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A Novel Concept of Complex Anti-Fuzzy Isomorphism over Groups

Bander Almutairi [1], Arshad Ali [2,*], Qin Xin [3] and Adnan Khan [2]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[2] Department of Mathematics, National College of Business Administration and Economics, Lahore 54660, Pakistan
[3] Faculty of Science and Technology, University of the Faroe Islands, FO 100 Torshavn, Faroe Islands, Denmark
* Correspondence: arshadali@ncbae.edu.pk

Abstract: In this research article, the fundamental properties of complex anti-fuzzy subgroups as well as the influence of group homomorphisms on their characteristics are investigated. Both the necessary and sufficient conditions for a complex anti-fuzzy subgroup are defined; additionally, the image, inverse image and some vital primary features of complex anti-fuzzy subgroups are examined. Moreover, the homomorphic and isomorphic relations of complex anti-fuzzy subgroups under group homomorphism are discussed, and numerical examples for various scenarios to describe complex anti-fuzzy symmetric groups are provided. Finally, it is proven that every homomorphic image of a complex anti-fuzzy cyclic group is cyclic, but the converse may not be true.

Keywords: complex anti-fuzzy homomorphism; complex anti-fuzzy cyclic group; complex anti-isomorphic images; anti-fuzzy subgroups

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1. Introduction

A homomorphism in algebra is a map that sustains the category relationship between two algebraic structures (such as two groups, two rings, or two vector spaces). An isomorphism preserves attributes including the group's order, whether it is abelian or not, the number of elements for each order, etc. Evaluating the homomorphisms inside the symmetric groups is a prominent way to examine a group structure. Using the features of homomorphisms, someone could easily and frequently demonstrate a fact regarding the presence of subgroups, the number of subgroups, the order of the elements, etc.

Fuzzy theory is a fascinating and interesting idea that is used in various departments of agricultural science, mathematics, environmental engineering, and space science to find desired results. Many disciplines have adopted this unique theory, including coding theory, transportation timelines, protein structure analysis, and diagnostic imaging techniques. Fuzzy theory also has many applications in algebraic structures. Group theory is an important part of algebra. It is used in physics, chemistry, computer science, picture creation, painting, diagnosis, and medicine [1–5]. Isomorphism is an absurdly general concept and is just a way of saying that two things are the same. The notions of images and pre-images are essential to classical homomorphism in group theory.

Life is fraught with uncertainty, which is inescapable. This universe was not created using precise dimensions or assertions. Obvious judgments are not always possible to achieve. Dealing with mistakes made when making decisions is a big difficulty for us. To deal with uncertainty in real-world circumstances, Zadeh [6] introduced the first concept of a fuzzy set, deviating from the traditional idea of yes or no in 1965. A function from $\mathbb{T} \to [0,1] \subseteq \mathbb{L}$ in classical fuzzy set theory represents the fuzzy subset of a crisp set T. A fuzzy subset T of a crisp set $\mathbb{L}$ is defined as $\{(l, \eta_T(l)) : l \in \mathbb{L}\}$; therefore $\eta_T : L \longrightarrow [0,1]$ η_T is membership function of T and $\eta_T(l)$ is degree of true value of l in $\mathbb{T}$. By enlarging the function limit from

$\{0, 1\}$ to $[0, 1]$, one can show that fuzzy sets are generalizations of the membership functions of classical sets. Several concepts have been created to deal with uncertain and ambiguous situations after the deployment of fuzzy sets. From its very beginnings, the idea of fuzzy sets has been expanded in lots of ways and across different applications such as artificial intelligence, data science, management science, electronics, statistical inference, optimization techniques, mathematical logic, industrial engineering, and automation, and the use of fuzzy technology is widespread in different industries. Al-Shami [7] introduced the concept of (2,1)-fuzzy sets, compared them to IFSs, and employed aggregation operators to solve multi-criteria decision-making issues. Al-Shami and Mhemdi [8] invented a successful method using aggregation operators for dealing with multi-criteria decision-making problems and introduced (m, n)-fuzzy sets, which are a useful tool for dealing with problems involving membership and non-membership. The idea of fuzzy subgroups was introduced by Rosenfeld [9], who also established the connection between group theory and fuzzy sets. Jin and Kim [10] investigated fuzzy symmetric subgroups of symmetric groups and discovered some of their features. A novel class of *t*-IF-subgroups was created by Gulzar et al. [11]. Additionally, they investigated the t-IF centralizer, normalizer, t-intuitionistic abelian subgroups of symmetric group with help of symmetric groups S_3 and V_4 as well as the dihedral group. Gulzar et al. [12] presented the novel concept of a Q-complex fuzzy sub-ring, investigated its various algebraic features, and extended this concept to propose the notion of the direct product of two Q-complex fuzzy sub-rings. Atanassov [13] presented the intuitionistic fuzzy set, which is an extension of fuzzy set. The elements of an intuitionistic fuzzy subset $\mathbb{T}$ of a crisp set $\mathbb{L}$ are defined as $\{(l,\ \eta_T(l),\ \overline{\eta_T}(l)) : l \in \mathbb{L}\}$, where $\eta_T(l)$ and $\overline{\eta}_T(l)$ are the affiliation and non-affiliation functions, respectively, such that $0 < \eta_T(l) + \overline{\eta}_T(l) \leq 1$ for all $l \in \mathbb{L}$. In comparison to traditional fuzzy sets, the favorable and unfavorable affiliation functions of the intuitionistic fuzzy sets indicate that they can tackle the ambiguous and uncertain situations in physical challenges, particularly in the area of decision-making, and categorization of the symmetries of molecules, atoms, and crystal structure [14]. Biswas [15] discussed the concept of fuzzy subgroups, anti-fuzzy subgroups, and established links among complements of the fuzzy subgroup and the anti-fuzzy subgroup as well as the basic algebraic principles and outcomes of the anti-fuzzy subgroup. After that, various researchers made some important findings on the theory of anti-fuzzy subgroups. Many researchers have discovered from their applications that none of these algorithms are effective for compensating for partial understanding and modification of data when they are used over a certain period of time. Furthermore, as data processing (regularity) changes, the uncertainty in our normal way of life and the ambiguity in the data also changes. As a consequence, there is a loss in data following the processing because the present theories are insufficient to account for the information. In order to overcome this, Ramot et al. [16,17] established the idea of a complex fuzzy set; the unit disc broadens the range of affiliation functions from real numbers to complex numbers. A complex fuzzy set is an object $\{l, \nu_T(l) = \mu_T(l)e^{i\eta_T(l)} : l \in \mathbb{L}\}$, where $\mu_T(l) \to [0, 1]$ is the membership value of the real part of a complex number and $\eta_T(l) \to [0, 2\pi]$ is a membership value of the imaginary part of a complex number. Therefore, a complex fuzzy set only examines the degree of affiliation rather than the degree of non-affiliation of data items, which in actuality play an equal role in the decision-making mechanism for analyzing the system, and it only gives weight to the degree of affiliation. Unfortunately, it is frequently challenging to represent affiliation degree approximation by a precise fuzzy set value in the real world. In certain circumstances, it could be simpler to express the real-world ambiguity and vagueness using two-dimensional information as compared to one-dimensional information. Furthermore, an expansion of scientific theories may be quite beneficial in explaining ambiguities based on apprehensive judgment in a difficult decision-making situation. Because of this, Alkouri and Salleh [18,19] enhanced the concept of the complex fuzzy set to complex intuitionistic fuzzy sets by integrating a complex degree of non-affiliation functions and their fundamental attributes; additionally, they explored the fundamental concepts of complex intuitionistic fuzzy sets. A complex intuitionistic fuzzy

set is an object $\{l, \nu_T(l) = \mu_T(l)e^{i\eta_T(l)}, \overline{\nu}_T(l) = \overline{\mu}_T(l)e^{i\overline{\eta}_T(l)} : l \in \mathbb{L}\}$, where $\mu_T(l) \to [0,1]$ is the membership value of the real part of a complex number, $\overline{\mu}_T(l) \to [0,1]$ is the non-membership value of the real part of a complex number, $\eta_T(l) \to [0, 2\pi]$ is the membership value of the imaginary part of a complex number and $\overline{\eta}_T(l) \to [0, 2\pi]$ is a non-membership value of the imaginary part of a complex number such that $0 < \mu_T(l) + \overline{\mu}_T(l) \leq 1$ and $0 < \eta_T(l) + \overline{\eta}_T(l) \leq 2\pi$ for all complex numbers $l \in L$.

Choudhury et al. [20] introduced the idea of a fuzzy homomorphism over two groups to investigate the effect on fuzzy subgroups. Fuzzy homomorphism and isomorphism between two fuzzy groups were presented by Fang [21] in a systematic way. The fuzzy homomorphism (isomorphism) theorem for fuzzy groups is presented using the idea of ϕ fuzzy quotient group. Dobritsa and Yakh [22] developed the concept of preserving a fuzzy operation in terms of its correctness to be clarified by a variety of homomorphisms of various characteristics of normal groups, such as features particular to systems with a fuzzy operation, were taken into consideration. Abdullah and Jeyaraman proposed a novel extension for the anti-homomorphism of two fuzzy groups, gave numerous findings that are similar to findings found for group homomorphism, and also described certain characteristics of level subgroups of fuzzy subgroups with regard to homomorphism and anti-homomorphism [23,24]. The concept of homomorphism is a structure-preserving map between two algebraic structures of the same type, this is essential to understand how molecules' symmetric bonds are formed [25,26]. Keeping the advantages of complex anti-fuzzy sets and taking the importance of group homomorphism, this article presents the homomorphic structure of a complex anti-fuzzy subgroup of abelian subgroups and a complex anti-fuzzy subgroup of cyclic subgroups.

The following is the motivation for this novel work:

1. Yun et al. [27] presented the idea of fuzzy homomorphisms and derived some fundamental results with the help of related examples. Harish and Rani [28] provided an idea of complex intuitionistic fuzzy sets that accommodate two-dimensional information in a single set and are characterized by complex-valued membership and non-membership functions of the unit disc on a complex plane.
2. Fang [21] presented the novel idea of fuzzy homomorphism and fuzzy isomorphism between two fuzzy groups and studied some of their fundamental characteristics. Palaniappan et al. [29] examined the fundamental results of an intuitionistic anti-fuzzy subgroup of a group with homomorphism and anti-homomorphism.
3. Ramot et al. [16,17] established the idea of a complex fuzzy set. Salleh [30] presented the concept of complex fuzzy subgroups. Our manuscript is motivated by the fact that a complex fuzzy set considers only the degree of membership but does not assign weight to the non-member portion of data entities, which also assume an equal part in the decision-making process. However, we apply the complex anti-fuzzy set to group homomorphism and isomorphism, which are the most important parts of algebraic structure.

Objectives

1. To propose the idea of a complex anti-fuzzy subgroup in a unit disk, various fundamental results, and a number of fundamental findings.
2. To introduce complex anti-fuzzy images and pre-images under classical group homomorphism and examine their primary features. Moreover, we define homomorphic and isomorphic relations over the complex anti-fuzzy subgroup using a relevant numerical illustration.
3. To demonstrate both necessary and sufficient conditions for a complex anti-fuzzy subgroup. We prove that every homomorphic image of a complex anti-fuzzy cyclic group is cyclic, but the converse may not be true and also shows that kernel (K) is a normal complex anti-fuzzy subgroup.

2. Preliminaries

In this section, we define some basic definitions and results which are essential to our intense argument and research analysis.

Definition 1. *If $\mathbb{H}$ is an universal set and* c *is an unspecified element of $\mathbb{H}$, then an anti-fuzzy set $\overline{\varphi}$ is defined as $\overline{\varphi} = \{(\mathrm{c}, 1-\lambda = \overline{\lambda}), \mathrm{c} \in \mathbb{H}\}$, where λ is a degree of belonging and $\overline{\lambda}$ is a degree of non-belonging such that λ, $\overline{\lambda} \in [0,1]$ and $\lambda + \overline{\lambda} = 1$.*

Definition 2. *A complex fuzzy set S of a universe set $\mathbb{L}$ denotes the degree of membership $\theta_S(\mathrm{c}) = \nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})}$ and is characterized as $\theta_S : \mathbb{L} \to \{c \in C : |\mathrm{c}| \leq 1\}$. This membership function obtains all truth degree from the complex plane's unit disc, when $i = \sqrt{-1}$, $\nu_S(\mathrm{c})$ and $\eta_S(\mathrm{c})$ are both true valued including $\nu_S(\mathrm{c}) \in [0,1]$ and $\eta_S(\mathrm{c}) \in [0, 2\pi]$ for all $\mathrm{c} \in \mathbb{L}$.*

Definition 3. *A complex fuzzy set $\overline{\mathbb{T}}$ of a universe set $\mathbb{L}$ denotes the degree of non-membership by $\overline{\theta}_{\overline{\mathbb{T}}}(\mathrm{c}) = \overline{\nu}_{\overline{\mathbb{T}}}(\mathrm{c})e^{i\overline{\eta}_{\overline{\mathbb{T}}}(\mathrm{c})}$ and defined as $\overline{\theta}_{\overline{\mathbb{T}}} : \mathbb{L} \to \{\mathrm{c} \in C : |\mathrm{c}| \leq 1\}$. This non-membership function obtains all fall degrees from the complex plane's unit disc, when $i = \sqrt{-1}$, $\overline{\nu}_S(\mathrm{c})$ and $\overline{\eta}_{\overline{\mathbb{T}}}(\mathrm{c})$ are both non-member values including $\overline{\nu}_{\overline{\mathbb{T}}}(\mathrm{c}) \in [0,1]$ and $\overline{\eta}_{\overline{\mathbb{T}}}(\mathrm{c}) \in [0, 2\pi]$ for all $\mathrm{c} \in \mathbb{L}$.*

Definition 4 ([13])**.** *Assume $S = \left\{\nu_S(l)e^{i\eta_S(l)}\right\}$ and $T = \left\{\nu_T(l)e^{i\eta_T(l)}\right\}$ are both complex anti-fuzzy sets of G. Then:*

1. *A complex anti-fuzzy set S is homogeneous complex anti-fuzzy set, if $\forall\, l, m \in G$. There will be $\nu_S(l) \leq \nu_S(m)$ if and only if $\eta_S(l) \leq \eta_S(m)$.*
2. *A complex anti-fuzzy set S is a homogeneous complex anti-fuzzy set with T if $\forall\, l, m \in G$. There will be $\nu_S(l) \leq \nu_T(l)$ if and only if $\eta_S(l) \leq \eta_T(l)$.*

Definition 5. *Assume that $S = \{(l,\ S(l)) :\ l \in G\}$ is an anti-fuzzy subset. Now, the set $S_\pi = \{(l,\ S_\pi(l)) : S_\pi(l) = 2\pi S(l), l \in G\}$ is known as a π-anti-fuzzy subset.*

Definition 6. *A π-anti-fuzzy set S_π of group G is known as a π-anti-fuzzy subgroup of G if the following conditions are satisfied*

1. $S_\pi(\mathrm{c}m) \leq \max\{S_\pi(\mathrm{c}), S_\pi(m)\}, \forall\, \mathrm{c}, m \in G,$
2. $S_\pi(\mathrm{c}^{-1}) \leq S_\pi(\mathrm{c}), \forall\, \mathrm{c}, m \in G.$

Definition 7. *Let S be a complex anti-fuzzy set of group G. Then, S is known as a complex anti-fuzzy subgroup of group G if the following criteria are fulfilled.*

1. $\nu_S(\mathrm{c}m)e^{i\eta_S(\mathrm{c}m)} \leq \max\left\{\nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})},\ \nu_S(m)e^{i\eta_S(m)}\right\},$
2. $\nu_S\left(\mathrm{c}^{-1}\right)e^{i\eta_S\left(\mathrm{c}^{-1}\right)} \leq \nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})}$, *for all* $\mathrm{c}, m \in G$.

The following example shows that T is a complex anti-fuzzy subgroup of the symmetric group S_3.

Example 1. *Suppose that $G = S_3 = \{i, (13), (12), (23), (132), (123)\}$ is a symmetric group. Consider T to be the complex anti-fuzzy set of G. This means that T is an anti-fuzzy set as well as a π-anti-fuzzy set*

$$T = \begin{cases} 0 & \text{if } u = i \\ 0.5e^{i\pi/2}, & \text{if } u^2 = i \\ 0.7e^{i3\pi/2}, & \text{if } u^3 = i \end{cases}$$

where $u \in G$ and i is the identity element. Clearly, by Definition (7), T satisfied the properties of a complex anti-fuzzy subgroup. So, T is a complex anti-fuzzy symmetric subgroup of group G.

Definition 8. *Let $S = \left\{\nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})}\right\}$ and $T = \left\{\nu_T(\mathrm{c})e^{i\eta_T(\mathrm{c})}\right\}$ be a complex anti-fuzzy subsets of set L. Then, the intersection and union of S and T are examined as:*

1. $(S \cap T)(\mathrm{c}) = \nu_{S\cap T}(\mathrm{c})e^{i\eta_{S\cap T}(\mathrm{c})} = \max\left\{\nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})},\ \nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})}\right\}$, $\forall\, \mathrm{c} \in L$.
2. $(S \cup T)(\mathrm{c}) = \nu_{S\cup T}(\mathrm{c})e^{i\eta_{S\cup T}(\mathrm{c})} = \min\left\{\nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})},\ \nu_S(\mathrm{c})e^{i\eta_S(\mathrm{c})}\right\}$, $\forall\, \mathrm{c} \in L$.

3. Homomorphism Under the Complex Anti-Fuzzy Subgroup

In this section, the basic characteristics of anti-fuzzy subgroups as well as the impact of group homomorphisms on those traits with the support of an illustrations are discussed.

Definition 9. *Let $\kappa : \mathbb{G} \longrightarrow \mathbb{H}$ be a group homomorphism. Let $\mathbb{T} = \{(u, \nu_T(u)) : u \in \mathbb{G}\}$ and $\mathbb{E} = \{(u, \nu_E(u)) : u \in \mathbb{H}\}$ be anti-fuzzy subgroups. Then, $\mathbb{C} = \{(v, \nu_T(v)) : v \in \mathbb{H}\}$ is known an* **image** *of $\mathbb{T}$, where*

$$\kappa(\nu_T)(v) = \begin{cases} \inf\{\nu_T(u) | u \in \mathbb{G}, \kappa(u) = v\} & \text{if } \kappa^{-1}(u) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all $v \in \mathbb{H}$. The set $\mathbb{D} = \{u, \kappa^{-1}(\nu_E)(u) : u \in \mathbb{G}\}$ is known an **inverse image** *of $\mathbb{T}$, where $\kappa^{-1}(\nu_E)(u) = (\nu_E)(\kappa(u))$ for all $u \in \mathbb{G}$.*

In the following example, a mapping between two anti-fuzzy sets is defined, and the images and inverse of this mapping are discussed.

Example 2. *Suppose that $\mathbb{Z}_{10} \longrightarrow \mathbb{Z}_5$ is a mapping and defined as $g(m) = 4m$. Let ν and η be two anti-fuzzy sets of $\mathbb{Z}_{10}$ and $\mathbb{Z}_5$, respectively.*

$\nu = \{(0,\ 0.4), (1,\ 0.5), (2,\ 0.7), (3,\ 0.8), (4,\ 0.2), (5,\ 0.5), (6,\ 0.7), (7,\ 0.1),$
$(8,\ 0.8), (9,\ 0.2)\},$
$\eta = \{(0,\ 0.8), (1,\ 0.7), (2,\ 0.5), (3,\ 0.3), (4, '0.7)\}.$
g is a mapping from $\mathbb{Z}_{10} \longrightarrow \mathbb{Z}_5$ and is defined as
$g(0) = 0 = g(5), \quad g(3) = 2 = g(8), \quad g(2) = 3 = g(7),$
$g(1) = 4 = g(6), \quad g(4) = 1 = g(9).$
Now, the set of pre-images of g of the element of $\mathbb{Z}_5$ is as follows
$g^{-1}(0) = \{0,\ 5\}, \quad g^{-1}(1) = \{4,\ 9\}, \quad g^{-1}(2) = \{3,\ 8\},$
$g^{-1}(3) = \{2,\ 7\}, \quad g^{-1}(4) = \{1,\ 6\}.$
By using Definition (9)

$$\kappa(\nu) = \begin{cases} \inf\{\nu_T(u) | u \in \mathbb{G}, \kappa(u) = v\} & \text{if } \kappa^{-1}(u) \neq \phi \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } v \in \mathbb{Z}_5.$$

The set of images of an anti-fuzzy set is

$$g(\nu) = \{(0,\ 0.4), (1,\ 0.2), (2,\ 0.8), (3,\ 0.1), (4,\ 0.5)\}.$$

Again, from Definition (9)

$$\kappa^{-1}(\eta)(u) = (\eta)(\kappa(u)) \quad \forall u \in \mathbb{Z}_{10}.$$

Now, we obtain the anti-fuzzy set η of pre-images as follows
$g^{-1}(\eta) = \{(0,\ 0.8),\ (1,\ 0.7),\ (2,\ 0.3),\ (3,\ 0.5),\ (4,\ 0.7),\ (5,\ 0.8),\ (6,\ 0.7),\ (7,\ 0.3),$
$(8,\ 0.5),\ (9,\ 0.7)\}.$

Theorem 1. *Let $f : \mathbb{G} \longrightarrow \mathbb{H}$ be a group homomorphism. Let $\mathbb{T}$ be an anti-fuzzy subgroup of $\mathbb{G}$ and let $\mathbb{E}$ be an anti-fuzzy subgroup of $\mathbb{H}$ with non-membership functions $\nu_T(u)$ and $\nu_E(u)$, respectively. Then, the inverse image of $\mathbb{E}$ is an anti-fuzzy subgroup of $\mathbb{G}$, and the image of $\mathbb{T}$ is an anti-fuzzy subgroup of $\mathbb{H}$.*

Proof. The proof of this theorem is obvious. □

The following theorem illustrates that every complex anti-fuzzy set generates two anti-fuzzy subgroups, namely, the complex anti-fuzzy subgroup and π-anti-fuzzy subgroup.

Theorem 2. *Let $\mathbb{G}$ is group and $\mathbb{T} = \{(l, \nu_T(l)) : l \in \mathbb{G}$ be a homogenous complex anti-fuzzy set with membership function $\nu_T(l) = \mu_T(l)e^{i\eta_T(l)}$. Then, $\mathbb{T}$ is a complex anti-fuzzy subgroup of $\mathbb{G}$ if and if only:*

1. *The set $\overline{\mathbb{T}} = \{(l,\ \mu_T(l)) : l \in \mathbb{G},\ \mu_T(l) \in\ [0,1]\}$ is an anti-fuzzy subgroup.*
2. *The set $\underline{\mathbb{T}} = \{(l,\ \eta_T(l)) : l \in \mathbb{G},\ \eta_T(l) \in\ [0,2\pi]\}$ is π-anti-fuzzy subgroup.*

Proof. Suppose that $\mathbb{T}$ is a complex anti-fuzzy subgroup and $l,\ m \in \mathbb{G}$. Then, we obtain

$$\begin{aligned} \mu_T(lm)\ e^{i\eta_T(lm)} &= \nu_T(lm) \\ &\leq \max\{\nu_T(l),\ \nu_T(l)\} \\ &= \max\{\mu_T(l)\ e^{i\eta_T(l)},\ \mu_T(m)\ e^{i\eta_T(m)}\} \\ &= \max\{\mu_T(l),\ \mu_T(m)\}\ e^{i\max\{\eta_T(l),\ \eta_T(m)\}}. \end{aligned}$$

Because $\mathbb{T}$ is homogenous, then
$\mu(lm) \leq \max\{\mu_T(l),\ \mu_T(m)\}$, and $\eta(lm) \leq \max\{\eta_T(l),\ \eta_T(m)\}$. Expressed in another way

$$\begin{aligned} \mu_T(l^{-1})\ e^{i\eta_T(l^{-1})} &= \nu(l^{-1}) \\ &\leq \nu_T(l) \\ &= \mu_T(l)\ e^{i\eta_T(l)}. \end{aligned}$$

Then, we can write
$\mu_T(l^{-1}) \leq \mu_T(l)$ and $\eta_T(l^{-1}) \leq \eta_T(l)$.

Thus, $\overline{\mathbb{T}}$ is an anti-fuzzy subgroup and $\underline{\mathbb{T}}$ is a π-**anti**-fuzzy subgroup.

Conversely, let $\overline{\mathbb{T}}$ be an anti-fuzzy subgroup, and let $\underline{\mathbb{T}}$ be a π-anti-fuzzy subgroup.

$$\begin{aligned} \mu_T(lm) &\leq \max\{\mu_T(l),\ \mu_T(m)\}, \max\{\eta_T(l),\ \eta_T(m)\} \\ \eta_T(lm) &\leq \max\{\eta_T(l),\ \eta_T(m)\} \\ \mu_T(l^{-1}) &\leq \mu_T(l) \\ \eta_T(l^{-1}) &\leq \eta_T(l). \end{aligned}$$

Now,

$$\begin{aligned} \nu_T(lm) = \mu_T(lm)\ e^{i\eta_T(lm)} &\leq \max\{\mu_T(l),\ \mu_T(m)\}\ e^{i\max\{\eta_T(l),\ \eta_T(m)\}} \\ &= \max\{\mu_T(l)\ e^{i\eta_T(l)},\ \mu_T(m)\ e^{i\eta_T(m)}\} \quad [\mathbb{T} \text{ is homogeneous}] \\ &= \max\{\mu_T(l),\ \mu_T(m)\}. \end{aligned}$$

In a similar fashion,

$$\begin{aligned} \nu_T(l^{-1}) &= \mu(l^{-1})\ e^{i\eta_T(l^{-1})} \\ &\leq \max\{\mu_T(l), e^{\eta_T(l)}\} \\ &= \mu_T(x). \end{aligned}$$

So, $\mathbb{T}$ is an anti-complex fuzzy subgroup. □

4. Homomorphism Under Complex Anti-Fuzzy Subgroup

In this section, we define the image and inverse image of complex anti-fuzzy subgroups that are sensitive to group homomorphism and examine their primary features as well as isomorphic relations of complex anti-fuzzy subgroups under group homomorphism.

Definition 10. *Suppose that* $f : \mathbb{G} \longrightarrow \mathbb{H}$ *is a group homomorphism. Let* $\mathbb{T} = \{(u, \nu_T(u)) : u \in \mathbb{G}\}$, *and let* $\mathbb{E} = \{(u, \nu_E(u)) : u \in \mathbb{H}\}$ *be complex anti-fuzzy subgroups, where* $\nu_T(u) = \mu_T(u)e^{i\eta_T(u)}$ *and* $\nu_E(u) = \mu_E(u)e^{i\eta_E(u)}$ *are their non-membership functions, respectively. Then,* $\mathbb{C} = \{(v, \nu_T(v)) : v \in \mathbb{H}\}$ *is known as an* **image** *of* $\mathbb{T}$, *where*

$$\kappa(\nu_T)(v) = \begin{cases} \inf\{\nu_T(u) | u \in \mathbb{G}, \kappa(u) = v\} & \text{if } \kappa^{-1}(u) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all $v \in \mathbb{H}$. *The set* $\mathbb{D} = \{u, \kappa^{-1}(\nu_E)(u) : u \in \mathbb{G}\}$ *is known as the* **inverse image** *of* $\mathbb{E}$, *where*

$$\kappa^{-1}(\nu_E)(u) = (\nu_E)(\kappa(u)) \quad \text{for all } u \in \mathbb{G}.$$

In the following example, a mapping between two complex anti-fuzzy subgroups will be defined and discussed, along with the images and inverses of the complex anti-fuzzy subgroups of the given mapping.

Example 3. *Suppose that a mapping g:* $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_6$ *is defined as* $g(l) = 2l$. *Let* ν *and* η *be two complex anti-fuzzy sets of* $\mathbb{Z}_{12}$ *and* $\mathbb{Z}_6$, *respectively.*

$\nu = \{(0,\ 0.4e^{i\pi/3}), (1,\ 0.5e^{i\pi/2}), (2,\ 0.6e^{i\pi/2}), (3,\ 0.8e^{i3\pi/2}), (4,\ 0.2e^{i\pi/3}), (5,\ 0.3e^{i\pi/4}),$
$(6,\ 0.7e^{i3\pi/2}), (7,\ 0.2e^{i\pi/3}), (8,\ 0.4e^{i\pi/3}), (9,\ 0.9e^{i\pi/2}), (10,\ 0.8e^{i3\pi/2}), (11,\ 1e^{i2\pi})\},$

$\eta = \{(0,\ 0.5e^{i\pi/2}), (1,\ 0.8e^{i3\pi/2}), (2,\ 0.4e^{i\pi/3}), (3,\ 0.1e^{i\pi/5}), (4, `0.7e^{i\pi}), (5, `0.2e^{i\pi/4})\}.$

Given that g is a mapping from $\mathbb{Z}_{12} \longrightarrow \mathbb{Z}_6$, *we obtain*

$g(0) = g(3) = g(6) = g(9) = 0, \quad g(1) = g(4) = g(7) = g(10) = 2,$
$g(2) = g(5) = g(8) = g(11) = 4.$

Now, the set of pre-images of mapping g of the element of $\mathbb{Z}_6$ *is as follows*

$g^{-1}(0) = \{0,3,6,9\}, \quad g^{-1}(2) = \{1,4,7,10\}, \quad g^{-1}(4) = \{2,5,8,11\},$
$g^{-1}(1) = \phi, \quad g^{-1}(4) = \phi, \quad g^{-1}(5) = \phi.$

By using Definition (10)

$$g(\nu) = \begin{cases} \inf\{\nu_T(l) | u \in \mathbb{G}, g(l) = m\} & \text{if } g^{-1}(l) \neq \phi \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } m \in \mathbb{Z}_6.$$

The set of images of a complex anti-fuzzy set is obtained with

$$g(\nu) = \{(0,\ 0.4e^{i\pi/3}), (1,\ 0), (2,\ 0.2e^{i\pi/4}), (3,\ 0), (4, `0.3e^{i\pi/4}), (5,0)\}.$$

Again, from Definition (10)

$$g^{-1}(\eta)(l) = (\eta)(g(l)), \qquad \forall\, l \in \mathbb{Z}_{12}.$$

Now, the set of pre-images of the complex anti-fuzzy set η *is obtained with* $g^{-1}(\eta) = \{(0,\ 0.5e^{i\pi/2}), (1,\ 0.4e^{i\pi/3}), (2,\ 0.7e^{i\pi}), (3,\ 0.5e^{i\pi/2}), (4,\ 0.4e^{i\pi/3}),$
$(5,\ 0.7e^{i\pi}), (6,\ 0.5e^{i\pi/2}), (7,\ 0.4e^{i\pi/3}), (8,\ 0.7e^{i\pi}), (9,\ 0.5e^{i\pi/2})(10,\ 0.4e^{i\pi/3}),$
$(11,\ 0.7e^{i\pi})\}.$

Lemma 1. *Suppose that* $f : \mathbb{G} \longrightarrow \mathbb{H}$ *is a group homomorphism. Let* $\mathbb{T}$ *be a complex anti-fuzzy subgroup of* $\mathbb{G}$, *and let* $\mathbb{E}$ *be a complex anti-fuzzy subgroup of* $\mathbb{H}$, *with the non-membership functions* $\nu_T(l) = \mu_T(l)e^{i\eta_T(l)}$ *and* $\nu_E(l) = \mu_E(l)e^{i\eta_E(l)}$, *respectively. Then,*

(i) $f(\nu_T)(v) = f(\mu_T)(v)e^{if(\eta_T)(v)}$,

(ii) $f^{-1}(\nu_E)(v) = f^{-1}(\mu_E)(v)e^{if^{-1}(\eta_E)(v)}$.

Proof.

$$\begin{aligned}
(i) \qquad f(\nu_T)(v) &= \min_{f(u)=v} \nu_T(v) \\
&= \min_{f(v)=v} \mu_T(v) e^{i\eta_T(v)} \\
&= \min_{f(v)=v} \mu_T(v) e^{\min_{\{f(u)=v\}} i\eta_T(v)} \\
&= f(\mu_T)(v) e^{if(\eta_T)(v)}. \\
(ii) \qquad f^{-1}(\nu_E)(v) &= \nu_E(f(v)) \\
&= \mu_E(f(v)) e^{i\eta_E(f(v))} \\
&= f^{-1}(\mu_E)(v) e^{if^{-1}(\eta_E)(v)}.
\end{aligned}$$

□

Remark 1. *The concept of an anti-fuzzy homomorphism has several applications in different social areas of life. For example, an anti-fuzzy homomorphism is used in the positioning of an image. A homomorphic picture of a person is actually captured in a photograph, which explains many of their genuine characteristics, including whether they are tall or short, male or female, and slim or heavy. Due to various lens errors, including scale and pincushion distortion, the homomorphic picture can be eradicated. Because anti-fuzzy homomorphism and isomorphism are effective at identifying and resolving the causes of distortion and pincushion, we can use it to restore a damaged photograph to its original form by reducing scale and pincushion distortion.*

The following theorem shows that if the image of $\mathbb{T}$ is a complex anti-fuzzy subgroup, then the images of μ_T and η_T will be an anti-fuzzy subgroup and a π-anti-fuzzy subgroup, respectively.

Theorem 3. *Suppose that $f : \mathbb{G} \longrightarrow \mathbb{H}$ is a group homomorphism. Let $\mathbb{T}$ be a complex anti-fuzzy subgroup of $\mathbb{G}$, and let $\mathbb{E}$ be a complex anti-fuzzy subgroup of $\mathbb{H}$ with the non-membership functions $\nu_T(\mathrm{c}) = \mu_T(\mathrm{c})e^{i\eta_T(\mathrm{c})}$ and $\nu_E(\mathrm{c}) = \mu_E(\mathrm{c})e^{i\eta_E(\mathrm{c})}$, respectively. Then, the image of T is a complex anti-fuzzy group of $\mathbb{H}$.*

Proof. Because $\mathbb{T}$ is a complex anti-fuzzy subgroup, μ_T is an anti-fuzzy subgroup and $\eta_T(\mathrm{c})$ is a π-anti-fuzzy subgroup. To show that the image of $\mathbb{T}$ is a complex anti-fuzzy subgroup, the images μ_T and $\eta_T(\mathrm{c})$ will be an anti-fuzzy subgroup and π-anti-fuzzy subgroup, respectively. Therefore, for all $c, m \in \mathbb{H}$, we obtain,

1. $f(\mu)_T(\mathrm{c}m) \leq \max\{ f(\mu)_T(\mathrm{c}), f(\mu)_T(m)\}$ and $f(\mu)_T(\mathrm{c}) \leq \{ f(\mu)_T(\mathrm{c}^{-1})\}$
2. $f(\eta)_T(\mathrm{c}m) \leq \max\{ f(\eta)_T(\mathrm{c}), f(\eta)_T(m)\}$ and $f(\eta)_T(\mathrm{c}) \leq \{ f(\eta)_T(\mathrm{c}^{-1})\}$.

By using Lemma (1), we have

$$\begin{aligned}
f(\nu_T)(\mathrm{c}m) &= f(\mu_T)(\mathrm{c}m) e^{if(\eta_T)(\mathrm{c}m)} \\
&\leq \max\{ f(\mu)_T(\mathrm{c}), f(\mu)_T(m)\} e^{i \max\{f(\eta)_T(\mathrm{c}), f(\eta)_T(m)\}} \\
&\leq \max\{ f(\mu)_T(\mathrm{c}) e^{if(\eta)_T(\mathrm{c})}, f(\mu)_T(m) e^{if(\eta)_T(m)}\} \\
&\leq \max\{ f(\nu)_T(m), f(\nu)_T(\mathrm{c})\}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
f(\nu_T)(\mathrm{c}^{-1}) &= f(\mu_T)(\mathrm{c}^{-1}) e^{if(\eta_T)(\mathrm{c}^{-1})} \\
&= f(\mu_T)(\mathrm{c}) e^{if(\eta_T)(\mathrm{c})} \\
&= f(\nu_T)(\mathrm{c}).
\end{aligned}$$

□

The following theorem represents that if a mapping between two groups is a homomorphism $\mathbb{G} \longrightarrow \mathbb{H}$, then the inverse of the image of E is a complex anti-fuzzy group of $\mathbb{G}$.

Theorem 4. *Let $\mathfrak{f} : \mathbb{G} \longrightarrow \mathbb{H}$ be a group homomorphism. Suppose that $\mathbb{T}$ is a complex anti-fuzzy subgroup of $\mathbb{G}$ and $\mathbb{E}$ is a complex anti-fuzzy subgroup of $\mathbb{H}$ with the non-membership functions $\nu_T(l) = \mu_T(l)e^{i\eta_T(l)}$ and $\nu_E(l) = \mu_E(l)e^{i\eta_E(l)}$, respectively. Then, the inverse of the image of E is a complex anti-fuzzy group of $\mathbb{G}$.*

Proof. Because $\mathbb{E}$ is a complex anti-fuzzy subgroup of $\mathbb{H}$, μ_E is an anti-fuzzy subgroup and $\eta_E(l)$ is a π-anti-fuzzy subgroup of $\mathbb{H}$. If the inverse of the image of E is a complex anti-fuzzy group of $\mathbb{G}$, then the inverse image of μ_E and $\eta_E(l)$ will be an anti-fuzzy subgroup and a π-anti-fuzzy subgroup, respectively. Therefore, for all $l, m \in \mathbb{H}$, we obtain,

1. $\mathfrak{f}^{-1}(\mu)_E(lm) \leq \max\{\mathfrak{f}^{-1}(\mu)_T(l), \mathfrak{f}^{-1}(\mu)_T(m)\}$ and $\mathfrak{f}^{-1}(\mu)_T(l) \leq \{\mathfrak{f}^{-1}(\mu)_T(l^{-1})\}$,
2. $\mathfrak{f}^{-1}(\eta)_E(lm) \leq \max\{\mathfrak{f}^{-1}(\eta)_E(l), f\mathfrak{f}-1(\eta)_E(m)\}$ and $\mathfrak{f}^{-1}(\eta)_E(l) \leq \{\mathfrak{f}^{-1}(\eta)_E(l^{-1})\}$.

From Lemma (1) we obtain

$$\begin{aligned}
\mathfrak{f}^{-1}(\nu_E)(lm) &= \mathfrak{f}^{-1}(\mu_E)(lm)e^{i\mathfrak{f}^{-1}(\eta_E)(lm)} \\
&\leq \max\{\mathfrak{f}^{-1}(\mu)_E(l), \mathfrak{f}^{-1}(\mu)_E(m)\}e^{i\max\{\mathfrak{f}^{-1}(\eta)_E(l), \mathfrak{f}^{-1}(\eta)_E(m)\}} \\
&\leq \max\{\mathfrak{f}^{-1}(\mu)_E(l)e^{i\mathfrak{f}^{-1}(\eta)_E(l)}, \mathfrak{f}^{-1}(\mu)_E(m)e^{i\mathfrak{f}^{-1}(\eta)_E(m)}\} \\
&\leq \max\{\mathfrak{f}^{-1}(\nu)_E(m), \mathfrak{f}^{-1}(\nu)_E(l)\}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\mathfrak{f}^{-1}(\nu_E)(l^{-1}) &= \mathfrak{f}^{-1}(\mu_E)(l^{-1})e^{i\mathfrak{f}^{-1}(\eta_E)(l^{-1})} \\
&= \mathfrak{f}^{-1}(\mu_E)(l)e^{i\mathfrak{f}^{-1}(\eta_E)(l)} \\
&= \mathfrak{f}^{-1}(\nu_E)(l).
\end{aligned}$$

□

The following theorem shows whether the reciprocal mapping of an abelian group is a homomorphism.

Theorem 5. *If $\mathbb{G}$ is an abelian group and $\kappa : \mathbb{G} \longrightarrow \mathbb{G}$ is a mapping defined as $\kappa(\mathrm{c}) = \mathrm{c}^{-1}\ \forall\, \mathrm{c} \in \mathbb{G}$, then κ is a homomorphism.*

Proof. Let a mapping $\kappa : \mathbb{G} \longrightarrow \mathbb{G}$ be defined as $\kappa(\mathrm{c}) = \mathrm{c}^{-1}$, $\forall\, \mathrm{c} \in \mathbb{G}$ a homomorphism. Let $\mathbb{T} = \{(\mathrm{c}, \nu_T(\mathrm{c})) : \mathrm{c} \in \mathbb{G}\}$ be the complex anti-fuzzy subgroup of $\mathbb{G}$, with the non-membership functions $\nu_T(\mathrm{c}) = \mu_T(\mathrm{c})e^{i\eta_T(\mathrm{c})}$, where $\mu_T(\mathrm{c}) \in [0,1]$ and $\eta_T(\mathrm{c}) \in [0, 2\pi]$. Let $\mathrm{c}, m \in \mathbb{G}$; then,

$$\begin{aligned}
\nu_T(\mathrm{c}m) &= \nu_T(m^{-1}\mathrm{c}^{-1})^{-1} = \mu_T(m^{-1}\mathrm{c}^{-1})^{-1}e^{i\eta_T(m^{-1}\mathrm{c}^{-1})^{-1}} \\
&= \mu_T(\kappa(m^{-1}\mathrm{c}^{-1}))e^{i\eta_T\kappa((m^{-1}\mathrm{c}^{-1}))} \qquad \text{Since : } \kappa(\mathrm{c}) = \mathrm{c}^{-1}\ \forall\, \mathrm{c} \in \mathbb{G} \\
&= \mu_T(\kappa(m^{-1}))e^{i\eta_T(\kappa(m^{-1}))}\mu_T(\kappa(\mathrm{c}^{-1}))e^{i\eta_T(\kappa(\mathrm{c}^{-1}))} \\
&\quad \because \kappa \text{ is homomorphism} \\
&= \nu_T(\kappa(m^{-1})\kappa(\mathrm{c}^{-1})) \\
&= \nu_T(m)(\mathrm{c}) \\
(\nu_T)(\mathrm{c}m) &= (\nu_T)(m\mathrm{c}).
\end{aligned}$$

Hence, if $\mathbb{G}$ is an abelian group, then any mapping $\mathbb{G} \longrightarrow \mathbb{G}$ is a homomorphism. □

The following example shows that if $\mathbb{G}$ is an abelian group, then the κ mapping $\mathbb{G} \longrightarrow \mathbb{G}$ is a homomorphism.

Example 4. *Let $\kappa : \mathbb{G} \longrightarrow \mathbb{G}$ be a mapping and defined as $\kappa(l) = l^5 \; \forall \; l \in \mathbb{G}$ and $\mathbb{G}$ is any abelian group. Show that κ is a homomorphism.*

Let $\kappa : \mathbb{G} \longrightarrow \mathbb{G}$ is itself mapping defined as $\kappa(l) = l^5 \; \forall \; l \in \mathbb{G}$, where $\mathbb{G}$ is an abelian group and assume that $\mathbb{T} = \{(l, \nu_T(l)) : l \in \mathbb{G}\}$ be complex anti-fuzzy subgroup of $\mathbb{G}$, with the non-membership functions $\nu_T(l) = \mu_T(l)e^{i\eta_T(l)}$, where $\mu_T(l) \in [0,1]$ and $\eta_T(l) \in [0, 2\pi]$. Let $l, m \in \mathbb{G}$; then, we have

$$\begin{aligned}
\kappa(\nu_T)(lm) &= \nu_T(lm)^5 = \mu_T(lm)^5 e^{i\eta_T(lm)^5} \\
&= \mu_T((lm.lm.lm.lm.lm))e^{i\eta_T((lm.lm.lm.lm.lm))} \\
&= \mu_T((l^5m^5))e^{i\eta_T((l^5m^5))} \qquad \therefore \; \mathbb{G} \; is \;\; abelian \\
&= \mu_T(\kappa(m^5))e^{i\eta_T(\kappa(m^5))}\mu_T(\kappa(l^5))e^{i\eta_T(\kappa(l^5))} \\
&= \nu_T(\kappa(m^5)\kappa(l^5)).
\end{aligned}$$

Hence, κ is a homomorphism.

Definition 11. *Suppose that $\mathbb{G}_1$ and $\mathbb{G}_2$ are any two groups, and θ and λ are complex anti-fuzzy subgroups of $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. A homomorphism $\kappa : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is known as a homomorphism from θ to λ. If $\kappa(\theta) = \lambda$ and κ is a homomorphism from $\theta \longrightarrow \lambda$, then we state that θ is a* **homomorphism** *to λ and it is represented by*

$$\theta \approx \lambda.$$

A mapping $\kappa : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$ is known as an isomorphism from θ to λ. If $\kappa(\theta) = \lambda$ and κ is isomorphism from $\theta \longrightarrow \lambda$, then we say that θ is **isomorphic** *to λ and it represented by*

$$\theta \cong \lambda$$

In the illustrations below, we describe homomorphism and isomorphism over complex anti-fuzzy subgroups.

Example 5. *Suppose that a mapping g: $(\mathbb{Z}_8, .) \longrightarrow (\mathbb{Z}_6, +)$ is a homomorphism and defined as $g(l) = 2l$. Let $(\nu, .)$ and $(\eta, +)$ be two complex anti-fuzzy subgroups of $\mathbb{Z}_8$ and $\mathbb{Z}_6$, respectively.*

$\nu = \{(1,\ 0.1e^{i\pi/3}), (3,\ 0.5e^{i\pi/2}), (5,\ 0.6e^{i\pi}), (7,\ 0.5e^{i3\pi/2})\}$,

$\eta = \{(0,\ 0.5e^{i\pi/2}), (2,\ 0.1e^{i\pi/3}), (4,\ 0.6e^{i\pi})\}$. *(1)*

Given that g is a mapping from $\mathbb{Z}_8 \longrightarrow \mathbb{Z}_6$, then we obtain

$$g(3) = 0, \quad g(7) = g(1) = 2, \quad g(5) = 4.$$

Now, the set of pre-images of g of the element of $\mathbb{Z}_6$ is as follows

$$g^{-1}(0) = \{3\}, \qquad g^{-1}(2) = \{1,7\}, \qquad g^{-1}(4) = \{5\},$$

By using Definition (9)

$$g(\nu) = \begin{cases} \inf\{\nu_T(l) | u \in \mathbb{G}, g(l) = m\} & \text{if } g^{-1}(l) \neq \phi \\ 0, & \text{otherwise} \end{cases} \quad \textit{for all } m \in \mathbb{Z}_6.$$

The set of images of a complex anti-fuzzy set is obtained by

$$g(\nu) = \{(0,\ 0.5e^{i\pi/2}), (2,\ 0.1e^{i\pi/3}), (4,\ 0.6e^{i\pi})\}. \qquad (2).$$

By using Equations (1) and (2) we obtain

$$g(\eta) = \mu.$$

Thus, a mapping g: $(\mathbb{Z}_8, .) \longrightarrow (\mathbb{Z}_6, +)$ *is a homomorphism from* $(\nu, .) \longrightarrow (\eta, +)$.

Example 6. *Suppose that a mapping* κ: $\mathbb{Z}_4 \longrightarrow \mathbb{U}(10)$ *is a homomorphism and defined as* $\kappa(l) = 7^l$. *Let* ν *and* η *be two complex anti-fuzzy subgroups of* $\mathbb{Z}_4$ *and* $\mathbb{U}(10)$, *respectively, and established by,*

$$\nu = \{(0,\ 0.8e^{i\pi/3}), (1,\ 0.7e^{i\pi/2}), (2,\ 0.6e^{i\pi}), (3,\ 0.7e^{i3\pi/2})\},$$
$$\eta = \{(0,\ 0.2e^{i\pi/5}), (3,\ 0.4e^{i\pi/4}), (7,\ 0.6e^{i\pi/2}), (9,\ 0.5e^{i\pi})\}. \quad (1)$$

As a result, κ *is an isomorphism mapping from* $\mathbb{Z}_4 \longrightarrow \mathbb{U}(10)$; *then, we obtain*

$$\kappa(1) = 7, \quad \kappa(0) = 1, \quad \kappa(3) = 3 \quad \kappa(2) = 9.$$

Now, the set of pre-images of κ *of the element of* $\mathbb{U}(10)$ *is as follows*

$$\kappa^{-1}(9) = \{2\}, \qquad \kappa^{-1}(1) = \{0\}, \qquad \kappa^{-1}(7) = \{1\}, \qquad \kappa^{-1}(3) = \{3\}.$$

By using Definition (9)

$$\kappa(\nu) = \begin{cases} \inf\{\nu_T(l) | u \in \mathbb{G}, \kappa(l) = m\} & \text{if } \kappa^{-1}(l) \neq \phi \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } m \in \mathbb{U}(10).$$

The set of images of a complex anti-fuzzy set is obtained with

$$\kappa(\nu) = \{(0,\ 0.2e^{i\pi/5}), (3,\ 0.4e^{i\pi/4}), (7,\ 0.6e^{i\pi/2}), (9,\ 0.5e^{i\pi})\}. \quad (2)$$

By using Equations (1) and (2) we obtain,

$$\kappa(\eta) = \mu.$$

Thus, a mapping κ: $\mathbb{Z}_4 \longrightarrow \mathbb{U}(10)$ *is an isomorphism from* $\nu \longrightarrow \eta$.

Theorem 6. *Let* ν *be a complex anti-fuzzy subgroup of* $\mathbb{G}$; *then,* θ *is a homomorphism from* $\mathbb{G}_1 \longrightarrow \mathbb{G}_2$ *with a kernel* (K) *defined as* $K(\nu)(l) = \{l, \nu(l) : \theta(l) = \nu(e_2),\ \forall\ l\ \in \mathbb{G}_1$ *and* $e_2\ \in \mathbb{G}_2\}$. *Then, K is a complex anti-fuzzy normal subgroup of* $\mathbb{G}$.

Proof. Let e_1 and e_2 are identities of group $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively; then, $Ker(\theta)$ is defined as $K(\nu)(l) = \{l, \nu(l) : \theta(l) = \nu(e_2),\ \forall\ l\ \in \mathbb{G}_1$and $e_2\ \in \mathbb{G}_2\}$ where ν is a complex anti-fuzzy subgroup of $\mathbb{G}$, $\nu(l) = \mu_K(l)e^{i\eta_K(l)}, \mu \in [0,1]$, and $\eta \in [0, 2\pi]$. Therefore, θ is a homomorphism from $\mathbb{G}_1 \longrightarrow \mathbb{G}_2$ with kernel (K), and then K will be a complex anti-fuzzy normal subgroup. Then,

$$\begin{aligned} Ker(\theta): \quad K &= \{l, \nu(l) : \theta(l) = \nu(e_2), l \in \mathbb{G}_1\} \subset \mathbb{G}_1 \\ \theta_K(\nu(e_1)) &= \nu(e_2) \Rightarrow e_1 \in K \Rightarrow K \neq \phi \\ \therefore \text{ Let } l, m \in K \text{ then } \quad \theta(\nu(l)) &= \nu(e_2) \text{ and } \theta(\nu(m)) = \nu(e_2). \end{aligned}$$

Again,

$$\begin{aligned} \theta_K(\nu(lm^{-1})) &= \theta_K(\nu(l))\theta_K(\nu(m^{-1})) \quad [\ \therefore\ \theta \text{ is homomorphism }] \\ &= \theta_K(\mu(l))e^{i\theta_K(\eta(l))}\ \theta_K(\mu(m^{-1}))e^{i\theta_K(\eta(m^{-1}))} \\ &= \theta_K(\mu(l))e^{i\theta_K(\eta(l))}\ [\theta_K(\mu(m))]^{-1}e^{i[\theta_K(\eta(m))]^{-1}} \\ &= \theta(\nu(l))\ [\theta(\nu(m))]^{-1} \\ &= (e_2)\ [e_2]^{-1} = e_2e_2 = \nu(e_2) \\ &= lm^{-1} \in K. \end{aligned}$$

So, K is a complex anti-fuzzy subgroup of $\mathbb{G}_1$. Now we show that K is a normal subgroup of $\mathbb{G}$.

Thus, we have $l \in K$, $m \in K$, $lm \in K$; therefore, $Ker(\theta)$ is a complex anti-fuzzy subgroup of $\mathbb{G}_1$. Suppose that $l \in \mathbb{G}_1$ and $c \in K$; then,

$$\begin{aligned}
\theta_K(\nu(l\ c\ l^{-1})) &= \theta_K(\nu(l))\ \theta_K(\nu(c))\ \theta_K(\nu(l^{-1})) \quad [\ \because\ \theta \text{ is homomorphism }] \\
&= \theta_K(\mu(l))e^{i\theta_K(\eta(l))}\ \theta_K(\mu(c))e^{i\theta_K(\eta(c))}\ \theta_K(\mu(l^{-1}))e^{i\theta_K(\eta(l^{-1}))} \\
&= \theta_K(\mu(l))e^{i\theta_K(\eta(l))}\ e_2\ [\theta_K(\mu(l))]^{-1}e^{i[\theta_K(\eta(l))]^{-1}} \\
&= \theta(\nu(l))\ [\theta(\nu(l))]^{-1} = \nu(e_2).
\end{aligned}$$

$\therefore\ l \in \mathbb{G}_1,\ c \in K, \Rightarrow lc\ l^{-1} \in K$. Hence, K is a complex anti-fuzzy normal subgroup of $\mathbb{G}$. □

In the following theorem, we first prove that every homomorphism image of a complex anti-fuzzy cyclic group is cyclic and show that the converse of this theorem does not hold by using a counterexample.

Theorem 7. *Every homomorphism image of a complex anti-fuzzy cyclic group is cyclic, but converse is not true.*

Proof. Suppose that κ is a homomorphism of a cyclic group $\mathbb{G}_1 =< \mathrm{c} > \longrightarrow \mathbb{G}_2$ and ν is a complex anti-fuzzy subgroup of $\mathbb{G}_2$, defined as $\nu(\kappa(\mathrm{c})) = \{m,\ \nu(m) : \kappa(\mathrm{c}) = m,\ \ m \in G_2\}$ where $\nu(\kappa(G_1)) = \mu(\kappa(G_1))\ e^{i\eta(\kappa(G_1))}$.
For $\kappa(\mathbb{G}_1)$ is a complex anti-cyclic subgroup:
suppose that $m \in \kappa(\mathbb{G}_1)$ then

$$\begin{aligned}
\nu(m) &= \nu(\kappa(\mathrm{c}^n)) = \mu(\kappa(\mathrm{c}^n))\ e^{i\eta(\kappa(\mathrm{c}^n))},\ \ \textit{where } n \in \mathbb{Z} \\
&= \mu(\kappa(\mathrm{c\ c\ c...c}\ n-\text{times}))\ e^{i\eta\kappa((\mathrm{c\ c\ c...c}\ n-\text{times}))} \\
&= \mu(\kappa(\mathrm{c})\kappa(\mathrm{c})\kappa(\mathrm{c})...\kappa(\mathrm{c})\ n-\text{times})\ e^{i\eta(\kappa(\mathrm{c})\kappa(\mathrm{c})\kappa(\mathrm{c})...\kappa(\mathrm{c})\ n-\text{times})} \\
&\quad \therefore\ \kappa \text{ is homomorphism} \\
&= \mu[\kappa(\mathrm{c})]^n\ e^{i\eta[\kappa(\mathrm{c})]^n} \\
&= \nu[\kappa(\mathrm{c})]^n.
\end{aligned}$$

It is demonstrated that each component of $\kappa(\mathbb{G}_1)$ is an integral power of $\kappa(\mathrm{c})$, that is,

$$\kappa(\mathbb{G}_1) = [\kappa(l)].$$

Hence, $\kappa(\mathbb{G}_1)$ is also a cyclic group.

Conversely: The following example demonstrates how the converse is not always true.

$\kappa : \mathbb{S}_3 \longrightarrow \{-1,\ 1\}$, $\nu(\kappa(\beta)) = \mu(\kappa(\beta))\ e^{i\eta(\kappa(\beta))} = \begin{cases} (1,\ 0.1e^{i\pi/2}), & \text{if } \beta \text{ is even} \\ (-1,\ 0.5e^{i\pi}), & \text{if } \beta \text{ is odd} \end{cases}$

is a homomorphism from $\mathbb{S}_3 \longrightarrow \{-1,\ 1\}$. Furthermore, the group under the multiplications $\mathbb{S}_3 = \{-1,\ 1\}$ is abelian cyclic but $\mathbb{S}_3$ is neither abelian nor cyclic. □

5. Conclusions

A useful generalization of the traditional anti-fuzzy set is a complex anti-fuzzy set, which plays an effective role in evaluating the ambiguity and uncertainty of data in different aspects. The aim of this research article is to introduce complex anti-homomorphism under groups, their features, and certain properties of complex anti-homomorphism under groups with the help of suitable examples. The necessary and suitable parameters for complex anti-homomorphisms under the group were determined. Additionally, we illustrated that every homomorphism image of a complex anti-fuzzy cyclic group is cyclic, but the converse is not true. Furthermore, homomorphic and isomorphic images under the complex anti-fuzzy subgroup were discussed. The possible future research will involve the application of these

algebraic structures to solve certain decision-making problems as an important addition to existing theories for dealing with uncertainty so that the complex anti-fuzzy set can be applied to the structure of ring theory, field theory, and the direct product of groups, rings, and fields.

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