

Article

Estimation and Prediction for Alpha-Power Weibull Distribution Based on Hybrid Censoring

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Abstract: This work discusses the issues of estimation and prediction when lifespan data following alpha-power Weibull distribution are observed under Type II hybrid censoring. We calculate point and related interval estimates for both issues using both non-Bayesian and Bayesian methods. Using the Newton–Raphson technique under the classical approach, we compute maximum likelihood estimates for point estimates in the estimation problem. Under the Bayesian approach, we compute Bayes estimates under informative and non-informative priors using the symmetric loss function. Using the Fisher information matrix under classical and Bayesian techniques, the corresponding interval estimates are derived. Additionally, using the best unbiased and conditional median predictors under the classical approach, as well as Bayesian predictive and associated Bayesian predictive interval estimates in the prediction approach, the predictive point estimates and associated predictive interval estimates are computed. We compare several suggested approaches of estimation and prediction using real data sets and Monte Carlo simulation studies. A conclusion is provided.

Keywords: hybrid Type II censored samples; one-sample prediction; two-sample prediction; simulation study; real-life data; maximum likelihood estimation



Citation: Almetwally, E.M.; Alotaibi, R.; Rezk, H. Estimation and Prediction for Alpha-Power Weibull Distribution Based on Hybrid Censoring. *Symmetry* **2023**, *15*, 1687. <https://doi.org/10.3390/sym15091687>

Academic Editor: Sergei D. Odintsov

Received: 15 June 2023

Revised: 29 July 2023

Accepted: 18 August 2023

Published: 2 September 2023



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1. Introduction

Using predictive analytics to foresee business outcomes may save time, money, and effort. More information will be accessible to help make better judgments. Organizations may also uncover possibilities and address their own challenges by providing precise and trustworthy insights. Aggregate data may be evaluated, using predictive analytics, to find new prospects for consumer acquisition. Prediction, which is important in many domains, has drawn greater attention in recent years. For instance, in business, an experimenter may wish to estimate the lifespan of an unseen future unit using data from a present sample. If so, the producer or the experimenter may release their goods into the market with the intention of capturing customers' attention and increasing demand by lowering the thresholds for their warranties. See [1–3] for additional details regarding the applicability of predictions in business and industry.

In cases involving parametric estimating, we usually assume the availability of a group of samples selected at random from a preset number of units (n). Furthermore, it is anticipated that random observations will fit a certain distribution. In practice, however, it is conceivable that not all of the unit observations are accessible, as some unit observations are either not captured or are lost during intermediary transmission. These data are referred to as censored samples. In the literature, there are techniques for estimating sample instances that have already been censored. Notably, from a theoretical standpoint, censoring may

not be the most efficient method of conducting experiments because it reduces the number of observations. However, due to administrative, logistical, and budgetary limitations, restricted observations must be available. In other words, censoring systems compensate for an estimator's loss of efficiency by providing administrative simplicity at a lower cost.

Numerous censoring methods have been examined in statistical literature, including time censoring (Type I censoring), item censoring (Type II censoring), Type I and Type II hybrid censoring, and progressive censoring. It should be noted that while the length of a test is assured if the experiment is stopped at a preset period (T), the results may differ if a censored sample's observations are randomly distributed. However, even if this censoring provides a certain efficiency, the test's duration becomes random if it is halted after a predefined number of observations. Hybrid censoring, created by combining the Type I and Type II censoring methods, provides a more flexible and useful life-testing technique.

After a predetermined time, the T of an experiment's duration or a predetermined number (R) of observations have been acquired, and the test is ended using Type I hybrid censoring. For example, if $X_{R:n}$ stands for the R^{th} ordered failure time, the experiment will end at $\min(T, X_{R:n})$. This ensures that the test will take less time than T , but the number of observations will fluctuate and be lower than R . Epstein [4] first presented the Type I hybrid censoring method, which was utilized in a subsequent reliability acceptance test [5]. With respect to Type I hybrid censored data, several authors have studied the estimation of the unknown parameters for different probability distributions (see, for example, [6–9]).

In several fields, such as medicine, the military, and aeronautics, efficiency levels are more important than experiment duration. Therefore, a censoring strategy may be proposed, in which the test is stopped when a predetermined number of observations are gathered and a predetermined time for the duration of the experiment is reached—i.e., the experiment is stopped at $\max(T, X_{R:n})$. This method is called the Type II hybrid censoring technique. It should be noted that the total number of observations in this censoring technique is random, but it will not be less than R . This ensures a minimal efficiency, although the length of the test can vary and continue beyond T . A thorough analysis of hybrid censoring systems, with generalizations and applications in competitive-risk and step-stress modeling, was provided by [10]. Additionally, for estimating the parameters under Type II hybrid censoring schemes, see [11–13].

Future-order statistics prediction arises readily in a variety of real-world circumstances. Here, we focus on future-order statistics estimates using the Bayesian paradigm. Predictive posterior distribution was first discussed by [14] in relation to prediction issues. Since then, other censoring techniques have been included in prediction tasks. Ebrahimi [15] provided two examples of Type I hybrid censoring prediction problems for exponential distributions. The one-sample and two-samples prediction problems, based on Type I hybrid censored samples for the general class of distribution and for generalized Lindley distribution, respectively, were studied by [16]. Based on a Type II hybrid censored sample for a generic class of distribution, Balakrishnan and Shafay [17] devised an estimation approach for one-sample and two-samples prediction issues.

Alpha-power Weibull (APW) distribution is significant because it extends the Weibull distribution and can model monotone and non-monotone failure rate functions, which are crucial in reliability research. In fact, the Weibull distribution's widespread application in reliability theory and the generalization's flexibility in lifetime data analysis served as the APW distribution's main sources of inspiration. Moreover, Nassar et al. [18] developed the APW model to provide a new generalization of the Weibull distribution based on the proposed Weibull model. The following is a list of the probability-density function (pdf), cumulative distribution (cdf), the associated hazard rate function (hrf), and the reversed hazard rate function (rhrf).

$$f(x) = \frac{\log \alpha}{\alpha - 1} \theta \lambda x^{\theta-1} e^{-\lambda x^\theta} \alpha^{1-e^{-\lambda x^\theta}}, \quad \alpha, \lambda, \theta > 0, x \geq 0, \quad (1)$$

$$F(x) = \frac{1}{1-\alpha} \left(1 - \alpha^{1-e^{-\lambda x^\theta}} \right), \alpha, \lambda, \theta > 0, x \geq 0, \quad (2)$$

$$h(x) = \frac{\log \alpha}{\left(\alpha^{-e^{-\lambda x^\theta}} - 1 \right)} \theta \lambda x^{\theta-1} e^{-\lambda x^\theta}, \alpha, \lambda, \theta > 0, x \geq 0, \quad (3)$$

and

$$rh(x) = \frac{\log \alpha}{\left(1 - \alpha^{1-e^{-\lambda x^\theta}} \right)} \theta \lambda x^{\theta-1} e^{-\lambda x^\theta} \alpha^{1-e^{-\lambda x^\theta}}, \alpha, \lambda, \theta > 0, x \geq 0, \quad (4)$$

The formula for the mean time to failure (MTF) is

$$\text{MTF} = E(X) = \int_0^\infty R(x) dx = \frac{\alpha}{1-\alpha} \int_0^\infty \left(1 - \alpha^{-e^{-\lambda x^\theta}} \right) dx. \quad (5)$$

Despite there being a wealth of research on estimates under hybrid censoring and the APW distribution being superior to many competing distributions, including the Weibull, alpha power exponential, McDonald Weibull, beta Weibull, transmuted Weibull, gamma Lomax, Zografos–Balakrishnan log-logistic, exponentiated Weibull, and exponentiated Weibull distributions, there is no body of work discussing the prediction of the future ordered statistics based on hybrid censored samples for the APW model. In addition, there is not much information available regarding research achieving classical and Bayesian estimates of the unknown parameters, reliability, hazard rate functions, and the MTF for APW distribution under hybrid Type II censoring. All of these gaps in the current literature prompted us to produce this work, which has three major purposes. The first goal is to examine the issue of estimating the unknown parameters, reliability, hazard rate functions, and the MTF of APW distribution using classical and Bayesian methods of estimation under Type II hybrid censoring. We also calculate the corresponding interval estimates for the model parameter. The second goal is to obtain one- and two-sample Bayesian prediction problems. Also, the one- and two-sample Bayesian prediction interval limits are constructed based on the Type II hybrid censored data. The third goal is to perform a simulation study to ensure the theoretical results of the paper and that an actual data set is given to indicate the proposed methodology in real life.

The rest of the article is organized as follows: Both classical and Bayesian approaches are used to estimate the parameters, reliability, hazard rate, and the MTF under Type II hybrid censoring, and the associated interval limits of the unknown parameters are determined in Section 2. In Section 3, it is discussed how to solve for one- and two-sample Bayesian predictive posterior density. Also built in this section are the prediction boundaries for next samples. The results of the Monte Carlo simulation are studied in Section 4 to allow for the comparison of the performance of the suggested estimators. In Section 5, an actual data set is examined as an example. Concluding remarks are provided in Section 6 at the end of this essay.

2. Estimation Based on Type II Hybrid Censored Samples

2.1. Maximum Likelihood

Assume that n identical units are put through a life test, and that they all have lives that follow the APW distribution specified in the pdf Equation (1). The life test is ended at $\max(X_{R:n}, T)$, where $X_{R:n}$ stands for the R^{th} order statistic, or at the later of a predetermined time (say, T) and a predetermined number of failures (say, $R \leq n$). The following scenarios would result in the random observations being obtained under this sort of censoring system (also called the Type II hybrid censoring method):

Case I : If $T < X_{R:n}$, then $(X_{1:n}, X_{2:n}, \dots, X_{R:n})$.

Case II : $(X_{1:n}, X_{2:n}, \dots, X_{k:n})$ if $X_{r:n} \leq T$, $R < k < n$.

Case III : if $X_{n:n} \leq T$, then $(X_{1:n}, X_{2:n}, \dots, X_{n:n})$.

According to the Type II hybrid censoring method, the likelihood function can be expressed as follows:

$$L(\alpha, \beta, \theta | \underline{x}) = \frac{n!}{(n-r_o)!} \prod_{i=1}^{r_o} f(x_i) [1 - F(t_o)]^{n-r_o};$$

$$= \frac{n!}{(n-r_o)!} \prod_{i=1}^{r_o} \frac{\log \alpha}{\alpha-1} \theta \lambda x_{(i)}^{\theta-1} e^{-\lambda x_{(i)}^\theta} \alpha^{1-e^{-\lambda x_{(i)}^\theta}} \left[\frac{1}{1-\alpha} \left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right) \right]^{n-r_o}, \quad (6)$$

where

$$r_o = \begin{cases} R & \text{Case I} \\ k & \text{Case II} \\ n & \text{Case III} \end{cases}, t_o = \begin{cases} X_{R:n} & \text{Case I} \\ T & \text{Case II} \\ X_{R:n} & \text{Case III} \end{cases}$$

The log-likelihood function, ℓ is provided by

$$\ell = \log \left(\frac{n!}{(n-r_o)!} \right) + r_o \log \left(\frac{\log \alpha}{\alpha-1} \right) + r_o \log \theta + r_o \log \lambda + (\theta-1) \sum_{i=1}^{r_o} \log(x_{(i)}) - \lambda \sum_{i=1}^{r_o} \log(x_{(i)})$$

$$+ \log(\alpha) \sum_{i=1}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta} \right) - (n-r_o) \log(1-\alpha) + (n-r_o) \log \left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right). \quad (7)$$

The following log-likelihood equations can be simultaneously solved to give the maximum likelihood estimates (MLEs), $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

$$\frac{\partial \ell}{\partial \alpha} = \frac{1}{\alpha \log \alpha} + \frac{1}{\alpha} \left[\sum_{i=1}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta} \right) - 1 \right] + \frac{(n-r_o)}{\alpha} + \frac{(n-r_o) \alpha^{-(e^{-\lambda t_o^\theta} + 1)}}{\alpha^{-e^{-\lambda t_o^\theta}} - 1}, \quad (8)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{r_o}{\theta} + \sum_{i=1}^{r_o} \log(x_{(i)}) - \lambda \log(\alpha) \sum_{i=1}^{r_o} e^{-\lambda x_{(i)}^\theta} x_{(i)}^\theta \log(x_{(i)}) + \frac{(n-r_o) \lambda t_o^\theta \log(t_o) e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta}}}{\alpha^{-e^{-\lambda t_o^\theta}} - 1}, \quad (9)$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{r_o}{\lambda} - \sum_{i=1}^{r_o} \log(x_{(i)}) + \log(\alpha) \sum_{i=1}^{r_o} e^{-\lambda x_{(i)}^\theta} (x_{(i)}^\theta) + \frac{(n-r_o) \alpha^{-e^{-\lambda t_o^\theta}} t_o^\theta e^{-\lambda t_o^\theta}}{\alpha^{-e^{-\lambda t_o^\theta}} - 1}, \quad (10)$$

As can be seen, the aforementioned equations cannot be solved explicitly and must instead be solved numerically using an iterative process. It is impossible to explicitly obtain the precise distribution of MLEs. Nonetheless, it is possible to build the confidence intervals for the parameters using the asymptotic characteristics of MLEs. The MLEs of the reliability (rf), hrf, and MTF may be directly calculated, respectively, by applying the invariance property of the maximum likelihood function, as shown below:

$$\hat{R}(x) = \frac{\hat{\alpha}}{\hat{\alpha}-1} \left(1 - \hat{\alpha}^{1-e^{-\lambda x^\theta}} \right), \quad (11)$$

$$\hat{h}(x) = \frac{\hat{\lambda} \hat{\theta} \log(\hat{\alpha}) x^{\hat{\theta}-1} e^{-\lambda x^\theta}}{\hat{\alpha} e^{-\lambda x^\theta} - 1}, \quad (12)$$

and

$$\widehat{\text{MTF}} = E(X) = \int_0^\infty R(x) dx = \frac{\alpha}{1-\alpha} \int_0^\infty \left(1 - \alpha^{-e^{-\lambda x^\theta}} \right) dx. \quad (13)$$

The MLEs $(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ roughly follow a bivariate normal distribution under some regularity criteria, with mean $\psi = (\alpha, \theta, \lambda)$ and variance matrix $\sigma = I^{-1}(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$, where $I(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is the observed Fisher's information matrix, which is given by

$$I(\hat{\alpha}, \hat{\theta}, \hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ell}{\partial \theta^2} & -\frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} & -\frac{\partial^2 \ell}{\partial \theta \partial \lambda} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\theta}, \hat{\lambda})},$$

where

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{(1 + \log(\alpha))}{\alpha^2 (\log(\alpha))^2} + \frac{1}{\alpha^2} \left[\sum_{i=0}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta} \right) - 1 \right] - \frac{(n - r_o)}{\alpha^2} - \left[\left(1 + e^{-\lambda t_o^\theta} \right) e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta} - 2} \right], \quad (14)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{1}{\alpha} \sum_{i=0}^{r_o} \lambda x_{(i)}^\theta \log(x_{(i)}) e^{-\lambda t_o^\theta} + \frac{(n - r_o) \lambda t_o^\theta e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta} - 1} \left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right) \left(-e^{-\lambda t_o^\theta} - 1 \right) - \alpha^{-e^{-\lambda t_o^\theta}}}{\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right)^2}, \quad (15)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = -\frac{1}{\alpha} \sum_{i=0}^{r_o} x_{(i)}^\theta e^{-\lambda t_o^\theta} + \frac{(n - r_o) t_o^\theta e^{-\lambda t_o^\theta} \left[\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right) - \alpha^{-e^{-\lambda t_o^\theta}} \right]}{\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right)^2}, \quad (16)$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \theta^2} = & -\frac{r_o}{\theta^2} + \lambda \log(\alpha) \sum_{i=0}^{r_o} \log(x_{(i)}) \left[e^{-\lambda x_{(i)}^\theta} x_{(i)}^\theta \log(x_{(i)}) (\lambda x_{(i)}^\theta - 1) \right] \\ & + \lambda (n - r_o) (\log(\alpha))^2 e^{-\lambda t_o^\theta} t_o^\theta \alpha^{-e^{-\lambda t_o^\theta}} \left[\frac{\left(1 - \lambda t_o^\theta - \lambda t_o^\theta e^{-\lambda t_o^\theta} \right) \left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right) - \lambda t_o^\theta e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta}}}{\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right)^2} \right], \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \theta \partial \lambda} = & \log(\alpha) \sum_{i=0}^{r_o} e^{-\lambda x_{(i)}^\theta} x_{(i)}^\theta (\lambda - \log(x_{(i)})) \\ & + \frac{(n - r_o) t_o^\theta \log(t_o) e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta}} \left\{ \left(1 - \lambda t_o^\theta + \lambda t_o^\theta e^{-\lambda t_o^\theta} \right) \left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right) - \lambda t_o^\theta e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta}} \right\}}{\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right)^2}, \end{aligned} \quad (18)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{r_o}{\lambda^2} - \log(\alpha) \sum_{i=0}^{r_o} x_{(i)}^{2\theta} e^{-\lambda x_{(i)}^\theta} - \frac{t_o^{2\theta} (n - r_o) e^{-\lambda t_o^\theta} \alpha^{-e^{-\lambda t_o^\theta}} \left[\alpha^{-e^{-\lambda t_o^\theta}} (1 - e^{-\lambda t_o^\theta}) \right]}{\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1 \right)^2}, \quad (19)$$

The asymptotic variances for the parameters $\psi = (\alpha, \theta, \lambda)$, are provided by the diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$, respectively. One may obtain the two-sided $(1 - \phi)100\%$ normal approximation confidence interval as

$$\hat{\psi} \mp Z_{\phi/2} \sqrt{\text{var}(\hat{\psi})},$$

where Z is the common normal variate.

2.2. Bayesian Estimation

On the basis of a Type II hybrid censored sample, we suggested Bayes estimators of the APW distribution's unknown parameters in this section. We must define the prior distributions for the parameters in Bayesian estimation. The authors of [19] proposed a number of methods for earlier definitions. Assuming the functional form for the prior density, eliciting its parameter(s), also known as the hyper-parameter(s), is the simplest method for prior specifications. The gamma distribution contains closed-form equations for moments and is highly versatile and able to accept different density forms. In some situations, it also offers conjugacy and mathematical simplicity. The gamma distribution

has been extensively used in Bayesian estimation as a prior distribution for the parameters of the other lifespan distributions. Moreover, we assume that $\psi = (\alpha, \theta, \lambda)$ are independent gamma random variables in this study. For more details, see [20–22]. The prior distribution of $\psi = (\alpha, \theta, \lambda)$ are provided by

$$\begin{aligned}\pi_1(\alpha) &\propto \alpha^{\nu_1-1} e^{-\nu_2 \alpha}; \alpha \geq 0, \nu_1, \nu_2 > 0, \\ \pi_2(\theta) &\propto \theta^{\nu_3-1} e^{-\nu_4 \theta}; \theta \geq 0, \nu_3, \nu_4 > 0, \\ \pi_3(\lambda) &\propto \lambda^{\nu_5-1} e^{-\nu_6 \lambda}; \lambda \geq 0, \nu_5, \nu_6 > 0,\end{aligned}\quad (20)$$

where it is assumed that the hyper-parameters $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$, and ν_6 are known and are selected to represent the previous assumption about the unknown parameters. These can be acquired if we can predict the parameter's anticipated values using the prior mean (M) and confidence in the predicted value using the prior variance (V). Consequently, by resolving the following previous moments equations for $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$, and ν_6 may be obtained: $M = \frac{\nu_2}{\nu_1}, V = \frac{\nu_2}{\nu_1^2}, M = \frac{\nu_4}{\nu_3}, V = \frac{\nu_4}{\nu_3^2}, M = \frac{\nu_6}{\nu_5}$, and $V = \frac{\nu_6}{\nu_5^2}$. The prior densities become non-informative inappropriate prior distributions as soon as $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5 = \nu_6 = 0$,

$$\pi_1(\alpha) \propto \alpha^{-1}; \alpha \geq 0, \pi_2(\theta) \propto \theta^{-1}; \theta \geq 0, \pi_3(\lambda) \propto \lambda^{-1}; \lambda \geq 0. \quad (21)$$

The prior distribution mentioned above occasionally matches the scale-invariant Jeffrey prior (see [23]). You can find the joint posterior density of $\psi = (\alpha, \theta, \lambda)$ as

$$\begin{aligned}\pi(\psi|\underline{x}) &\propto \pi_1(\alpha)\pi_2(\theta)\pi_3(\lambda)L(\alpha, \theta, \lambda|\underline{x}), \\ \pi(\psi|\underline{x}) &\propto \alpha^{\nu_1-1} e^{-\nu_2 \alpha + \nu_4 \theta + \nu_6 \lambda} \theta^{\nu_3-1} \lambda^{\nu_5-1} r_o \log\left(\frac{\log \alpha}{\alpha-1}\right) + r_o \log \theta + r_o \log \lambda + (\theta-1) \sum_{i=0}^{r_o} \log(x_{(i)}) - \lambda \sum_{i=0}^{r_o} \log(x_{(i)}) + \\ &\log(\alpha) \sum_{i=0}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta}\right) - (n-r_o) \log(1-\alpha) + (n-r_o) \log\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1\right).\end{aligned}\quad (22)$$

It is generally known that the expected value of any parametric function using the posterior distribution, or $E_\pi[\psi(\alpha, \theta, \lambda)]$, is the Bayes estimator of that function, $\psi = (\alpha, \theta, \lambda)$, under squared error loss. Consequently, the Bayes estimates of α, θ , and λ are the means of their posteriors. Similar to this, it is simple to construct the Bayes estimates of the reliability and hazard rate under squared error loss as

$$\begin{aligned}\hat{R}(t) &\propto \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} \alpha^{\nu_1-1} e^{-\nu_2 \alpha + \nu_4 \theta + \nu_6 \lambda} \theta^{\nu_3-1} \lambda^{\nu_5-1} \frac{1}{1-\alpha} \left(\alpha^{-e^{-\lambda x_{(i)}^\theta}} - 1\right) \left(r_o \log\left(\frac{\log \alpha}{\alpha-1}\right) + r_o \log \theta + r_o \log \lambda \right. \\ &+ (\theta-1) \sum_{i=0}^{r_o} \log(x_{(i)}) - \lambda \sum_{i=0}^{r_o} \log(x_{(i)}) + \log(\alpha) \sum_{i=0}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta}\right) - (n-r_o) \log(1-\alpha) \\ &\left. + (n-r_o) \log\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1\right)\right) d\lambda d\theta d\alpha,\end{aligned}\quad (23)$$

$$\begin{aligned}\hat{h}(t) &\propto \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} \alpha^{\nu_1-1} e^{-\nu_2 \alpha + \nu_4 \theta + \nu_6 \lambda} \theta^{\nu_3-1} \lambda^{\nu_5-1} \frac{\log \alpha}{\left(\alpha^{-e^{-\lambda x_{(i)}^\theta}} - 1\right)} \theta \lambda x_{(i)}^{\theta-1} e^{-\lambda x_{(i)}^\theta} \left(r_o \log\left(\frac{\log \alpha}{\alpha-1}\right) + r_o \log \theta \right. \\ &+ r_o \log \lambda + (\theta-1) \sum_{i=0}^{r_o} \log(x_{(i)}) - \lambda \sum_{i=0}^{r_o} \log(x_{(i)}) + \log(\alpha) \sum_{i=0}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta}\right) \\ &\left. - (n-r_o) \log(1-\alpha) + (n-r_o) \log\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1\right)\right) d\lambda d\theta d\alpha,\end{aligned}\quad (24)$$

similar to how the MTF's Bayes estimate may be determined by

$$\begin{aligned}\widehat{MTF}(t) &\propto \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} \int_{x=0}^{\infty} \frac{\alpha}{1-\alpha} \left(1 - \alpha^{-e^{-\lambda x_{(i)}^\theta}}\right) \alpha^{\nu_1-1} e^{-\nu_2 \alpha + \nu_4 \theta + \nu_6 \lambda} \theta^{\nu_3-1} \lambda^{\nu_5-1} \left(r_o \log\left(\frac{\log \alpha}{\alpha-1}\right) + r_o \log \theta \right. \\ &+ r_o \log \lambda + (\theta-1) \sum_{i=0}^{r_o} \log(x_{(i)}) - \lambda \sum_{i=0}^{r_o} \log(x_{(i)}) + \log(\alpha) \sum_{i=0}^{r_o} \left(1 - e^{-\lambda x_{(i)}^\theta}\right) \\ &\left. - (n-r_o) \log(1-\alpha) + (n-r_o) \log\left(\alpha^{-e^{-\lambda t_o^\theta}} - 1\right)\right) dx d\lambda d\theta d\alpha,\end{aligned}\quad (25)$$

It is clear that the aforementioned expressions, Equations (23)–(25), cannot be resolved in a pleasing closed form. To take a random sample from the joint posterior and enable sample-based inference, we proposed using Markov chain Monte Carlo (MCMC) techniques, such as the Gibbs sampler (see [24]) and Metropolis Hastings algorithm (see [25,26]).

3. Predictive Posterior Density

In this part, we have used a Type II hybrid censored sample to determine the density of future order statistics from the APW distribution. Predictive density is a term used to describe the pdf of the future sample. There are two categories for the prediction problems, with two sample predictions being covered in the following two successive subsections:

- 1- One-sample prediction;
- 2- Two-sample prediction.

3.1. One Sample Prediction

Out of all the units/items tested, only a small number of them were noticed in the Type II hybrid censoring scheme. In such cases, the investigator could be curious to learn how long the removed surviving units had lived based on the sample at hand.

Let $r_0 < n$ and $X_{s:n}(r_0 < s \leq n)$ be the prediction. Given the Type II hybrid censored sample $\underline{x}$, the conditional distribution of sth order statistics should be as follows:

Case I:

$$f_1(x_{s:n}|\underline{x};\psi) = f(x_{s:n}|x_{R:n};\psi) = \frac{f(X_{s:n} = x_{s:n}, X_{R:n} = x_{R:n}|\psi)}{f(X_{R:n} = x_{R:n}|\psi)}, \quad (26)$$

so

$$f_1(x_{s:n}|\underline{x};\psi) = \frac{(n-R)! [F(x_{s:n}) - F(x_{R:n})]^{s-R-1} [1 - F(x_{s:n})]^{n-s} f(x_{s:n})}{(n-s)!(s-R-1)! [1 - F(x_{R:n})]^{n-R}}, \quad (27)$$

where $(R < s \leq n)$, and $\underline{x} = \{x_{1:n}, x_{2:n}, \dots, x_{R:n}\}$ $x_{s:n} > x_{R:n}$. By adding Equations (1) and (2) to Equation (27), we have

$$f_1(x_{s:n}|\underline{x};\psi) = \frac{(n-R) \log(\alpha)(1-\alpha) \left[\alpha^{1-e^{-\lambda x_{R:n}^\theta}} - \alpha^{1-e^{-\lambda x_{s:n}^\theta}} \right]^{s-R-1} \left[\alpha^{1-e^{-\lambda x_{s:n}^\theta}} \right]^{n-s+1} \theta \lambda x_{s:n}^{\theta-1} e^{-\lambda x_{s:n}^\theta}}{(\alpha-1)(n-s)! \left[\alpha^{1-e^{-\lambda x_{R:n}^\theta}} \right]^{n-R}}. \quad (28)$$

Case II:

$$f_2(x_{s:n}|\underline{x}, K=k; \psi) = \frac{1}{P(R \leq K \leq s-1)} \sum_{k=R}^{s-1} \frac{f(X_{s:n} = x_{s:n}, X_{K:n} = x_{k:n} | K=k, \psi)}{f(X_{K:n} = x_{k:n} | \psi)} P(K=k), \quad (29)$$

where

$$f(X_{s:n} = x_{s:n}, X_{K:n} = x_{k:n} | K=k, \psi) = \frac{n! (F(T))^{k-1} f(T) (1 - F(x_{s:n}))^{n-s} f(x_{s:n}) (F(x_{s:n}) - F(T))^{s-k-1}}{(k-1)!(s-k-1)!(n-s)!},$$

and

$$f(X_{K:n} = x_{k:n} | \psi) = \frac{n! (F(T))^{k-1} f(T) (1 - F(T))^{n-k}}{(k-1)!(n-k)!}.$$

So,

$$f_2(x_{s:n}|\underline{x}, K=k; \psi) = \frac{1}{P(R \leq K \leq s-1)} \sum_{k=R}^{s-1} \frac{(n-k)! (1 - F(x_{s:n}))^{n-s} f(x_{s:n}) (F(x_{s:n}) - F(T))^{s-k-1}}{(s-k-1)!(n-s)!(1 - F(T))^{n-k}}, \quad (30)$$

where $\underline{x} = \{x_{1:n}, x_{2:n}, \dots, x_{k:n}\}; k < s \leq n, x_{s:n} > T$; and K is a random variable. The value T denotes the maximum number of x_i 's. Then,

$$P(K = k) = \binom{n}{k} (F(T))^k (1 - F(T))^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

Substituting Equations (1) and (2) into Equation (30), we obtain

$$\begin{aligned} f_2(x_{s:n} | \underline{x}, K = k; \psi) \\ = \frac{\theta \lambda \log \alpha (n-k)!}{P(R \leq K \leq s-1)} \sum_{k=R}^{s-1} \frac{\left(\frac{\alpha}{1-\alpha} \left(\alpha^{1-e^{-\lambda x_{s:n}^\theta}} - 1 \right) \right)^{n-s} x_{s:n}^{\theta-1} e^{-\lambda x_{s:n}^\theta} \alpha^{1-e^{-\lambda x_{s:n}^\theta}} \left(\alpha^{1-e^{-\lambda T^\theta}} - \alpha^{1-e^{-\lambda x_{s:n}^\theta}} \right)^{s-k-1}}{(\alpha-1)(s-k-1)!(n-s)! \left(\frac{\alpha}{1-\alpha} \left(\alpha^{1-e^{-\lambda T^\theta}} - 1 \right) \right)^{n-k}}. \end{aligned} \quad (31)$$

The following formula defines the one-sample predictive posterior density of future observables:

Case I:

$$f_1^*(x_{s:n} | \underline{x}) = \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} f_1(x_{s:n} | \underline{x}; \psi) \pi(\psi | \underline{x}) d\lambda d\theta d\alpha. \quad (32)$$

Case II:

$$f_2^*(x_{s:n} | \underline{x}) = \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} f_2(x_{s:n} | \underline{x}; \psi) \pi(\psi | \underline{x}) d\lambda d\theta d\alpha. \quad (33)$$

Numerical approaches are thus needed to investigate the posteriors' characteristics as the aforementioned predictive posteriors cannot be reduced to any standard distribution. Using the method described in the previous section, we can utilize the M-H algorithm to extract a sample from the predicted posteriors and provide estimates and forecast intervals for upcoming data. By resolving the following equations, we can also obtain the two-sided $100(1 - \phi)\%$ prediction intervals (L_s, U_s) for $y_{(s)}$.

$$P(y_s > U_s | \underline{x}) = \frac{\phi}{2}, \quad P(y_s > L_s | \underline{x}) = 1 - \frac{\phi}{2}.$$

As the aforementioned equations cannot be solved directly, confidence intervals can be calculated by utilizing any acceptable iterative approach.

3.2. Two-Sample Prediction

In another circumstance, we may be interested in the failure time of the k th ordered observation from a future sample of size N from the same lifespan distribution, i.e., $P(y_{(s)} | \psi, \underline{x}) = P(y_{(s)} | \psi)$. The two sample prediction issues are the result of this. The s th order statistic's pdf is provided by

$$P(y_{(s)} | \psi) = \frac{N!}{(s-1)!(N-s)!} [F(y_{(s:N)})]^{k-1} [1 - F(y_{(s:N)})]^{N-k} f(y_{(s:N)}), \quad (34)$$

And when we add Equations (1) and (2) to Equation (34), we obtain

$$P(y_{(s)} | \psi) = \frac{\theta \lambda N! \log \alpha}{(\alpha-1)(s-1)!(N-s)!} \left[\frac{1}{1-\alpha} \left(1 - \alpha^{1-e^{-\lambda y_{(s:N)}^\theta}} \right) \right]^{k-1} \left[\frac{\alpha}{1-\alpha} \left(\alpha^{1-e^{-\lambda y_{(s:N)}^\theta}} - 1 \right) \right]^{N-k} y_{(s:N)}^{\theta-1} e^{-\lambda y_{(s:N)}^\theta} \alpha^{1-e^{-\lambda y_{(s:N)}^\theta}}. \quad (35)$$

As stated below, the two-sample predictive posterior density of a future observation is

$$P^*(y_{(s)} | \psi) = \int_{\alpha=0}^{\infty} \int_{\theta=0}^{\infty} \int_{\lambda=0}^{\infty} P(y_{(s)} | \psi, \underline{x}) \pi(\psi | \underline{x}) d\lambda d\theta d\alpha. \quad (36)$$

As the density in Equation (34) is independent of Type II censoring, it is the same for both instances (Case I and Case II). We can also obtain the predictive density for Case I and Case II by replacing the posterior $\pi(\psi|\underline{x})$ of the corresponding case. Using the MCMC techniques covered in the preceding section, it is possible to quantitatively investigate the aforementioned posterior density. By resolving the following two equations, we can also obtain the two-sided $100(1 - \phi)\%$ prediction intervals (L_k, U_k) for $y_{(k)}$

$$P(y_{(k)} > U_k | \underline{x}) = \frac{\phi}{2}, P(y_{(k)} > L_k | \underline{x}) = 1 - \frac{\phi}{2}.$$

As the aforementioned equations cannot be solved directly, confidence intervals can be calculated by utilizing any acceptable iterative approach.

4. Simulation

In this section, we give some experimental findings, mostly for the purpose of seeing how the various approaches behave for various sample sizes and time censoring techniques. We use the MLE and Bayes estimators developed by employing the MCMC approach to estimate the unknown parameters. We contrast how well each estimator performs in terms of relative absolute bias (RAB) and MSE. Additionally, as the aim of this section, Bayes prediction has been obtained from the s -th observation, where $r_0 < s \leq n$ for a one-sample and two-sample prediction problem, and is associated with the inference based on the available data, namely $x_{(1)} < x_{(2)} < \dots < x_{(r_0)}$. Specifically, we wish to provide an estimate of the posterior density function of $x_{(s)}$ given the data and also construct a $100(1 - \phi)\%$ predictive interval of $x_{(s)}$. We consider these two cases separately.

When the parameters α, θ , and λ are supposed to have true values with the time value of $\alpha = 0.5, \theta = 0.6, \lambda = 0.4, T = 3$ and 4; $\alpha = 0.5, \theta = 1.3, \lambda = 3, T = 0.3$ and 0.6; $\alpha = 0.5, \theta = 0.6, \lambda = 3, T = 0.15$ and 0.4; and $\alpha = 2, \theta = 1.3, \lambda = 3, T = 0.5$ and 0.8, we repeat hybrid censoring scheme data from an APW distribution 5000 times. We chose various sample sizes, such as $n = 50$ and 100, as well as hybrid censored sample sizes, such as $r = 35$ and 45 for $n = 50$, and $r = 70$ and 85 for $n = 100$. For Bayes prediction, $s_1 = 40$ and $s_2 = 45$ for $r = 35$, $s_1 = 46$ and $s_2 = 48$ for $r = 45$, $s_1 = 80$ and $s_2 = 85$ for $r = 70$, and $s_1 = 88$ and $s_2 = 95$ for $r = 85$.

The estimates of the parameters in the Bayes technique are generated based on informative priors in order to evaluate the type of prior. In the case of informative priors, the hyper-parameters are selected by elective hyper-parameters utilizing MLE information to display the outcomes of estimated parameters with the help of the MLE's asymptotic distribution and Gibbs samples. For confidence intervals, we compare the average intervals (lower and upper) of the asymptotic confidence intervals for the MLE and HPD for the Bayesian estimators based on their coverage percentages of 95%. Also, we have calculated approximate 95% confidence intervals for the unknown parameters. The procedure is repeated 1000 times, and the average confidence/credibility is reported. Tables 1–4 present the complete results. The acquired estimates were calculated using the “maxLik” and “coda” packages of the R programming language because the theoretical findings of d and q obtained using the suggested estimation methods cannot be represented in closed form. For these estimates, the “cmaxLik” package was used for MLE, while the “coda” package was used to obtain the Bayesian estimation for MCMC based on MH algorithms.

The simulation results concluded the following findings, which we highlight:

- As the sample size increases for the parameters of APW α, θ , and λ , the minimum RAB and MSE decline for the estimated parameters of MLE and Bayes estimates;
- The Bayes estimates consistently outperform the MLE in terms of RAB, MSE, and interval values;
- In most cases, the HPD intervals are shorter than ACI;
- When the censored sample size r increases while keeping the hybrid censored sample's sample size n and time constant, and the performance improves.

When sample size n and censored sample size r are kept constant, performance improves as the time of the hybrid censored sample lengthens.

Table 1. MLE and Bayesian; $\alpha = 0.5, \theta = 0.6, \lambda = 0.4$.

n	T	r	MLE				Bayesian				Prediction	One		Two			
			RAB	MSE	Lower	Upper	RAB	MSE	Lower	Upper	$s1, s2$	$x_{(s1)}$	$x_{(s2)}$	$y_{(s1)}$	$y_{(s2)}$		
50	3	35	α	0.2026	0.3199	0.0517	1.6925	0.0277	0.0186	0.2079	0.7426	40, 45	Point	2.1031	3.2638	1.8775	2.9404
			θ	0.0390	0.0112	0.4213	0.8254	0.0188	0.0064	0.4599	0.7574		Lower	0.2534	0.5543	0.6219	0.8281
			λ	0.0133	0.0254	0.0824	0.7069	0.0049	0.0078	0.2321	0.5686		Upper	4.7466	7.2531	3.5614	5.5412
		45	α	0.1442	0.2445	0.1052	1.5314	0.0126	0.0078	0.3268	0.6620	46, 48	Point	3.5224	4.4166	3.3800	4.2708
			θ	0.0326	0.0102	0.4311	0.8283	0.0072	0.0046	0.4760	0.7384		Lower	0.8367	1.1031	0.9917	1.4414
			λ	0.0311	0.0218	0.0994	0.6758	0.0022	0.0049	0.2744	0.5375		Upper	7.2299	9.0887	6.6257	8.8478
	4	35	α	0.1934	0.3069	0.1152	1.8134	0.0251	0.0172	0.2146	0.7673	40, 45	Point	2.0594	3.1748	1.8931	2.9502
			θ	0.0318	0.0108	0.4191	0.8190	0.0210	0.0057	0.4872	0.7661		Lower	0.3788	0.6340	0.6475	0.9533
			λ	0.0139	0.0248	0.0865	0.7451	0.0040	0.0077	0.2236	0.5583		Upper	4.4258	6.9577	3.5605	5.6441
		45	α	0.1419	0.2329	0.1263	1.5070	0.0011	0.0073	0.3183	0.6554	46, 48	Point	3.5462	4.4823	3.3446	4.2395
			θ	0.0271	0.0099	0.4233	0.8092	0.0001	0.0040	0.4810	0.7266		Lower	0.9232	1.1323	1.0593	1.2635
			λ	0.0132	0.0203	0.0993	0.6815	0.0007	0.0044	0.2749	0.5220		Upper	7.5150	9.4798	6.4960	8.4014
100	3	70	α	0.0445	0.1599	-0.2607	1.3052	0.0254	0.0170	0.2137	0.7535	80, 85	Point	1.8986	2.3406	1.8074	2.2342
			θ	0.0099	0.0048	0.4699	0.7419	0.0103	0.0032	0.4921	0.7114		Lower	0.6983	0.7122	0.8720	0.9590
			λ	0.0388	0.0174	0.1277	0.6413	0.0104	0.0054	0.2510	0.5347		Upper	3.6524	4.3399	3.0887	3.6875
		85	α	0.0229	0.1362	-0.2121	1.2349	0.0011	0.0071	0.3414	0.6627	88, 95	Point	2.7074	3.8979	2.6593	3.8467
			θ	0.0081	0.0045	0.4717	0.7477	0.0030	0.0028	0.5044	0.7053		Lower	1.0529	1.5861	1.2280	1.7688
			λ	0.0431	0.0163	0.1348	0.6307	0.0010	0.0033	0.2972	0.5246		Upper	5.0868	7.3680	4.5357	6.6665
	4	70	α	0.0556	0.1600	-0.2546	1.3102	0.0191	0.0170	0.2230	0.7561	80, 85	Point	1.9252	2.3701	1.8398	2.2718
			θ	0.0132	0.0047	0.4746	0.7413	0.0095	0.0028	0.5041	0.7086		Lower	0.7294	0.7921	0.8472	1.0814
			λ	0.0426	0.0172	0.1216	0.6405	0.0131	0.0049	0.2618	0.5290		Upper	3.5883	4.2110	2.9375	3.6616
		85	α	0.0118	0.1188	-0.1816	1.1698	0.0010	0.0069	0.3530	0.6567	88, 95	Point	2.7373	3.9274	2.6134	3.7641
			θ	0.0121	0.0045	0.4761	0.7384	0.0039	0.0027	0.4908	0.6919		Lower	0.7406	1.2681	1.2032	1.8186
			λ	0.0700	0.0152	0.1365	0.6076	0.0014	0.0031	0.2967	0.5175		Upper	4.9428	7.2694	4.2727	6.4409

Table 2. MLE and Bayesian $\alpha = 0.5, \theta = 1.3, \lambda = 3$.

			MLE				Bayesian				Prediction	One		Two			
n	T	r	RAB	MSE	Lower	Upper	RAB	MSE	Lower	Upper	s_1, s_2	$x_{(s_1)}$	$x_{(s_2)}$	$y_{(s_1)}$	$y_{(s_2)}$		
50	0.3	35	α	0.1108	0.2678	0.0982	1.5644	0.0177	0.0193	0.2342	0.7658	40, 45	Point	0.2291	0.2724	0.2286	0.2718
			θ	0.0390	0.0669	0.8535	1.8480	0.0055	0.0115	1.1196	1.5350		Lower	0.1473	0.1715	0.1543	0.1913
			λ	0.0242	0.2761	1.9069	3.9481	0.0031	0.0225	2.7080	3.2779		Upper	0.3329	0.3775	0.3092	0.3620
		45	α	0.0936	0.2488	0.1259	1.5207	0.0054	0.0076	0.3329	0.6741	46, 48	Point	0.2832	0.3073	0.2819	0.3061
			θ	0.0343	0.0634	0.8585	1.8307	0.0010	0.0063	1.1409	1.4518		Lower	0.1958	0.2132	0.2096	0.2309
			λ	0.0315	0.2284	1.9867	3.8240	0.0005	0.0079	2.8268	3.1734		Upper	0.3740	0.3956	0.3648	0.3943
	0.6	35	α	0.1105	0.2430	0.1925	1.6476	0.0050	0.0187	0.2276	0.7550	40, 45	Point	0.2295	0.2732	0.2278	0.2709
			θ	0.0390	0.0506	0.9212	1.7803	0.0043	0.0115	1.0881	1.5124		Lower	0.1359	0.1789	0.1479	0.1828
			λ	0.0008	0.2399	1.9301	4.0748	0.0017	0.0213	2.7682	3.2844		Upper	0.3215	0.3818	0.3025	0.3534
		45	α	0.0924	0.2375	0.2055	1.7988	0.0032	0.0072	0.3416	0.6647	46, 48	Point	0.2829	0.3072	0.2866	0.3109
			θ	0.0232	0.0385	0.9497	1.7105	0.0003	0.0058	1.1391	1.4333		Lower	0.2039	0.2227	0.2086	0.2259
			λ	0.0037	0.2051	1.9941	4.3838	0.0005	0.0061	2.8327	3.1657		Upper	0.3741	0.3998	0.3699	0.3945

Table 2. Cont.

			MLE				Bayesian				Prediction		One		Two		
<i>n</i>	<i>T</i>	<i>r</i>	RAB	MSE	Lower	Upper	RAB	MSE	Lower	Upper	<i>s</i> 1, <i>s</i> 2		<i>x</i> (<i>s</i> 1)	<i>x</i> (<i>s</i> 2)	<i>y</i> (<i>s</i> 1)	<i>y</i> (<i>s</i> 2)	
100	0.3	70	<i>α</i>	0.0835	0.2363	0.1985	2.1887	0.0083	0.0183	0.2586	0.7596	80, 85	Point	0.2303	0.2507	0.2300	0.2505
			<i>θ</i>	0.0094	0.0315	0.9650	1.6593	0.0062	0.0079	1.1396	1.4814		Lower	0.1654	0.1809	0.1744	0.1890
			<i>λ</i>	0.0101	0.2041	1.9722	4.2175	0.0017	0.0219	2.7064	3.2757		Upper	0.3020	0.3209	0.2944	0.3157
		85	<i>α</i>	0.0932	0.2261	0.2167	2.3658	0.0033	0.0068	0.3406	0.6703	88, 95	Point	0.2625	0.3003	0.2620	0.3001
			<i>θ</i>	0.0089	0.0306	0.9448	1.6897	0.0029	0.0049	1.1723	1.4382		Lower	0.1968	0.2322	0.2076	0.2397
			<i>λ</i>	0.0035	0.1939	1.7863	4.2346	0.0000	0.0061	2.8302	3.1799		Upper	0.3258	0.3671	0.3207	0.3616
	0.6	70	<i>α</i>	0.0750	0.1975	0.2194	2.3796	0.0009	0.0169	0.2628	0.7679	80, 85	Point	0.2294	0.2498	0.2289	0.2492
			<i>θ</i>	0.0094	0.0284	0.9824	1.6420	0.0052	0.0081	1.1370	1.4780		Lower	0.1497	0.1688	0.1710	0.1898
			<i>λ</i>	0.0227	0.1944	1.9776	4.1236	0.0014	0.0216	2.7202	3.3071		Upper	0.2959	0.3201	0.2855	0.3096
		85	<i>α</i>	0.0695	0.1908	0.3182	2.3856	0.0060	0.0068	0.3326	0.6509	88, 95	Point	0.2618	0.2999	0.2610	0.2989
			<i>θ</i>	0.0038	0.0212	1.0194	1.5906	0.0023	0.0045	1.1524	1.4409		Lower	0.1942	0.2343	0.2009	0.2351
			<i>λ</i>	0.0149	0.1855	1.9959	4.1502	0.0001	0.0059	2.8327	3.1810		Upper	0.3292	0.3779	0.3249	0.3643

Table 3. MLE and Bayesian $\alpha = 0.5$, $\theta = 0.6$, $\lambda = 3$.

			MLE				Bayesian				Prediction		One		Two		
<i>n</i>	<i>T</i>	<i>r</i>	RAB	MSE	Lower	Upper	RAB	MSE	Lower	Upper	s1, s2		<i>x</i> (s1)	<i>x</i> (s2)	<i>y</i> (s1)	<i>y</i> (s2)	
50	0.15	35	<i>α</i>	0.0269	0.1426	0.0023	1.2534	0.0096	0.0194	0.2273	0.7593	40, 45	Point	0.0659	0.1003	0.0635	0.0976
			<i>θ</i>	0.0446	0.0131	0.4087	0.8449	0.0160	0.0032	0.5167	0.7266		Lower	0.0203	0.0355	0.0251	0.0465
			<i>λ</i>	0.0110	0.1541	2.2001	3.7341	0.0006	0.0207	2.7075	3.2595		Upper	0.1257	0.1741	0.1090	0.1666
		45	<i>α</i>	0.0353	0.1360	0.0266	1.3011	0.0015	0.0080	0.3377	0.6822	46, 48	Point	0.1095	0.1368	0.1085	0.1363
			<i>θ</i>	0.0380	0.0109	0.4226	0.8230	0.0060	0.0027	0.5121	0.7112		Lower	0.0460	0.0630	0.0542	0.0682
			<i>λ</i>	0.0121	0.1472	2.3318	3.7244	0.0007	0.0087	2.8271	3.1841		Upper	0.1777	0.2251	0.1816	0.2218
	0.4	35	<i>α</i>	0.0256	0.1361	0.0125	1.1315	0.0043	0.0191	0.2171	0.7526	40, 45	Point	0.0658	0.1004	0.0644	0.0989
			<i>θ</i>	0.0438	0.0117	0.4207	0.8319	0.0146	0.0030	0.4999	0.7135		Lower	0.0152	0.0297	0.0231	0.0457
			<i>λ</i>	0.0053	0.1489	2.2598	3.7720	0.0028	0.0208	2.7241	3.2833		Upper	0.1170	0.1731	0.1091	0.1695
		45	<i>α</i>	0.0184	0.1277	0.0279	1.3630	0.0017	0.0077	0.3479	0.6954	46, 48	Point	0.1103	0.1375	0.1090	0.1368
			<i>θ</i>	0.0336	0.0088	0.4407	0.7996	0.0065	0.0025	0.5013	0.7048		Lower	0.0451	0.0621	0.0520	0.0597
			<i>λ</i>	0.0050	0.1230	2.3208	3.6910	0.0001	0.0080	2.8193	3.1712		Upper	0.1854	0.2264	0.1801	0.2166
100	0.15	70	<i>α</i>	0.0405	0.1251	0.0196	2.0385	0.0081	0.0184	0.2616	0.7778	80, 85	Point	0.0637	0.0781	0.0622	0.0765
			<i>θ</i>	0.0114	0.0062	0.4527	0.7610	0.0140	0.0019	0.5217	0.6862		Lower	0.0243	0.0319	0.0325	0.0423
			<i>λ</i>	0.0282	0.1357	1.6092	4.5602	0.0006	0.0215	2.7226	3.2838		Upper	0.1039	0.1250	0.0952	0.1166
		85	<i>α</i>	0.0129	0.1244	0.0352	1.9122	0.0074	0.0078	0.3397	0.6752	88, 95	Point	0.0888	0.1261	0.0880	0.1255
			<i>θ</i>	0.0029	0.0055	0.4561	0.7474	0.0073	0.0016	0.5279	0.6858		Lower	0.0444	0.0688	0.0506	0.0758
			<i>λ</i>	0.0174	0.1225	1.6836	4.2595	0.0009	0.0080	2.8351	3.1794		Upper	0.1331	0.1828	0.1277	0.1800
	0.4	70	<i>α</i>	0.0137	0.1147	0.0199	1.9815	0.0219	0.0171	0.2630	0.7562	80, 85	Point	0.0632	0.0775	0.0624	0.0768
			<i>θ</i>	0.0148	0.0057	0.4623	0.7554	0.0159	0.0020	0.5262	0.6941		Lower	0.0272	0.0325	0.0304	0.0423
			<i>λ</i>	0.0305	0.1145	1.7882	4.3949	0.0010	0.0213	2.6968	3.2729		Upper	0.1021	0.1207	0.0941	0.1170
		85	<i>α</i>	0.0124	0.1004	0.0830	1.9730	0.0058	0.0075	0.3383	0.6822	88, 95	Point	0.0890	0.1265	0.0874	0.1248
			<i>θ</i>	0.0040	0.0049	0.4655	0.7393	0.0057	0.0016	0.5261	0.6822		Lower	0.0460	0.0668	0.0494	0.0770
			<i>λ</i>	0.0080	0.1043	1.6964	4.3514	0.0005	0.0079	2.8234	3.1685		Upper	0.1427	0.1905	0.1254	0.1785

Table 4. MLE and Bayesian $\alpha = 2$, $\theta = 1.3$, $\lambda = 3$.

<i>n</i>	<i>T</i>	<i>r</i>		MLE				Bayesian				Prediction		One		Two	
				RAB	MSE	Lower	Upper	RAB	MSE	Lower	Upper	s1, s2		$x_{(s1)}$	$x_{(s2)}$	$y_{(s1)}$	$y_{(s2)}$
50	0.5	35	α	0.1689	1.9609	0.5798	4.3272	0.0005	0.0239	1.6725	2.2775	40, 45	Point	0.3293	0.3807	0.3283	0.3794
			θ	0.0792	0.0775	0.8957	1.9101	0.0023	0.0148	1.0816	1.5504		Lower	0.2281	0.2727	0.2303	0.2813
			λ	0.0532	0.9964	0.9083	4.7728	0.0009	0.0241	2.7058	3.3127		Upper	0.4491	0.4927	0.4193	0.4791
		45	α	0.1113	1.8086	0.6942	4.1576	0.0009	0.0082	1.8284	2.1785	46, 48	Point	0.3927	0.4196	0.3920	0.4188
			θ	0.0799	0.0719	0.8996	1.9522	0.0017	0.0066	1.1541	1.4606		Lower	0.2914	0.3184	0.2995	0.3233
			λ	0.0422	0.9500	0.9783	4.7685	0.0004	0.0085	2.8295	3.1817		Upper	0.4876	0.5163	0.4859	0.5114
	0.8	35	α	0.0049	1.9653	0.6091	4.7392	0.0044	0.0233	1.6962	2.2756	40, 45	Point	0.3293	0.3799	0.3292	0.3802
			θ	0.0796	0.0730	0.9142	1.8929	0.0032	0.0138	1.0618	1.5156		Lower	0.2153	0.2673	0.2297	0.2880
			λ	0.0198	0.8697	1.2345	4.8846	0.0025	0.0215	2.7304	3.2945		Upper	0.4284	0.4844	0.4234	0.4858
		45	α	0.1016	1.7415	0.6716	4.4127	0.0007	0.0080	1.8287	2.1751	46, 48	Point	0.3893	0.4159	0.3904	0.4172
			θ	0.0856	0.0707	0.9378	1.8849	0.0027	0.0062	1.1466	1.4555		Lower	0.2941	0.3186	0.3042	0.3249
			λ	0.0552	0.7473	1.1706	4.4980	0.0018	0.0084	2.8266	3.1809		Upper	0.4875	0.5151	0.4870	0.5083
100	0.5	70	α	0.0267	0.8196	0.1745	3.7189	0.0002	0.0231	1.7032	2.2969	80, 85	Point	0.3280	0.3524	0.3290	0.3533
			θ	0.0237	0.0312	0.9896	1.6721	0.0002	0.0093	1.0998	1.4748		Lower	0.2527	0.2717	0.2602	0.2829
			λ	0.0020	0.3551	1.8255	4.1626	0.0022	0.0202	2.6934	3.2714		Upper	0.4160	0.4390	0.4054	0.4318
		85	α	0.0021	0.7953	0.2354	4.0292	0.0006	0.0079	1.8413	2.1842	88, 95	Point	0.3691	0.4117	0.3673	0.4101
			θ	0.0227	0.0306	0.9685	1.7027	0.0012	0.0058	1.1422	1.4460		Lower	0.2972	0.3343	0.2976	0.3427
			λ	0.0013	0.3211	1.8850	4.1075	0.0001	0.0078	2.8431	3.1833		Upper	0.4517	0.4913	0.4324	0.4807
	0.8	70	α	0.0217	0.7381	0.2741	3.6392	0.0006	0.0228	1.7298	2.2978	80, 85	Point	0.3295	0.3538	0.3287	0.3530
			θ	0.0409	0.0303	1.0151	1.6912	0.0016	0.0091	1.1065	1.4726		Lower	0.2479	0.2767	0.2545	0.2784
			λ	0.0199	0.3158	1.9638	4.1555	0.0015	0.0201	2.7075	3.2674		Upper	0.4237	0.4531	0.3968	0.4225
		85	α	0.0261	0.7182	0.3740	3.7218	0.0019	0.0069	1.8401	2.2069	88, 95	Point	0.3704	0.4132	0.3693	0.4123
			θ	0.0352	0.0254	1.0461	1.6454	0.0011	0.0056	1.1564	1.4419		Lower	0.2934	0.3328	0.3040	0.3364
			λ	0.0105	0.2677	1.9956	3.9813	0.0010	0.0078	2.8175	3.1515		Upper	0.4475	0.4903	0.4348	0.4738

5. Application

Reference [18] used two real-life data sets and confirmed that APW distribution is better than many competitive distributions such as: Weibull, alpha power exponential, McDonald Weibull, beta Weibull, transmuted Weibull, gamma Lomax, Zografos–Balakrishnan log-logistic, exponentiated Weibull, and exponentiated Weibull distributions. The APW distribution has more different applications, such as inferences and engineering applications based on progressive Type II censoring (see [27]); the optimal test plan of step-stress models under progressively Type II censored samples (see [28]); and discrete APW and its applications (see [29]). In this section, we consider two real-life data sets, which are discussed by [18], and illustrate the methods proposed in the previous sections.

First, the data set is from [18] for the application to the APW distribution, and it represents the survival times (in days) of 109 successive coal-mining disasters in Great Britain, within the period of 1875–1951. The statistical summarized measures of this data are given as follows: minimum value is 1; first quartile is 54; median is 145; mean is 233.3; third quartile is 312; and maximum value is 1630. Before moving on, we want to look at the data set using a scaled Total Time on Test (TTT) plot, a strip plot, a violin plot, and an empirical hazard function of the observed data. Figure 1 gives a clear picture of the distribution's hazard function's shape, which has a decreasing shape and a TTT line under half line. In Figures 1 and 2, the APW is good model to describe these data. Figure 2 illustrates the APW distribution's theoretical and empirical pdf, QQ, CDF, and P-P plot using the data set of the survival times data, and it can be seen that the APW is suitable and reliable for fitting the survival times data set.

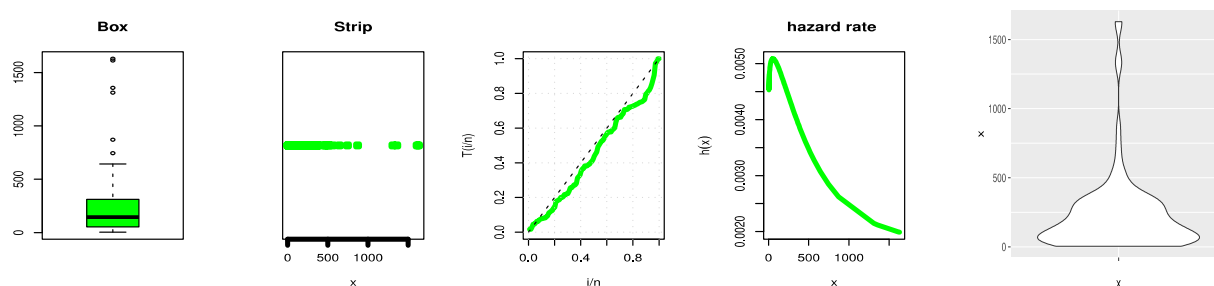


Figure 1. Classical examination of the survival times.

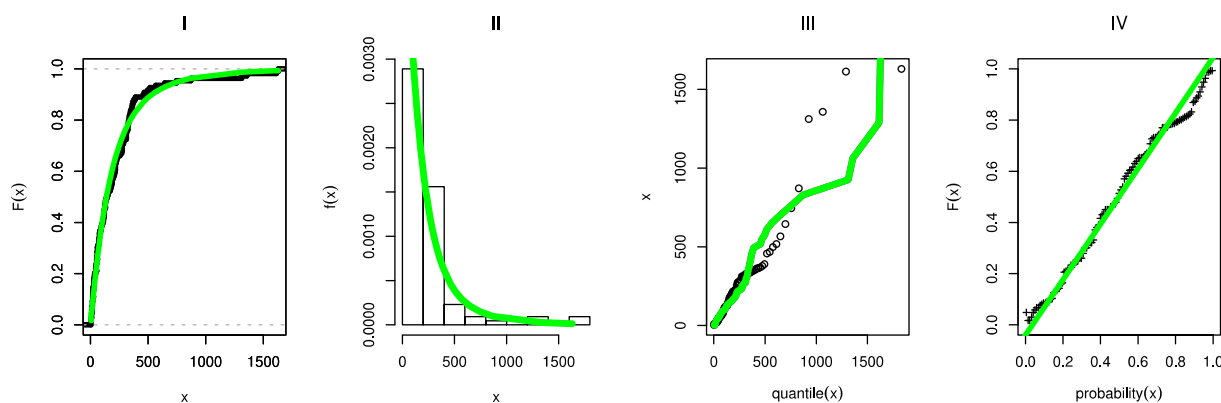


Figure 2. APW estimated examination of the survival times.

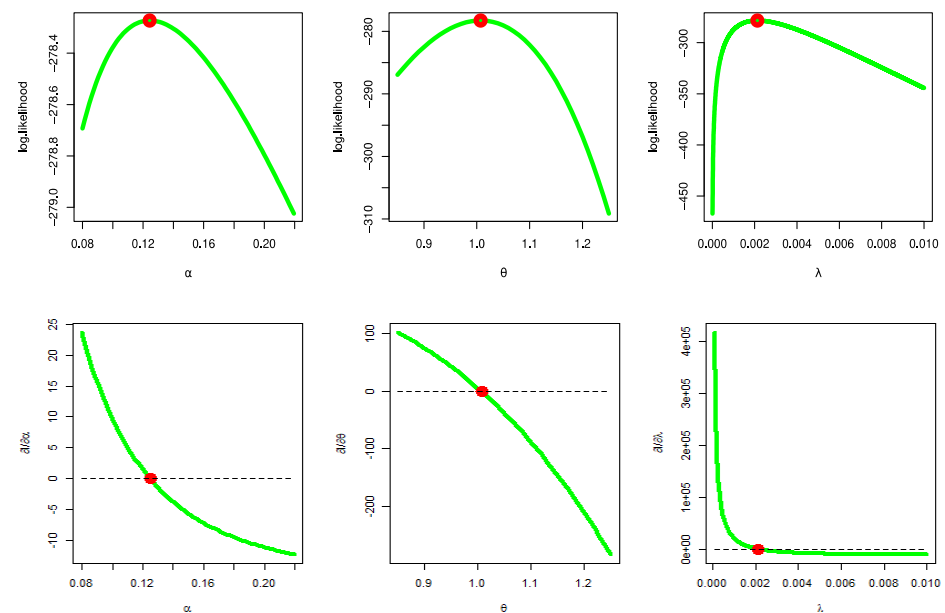
Reference [18] obtained the MLE parameters of APW as $\alpha = 0.01956$, $\theta = 1.04985$, and $\lambda = 0.00106$, the KSD is 0.0592, and its P-value is 0.8383. Table 5 discussed MLE and Bayesian estimation to estimate parameters of APW based on hybrid censoring with different size of samples, which different measures have been obtained as the MLE of the parameters with standard error (SE), lower, upper for confidence intervals with 95%, reliability, and hazard value with t_0 . Figure 3 discussed the existence and uniqueness plot of maximum likelihood estimates for survival times data to check the existence and uniqueness estimators. The maximum likelihood values for survival time data set, where $r = 60$ and $T = 110$ for the estimated parameter values that coincide with the MLE estimates in Table 5, are shown in Figure 3, which also supports the MLE estimates.

Table 5. MLE and Bayesian with different measures for survival times.

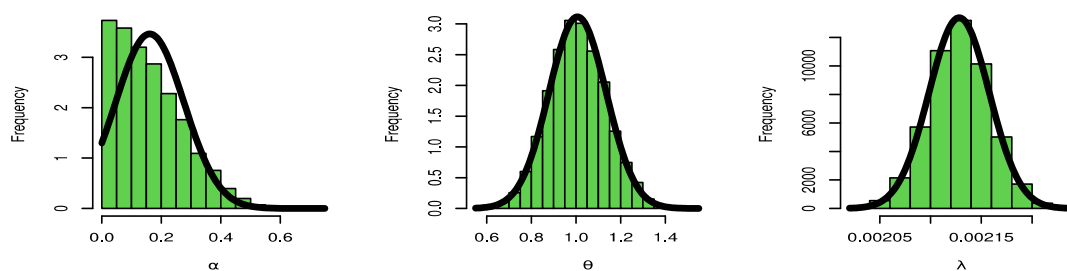
r	T		Estimates	SE	Lower	Upper	R	h
60	110	MLE	0.12432	0.06293	0.00098	0.24767	0.58740	0.00450
			1.00717	0.12820	0.75590	1.25844		
			0.00213	0.00027	0.00160	0.00265		
		Bayesian	0.16131	0.05115	0.00019	0.20380		
			1.00678	0.12802	0.76875	1.26696		
			0.00213	0.00021	0.00207	0.00219		
	235	MLE	0.11153	0.05176	0.00233	0.24563	0.43888	0.00395
			0.98353	0.10345	0.78076	1.18629		
			0.00226	0.00023	0.00167	0.00259		
		Bayesian	0.13639	0.04956	0.00022	0.23157		
			0.98322	0.10331	0.79113	1.19317		
			0.00226	0.00003	0.00220	0.00233		

Table 5. Cont.

r	T		Estimates	SE	Lower	Upper	R	h
90	250	MLE	0.11035	0.04169	0.00265	0.44420	0.33250	0.00375
			0.99656	0.09374	0.81284	1.18029		
			0.00216	0.00023	0.00172	0.00261		
		Bayesian	0.13494	0.03935	0.00023	0.30928	0.35126	0.00360
			0.99628	0.09361	0.82224	1.18652		
			0.00215	0.00025	0.00211	0.00221		
	300	MLE	0.14193	0.03919	0.00294	0.45158	0.27213	0.00366
			1.00695	0.08421	0.84190	1.17200		
			0.00223	0.00025	0.00174	0.00272		
		Bayesian	0.16166	0.02961	0.00012	0.30063	0.28566	0.00355
			1.00669	0.08409	0.85034	1.17760		
			0.00222	0.00021	0.00218	0.00229		

Figure 3. Existence and uniqueness plot for MLE of survival times: $r = 60$, and $T = 110$.

The MCMC results are normal, exhibit symmetric posterior density histograms, and have convergence measures by using the Brooks–Gelman–Rubin (BGR) statistic, as shown in Figures 4 and 5. According to Figure 6, the values for the MCMC series, which began with zero and ended with one, do not exhibit any auto-correlation.

Figure 4. Histogram of posterior for APW parameters: $r = 60$, and $T = 110$.

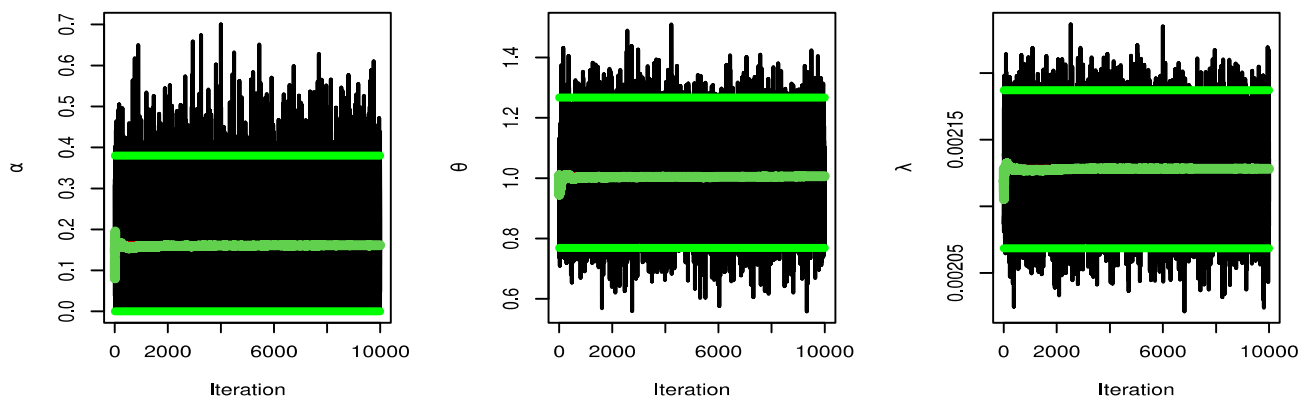


Figure 5. Trace and BGR lines of MCMC results for APW parameters. Survival times: $r = 60$, and $T = 110$.

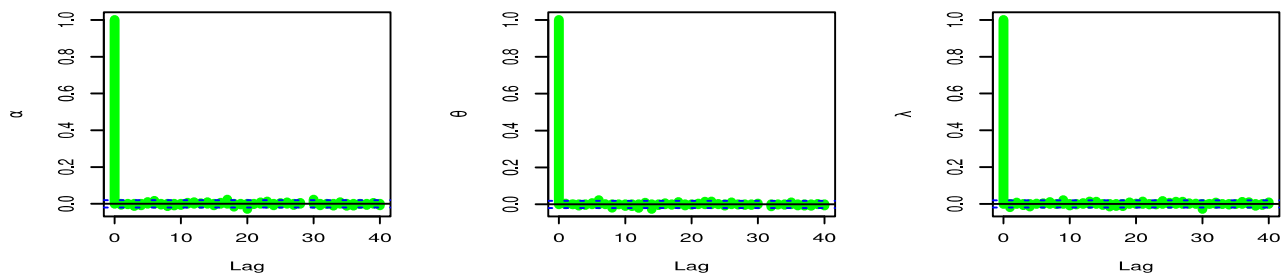


Figure 6. Auto-correlation test with different lags of MCMC results for APW parameters. Survival times: $r = 60$ and $T = 110$.

Now let us look at the sample prediction problems for one and two. Based on the observed sample, we present in Figure 7 the predictive point for one sample and two sample, and the predictive point of the s -st order statistic. Accordingly, the s -st failure will occur between r and 109 days based on the observed sample. Based on the results in Table 6 and Figure 7, we note the closeness of the results of the two methods for predicting values, but to determine the best method, the key is the KS test for two independent samples. It has been noted that the first method is the best because it contains the least distance between the data and also has the largest p -value of KS test.

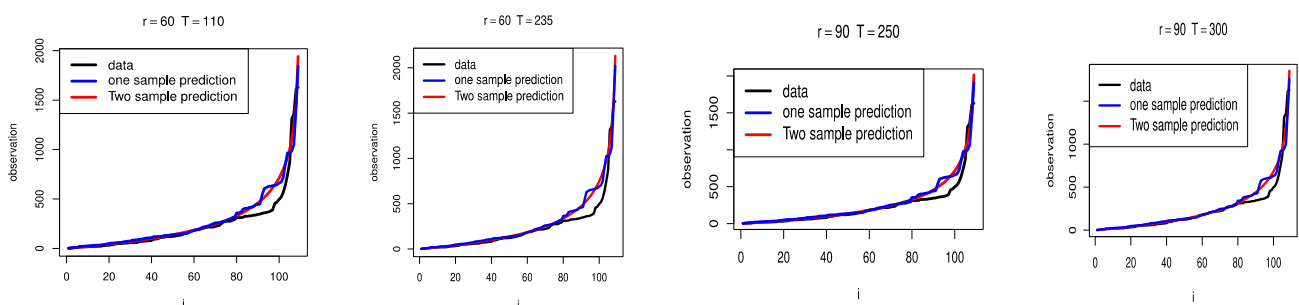


Figure 7. Bayes prediction by one-sample prediction and two-sample prediction: survival times.

Table 6. Bayes prediction with different size and two sample KS test: survival times.

r	T	110		235	
		One	Two	One	Two
60	$x_{(s1:n)}$	213.592	227.753	215.257	220.946
	$x_{(s2:n)}$	330.184	356.529	336.810	366.244
	KSD	0.110	0.128	0.119	0.138
	p value	0.524	0.330	0.420	0.253
90		250		300	
	$x_{(s1:n)}$	209.059	214.511	203.959	209.208
	$x_{(s2:n)}$	325.225	353.268	315.296	342.015
	KSD	0.110	0.138	0.101	0.128
	p value	0.524	0.253	0.636	0.330

The authors of [30] provided the second data set. The statistics are 1.5 cm glass fiber strengths, as determined by the National Physical Laboratory in England. The statistically summarized measures of these data are given as follows: minimum value is 0.55; first quartile is 1.375; median is 1.590; mean is 1.507; third quartile is 1.685; and maximum value is 2.240. Reference [18] obtained the MLE parameters of APW for glass fiber data as $\alpha = 10.8558$, $\theta = 4.48362$, and $\lambda = 0.194777$, the KSD is 0.10661, and its p -value is 0.47107. Figure 8 gives a clear picture of the distribution's hazard function's shape, which has a decreasing shape, and the TTT line is under half line. Based on Figures 8 and 9, the APW is a good model to describe these data. Figure 9 illustrates the APW distribution's theoretical and empirical pdf, QQ, CDF, and P-P plot using the glass fiber data, and it can be seen that the APW is suitable and reliable for the fitting glass fiber data.

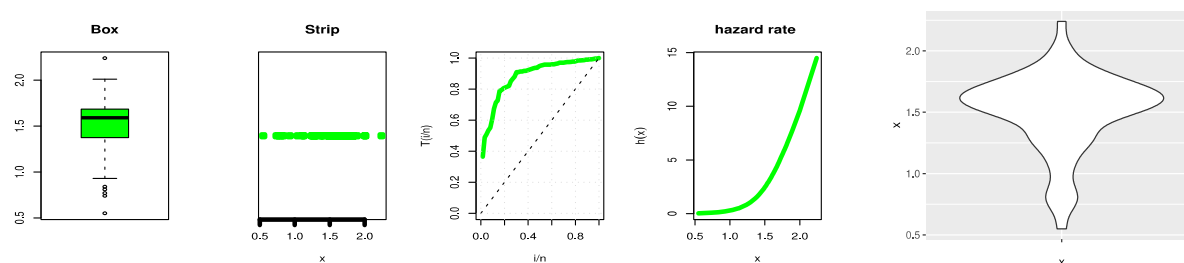
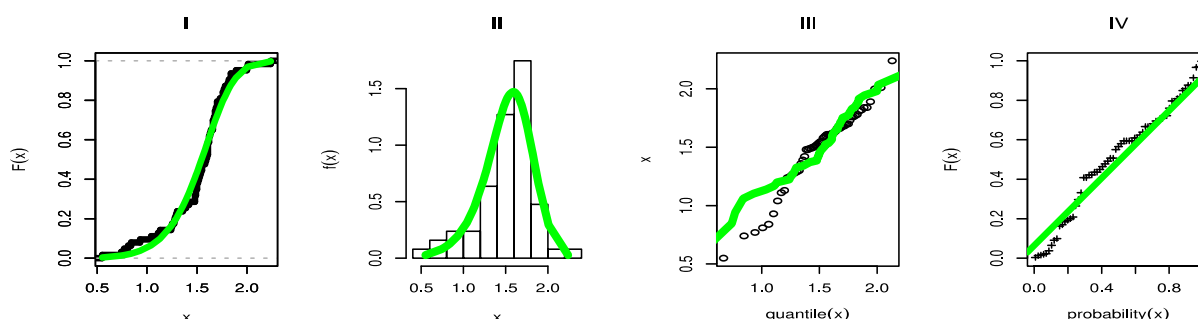
**Figure 8.** Classical examination of glass fiber data.**Figure 9.** Classical examination of Fibre data.

Table 7 discussed the MLE and Bayesian approaches to estimate the parameters of APW based on hybrid censoring with different size of samples, for which different measures have been obtained as the MLE of parameters with SE, lower, upper for confidence intervals with 95%, reliability, and hazard value with t_0 . Figure 10 discussed the existence and

uniqueness plot of maximum likelihood estimates for glass fiber data to check for existence and uniqueness estimators. The maximum likelihood values for the glass fiber data set, where $r = 40$ and $T = 1.5$ for the estimated parameter values that coincide with the MLE estimates in Table 7, are shown in Figure 10, which also supports the MLE estimates.

Table 7. MLE and Bayesian with different measures for glass fibers data.

r	T		Estimates	SE	Lower	Upper	R	h
40	1.5	MLE	6.38471	4.04607	1.14558	33.91500	0.62580	1.46420
			3.61683	1.28464	1.09894	6.13471		
			0.20871	0.13080	0.02465	0.66107		
		Bayesian	6.37244	0.97457	4.42246	8.24316	0.62825	1.44788
			3.60977	0.64323	2.40183	4.87414		
			0.20784	0.04637	0.12152	0.30367		
	1.6	MLE	8.13212	3.88604	1.08452	35.34876	0.48970	2.36942
			3.85947	1.15056	1.60436	6.11457		
			0.21478	0.18670	0.05115	0.58071		
		Bayesian	8.13443	0.96970	6.16518	9.96940	0.48921	2.37509
			3.86299	0.65603	2.61326	5.15972		
			0.21465	0.03762	0.14443	0.29162		
50	1.65	MLE	8.89171	3.80317	2.24250	32.02592	0.37761	3.57436
			4.34119	1.02724	2.32779	6.35459		
			0.19080	0.10426	0.02234	0.45395		
		Bayesian	8.89902	0.82629	7.29770	10.54122	0.37968	3.54935
			4.33410	0.59919	3.17194	5.50907		
			0.19074	0.02704	0.14021	0.24606		
	1.7	MLE	8.69209	1.46131	0.85208	27.23627	0.25907	5.41665
			4.96924	0.95637	3.09475	6.84374		
			0.15153	0.09348	0.03169	0.33475		
		Bayesian	8.69814	0.66231	7.41435	10.01433	0.26094	5.38285
			4.96264	0.55785	3.88066	6.05655		
			0.15149	0.01883	0.11631	0.19001		

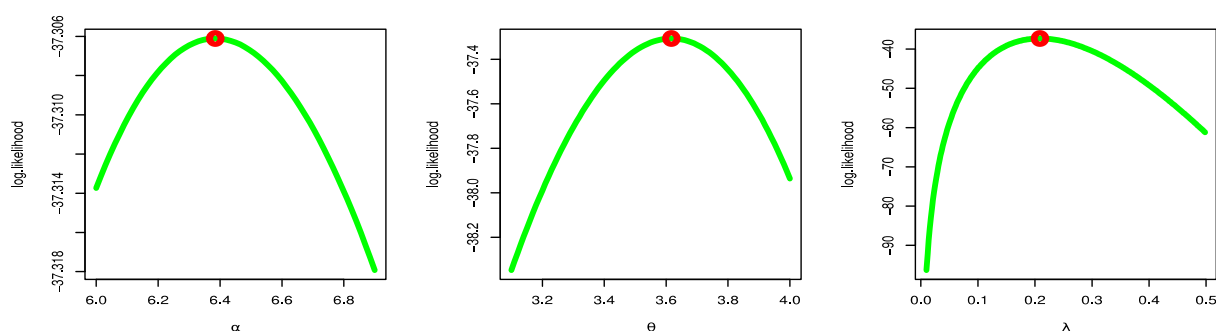


Figure 10. Existence and uniqueness plot for MLE of glass fibers data: $r = 40$, and $T = 1.5$.

The MCMC results are normal, exhibit symmetric posterior density histograms, and have convergence measures for APW parameters with the glass fiber data, as shown in Figures 11 and 12. According to Figure 13, the values for the MCMC series, which began

with zero and ended with one, do not exhibit any auto-correlation for APW parameters with glass fibers data.

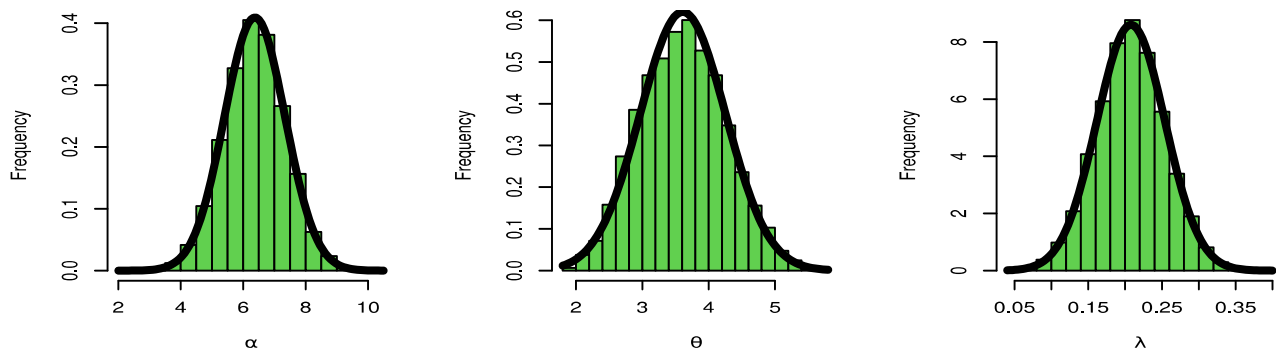


Figure 11. Histogram of posterior for APW parameters of glass fiber data: $r = 40$, and $T = 1.5$.

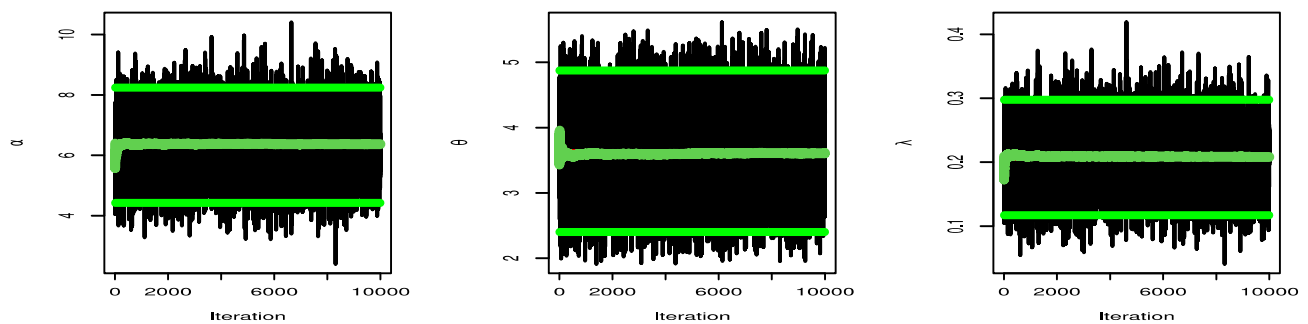


Figure 12. Trace and BGR lines of MCMC results for APW parameters of glass fiber data: $r = 40$, and $T = 1.5$.

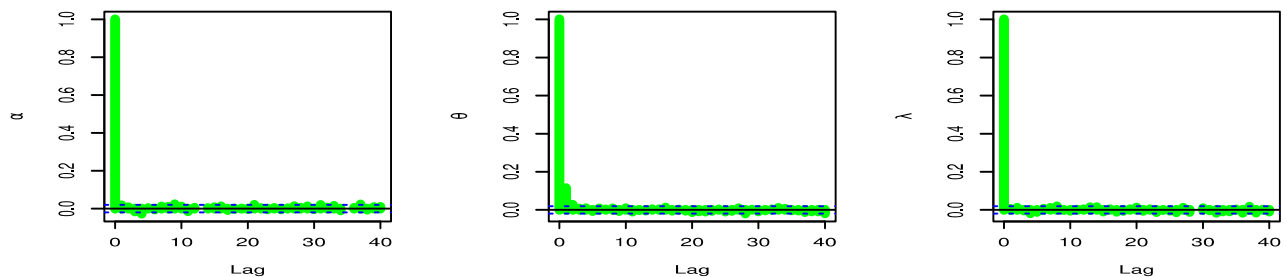


Figure 13. Auto-correlation test with different lags of MCMC results for APW parameters of glass fiber data: $r = 40$, and $T = 1.5$.

Now let us look at the sample prediction problems for one and two. Based on the observed sample, we present in Figure 14 the predictive point for the one-sample and two-sample prediction of the s -st order statistic for glass fiber data. Accordingly, the s -st failure will occur between the r and 63rd observation based on the observed sample. Based on the results in Table 8 and Figure 7, we note the closeness of the results of the two methods for predicting values, but to suggest the best method, the key is the KS test for two independent samples. It has been noted that the first method is the best because it contains the least distance between the data and also has the largest p -value of KS test.

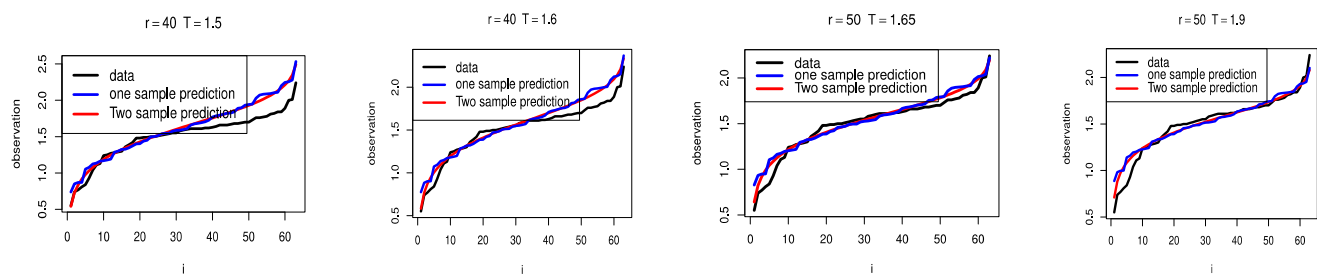


Figure 14. Bayes prediction by one-sample prediction and two-sample prediction: glass fiber data.

Table 8. Bayes prediction with different size and two sample KS test: glass fiber data.

r	T	1.4		1.6	
		One	Two	One	Two
40	$x_{(s1:n)}$	1.930	1.942	1.850	1.860
	$x_{(s2:n)}$	2.041	2.089	1.946	1.988
	KSD	0.222	0.238	0.159	0.175
	p value	0.085	0.053	0.398	0.284
r	T	1.65		1.7	
		One	Two	One	Two
50	$x_{(s1:n)}$	1.783	1.792	1.735	1.742
	$x_{(s2:n)}$	1.865	1.900	1.804	1.834
	KSD	0.111	0.127	0.111	0.127
	p value	0.828	0.681	0.826	0.686

6. Concluding Remarks

Under the hybrid Type II censoring schemes, the maximum likelihood estimation and Bayesian estimation techniques for the alpha power Weibull distribution are devised. Both paradigms take into account parameter estimates as well as reliability, hazard rate functions and mean failure time. On the basis of simulated samples produced from the distribution for various hybrid Type II censoring techniques, the long-term performances of the suggested estimators are compared. Under the symmetric loss function, Bayes estimators are built. In reality, Bayes estimators give accurate estimates of the parameters because, as predicted, they have mean squared errors that are lower than those of MLEs. Additionally, the issue of predicted future observables was also covered, and one- and two-sample predictive posteriors of the future order statistics were obtained. The posteriors become quite complex and are impossible to simplify into any clear forms. The samples are extracted from the posteriors using the Metropolis–Hastings method, and a sample-based summary of the predicted posteriors is then provided. Because of reasonable results of the APW model in application and its flexibility in this study, in the future, researchers should discuss several new issues for the APW model. For instance, the Bayesian estimation of one- and two-sample prediction based on different censored samples and constant/step stress for APW will be studied. Also, the classical two-sample prediction of APW distribution based on different censored samples can be obtained. Finally, we think that academics, reliability experts, and scientists in the engineering and manufacturing industries will find the approaches outlined in this work to be of great use.

Author Contributions: Methodology, R.A. and H.R.; Software, E.M.A.; Writing—original draft, E.M.A., R.A. and H.R.; Funding acquisition, R.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R50), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are included within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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