d}(\mathtt{x_n},\mathtt{x_{n+1}})=0. \tag{2}$$

Now, we will show that $\{\mathtt{x_n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-Cauchy sequence. Suppose that $\{\mathtt{x_n}\}_{\mathtt{n}=1}^{\infty}$ is not an $\mathscr{O}$-Cauchy sequence; if there exist $\varepsilon>0$ and sequences $\{\eta(\mathtt{n})\}_{\mathtt{n}=1}^{\infty}$ and $\{\mu(\mathtt{n})\}_{\mathtt{n}=1}^{\infty}$ of $\mathscr{N}$ such that for $\eta(\mathtt{n})>\mu(\mathtt{n})>\mathtt{n}$, we have

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\mu(\mathtt{n})})\ge\varepsilon.$$

Then,

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})-1},\mathtt{x}_{\mu(\mathtt{n})})<\varepsilon, \tag{3}$$

for all $\mathtt{n}\in\mathscr{N}$. So, by triangle inequality and (3), we have

$$\varepsilon\le\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\mu(\mathtt{n})})\le\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\eta(\mathtt{n})-1})+\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})-1},\mathtt{x}_{\mu(\mathtt{n})})\le\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\eta(\mathtt{n})-1})+\varepsilon.$$

By letting the limit and using (3), we obtain

$$\lim_{\mathtt{n}\to\infty}\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\mu(\mathtt{n})})=\varepsilon. \tag{4}$$

From (2), choose a natural number $\mathtt{n_0}\in\mathscr{N}$ such that

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\eta(\mathtt{n})+1})<\frac{\varepsilon}{4}\text{ and }\mathtt{d}(\mathtt{x}_{\mu(\mathtt{n})},\mathtt{x}_{\mu(\mathtt{n})+1})<\frac{\varepsilon}{4}, \tag{5}$$

for all $\mathtt{n}\ge\mathtt{n_0}$. Next, we claim that $\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})}\ne\mathbb{M}\mathtt{x}_{\mu(\mathtt{n})}$ for all $\mathtt{n}\ge\mathtt{n_0}$, that is,

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1},\mathtt{x}_{\mu(\mathtt{n})+1})=\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})},\mathbb{M}\mathtt{x}_{\mu(\mathtt{n})})>0. \tag{6}$$

Arguing by contradiction, there exists $\mathscr{N}_0\ge\mathtt{n_0}$ such that $\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1},\mathtt{x}_{\mu(\mathtt{n})+1})=0$. It follows from (2), (5), and (6) that

$$\begin{aligned}\varepsilon &\le \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\mu(\mathtt{n})})\\ &\le \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})},\mathtt{x}_{\eta(\mathtt{n})+1})+\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1},\mathtt{x}_{\mu(\mathtt{n})+1})+\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1},\mathtt{x}_{\mu(\mathtt{n})}))]\\ &\le \frac{\varepsilon}{4}+0+\frac{\varepsilon}{4}\\ &= \frac{\varepsilon}{2},\end{aligned}$$

a contradiction. Therefore, (5) holds. Then, by the axiom, we obtain

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\mu(\mathtt{n})}))]^{\theta}. \tag{7}$$

By the limit as $\mathtt{n} \to +\infty$ and condition $(\mathbb{G}_3')$, (4) and (7), we obtain $\mathbb{G}(\varepsilon) \leq [\mathbb{G}(\varepsilon)]^{\theta}$, a contradiction. Therefore, $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-Cauchy sequence. By $\mathscr{O}$-completeness of $\mathfrak{U}$, there exists $\mathtt{z} \in \mathfrak{U}$ such that $\mathtt{x}_\mathtt{n} \to \mathtt{z}$ as $\mathtt{n} \to \infty$. Now, since $\mathbb{M}$ is $O - (\alpha - \mathfrak{y})$-continuous and $\mathfrak{y}(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_\mathtt{n}) \leq \alpha(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_\mathtt{n})$, so

$$\mathtt{d}(\mathtt{z}, \mathbb{M}\mathtt{z}) = \lim_{\mathtt{n}\to\infty} \mathtt{d}(\mathtt{x}_\mathtt{n}, \mathbb{M}\mathtt{x}_\mathtt{n}) = \lim_{\mathtt{n}\to\infty} \mathtt{d}(\mathtt{x}_\mathtt{n}, \mathtt{x}_{\mathtt{n}+1}) = \mathtt{d}(\mathtt{z}, \mathtt{z}) = 0.$$

Hence, $\mathtt{z}$ is a fixed point of $\mathbb{M}$.

Now, we prove that $\mathtt{z}$ is a unique fixed point of $\mathbb{M}$. Let $\sigma$ be another fixed point of $\mathbb{M}$. If $\mathtt{x}_\mathtt{n} \to \sigma$ as $\mathtt{n} \to \infty$, we have $\mathtt{z} = \sigma$. If $\mathtt{x}_\mathtt{n}$ does not converge to $\sigma$ as $\mathtt{n} \to \infty$, there is a subsequence $\{\mathtt{x}_{\mathtt{n}_\theta}\}$ such that $\mathbb{M}\mathtt{x}_{\mathtt{n}_\theta} \neq \sigma$ for all $\theta \in \mathscr{N}$. By the choice of $\mathtt{x}_0$, in the proof of the first part, thus, we have $(\mathtt{x}_0 \perp \sigma)$ or $(\sigma \perp \mathtt{x}_0)$. Since $\mathbb{M}$ is $\perp$-preserving and $\mathbb{M}^\mathtt{n}\sigma = \sigma$ for all $\mathtt{n} \in \mathscr{N}$, we have $(\mathbb{M}^\mathtt{n}\mathtt{x}_0 \perp \sigma)$ or $(\sigma \perp \mathbb{M}^\mathtt{n}\mathtt{x}_0)$ for all $\mathtt{n} \in \mathscr{N}$. Since $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contractive map, we have

$$\mathbb{G}(\mathtt{d}(\mathtt{z}, \sigma)) = [\mathbb{G}(\mathtt{d}(\mathtt{z}, \sigma))]^{\theta},$$

a contradiction because $\theta \in \mathfrak{R}^+$. Thus, $\mathtt{z}$ is the unique fixed point of $\mathbb{M}$. □

**Theorem 4.** *Let $(\mathfrak{U}, \perp, \mathtt{d})$ be an orthogonal complete metric space. Let $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ be a self-map postulating the axioms:*

*(i) $\mathbb{M}$ is $\alpha$-admissible map with respect to $\mathfrak{y}$,*
*(ii) $\mathbb{M}$ is an $O - \alpha - \mathfrak{y} - \mathbb{G}$-contraction,*
*(iii) There exists $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) \geq \mathfrak{y}(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0)$,*
*(iv) if $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-sequence in $\mathfrak{U}$ such that $\alpha(\mathtt{x}_\mathtt{n}, \mathtt{x}_{\mathtt{n}+1}) > \mathfrak{y}(\mathtt{x}_\mathtt{n}, \mathtt{x}_{\mathtt{n}+1})$ with $\mathtt{x}_\mathtt{n} \to \mathtt{x}$ as $\mathtt{n} \to \infty$, then, either $\mathfrak{y}(\mathbb{M}\mathtt{x}_\mathtt{n}, \mathbb{M}^2\mathtt{x}_\mathtt{n}) \leq \alpha(\mathbb{M}\mathtt{x}_\mathtt{n}, \mathtt{x})$ or $\mathfrak{y}(\mathbb{M}^2\mathtt{x}_\mathtt{n}, \mathbb{M}^3\mathtt{x}_\mathtt{n}) \leq \alpha(\mathbb{M}^2\mathtt{x}_\mathtt{n}, \mathtt{x})$ holds for all $\mathtt{n} \in \mathscr{N}$.*

*Then, $\mathbb{M}$ has a fixed point. Moreover, $\mathbb{M}$ has a unique fixed point whenever $\alpha(\mathtt{x}, \mathtt{y}) \geq \mathfrak{y}(\mathtt{x}, \mathtt{x})$ for all $\mathtt{x}, \mathtt{y} \in Fix(\mathbb{M})$.*

**Proof.** From orthogonality, it follows that

$$\mathtt{x}_0 \perp \mathbb{M}(\mathtt{x}_0) \text{ or } \mathbb{M}(\mathtt{x}_0) \perp \mathtt{x}_0.$$

Let

$$\mathtt{x}_1 = \mathbb{M}(\mathtt{x}_0),\ \mathtt{x}_2 = \mathbb{M}(\mathtt{x}_1) = \mathbb{M}^2\mathtt{x}_0,\ \ldots,\ \mathtt{x}_\mathtt{n} = \mathbb{M}\mathtt{x}_{\mathtt{n}-1} = \mathbb{M}^\mathtt{n}\mathtt{x}_0$$

for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. If $\mathtt{x}_{\mathtt{n}^*} = \mathtt{x}_{\mathtt{n}^*+1}$ for some $\mathtt{n}^* \in \mathscr{N} \cup \{0\}$, then, $\mathtt{x}_{\mathtt{n}^*}$ is a fixed point of $\mathbb{M}$ and so the proof is obvious. Suppose it is not true, then $\mathtt{x}_\mathtt{n} \neq \mathtt{x}_{\mathtt{n}+1}$ for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. Thus, we have $\mathtt{d}(\mathbb{M}\mathtt{x}_{\mathtt{n}-1}, \mathbb{M}\mathtt{x}_\mathtt{n}) > 0$ for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. Since $\mathbb{M}$ is $\perp$-preserving, we have

$$\mathtt{x}_\mathtt{n} \perp \mathtt{x}_{\mathtt{n}+1} \text{ or } \mathtt{x}_{\mathtt{n}+1} \perp \mathtt{x}_\mathtt{n} \tag{8}$$

for all $\mathtt{n} \in \mathscr{N} \cup \{0\}$. It provides that $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-sequence. Let $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0) > \mathfrak{y}(\mathtt{x}_0, \mathbb{M}\mathtt{x}_0)$. Proof of Theorem 3, it implies that

$$\alpha(\mathtt{x}_\mathtt{n}, \mathtt{x}_{\mathtt{n}+1}) > \mathfrak{y}(\mathtt{x}_\mathtt{n}, \mathtt{x}_{\mathtt{n}+1}) \text{ and } \mathtt{x}_\mathtt{n} \to \mathtt{x}^* \text{ as } \mathtt{n} \to \infty,$$

where $\mathtt{x}_{\mathtt{n}+1} = \mathbb{M}\mathtt{x}_\mathtt{n}$. So, axiom (iv),

$$\text{either } \mathfrak{y}(\mathbb{M}\mathtt{x}_\mathtt{n}, \mathbb{M}^2\mathtt{x}_\mathtt{n}) \leq \alpha(\mathbb{M}\mathtt{x}_\mathtt{n}, \mathtt{x}^*) \text{ or } \mathfrak{y}(\mathbb{M}^2\mathtt{x}_\mathtt{n}, \mathbb{M}^3\mathtt{x}_\mathtt{n}) \leq \alpha(\mathbb{M}^2\mathtt{x}_\mathtt{n}, \mathtt{x}^*) \text{ true for all } \mathtt{n} \in \mathscr{N}.$$

This shows that $\mathfrak{y}(\mathtt{x}_{\mathtt{n}_\theta+1},\mathtt{x}_{\mathtt{n}_\theta+2}) \leq \alpha(\mathtt{x}_{\mathtt{n}_\theta+1},\mathtt{x})$ or $\mathfrak{y}(\mathtt{x}_{\mathtt{n}_\theta+2},\mathtt{x}_{\mathtt{n}_\theta+3}) \leq \alpha(\mathtt{x}_{\mathtt{n}_\theta+2},\mathtt{x})$ is true for all $\mathtt{n} \in \mathscr{N}$. Consequently, there exists a subsequence $\{\mathtt{x}_{\mathtt{n}_\theta}\}$ of $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ such that

$$\mathfrak{y}(\mathtt{x}_{\mathtt{n}_\theta},\mathbb{M}\mathtt{x}_{\mathtt{n}_\theta}) = \mathfrak{y}(\mathtt{x}_{\mathtt{n}_\theta},\mathtt{x}_{\mathtt{n}_\theta+1}) \leq \alpha(\mathtt{x}_{\mathtt{n}_\theta},\mathtt{x}^*), \tag{9}$$

and so from (7), we conclude that

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}_{\mathtt{n}_\theta},F\mathtt{x}^*)) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}_\theta},\mathtt{x}^*))]^{\lambda} < \mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}_\theta},\mathtt{x}^*)).$$

From ($\mathbb{G}_1$), we have

$$\mathtt{d}(\mathtt{x}_{\mathtt{n}_\theta+1},\mathbb{M}\mathtt{x}^*) < \mathtt{d}\mathtt{x}_{\mathtt{n}_\theta},\mathtt{x}^*).$$

Letting the limit as $\theta \to \infty$ in the above inequality, we have $\mathtt{d}(\mathtt{x}^*,\mathbb{M}\mathtt{x}^*) = 0$, i.e., $\mathtt{x}^* = \mathbb{M}\mathtt{x}^*$. Similarly, uniqueness follows in the same way as Theorem 3. □

Let $\alpha(\mathtt{x},\mathtt{y}) = \mathfrak{y}(\mathtt{x},\mathtt{y}) = 1$ for all $\mathtt{x},\mathtt{y} \in \mathfrak{U}$, then we provide the following result as a corollary.

**Corollary 1.** *Let $(\mathfrak{U},\perp,\mathtt{d})$ be an orthogonal complete metric space and $\mathbb{M}:\mathfrak{U}\to\mathfrak{U}$ be a self-map. If for all $\mathtt{x},\mathtt{y} \in \mathfrak{U}, \mathtt{x} \perp \mathtt{y}$ with $\mathtt{d}(\mathbb{M}\mathtt{x},\mathbb{M}\mathtt{y}) > 0$, we obtain $\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x},\mathbb{M}\mathtt{y})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x},\mathtt{y}))]^{\theta}$, for each $\mathbb{M} \in \Omega$. Then, $\mathbb{M}$ has a unique fixed point.*

A self-map $\mathbb{M}$ has the property $\mathcal{P}$, if Fix $(\mathbb{M}^{\mathtt{n}}) = \check{\mathcal{F}}(\mathbb{M})$ for every $\mathtt{n} \in \mathscr{N}$.

**Theorem 5.** *Let $(\mathfrak{U},\perp,\mathtt{d})$ be an orthogonal complete metric space and $\mathbb{M}:\mathfrak{U}\to\mathfrak{U}$ be a $\alpha$-continuous self-map. Suppose there exists some $\theta \in \mathfrak{R}^+$ such that*

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x},\mathbb{M}^2\mathtt{x})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x},\mathbb{M}\mathtt{x}))]^{\theta} \tag{10}$$

*holds for all $\mathtt{x} \in \mathfrak{U}$ with $\mathtt{d}(\mathbb{M}\mathtt{x},\mathbb{M}^2\mathtt{x}) > 0$ for each $\mathbb{G} \in \Omega$. If $\mathbb{M}$ is an $\alpha$-admissible and there exist $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0,\mathbb{M}\mathtt{x}_0) > 1$, then $\mathbb{M}$ has the property $\mathcal{P}$.*

**Proof.** From orthogonality, it follows that

$$\mathtt{x}_0 \perp \mathbb{M}(\mathtt{x}_0) \text{ or } \mathbb{M}(\mathtt{x}_0) \perp \mathtt{x}_0.$$

Let

$$\mathtt{x}_1 = \mathbb{M}(\mathtt{x}_0),\ \mathtt{x}_2 = \mathbb{M}(\mathtt{x}_1) = \mathbb{M}^2\mathtt{x}_0,\ \ldots,\ \mathtt{x}_\mathtt{n} = \mathbb{M}\mathtt{x}_{\mathtt{n}-1} = \mathbb{M}^\mathtt{n}\mathtt{x}_0$$

for all $\mathtt{n} \in \mathscr{N}\cup\{0\}$. If $\mathtt{x}_{\mathtt{n}^*} = \mathtt{x}_{\mathtt{n}^*+1}$ for some $\mathtt{n}^* \in \mathscr{N}\cup\{0\}$, then $\mathtt{x}_{\mathtt{n}^*}$ is a fixed point of $\mathbb{M}$ and so the proof is obvious. Suppose it is not true, then $\mathtt{x}_\mathtt{n} \neq \mathtt{x}_{\mathtt{n}+1}$ for all $\mathtt{n} \in \mathscr{N}\cup\{0\}$. Thus, we have $\mathtt{d}(\mathbb{M}\mathtt{x}_{\mathtt{n}-1},\mathbb{M}\mathtt{x}_\mathtt{n}) > 0$ for all $\mathtt{n} \in \mathscr{N}\cup\{0\}$. Since $\mathbb{M}$ is $\perp$-preserving, we have

$$\mathtt{x}_\mathtt{n} \perp \mathtt{x}_{\mathtt{n}+1} \text{ or } \mathtt{x}_{\mathtt{n}+1} \perp \mathtt{x}_\mathtt{n} \tag{11}$$

for all $\mathtt{n} \in \mathscr{N}\cup\{0\}$. We conclude that $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-sequence. Let $\mathtt{x}_0 \in \mathfrak{U}$ such that $\alpha(\mathtt{x}_0,\mathbb{M}\mathtt{x}_0) > 1$. Now, since $\mathbb{M}$ is $\alpha$-admissible map, so $\alpha(\mathtt{x}_1,\mathtt{x}_2) = \alpha(\mathbb{M}\mathtt{x}_0,\mathbb{M}\mathtt{x}_1) > 1$. Proceeding in this way, we obtain

$$\alpha(\mathtt{x}_{\mathtt{n}-1},\mathtt{x}_\mathtt{n}) \geq 1$$

for all $\mathtt{n} \in \mathscr{N}$. From (10), we have

$$1 < \mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}_{\mathtt{n}-1},\mathbb{M}^2\mathtt{x}_{\mathtt{n}-1})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}-1},\mathbb{M}\mathtt{x}_{\mathtt{n}-1}))]^{\theta},$$

which implies

$$1 < \mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_{\mathtt{n}}))]^{\theta},$$

and so

$$1 < \mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_{\mathtt{n}}))]^{\theta}.$$

Therefore,

$$1 < \mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_{\mathtt{n}}))]^{\theta} \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}-2}, \mathtt{x}_{\mathtt{n}-1}))]^{\theta 2} \leq \cdots \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{o}}, \mathtt{x}_{\mathtt{1}}))]^{\theta^{\mathtt{n}}}.$$

By the limit as $\mathtt{n} \to \infty$ in the above inequality, we obtain $\lim_{\mathtt{n}\to\infty} \mathbb{G}(\mathtt{d}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1})) = 1$, and from $\mathbb{G} \in \Omega$ we have

$$\lim_{\mathtt{n}\to\infty} \mathtt{d}(\mathtt{x}_{\mathtt{n}}, \mathtt{x}_{\mathtt{n}+1}) = 0. \tag{12}$$

Now, we show that $\{\mathtt{x}_{\mathtt{n}}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-Cauchy sequence. Suppose $\{\mathtt{x}_{\mathtt{n}}\}_{\mathtt{n}=1}^{\infty}$ is not an $\mathscr{O}$-Cauchy sequence, there exists $\varepsilon > 0$ and $\mathscr{O}$-sequences $\{\eta(\mathtt{n})\}_{\mathtt{n}=1}^{\infty}$ and $\{\mu(\mathtt{n})\}_{\mathtt{n}=1}^{\infty}$ of natural numbers such that for $\eta(\mathtt{n}) > \mu(\mathtt{n}) > \mathtt{n}$, we have

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}) = \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\mu(\mathtt{n})}) \geq \varepsilon. \tag{13}$$

Then

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})-1}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}) < \varepsilon$$

for all $\mathtt{n} \in \mathscr{N}$. So, by triangle inequality and by Equation (13), we have

$$\begin{aligned}\varepsilon \leq \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}) &\leq \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\eta(\mathtt{n})-1}) + \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})-1}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}) \\ &\leq \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\eta(\mathtt{n})-1}) + \varepsilon.\end{aligned}$$

By applying the limit and by Equation (12), we have

$$\lim_{\mathtt{n}\to\infty} \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}) = \varepsilon.$$

On the other hand, by (12), there exists a natural number $\mathtt{n}_0 \in \mathscr{N}$ such that

$$\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\eta(\mathtt{n})+1}) < \frac{\varepsilon}{4} \text{ and } \mathtt{d}(\mathtt{x}_{\mu(\mathtt{n})}, \mathtt{x}_{\mu(\mathtt{n})+1}) < \frac{\varepsilon}{4} \tag{14}$$

for all $\mathtt{n} \geq \mathtt{n}_0$. Next, we claim that

$$\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}^2\mathtt{x}_{\mu(\mathtt{n})-1}) = \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})}) > 0 \tag{15}$$

for all $\mathtt{n} \geq \mathtt{n}_0$. On the contrary, assume that there exists $\varrho > \mathtt{n}_0$ such that

$$\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\varrho)}, \mathbb{M}^2\mathtt{x}_{\mu(\varrho)-1}) = \mathtt{d}(\mathtt{x}_{\eta(\varrho)+1}, \mathbb{M}\mathtt{x}_{\mu(\varrho)}) = 0. \tag{16}$$

Then, from (14)–(16), we obtain

$$\begin{aligned}
\varepsilon \le \mathtt{d}(\mathtt{x}_{\eta(\varrho)}, \mathbb{M}\mathtt{x}_{\mu(\varrho)-1}) &\le \mathtt{d}(\mathtt{x}_{\eta(\varrho)}, \mathtt{x}_{\eta(\varrho)+1}) + \mathtt{d}(\mathtt{x}_{\eta(\varrho)+1}, \mathbb{M}\mathtt{x}_{\mu(\varrho)-1}) \\
&\le \mathtt{d}(\mathtt{x}_{\eta(\varrho)}, \mathtt{x}_{\eta(\varrho)+1}) + \mathtt{d}(\mathtt{x}_{\eta(\varrho)+1}, \mathtt{x}_{\mu(\varrho)+1}) + \mathtt{d}(\mathtt{x}_{\mu(\varrho)+1}, \mathbb{M}\mathtt{x}_{\mu(\varrho)-1}) \\
&= \mathtt{d}(\mathtt{x}_{\eta(\varrho)}, \mathtt{x}_{\eta(\varrho)+1}) + \mathtt{d}(\mathtt{x}_{\eta(\varrho)+1}, \mathfrak{F}\mathtt{x}_{\mu(\varrho)}) + \mathtt{d}(\mathtt{x}_{\mu(\varrho)+1}, \mathtt{x}_{\mu(\varrho)} \\
&< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2},
\end{aligned}$$

a contradiction. Thus,

$$\begin{aligned}
\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}^2\mathtt{x}_{\mu(\mathtt{n})-1}) = \mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})}) > 0, \\
\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}^2\mathtt{x}_{\mu(\mathtt{n})-1})) \le [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathbb{M}\mathtt{x}_{\mu(\mathtt{n})-1}))]^{\theta},
\end{aligned} \tag{17}$$

are established, which shows that

$$\mathbb{G}(\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})+1}, \mathtt{x}_{\mu(\mathtt{n})+1})) \le [\mathbb{G}(\mathtt{d}(\mathtt{x}_{\eta(\mathtt{n})}, \mathtt{x}_{\mu(\mathtt{n})}))]^{\theta}.$$

From $(\mathbb{G}_3)$, (13) and (17), we have $\mathbb{G}(\varepsilon) \le [\mathbb{G}(\varepsilon)]^{\theta}$, a contradiction because $\theta \in \mathfrak{R}^+$. Therefore, $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is an $\mathscr{O}$-Cauchy sequence. By $\mathscr{O}$-completeness of $\mathfrak{U}$, there exists $\mathtt{x}^* \in \mathfrak{U}$ such that $\mathtt{x}_\mathtt{n} \to \mathtt{x}^*$ as $\mathtt{n} \to \infty$. Now, since $\mathbb{M}$ is $\alpha$-continuous and $\alpha(\mathtt{x}_{\mathtt{n}-1}, \mathtt{x}_\mathtt{n}) > 1$, we have, $\mathtt{x}_{\mathtt{n}+1} = \mathbb{M}\mathtt{x}_\mathtt{n} \to \mathbb{M}\mathtt{x}^*$ as $\mathtt{n} \to \infty$, i.e., $\mathtt{x}^* = \mathbb{M}\mathtt{x}^*$. Therefore, $\mathbb{M}$ has a fixed point and $\mathbb{M}(\mathbb{M}^\mathtt{n}) = \mathbb{M}(\mathbb{M})$ for $\mathtt{n} = 1$. Let $\mathtt{n} > 1$. On the contrary, suppose that $\vartheta \in \mathbb{M}(\mathbb{M}^\mathtt{n})$ and $\vartheta \notin \mathbb{M}(\mathbb{M})$. Then, $\mathtt{d}(\vartheta, \mathbb{M}\vartheta) > 0$. Now, we obtain

$$\begin{aligned}
1 < \mathbb{G}(\mathtt{d}(\vartheta, \mathbb{M}\vartheta)) &= \mathbb{G}(\mathtt{d}(\mathbb{M}(\mathbb{M}^{\mathtt{n}-1}\vartheta)), \mathbb{M}^2(\mathbb{M}^{\mathtt{n}-1}\vartheta)) \\
&\le [\mathbb{G}(\mathtt{d}(\mathbb{M}^{\mathtt{n}-1}\vartheta, \mathbb{M}^{\mathtt{n}}\vartheta))]^{\theta} \\
&\le [\mathbb{G}(\mathtt{d}(\mathbb{M}^{\mathtt{n}-2}\vartheta, \mathbb{M}^{\mathtt{n}-1}\vartheta))]^{\theta^2} \\
&\le \ldots \\
&\le [\mathbb{G}(\mathtt{d}(\vartheta, \mathbb{M}\vartheta))]^{\theta^{\mathtt{n}}}.
\end{aligned}$$

By the limit as $\mathtt{n} \to \infty$ in the above inequality, we conclude that $\mathbb{G}(\mathtt{d}(\vartheta, \mathbb{M}\vartheta)) = 1$. Hence, from $(\mathbb{G}_2)$, $\mathtt{d}(\vartheta, \mathbb{M}\vartheta) = 0$, a contradiction. Hence, $\mathbb{M}(\mathbb{M}^\mathtt{n}) = \mathbb{M}(\mathbb{M})$ for all $\mathtt{n} \in \mathscr{N}$. □

**Example 5.** *Let* $\mathfrak{U} = [0, \infty)$ *and* $\mathtt{d} : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ *be a map defined by*

$$\mathtt{d}(\mathtt{x}, \mathtt{y}) = |\mathtt{x} - \mathtt{y}| \ \textit{for all} \ \mathtt{x}, \mathtt{y} \in \mathfrak{U}.$$

*Consider the sequence* $\{\mathcal{S}_\mathtt{n}\}_{\mathtt{n} \in \mathscr{N}}$ *defined as*

$$\mathcal{S}_\mathtt{n} = \frac{\mathtt{n}}{6}(2\mathtt{n} + 1)(\mathtt{n} + 1) \ \textit{for all} \ \mathtt{n} \in \mathscr{N} \cup \{0\}.$$

*Define a relation* $\perp$ *on* $\mathfrak{U}$ *by*

$$\mathtt{x} \perp \mathtt{y} \Longleftrightarrow \mathtt{x}\mathtt{y} \in \{\mathtt{x}, \mathtt{y}\} \subseteq \{\mathcal{S}_\mathtt{n}\}.$$

*Thus,* $(\mathfrak{U}, \perp, \mathtt{d})$ *is an orthogonal complete metric space. Now, we will define a map* $\mathbb{M} : \mathfrak{U} \to \mathfrak{U}$ *by*

$$\mathbb{M}\mathtt{x} = \begin{cases} \mathcal{S}_0, & \textit{if } \mathcal{S}_0 \le \mathtt{x} \le \mathcal{S}_1, \\ \frac{\mathcal{S}_{\mathtt{n}-1}(\mathcal{S}_{\mathtt{n}+1} - \mathtt{x}) + \mathcal{S}_\mathtt{n}(\mathtt{x} - \mathcal{S}_\mathtt{n})}{\mathcal{S}_{\mathtt{n}+1} - \mathcal{S}_\mathtt{n}}, & \textit{if } \mathcal{S}_\mathtt{n} \le \mathtt{x} \le \mathcal{S}_{\mathtt{n}+1}, \textit{for each } \mathtt{n} \ge 1. \end{cases}$$

*Now, we show that $\mathbb{M}$ is $\perp$-preserving. Let $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$ and $\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) > 0$. Then, $\mathbb{M}$ is an orthogonal-preserving. Now, we show that $\mathbb{M}$ is an $O-(\alpha-\mathfrak{y}-\mathbb{G})$-contractive map. Define $\alpha, \mathfrak{y} : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ by $\alpha(\mathtt{x}, \mathtt{y}) = \mathtt{d}(\mathtt{x}, \mathtt{y})$ and $\mathfrak{y}(\mathtt{x}, \mathtt{y}) = \frac{1}{2(1+\delta)}\mathtt{d}(\mathtt{x}, \mathtt{y})$ for all $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$, where $\delta > 0$. Now, assume that there exists some $\theta \in (0, 1)$ such that*

$$\frac{1}{2(1+\delta)}\mathtt{d}(\mathtt{x}, \mathbb{M}\mathtt{x}) \leq \mathtt{d}(\mathtt{x}, \mathtt{y}) \textit{ implies } \mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^{\theta} \tag{18}$$

*for $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$, $\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) > 0$, where $\mathbb{G} \in \Psi$. Since, $\frac{1}{2(1+\delta)}\mathtt{d}(\mathtt{x}, \mathtt{x}) \leq \mathtt{d}(\mathtt{x}, \mathtt{y})$ for all $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$, so $\mathfrak{y}(\mathtt{x}, \mathtt{y}) \leq \alpha(\mathtt{x}, \mathtt{y})$ for all $\mathtt{x}, \mathtt{y} \in \mathfrak{U}$ with $\mathtt{x} \perp \mathtt{y}$. Let $\mathfrak{y}(\mathtt{x}, \mathbb{M}\mathtt{x}) \leq \alpha(\mathtt{x}, \mathtt{y})$. So, $\frac{1}{2(1+\delta)}\mathtt{d}(\mathtt{x}, \mathbb{M}\mathtt{x}) \leq \mathtt{d}(\mathtt{x}, \mathtt{y})$. Then, from (18), we obtain*

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^{\theta}. \tag{19}$$

*Hence, all conditions of Theorem 3 hold and $\mathbb{M}$ has a unique fixed point.*

## 4. Application

Consider the ordinary differential equation

$$\begin{cases} -\frac{\mathtt{d}^2\mathtt{x}}{\mathtt{d}\delta^2} = \mathbb{I}(\delta, \mathtt{x}(\delta)), \delta \in [0, 1], \\ \mathtt{x}(0) = \mathtt{x}(1) = 0. \end{cases} \tag{20}$$

where $\mathbb{I} : [0, 1] \times \mathfrak{R} \to \mathfrak{R}$ is a continuous function and the space of all continuous functions $\mathbb{C}(\mathbb{I})$ be defined on $\mathbb{I} = [0, 1]$. Assume that $\mathtt{d}(\mathtt{x}, \mathtt{y}) = |\mathtt{x} - \mathtt{y}|$ for all $\mathtt{x}, \mathtt{y} \in \mathbb{C}(\mathbb{I})$. Clearly, $(\mathbb{C}(\mathbb{I}), \mathtt{d})$ is an orthogonal complete metric space. Assume that the following conditions hold:

($i$) There exists a map $\xi : \mathfrak{R}^2 \to \mathfrak{R}$ such that for all $\mathfrak{w}, \varkappa \in \mathfrak{R}$ with $\xi(\mathfrak{w}, \varkappa) \geq 0$, we have $|\mathbb{I}(\delta, \mathfrak{w}) - \mathbb{I}(\delta, \varkappa)| \leq (|\mathfrak{w} - \varkappa|)^{\theta}$ for all $\delta \in \mathbb{I}, \theta \in (0, 1)$,

($ii$) There exists $\mathtt{x}_1 \in \mathbb{C}(\mathbb{I})$ such that for all $\delta \in \mathbb{I}$, $\xi(\mathtt{x}_1(\delta), \int_0^1 \mathbb{I}(\delta, \mathtt{x}_1(\delta))\mathtt{d}\delta) \geq 0$,

($iii$) For all $\delta \in \mathbb{I}$ and for all $\mathtt{x}, \mathtt{y}, \mathtt{z} \in \mathbb{C}(\mathbb{I}), \xi(\mathtt{x}(\delta), \mathtt{y}(\delta)) \geq 0$ and $\xi(\mathtt{y}(\delta), \mathtt{z}(\delta)) \geq 0$ imply $\xi(\mathtt{x}(\delta), \mathtt{z}(\delta)) \geq 0$,

($iv$) For all $\delta \in \mathbb{I}$ and for all $\mathtt{x}, \mathtt{y} \in \mathbb{C}(\mathbb{I})$.

We can now guarantee that the prescribed second order differential equation has a solution. The above procedure demonstrates similar results, but differs from [18].

**Theorem 6.** *Assume the conditions (i)–(iv) are satisfied. Then,* (20) *has at least one solution* $\mathtt{x}^* \in \mathbb{C}(\mathbb{I})$.

**Proof.** Let $\mathfrak{U} = \{\mathtt{x} \in \mathbb{C}(\mathbb{I}, \mathfrak{R}) : \mathtt{x}(\delta) > 0\} \forall \delta \in \mathbb{I}$. We consider the following orthogonality relation in $\mathfrak{U}$:

$$\mathtt{x} \perp \mathtt{y} \Longleftrightarrow \mathtt{x}(\delta)\mathtt{y}(\delta) \geq (\mathtt{x}(\delta) \vee \mathtt{y}(\delta))$$

for all $\delta \in \mathbb{I}$. Clearly, $(\mathfrak{U}, \perp, \mathtt{d})$ is an orthogonal metric space.

Next, we will prove that $\mathfrak{U}$ is an $\mathscr{O}$-complete (not necessarily complete). Consider the $\mathscr{O}$-Cauchy sequence $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ in $\mathfrak{U}$. Easily, we demonstrate that $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ is convergent to a point $\mathtt{x} \in \mathbb{C}(\mathbb{I})$. It is enough to prove that $\mathtt{x} \in \mathfrak{U}$. Fix $\delta \in \mathbb{I}$. Since $\perp$-preserving, we have

$$\mathtt{x}_\mathtt{n}(\delta)\mathtt{x}_{\mathtt{n}+1}(\delta) \geq (\mathtt{x}_\mathtt{n}(\delta) \vee \mathtt{x}_{\mathtt{n}+1}(\delta))$$

for each $\mathtt{n} \in \mathscr{N}$. Since $\mathtt{x}_\mathtt{n}(\delta) > 0$ for all $\mathtt{n} \in \mathscr{N}$, there exists a subsequence $\{\mathtt{x}_{\mathtt{n}(\theta)}\}$ in $\{\mathtt{x}_\mathtt{n}\}_{\mathtt{n}=1}^{\infty}$ for which $\{\mathtt{x}_{\mathtt{n}(\theta)}(\delta)\} \geq 1$ for each $\theta \in \mathscr{N}$. It is convergence to real numbers $\mathtt{x}(\delta) \implies \mathtt{x}(\delta) \geq 1$. But since $\delta \in \mathbb{I}$ is arbitrary, it shows that $\mathtt{x} \geq 1$ and hence $\mathtt{x} \in \mathfrak{U}$.

It is easily shown that $\mathtt{x}^* \in \mathbb{C}(\mathbb{I})$ is a solution of (20) if $\mathtt{x}^* \in \mathbb{C}(\mathbb{I})$ is a solution of the integral equation. A self-map $\mathbb{M} : \mathbb{C}(\mathbb{I}) \to \mathbb{C}(\mathbb{I})$ is defined by

$$\mathbb{M}\mathtt{x}(\delta) = \int_0^1 \mathbb{I}(\delta, \mathtt{x}(\delta))\mathtt{d}\delta \ \text{ for all } \ \delta \in \mathbb{I}.$$

Therefore, the differential equation (20) makes it easy to find $\mathtt{x}^* \in \mathbb{C}(\mathbb{I})$, i.e., a fixed point of $\mathbb{M}$. Let $\mathtt{x}, \mathtt{y} \in \mathbb{C}(\mathbb{I})$ such that $\xi(\mathtt{x}(\delta), \mathtt{y}(\delta)) \geq 0$ for all $\delta \in \mathbb{I}$. From (i), we have

$$\begin{aligned}
|\mathbb{M}\mathtt{x}(\delta) - \mathbb{M}\mathtt{y}(\delta)| &= \left| \int_0^1 [\mathbb{I}(\delta, \mathtt{x}(\delta)) - \mathbb{I}(\delta, \mathtt{y}(\delta))]\mathtt{d}\delta \right| \\
&\leq \int_0^1 |\mathbb{I}(\delta, \mathtt{x}(\delta)) - \mathbb{I}(\delta, \mathtt{y}(\delta))|\mathtt{d}\delta \\
&\leq \int_0^1 |\mathtt{x}(\delta) - \mathtt{y}(\delta)|^\theta \mathtt{d}\delta \\
&= |\mathtt{x}(\delta) - \mathtt{y}(\delta)|^\theta \int_0^1 \mathtt{d}\delta \\
&= |\mathtt{x}(\delta) - \mathtt{y}(\delta)|^\theta.
\end{aligned}$$

This implies that

$$\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y}) \leq (\mathtt{d}(\mathtt{x}, \mathtt{y}))^\theta.$$

Moreover, we find that

$$\mathbb{G}(\mathtt{d}(\mathbb{M}\mathtt{x}, \mathbb{M}\mathtt{y})) \leq \mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))^\theta \leq [\mathbb{G}(\mathtt{d}(\mathtt{x}, \mathtt{y}))]^\theta$$

for all $\mathtt{x}, \mathtt{y} \in \mathbb{C}(\mathbb{I})$ such that $\xi(\mathtt{x}(\delta), \mathtt{y}(\delta)) \geq 0$ for all $\delta \in \mathbb{I}$.

Therefore, all the conditions of the Theorem 6 are satisfied. Hence, $\mathbb{M}$ has a fixed point $\mathtt{x}^* \in \mathbb{C}(\mathbb{I})$ such that $\mathbb{M}\mathtt{x}^* = \mathtt{x}^*$ is a solution of (20). □

## 5. Conclusions

In this manuscript, we established the notion of $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contraction with an orthogonal metric space. We established certain fixed point theorems in these $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contractions on an orthogonal metric space. We gave an application of differential equations to support our finding fixed point results via $O - (\alpha - \mathfrak{y} - \mathbb{G})$-contraction on an orthogonal metric space.

**Author Contributions:** Conceptualization, G.N., A.J.G. and I.A.B.; methodology, G.N., A.J.G. and I.A.B.; validation, X.L., G.N., A.U.H., A.J.G. and I.A.B.; formal analysis, G.N., A.J.G. and I.A.B.; investigation, X.L., G.N., A.U.H., A.J.G. and I.A.B.; writing—original draft preparation, G.N., A.J.G. and I.A.B.; writing—review and editing, X.L., G.N., A.U.H., A.J.G. and I.A.B. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** No data were used to support this study.

**Acknowledgments:** The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article. The work is partially supported by National Natural Science Foundation of China (Grant No. 11872043), Natural Science Foundation of Sichuan Province (Grant No. 2023NSFSC1299), Fund Project of Sichuan University of Science and Engineering in hit-haunting for talents (Grant No. 2022RC04), 2021 Innovation and Entrepreneurship Training Program for College Students of Sichuan University of Science and Engineering(Grant No. cx2021150), 2022 Graduate Innovation Project of Sichuan University of Science and Engineering (Grant No. Y2022190). The work was also partially supported by the Higher Education Commission of Pakistan and the author, Imran Abbas Baloch would like to thanks HEC for this support.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Droz, M.; Stierlin, R.; Hersch, J.; Gutknecht, M.; Henrici, P.; Boutellier, R.; Burckhardt, J.J.; Gander, W. Buchbesprechungen. *Z. Angew. Math. Phys.* **1985**, *36*, 784–788. [CrossRef]
3. Emmanouil, Z. An extension of Banach's contraction principle. *Proc. Am. Math. Soc.* **2010**, *12*, 7–10.
4. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha - \psi$ -contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
5. Salimi, P.; Latif, A.; Hussain, N. Modified $\alpha - \Psi$-contractive mappings with applications. *Fixed Point Theory Appl.* **2013**, *2013*, 19. [CrossRef]
6. Hussain, N.; Kutbi, M.A.; Salimi, P. Fixed point theory in $\alpha$-complete metric spaces with applications. *Abstr. Appl. Anal.* **2014**, *2014*, 11. [CrossRef]
7. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequal. Appl.* **2014**, *2014*, 8. [CrossRef]
8. Zheng, D.; Cai, Z.; Wang, P. New fixed point theorems for $\theta - \phi$-contraction in complete metric spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 2662–2670. [CrossRef]
9. Gordji, M.E.; Rameani, M.; Sen, M.D.L.; Cho, Y.J. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory* **2017**, *18*, 569–578. [CrossRef]
10. Hussain, N.; Parvaneh, V.; Samet, B.; Vetro, C. Some fixed point theorems for generalized contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 17. [CrossRef]
11. Ahmad, J.; Al-Mazrooei, A.E.; Cho, Y.J.; Yang, Y.O. Fixed point results for generalized Θ-contractions, *J. Nonlinear Sci. Appl.* **2017**, *10*, 2350–2358. [CrossRef]
12. Sawangsup, K.; Sintunavarat, W.; Cho, Y.J. Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces. *J. Fixed Point Theory Appl.* **2020**, *22*, 1–14 . [CrossRef]
13. Hussain, N.; Al-Mazrooei, A.E.; Ahmad, J. Fixed point results for generalized $(\alpha - \eta) - \Theta$-contractions with applications. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4197–4208. [CrossRef]
14. Karapinar, E. $\alpha - \psi$-Geraghty contraction type mappings and some related fixed point results. *Filomat* **2014**, *28*, 37–48. [CrossRef]
15. Prakasam, S.K.; Gnanaprakasam, A.J.; Mani, G.; Jarad, F. Solving an integral equation via orthogonal generalized $\alpha - \psi$-Geraghty contractions. *AIMS Math.* **2023**, *8*, 5899–917. [CrossRef]
16. Nallaselli, G.; Gnanaprakasam, A.J.; Mani, G.; Ege, O. Solving integral equations via admissible contraction mappings. *Filomat* **2022**, *36*, 4947–4961. [CrossRef]
17. Nallaselli, G.; Gnanaprakasam, A.J.; Mani, G.; Javed, K.; Almalki, Y. Solution of nonlinear Fredholm integral equations on almost Z-contraction. *J. Math.* **2023**, *2023*, 1–9. [CrossRef]
18. Nazam, M.; Aydi, H.; Hussain, A. Existence theorems for $(\psi, \phi)$-orthogonal interpolative contractions and an application to fractional differential equations. *Optimization* **2023**, *72*, 1899–1929. [CrossRef]
19. Nazam, M.; Javed, K.; Arshad, M. The $(\psi, \phi)$-orthogonal interpolative contractions and an application to fractional differential equations. *Filomat* **2023**, *37*, 1167–1185. [CrossRef]
20. Nazam, M.; Chandok, S.; Hussain, A.; Sulmi, H.H.A. On the iterative multivalued $\perp$-preserving mappings and an application to fractional differential equation. *Axioms* **2023**, *12*, 53. [CrossRef]