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Two Forms for Maclaurin Power Series Expansion of Logarithmic Expression Involving Tangent Function

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Abstract: In view of a general formula for higher order derivatives of the ratio of two differentiable functions, the authors establish the first form for the Maclaurin power series expansion of a logarithmic expression in term of determinants of special Hessenberg matrices whose elements involve the Bernoulli numbers. On the other hand, for comparison, the authors recite and revise the second form for the Maclaurin power series expansion of the logarithmic expression in terms of the Bessel zeta functions and the Bernoulli numbers.

Keywords: maclaurin power series expansion; hessenberg matrix; determinant; bernoulli number; bessel zeta function; logarithmic expression; coefficient; derivative formula; tangent function

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1. Introduction

In the mathematical handbook [1] (pp. 42, 55) and the paper [2] (p. 798), we find the Maclaurin power series expansions

$$\begin{aligned} \tan u &= \sum_{\ell=1}^{\infty} \frac{2^{2\ell}\left(2^{2\ell}-1\right)}{(2\ell)!}|B_{2\ell}|u^{2\ell-1} \\ &= u + \frac{u^3}{3} + \frac{2u^5}{15} + \frac{17u^7}{315} + \frac{62u^9}{2835} + \frac{1382u^{11}}{155{,}925} + \frac{21{,}844u^{13}}{6{,}081{,}075} + \cdots, \end{aligned} \tag{1}$$

$$\begin{aligned} \tan^2 u &= \sum_{\ell=1}^{\infty} \frac{2^{2\ell+2}\left(2^{2\ell+2}-1\right)(2\ell+1)}{(2\ell+2)!}|B_{2\ell+2}|u^{2\ell} \\ &= u^2 + \frac{2u^4}{3} + \frac{17u^6}{45} + \frac{62u^8}{315} + \frac{1382u^{10}}{14{,}175} + \frac{21{,}844u^{12}}{467{,}775} + \cdots \end{aligned} \tag{2}$$

for $|u| < \frac{\pi}{2}$ and the series expansion

$$\begin{aligned} \ln \tan u &= \ln u + \sum_{\ell=1}^{\infty} \frac{2^{2\ell}\left(2^{2\ell-1}-1\right)}{\ell(2\ell)!}|B_{2\ell}|u^{2\ell} \\ &= \ln u + \frac{u^2}{3} + \frac{7u^4}{90} + \frac{62u^6}{2835} + \frac{127u^8}{18{,}900} + \frac{146u^{10}}{66{,}825} + \cdots \end{aligned} \tag{3}$$

for $0 < u < \frac{\pi}{2}$, where $B_{2\ell}$ can be generated in [3] (p. 3) by

$$\frac{w}{e^w - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{w^\ell}{\ell!} = 1 - \frac{w}{2} + \sum_{\ell=1}^{\infty} B_{2\ell}\frac{w^{2\ell}}{(2\ell)!}, \quad |w| < 2\pi.$$

By the series expansion (3), we acquire the Maclaurin power series expansion

$$\begin{aligned} \ln \frac{\tan u}{u} &= \sum_{\ell=1}^{\infty} \frac{2^{2\ell}(2^{2\ell-1}-1)}{\ell(2\ell)!}|B_{2\ell}|u^{2\ell} \\ &= \frac{u^2}{3} + \frac{7u^4}{90} + \frac{62u^6}{2835} + \frac{127u^8}{18{,}900} + \frac{146u^{10}}{66{,}825} + \cdots, \quad 0 < |u| < \frac{\pi}{2}. \end{aligned} \tag{4}$$

At the site https://mathoverflow.net/q/444321 (accessed on 7 April 2023), the second author proposed the following question: What and where is the Maclaurin power series expansion of the function

$$\ln(\tan u - u), \quad 0 < u < \frac{\pi}{2} \tag{5}$$

or

$$\ln\left(\frac{\tan u}{u} - 1\right), \quad 0 < |u| < \frac{\pi}{2} \tag{6}$$

around $u = 0$? Equivalently speaking, what is the Maclaurin power series expansion of the function

$$\ln \frac{3(\tan u - u)}{u^3} = \ln\left[\frac{3}{u^2}\left(\frac{\tan u}{u} - 1\right)\right], \quad 0 < |u| < \frac{\pi}{2}$$

around $u = 0$? These questions are fundamental and significant in the theory of series and in the theory of generating functions of analytic combinatorics.

In what follows, we consider the logarithmic expression

$$F(u) = \begin{cases} \ln \dfrac{3(\tan u - u)}{u^3}, & 0 < |u| < \dfrac{\pi}{2}; \\ 0, & u = 0. \end{cases} \tag{7}$$

It is easy to see that the function $F(u)$ is even on the symmetric interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

It is well known that, by virtue of any computer software (or say, any computer algebra system) such as the Wolfram Mathematica, the Maple, and the MATLAB, one can compute great deal more but finitely many terms of the coefficients in the power series expansions of the functions formulated in (5)–(7). One problem is, what are the closed-form expressions for the general terms of the coefficients in the power series expansions of the functions in (5), (6), and (7), respectively?

In this paper, we adopt the definition of closed-form expressions at https://en.wikipedia.org/wiki/Closed-form_expression (accessed on 7 May 2023): "In mathematics, a closed-form expression is a mathematical expression that uses a finite number of standard operations. It may contain constants, variables, certain well-known operations (e.g., $+\ -\ \times\ \div$), and functions (e.g., nth root, exponent, logarithm, trigonometric functions, and inverse hyperbolic functions), but usually no limit, or integral". In the article [4], there is a special and systematic review and survey on "closed forms: what they are and why we care".

An upper (or a lower, respectively) Hessenberg matrix is an $m \times m$ matrix $H_m = (h_{r,s})_{1 \le r,s \le m}$, whose elements $h_{r,s} = 0$ for all tuples (r,s) such that $s + 1 < r$ (or $r + 1 < s$, respectively). See [5] (Chapter 10).

From the Maclaurin power series expansion (1), it follows that

$$F(u) = \ln\left[3 \sum_{k=0}^{\infty} \frac{2^{2k+4}(2^{2k+4}-1)}{(2k+4)!}|B_{2k+4}|u^{2k}\right], \quad |u| < \frac{\pi}{2}, \tag{8}$$

which is analytic in $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Because $F(u)$ is an elementary function, all of its derivatives are elementary, and in theory there must exist a closed-form expression for the general term of all coefficients in the Maclaurin power series expansion of the elementary function $F(u)$ around $u = 0$.

In this paper, motivated by the above ideas and observations and stimulated by the results in the newly-published papers [6,7], we will pay our main attention on expanding the even function $F(u)$ into its Maclaurin power series expansion around $u = 0$.

In the theory of series, the Maclaurin power series expansion of an analytic function at the origin 0 is unique. In this paper, we will provide and compare two forms for the Maclaurin power series expansion of $F(u)$ around $u = 0$.

2. Preliminaries

For attaining our main aims of this paper, we need the following preliminaries.

Let $\mu(y)$ and $\nu(y) \neq 0$ be two n-time differentiable functions on an interval I for a given integer $n \geq 0$. Then, the nth derivative of the ratio $\frac{\mu(y)}{\nu(y)}$ is

$$\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[\frac{\mu(y)}{\nu(y)}\right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(y)|}{\nu^{n+1}(y)}, \quad n \geq 0, \tag{9}$$

where the matrix

$$W_{(n+1)\times(n+1)}(y) = \begin{pmatrix} U_{(n+1)\times 1}(y) & V_{(n+1)\times n}(y) \end{pmatrix}_{(n+1)\times(n+1)},$$

the matrix $U_{(n+1)\times 1}(y)$ is an $(n+1)\times 1$ matrix whose elements satisfy $\mu_{k,1}(y) = \mu^{(k-1)}(y)$ for $1 \leq k \leq n+1$, the matrix $V_{(n+1)\times n}(y)$ is an $(n+1)\times n$ matrix whose elements are

$$\nu_{\ell,j}(y) = \begin{cases} \dbinom{\ell-1}{j-1}\nu^{(\ell-j)}(y), & \ell - j \geq 0 \\ 0, & \ell - j < 0 \end{cases}$$

for $1 \leq \ell \leq n+1$ and $1 \leq j \leq n$, and the notation $|W_{(n+1)\times(n+1)}(y)|$ denotes the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(y)$. The Formula (9) is a reformulation of Exercise 5 in [8] (p. 40).

It is common knowledge [3] (p. 51, (3.9)) that the classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{\ell=0}^{n}(z+\ell)}, \quad z \neq 0, -1, -2, \dots .$$

It is also well known that the Bessel function of the first kind $J_\lambda(w)$ can be represented [9] (p. 360, Entry 9.1.10) as

$$J_\lambda(w) = \left(\frac{w}{2}\right)^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\lambda+n+1)} \frac{1}{n!}\left(\frac{w}{2}\right)^{2n}, \quad w \in \mathbb{C},$$

where $\lambda \in \mathbb{C} \setminus \{-1, -2, \dots\}$ is called the order of $J_\lambda(w)$.

Let $j_{\lambda,n}$ for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{-1, -2, \dots\}$ denote the zeros of the function $\frac{J_\lambda(z)}{z^v}$. The Bessel zeta function

$$\zeta_\lambda(q) = \sum_{n=1}^{\infty} \frac{1}{j_{\lambda,n}^q}, \quad q > 1 \tag{10}$$

was originally introduced and studied in [10–13]. In [14] (p. 1251), the special values

$$\zeta_\lambda(2) = \frac{1}{4(\lambda+1)}, \qquad \zeta_\lambda(4) = \frac{1}{16(\lambda+1)^3},$$

$$\zeta_\lambda(6) = \frac{1}{16(\lambda+1)^4(2\lambda+3)}, \qquad \zeta_\lambda(8) = \frac{10\lambda+11}{256(\lambda+1)^6(2\lambda^2+7\lambda+6)}$$

are listed. These values were recited in [15] (p. 411, Equation (6)). However, as listed at the web site https://math.stackexchange.com/a/2911997 (accessed on 8 May 2023), the special values $\zeta_\lambda(4)$, $\zeta_\lambda(6)$, and $\zeta_\lambda(8)$ should be corrected as

$$\zeta_\lambda(4) = \frac{1}{16(\lambda+1)^2(\lambda+2)}, \quad \zeta_\lambda(6) = \frac{1}{32(\lambda+1)^3(\lambda+2)(\lambda+3)},$$

and

$$\zeta_\lambda(8) = \frac{5\lambda+11}{256(\lambda+1)^4(\lambda+2)^2(\lambda+3)(\lambda+4)}.$$

In the paper [16], three more special values

$$\zeta_\lambda(12) = \frac{21\lambda^3+181\lambda^2+513\lambda+473}{2^{11}(\lambda+1)^6(\lambda+2)^3(\lambda+3)^2(\lambda+4)(\lambda+5)(\lambda+6)},$$
$$\zeta_\lambda(14) = \frac{33\lambda^3+329\lambda^2+1081\lambda+1145}{2^{12}(\lambda+1)^7(\lambda+2)^3(\lambda+3)^2(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)},$$

and

$$\zeta_\lambda(18) = \frac{715\lambda^6+16{,}567\lambda^5+158{,}568\lambda^4+798{,}074\lambda^3+2{,}217{,}079\lambda^2+3{,}212{,}847\lambda+1{,}893{,}046}{2^{17}(\lambda+1)^9(\lambda+2)^4(\lambda+3)^3(\lambda+4)^2\prod_{j=5}^{9}(\lambda+j)}$$

were derived.

Making use of the general Formula (9) for the nth derivative of the ratio of two nth differentiable functions, the authors established in [15] (Theorems 2.3 and 3.2) two closed-form formulas

$$\zeta_\lambda(2k) = (-1)^{k+1}\frac{[\Gamma(\lambda+1)]^{2k+1}}{(2k)!}\begin{vmatrix} P_{2k+1,1}(\lambda) & Q_{2k+1,2k}(\lambda) \end{vmatrix} \tag{11}$$

and

$$\zeta_\lambda(2k) = \frac{(-1)^{k+1}\lambda}{(2k)!}\begin{vmatrix} R_{2k+1,1}(\lambda) & S_{2k+1,2k}(\lambda) \end{vmatrix}_{(2k+1)\times(2k+1)} \tag{12}$$

for $\lambda \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$ and $k \in \mathbb{N}$, where the $(2k+1)\times 1$ matrix $P_{2k+1,1}(\lambda)$ is defined by

$$P_{2k+1,1}(\lambda) = (p_{i,j})_{1\le i\le 2k+1, j=1}, \quad p_{i,j} = \frac{1-(-1)^i}{2}\frac{(i-2)!!}{2^{(i-1)/2}}\frac{1}{\Gamma(\lambda+\frac{i-1}{2})},$$

the $(2k+1)\times(2k)$ matrix $Q_{2k+1,2k}(\lambda)$ is defined by

$$Q_{2k+1,2k}(\lambda) = (q_{i,j})_{1\le i\le 2k+1, 1\le j\le 2k},$$
$$q_{i,j} = \frac{1+(-1)^{i-j}}{2}\binom{i-1}{j-1}\frac{(i-j-1)!!}{2^{(i-j)/2}}\frac{1}{\Gamma(\lambda+\frac{|i-j|}{2}+1)},$$

the $(2k+1)\times 1$ matrix $R_{2k+1,1}(\lambda)$ is defined by

$$R_{2k+1,1}(\lambda) = (r_{i,j})_{1\le i\le 2k+1, j=1}, \quad r_{i,j} = \frac{1-(-1)^i}{2}\frac{(i-2)!!}{2^{(i-1)/2}}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\frac{i-1}{2})},$$

and the $(2k+1)\times(2k)$ matrix $S_{2k+1,2k}(\lambda)$ is defined by

$$S_{2k+1,2k}(\lambda) = (s_{i,j})_{1\le i\le 2k+1, 1\le j\le 2k}, \quad s_{i,j} = \frac{1+(-1)^{i-j}}{2}\binom{i-1}{j-1}\frac{(i-j-1)!!}{2^{(i-j)/2}}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\frac{i-j}{2})}.$$

We note that the determinantal expressions (11) and (12) can be transferred to each other by simple linear algebraic operations on determinants.

3. The First Form of Maclaurin Power Series Expansion

In this section, making use of the general Formula (9), we now start out by establishing the first closed-form Maclaurin power series expansion of the function $F(u)$ defined by (7).

Theorem 1. *The even function $F(u)$ defined by (7) has the Maclaurin power series expansion*

$$\begin{aligned} F(u) &= -\sum_{k=1}^{\infty} \frac{3^{2k}}{(2k)!} D_{2k} u^{2k} \\ &= \frac{2}{5}u^2 + \frac{43}{525}u^4 + \frac{524}{23{,}625}u^6 + \frac{40{,}897}{6{,}063{,}750}u^8 + \cdots, \quad |u| < \frac{\pi}{2} \end{aligned} \tag{13}$$

where

$$\begin{aligned} D_{2k} &= \begin{vmatrix} 0 & Q_0 & 0 & \cdots & 0 & 0 & 0 \\ Q_1 & 0 & Q_0 & \cdots & 0 & 0 & 0 \\ 0 & Q_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ Q_{k-1} & 0 & \binom{2k-3}{1}Q_{k-2} & \cdots & 0 & Q_0 & 0 \\ 0 & Q_{k-1} & 0 & \cdots & \binom{2k-2}{2k-4}Q_1 & 0 & Q_0 \\ Q_k & 0 & \binom{2k-1}{1}Q_{k-1} & \cdots & 0 & \binom{2k-1}{2k-3}Q_1 & 0 \end{vmatrix} \\ &= \left|e_{i,j}\right|_{2k\times 2k}, \quad k \ge 1, \end{aligned}$$

$$e_{i,j} = \begin{cases} 0, & (i,j) = (2\ell-1,1), \quad 1 \le \ell \le k; \\ Q_\ell, & (i,j) = (2\ell,1), \quad 1 \le \ell \le k; \\ 0, & 1 \le i \le j-2 \le 2k-2; \\ 0, & i-j = 2\ell, \quad 0 \le \ell \le k-1; \\ \dbinom{i-1}{j-2} Q_\ell, & i-j = 2\ell-1, \quad 0 \le \ell \le k-1, \quad j \ge 2, \end{cases}$$

and

$$Q_m = \frac{2^{2m+2}(2^{2m+4}-1)}{(m+1)(m+2)(2m+1)(2m+3)}|B_{2m+4}|, \quad m \ge 0.$$

Proof. The first derivative of $F(u)$ is

$$F'(u) = \frac{\dfrac{u\tan^2 u - 3(\tan u - u)}{u^4}}{\dfrac{\tan u - u}{u^3}} \triangleq \frac{\mu(u)}{\nu(u)} \to 0, \quad u \to 0.$$

By virtue of the Marclaurin power series expansions (1) and (2), we derive

$$\mu(u) = \frac{u\tan^2 u - 3(\tan u - u)}{u^4} = \sum_{\ell=0}^{\infty} \frac{2^{2\ell+7}(\ell+1)(2^{2\ell+6}-1)}{(2\ell+6)!}|B_{2\ell+6}|u^{2\ell+1}$$

and

$$\nu(u) = \frac{\tan u - u}{u^3} = \sum_{\ell=0}^{\infty} \frac{2^{2\ell+4}(2^{2\ell+4}-1)}{(2\ell+4)!}|B_{2\ell+4}|u^{2\ell} \tag{14}$$

for $0 < |u| < \frac{\pi}{2}$. Accordingly, we obtain

$$\mu^{(n)}(0) = \begin{cases} 0, & n = 2\ell \\ \dfrac{2^{2\ell+4}(2^{2\ell+6}-1)}{(\ell+2)(\ell+3)(2\ell+3)(2\ell+5)}|B_{2\ell+6}| = Q_{\ell+1}, & n = 2\ell+1 \end{cases}$$

and

$$\nu^{(n)}(0) = \begin{cases} \dfrac{2^{2\ell+2}(2^{2\ell+4}-1)}{(\ell+1)(\ell+2)(2\ell+1)(2\ell+3)}|B_{2\ell+4}| = Q_\ell, & n = 2\ell \\ 0, & n = 2\ell+1 \end{cases}$$

for $\ell, n \geq 0$. Hence, by the Formula (9), a general formula for the nth derivative of the ratio of two n-time differentiable functions, we arrive at

$$\begin{aligned}
F^{(2\ell)}(0) &= \lim_{u\to 0}\left[\frac{\mu(u)}{\nu(u)}\right]^{(2\ell-1)} \\
&= -\frac{1}{[\nu(0)]^{2\ell}} \begin{vmatrix}
\mu(0) & \nu(0) & 0 & \cdots & 0 & 0 & 0 \\
\mu'(0) & \nu'(0) & \binom{1}{1}\nu(0) & \cdots & 0 & 0 & 0 \\
\mu''(0) & \nu''(0) & \binom{2}{1}\nu'(0) & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\mu^{(2\ell-3)}(0) & \nu^{(2\ell-3)}(0) & \binom{2\ell-3}{1}\nu^{(2\ell-4)}(0) & \cdots & \binom{2\ell-3}{2\ell-4}\nu'(0) & \binom{2\ell-3}{2\ell-3}\nu(0) & 0 \\
\mu^{(2\ell-2)}(0) & \nu^{(2\ell-2)}(0) & \binom{2\ell-2}{1}\nu^{(2\ell-3)}(0) & \cdots & \binom{2\ell-2}{2\ell-4}\nu''(0) & \binom{2\ell-2}{2\ell-3}\nu'(0) & \binom{2\ell-2}{2\ell-2}\nu(0) \\
\mu^{(2\ell-1)}(0) & \nu^{(2\ell-1)}(0) & \binom{2\ell-1}{1}\nu^{(2\ell-2)}(0) & \cdots & \binom{2\ell-1}{2\ell-4}\nu'''(0) & \binom{2\ell-1}{2\ell-3}\nu''(0) & \binom{2\ell-1}{2\ell-2}\nu'(0)
\end{vmatrix} \\
&= -3^{2\ell} \begin{vmatrix}
0 & Q_0 & 0 & \cdots & 0 & 0 & 0 \\
Q_1 & 0 & Q_0 & \cdots & 0 & 0 & 0 \\
0 & Q_1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q_{\ell-1} & 0 & \binom{2\ell-3}{1}Q_{\ell-2} & \cdots & 0 & Q_0 & 0 \\
0 & Q_{\ell-1} & 0 & \cdots & \binom{2\ell-2}{2\ell-4}Q_1 & 0 & Q_0 \\
Q_\ell & 0 & \binom{2\ell-1}{1}Q_{\ell-1} & \cdots & 0 & \binom{2\ell-1}{2\ell-3}Q_1 & 0
\end{vmatrix} \\
&= -3^{2\ell}D_{2\ell}
\end{aligned}$$

for $\ell \geq 1$. Considering that the function $F(u)$ is even, we thus derive the Maclaurin power series expansion (13). The proof of Theorem 1 is complete. □

4. The Second Form of Maclaurin Power Series Expansion

At the site https://mathoverflow.net/a/444638 (accessed on 12 April 2023), Paul Enta (a physicist in France, https://stackexchange.com/users/3991904/paul-enta, accessed on 12 April 2023) outlined the second closed-form Maclaurin power series expansion of the function $F(u)$ around $u = 0$. In this section, for comparison, we recite and revise Paul Enta's result and its proof.

Theorem 2. *The even function $F(u)$ defined by (7) has the Maclaurin power series expansion*

$$F(u) = \sum_{\ell=1}^{\infty}\left[\frac{2^{2\ell-1}(2^{2\ell}-1)}{(2\ell)!}|B_{2\ell}| - \zeta_{3/2}(2\ell)\right]\frac{u^{2\ell}}{\ell}, \quad |u| < \frac{\pi}{2}, \tag{15}$$

where the Bessel zeta function $\zeta_{3/2}(2\ell)$ is defined by (10).

Proof. From the integral representation

$$J_\lambda(w) = \frac{1}{\sqrt{\pi}\,\Gamma(\lambda + \frac{1}{2})}\left(\frac{w}{2}\right)^{\lambda} \int_0^\pi \cos(w \sin\theta) \sin^{2\lambda}\theta \, \mathrm{d}\,\theta, \quad \Re(\lambda) > -\frac{1}{2}$$

in [9] (p. 360, Entry 9.1.20), it follows that

$$J_{3/2}(w) = \sqrt{\frac{2}{\pi}}\,\frac{\sin w - w\cos w}{w^{3/2}}. \tag{16}$$

The Formula (16) also appeared in [17] (p. 43).

Entry 7 in [18] (p. 26) reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}\,u^n}\big[w^{(2n+1)/4} J_{-n-1/2}(a\sqrt{w}\,)\big] = \frac{(-1)^n}{\sqrt{\pi}}\left(\frac{a}{2}\right)^{n-1/2} \cos(a\sqrt{w}\,), \quad n \ge 0.$$

Taking $n = 0$, letting $a = 1$, and replacing $\sqrt{w}$ by w lead to

$$J_{-1/2}(w) = \sqrt{\frac{2}{\pi}}\,\frac{\cos w}{\sqrt{w}}. \tag{17}$$

Making use of the results in (16) and (17), we write the function $F(u)$ as

$$F(u) = \ln\frac{3(\sin u - u\cos u)}{u^3 \cos u} = \ln\frac{3J_{3/2}(u)}{u^2 J_{-1/2}(u)}.$$

Basing on [19] (§15.41), Dickinson obtained in [20] (p. 949, Equation (14)) the formulas

$$J_{\lambda-1}(u) = \frac{1}{\Gamma(\lambda)}\left(\frac{u}{2}\right)^{\lambda-1} \prod_{n=1}^{\infty}\left(1 - \frac{u^2}{j_{\lambda-1,n}}\right)$$

and

$$\ln J_{\lambda-1}(u) = \ln\left[\frac{1}{\Gamma(\lambda)}\left(\frac{u}{2}\right)^{\lambda-1}\right] + \sum_{n=1}^{\infty} \ln\left(1 - \frac{u^2}{j_{\lambda-1,n}}\right)$$

for $\lambda > 0$ and $0 < u < j_{\lambda-1,1}$, see also [9] (p. 370, Entry 9.5.10). Further considering the series expansion

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{u^n}{n}, \quad |u| < 1,$$

we derive

$$\begin{aligned} \ln J_\lambda(u) &= \ln\left[\frac{1}{\Gamma(\lambda+1)}\left(\frac{u}{2}\right)^{\lambda}\right] - \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{\ell}\frac{u^{2\ell}}{j_{\lambda,n}^{2\ell}} \\ &= \ln\left[\frac{1}{\Gamma(\lambda+1)}\left(\frac{u}{2}\right)^{\lambda}\right] - \sum_{\ell=1}^{\infty}\frac{u^{2\ell}}{\ell}\sum_{n=1}^{\infty}\frac{1}{j_{\lambda,n}^{2\ell}}. \end{aligned} \tag{18}$$

Thus, we acquire

$$\begin{aligned} F(u) &= \ln 3 - 2\ln u + \ln J_{3/2}(u) - \ln J_{-1/2}(u) \\ &= \sum_{\ell=1}^{\infty}\frac{u^{2\ell}}{\ell}\left(\sum_{n=1}^{\infty}\frac{1}{j_{-1/2,n}^{2\ell}} - \sum_{n=1}^{\infty}\frac{1}{j_{3/2,n}^{2\ell}}\right) \\ &= \sum_{\ell=1}^{\infty}\frac{\zeta_{-1/2}(2\ell) - \zeta_{3/2}(2\ell)}{\ell}u^{2\ell}. \end{aligned} \tag{19}$$

From the formula

$$j_{-1/2,n} = \frac{(2n-1)\pi}{2}, \quad n \ge 1,$$

see the paper [21], for example, we arrive at

$$\zeta_{-1/2}(2\ell) = \sum_{n=1}^{\infty} \frac{1}{j_{-1/2,n}^{2\ell}} = \sum_{n=1}^{\infty} \left[\frac{2}{(2n-1)\pi}\right]^{2\ell} = \frac{2^{2\ell}-1}{\pi^{2\ell}}\zeta(2\ell) = (2^{2\ell}-1)\frac{2^{2\ell-1}}{(2\ell)!}|B_{2\ell}| \quad (20)$$

for $\ell \geq 1$, where we use the relation

$$|B_{2\ell}| = \frac{2(2\ell)!}{(2\pi)^{2\ell}}\zeta(2\ell), \quad \ell \in \mathbb{N}$$

in [3] (p. 5, (1.14)) and $\zeta(2\ell)$ stands for the Riemann zeta function.

Finally, substituting (20) into (19), we conclude the closed-form Maclaurin power series expansion (15). The required proof is thus complete. □

5. Applications of Theorems 1 and 2 and Remarks

In this section, applying our main results, the Maclaurin power series expansions (13) and (15) in Theorems 1 and 2, respectively, we can derive some new results and important remarks.

As direct consequences of Theorem 1, we derive the following series expansions.

Corollary 1. *The series expansions*

$$\ln(|\tan u| - |u|) = -\ln 3 + 3\ln|u| - \sum_{\ell=1}^{\infty} \frac{3^{2\ell} D_{2\ell}}{(2\ell)!} u^{2\ell}$$

and

$$\ln\left(\frac{\tan u}{u} - 1\right) = -\ln 3 + 2\ln|u| - \sum_{\ell=1}^{\infty} \frac{3^{2\ell} D_{2\ell}}{(2\ell)!} u^{2\ell}$$

hold for $0 < |u| < \frac{\pi}{2}$.

Remark 1. *These two series expansions in Corollary 1 and the Maclaurin power series expansion (13), not including their proof and deduction, have been announced at the site https://mathoverflow.net/a/444485 (accessed on 10 April 2023) as the best answer to the question at https://mathoverflow.net/q/444321 (accessed on 7 April 2023).*

As direct consequences of Theorem 2, the following series expansions can be deduced.

Corollary 2. *The series expansions*

$$\ln(|\tan u| - |u|) = 3\ln|u| - \ln 3 + \sum_{\ell=1}^{\infty} \left[\frac{2^{2\ell-1}(2^{2\ell}-1)}{(2\ell)!}|B_{2\ell}| - \zeta_{3/2}(2\ell)\right] \frac{u^{2\ell}}{\ell}$$

and

$$\ln\left(\frac{\tan u}{u} - 1\right) = 2\ln|u| - \ln 3 + \sum_{\ell=1}^{\infty} \left[\frac{2^{2\ell-1}(2^{2\ell}-1)}{(2\ell)!}|B_{2\ell}| - \zeta_{3/2}(2\ell)\right] \frac{u^{2\ell}}{\ell}$$

hold for $0 < |u| < \frac{\pi}{2}$.

Remark 2. *Corollary 2 can be regarded as an alternative answer to the question at https://mathoverflow.net/q/444321 (accessed on 7 April 2023).*

Remark 3. *For* $\ell = 3$*, the determinant* D_6 *is*

$$D_6 = \begin{vmatrix} 0 & Q_0 & 0 & 0 & 0 & 0 \\ Q_1 & 0 & Q_0 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & Q_0 & 0 & 0 \\ Q_2 & 0 & \binom{3}{1}Q_1 & 0 & Q_0 & 0 \\ 0 & Q_2 & 0 & \binom{4}{2}Q_1 & 0 & Q_0 \\ Q_3 & 0 & \binom{5}{1}Q_2 & 0 & \binom{5}{3}Q_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{4}{15} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{4}{15} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{136}{105} & 0 & \frac{4}{5} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{136}{105} & 0 & \frac{8}{5} & 0 & \frac{1}{3} \\ \frac{992}{63} & 0 & \frac{136}{21} & 0 & \frac{8}{3} & 0 \end{vmatrix} = -\frac{8384}{382{,}725}$$

and $-\frac{3^6}{6!}D_6 = \frac{524}{23{,}625}$. *This number coincides with the coefficient of the term* u^6 *in the Maclaurin power series expansion* (13). *This coincidence implies that the Maclaurin power series expansion* (13) *is generally correct.*

Remark 4. *Due to the evenness of te function* $F(u)$, *it follows that* $F^{(2\ell+1)}(0) = 0$ *for* $\ell \ge 0$. *By virtue of the Formula* (9), *as in the proof of Theorem 1, we can derive*

$$\begin{vmatrix} 0 & Q_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ Q_1 & 0 & Q_0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & Q_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ Q_2 & 0 & \binom{3}{1}Q_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ Q_{\ell-1} & 0 & \binom{2\ell-3}{1}Q_{\ell-2} & \cdots & 0 & Q_0 & 0 & 0 \\ 0 & Q_{\ell-1} & 0 & \cdots & \binom{2\ell-2}{2\ell-4}Q_1 & 0 & Q_0 & 0 \\ Q_\ell & 0 & \binom{2\ell-1}{1}Q_{\ell-1} & \cdots & 0 & \binom{2\ell-1}{2\ell-3}Q_1 & 0 & Q_0 \\ 0 & Q_\ell & 0 & \cdots & \binom{2\ell}{2\ell-4}Q_2 & 0 & \binom{2\ell}{2\ell-2}Q_1 & 0 \end{vmatrix} = 0$$

for $\ell \ge 1$. *We believe that it would be very difficult to calculate this equality by operations of the linear algebra.*

Proposition 1. *For* $n \ge 1$, *we have*

$$D_{2n} = \frac{1}{3^{2n}} \sum_{\ell=1}^{2n} (-1)^{\ell-1}(\ell-1)! B_{2n,\ell}\left(0, \frac{4}{5}, 0, \frac{136}{35}, \dots, \lim_{u\to0}\left[e^{F(u)}\right]^{(2n-\ell+1)}\right) \tag{21}$$

and

$$\begin{aligned} &2^{2n-1}(2^{2n}-1)|B_{2n}| - (2n)!\zeta_{3/2}(2n) \\ &= n\sum_{\ell=1}^{2n} (-1)^{\ell-1}(\ell-1)! B_{2n,\ell}\left(0, \frac{4}{5}, 0, \frac{136}{35}, \dots, \lim_{u\to0}\left[e^{F(u)}\right]^{(2n-\ell+1)}\right). \end{aligned} \tag{22}$$

Proof. From the equation (8), it follows that

$$e^{F(u)} = \frac{3(\tan u - u)}{u^3} = 3\sum_{\ell=0}^{\infty} \frac{2^{2\ell+4}(2^{2\ell+4}-1)}{(2\ell+4)!}|B_{2\ell+4}|u^{2\ell}, \quad |u| < \frac{\pi}{2}.$$

This means that the limits

$$\lim_{u\to0}\left[e^{F(u)}\right]^{(2\ell+1)} = 0 \quad \text{and} \quad \lim_{u\to0}\left[e^{F(u)}\right]^{(2\ell)} = 3(2\ell)!\frac{2^{2\ell+4}(2^{2\ell+4}-1)|B_{2\ell+4}|}{(2\ell+4)!}$$

hold for $\ell \ge 0$. Hence, by virtue of the Faá di Bruno formula [22], we obtain

$$\begin{aligned} F^{(n)}(u) &= \left(\ln\left[e^{F(u)}\right]\right)^{(n)} \\ &= \sum_{\ell=1}^{n} \frac{(-1)^{\ell-1}(\ell-1)!}{e^{\ell F(u)}} B_{n,\ell}\left(\left[e^{F(u)}\right]', \left[e^{F(u)}\right]'', \dots, \left[e^{F(u)}\right]^{(n-\ell+1)}\right) \end{aligned}$$

for $n \geq 1$, where $B_{n,\ell}$ denotes the Bell polynomials of the second kind and satisfies

$$B_{2n+1,\ell}\left(0, u_2, 0, u_4, \dots, \frac{1+(-1)^\ell}{2} u_{2n-\ell+2}\right) = 0$$

for $\ell, n \geq 0$, $u_m \in \mathbb{C}$, and $m \in \mathbb{N}$, see ([22], Remark 7.4). Accordingly, we arrive at $F^{(2n+1)}(0) = 0$ for $n \geq 0$ and

$$F^{(2n)}(0) = \sum_{\ell=1}^{2n} (-1)^{\ell-1}(\ell-1)! B_{2n,\ell}\left(0, \lim_{u\to 0}[e^{F(u)}]'', 0, \lim_{u\to 0}[e^{F(u)}]^{(4)}, \dots, \lim_{u\to 0}[e^{F(u)}]^{(2n-\ell+1)}\right)$$

for $n \geq 1$. Consequently, we conclude

$$\begin{aligned} F(u) &= \sum_{n=0}^{\infty} F^{(2n)}(0) \frac{u^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty} \frac{u^{2n}}{(2n)!} \left[\sum_{\ell=1}^{2n} (-1)^{\ell-1}(\ell-1)! B_{2n,\ell}\left(0, \frac{4}{5}, 0, \frac{136}{35}, \dots, \lim_{u\to 0}[e^{F(u)}]^{(2n-\ell+1)}\right)\right]. \end{aligned}$$

Comparing this with series expansions (13) and (15) leads to (21) and (22) for $n \geq 1$. □

Remark 5 (Computation of determinants)**.** *After obtaining the Maclaurin power series expansion* (13)*, which is very symmetric and beautiful in mathematics, the next problem is: how to compute or expand the determinant* $D_{2\ell}$ *into simpler forms?*

Comparing the series expansions (13) *and* (15)*, we find the relation*

$$D_{2\ell} = -\frac{(2\ell)!}{3^{2\ell}\ell}\left[\frac{2^{2\ell-1}(2^{2\ell}-1)}{(2\ell)!}|B_{2\ell}| - \zeta_{3/2}(2\ell)\right], \quad \ell \geq 1.$$

At the site https://mathoverflow.net/q/444321#comment1147933_444336 *(accessed on 11 April 2023), Fred Hucht (University of Duisburg-Essen) pointed out that "you can reduce the matrix size from* $2\ell \times 2\ell$ *to* $\ell \times \ell$ *by rearranging the rows and columns, first the odd and then the even rows/cols. The resulting* 2×2 *block diagonal matrix has one trivial and one non-trivial determinant".*

Remark 6 (Closed-form expressions)**.** *From the determinantal representations* (11) *and* (12)*, we see that the quantity* $\zeta_{3/2}(2\ell)$ *in the Maclaurin power series expansion* (15) *is of closed form. In* [23] *(Theorem 4.1), the closed-form formula*

$$B_{2m} = \frac{2^{2m-1}}{2^{2m-1}-1} \sum_{\ell=1}^{2m} \sum_{j=1}^{\ell} (-1)^{j+1} \binom{\ell}{j} \frac{T(2m+j, j)}{\binom{2m+j}{j}}, \quad m \in \mathbb{N}$$

was discovered for the Bernoulli numbers B_{2m}*, where* $T(0,0) = 1$ *and*

$$T(n, \ell) = \frac{1}{\ell!} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} \left(\frac{\ell}{2} - q\right)^n$$

for $\ell, n \geq 0$ *such that* $(n, \ell) \neq (0, 0)$*. Consequently, the general expression for coefficients in the Maclaurin power series expansion* (15) *in Theorem* 2 *is of closed form, essentially.*

In fact, in theory, because all derivatives of analytic elementary functions are analytic and elementary, all analytic elementary functions must have unique series expansions whose coefficients are of closed form.

Remark 7. *One anonymous referee pointed out that the following result can be found in* [24]*.*

If

$$\sum_{n=0}^{\infty} c_n u^n = \frac{\sum_{n=0}^{\infty} a_n u^n}{\sum_{n=0}^{\infty} b_n u^n}$$

and $b_0 \neq 0$, *then* c_n *can be written as*

$$c_n = \frac{(-1)^n}{b_0^{n+1}} \begin{vmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & b_1 & b_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & b_2 & b_1 & b_0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & b_3 & b_2 & b_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & b_{n-3} & b_{n-4} & b_{n-5} & \cdots & b_1 & b_0 & 0 & 0 \\ a_{n-2} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_2 & b_1 & b_0 & 0 \\ a_{n-1} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_3 & b_2 & b_1 & b_0 \\ a_n & b_n & b_{n-1} & b_{n-2} & \cdots & b_4 & b_3 & b_2 & b_1 \end{vmatrix}. \tag{23}$$

This result can be used to expand the function $F(u)$ *into a Maclaurin power series around* $u = 0$, *as achieved in the proof of Theorem 1.*

As demonstrated in [25] (Section 5), the determinantal expression (23) can also be derived from the derivative Formula (9) as follows.

It is easy to see that

$$n!c_n = \lim_{u \to 0} \left(\frac{\sum_{n=0}^{\infty} a_n u^n}{\sum_{n=0}^{\infty} b_n u^n} \right)^{(n)}, \quad n \geq 0.$$

Let

$$\mu(u) = \sum_{n=0}^{\infty} a_n u^n \quad \text{and} \quad \nu(u) = \sum_{n=0}^{\infty} b_n u^n.$$

Then

$$\mu^{(n)}(0) = n!a_n \quad \text{and} \quad \nu^{(n)}(0) = n!b_n.$$

From the derivative Formula (9) and in view of algebraic operations for determinants, it follows that

$$\lim_{u \to 0} \left(\frac{\sum_{n=0}^{\infty} a_n u^n}{\sum_{n=0}^{\infty} b_n u^n} \right)^{(n)} = \frac{(-1)^n}{b_0^{n+1}}$$

$$\times \begin{vmatrix} a_0 & \frac{0!}{0!}b_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & \frac{1!}{0!}b_1 & \frac{1!}{1!}b_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2!a_2 & \frac{2!}{0!}b_2 & \frac{2!}{1!}b_1 & \frac{2!}{2!}b_0 & \cdots & 0 & 0 & 0 & 0 \\ 3!a_3 & \frac{3!}{0!}b_3 & \frac{3!}{1!}b_2 & \frac{3!}{2!}b_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (n-3)!a_{n-3} & \frac{(n-3)!}{0!}b_{n-3} & \frac{(n-3)!}{1!}b_{n-4} & \frac{(n-3)!}{2!}b_{n-5} & \cdots & \frac{(n-3)!}{(n-4)!}b_1 & \frac{(n-3)!}{(n-3)!}b_0 & 0 & 0 \\ (n-2)!a_{n-2} & \frac{(n-2)!}{0!}b_{n-2} & \frac{(n-2)!}{1!}b_{n-3} & \frac{(n-2)!}{2!}b_{n-4} & \cdots & \frac{(n-2)!}{(n-4)!}b_2 & \frac{(n-2)!}{(n-3)!}b_1 & \frac{(n-2)!}{(n-2)!}b_0 & 0 \\ (n-1)!a_{n-1} & \frac{(n-1)!}{0!}b_{n-1} & \frac{(n-1)!}{1!}b_{n-2} & \frac{(n-1)!}{2!}b_{n-3} & \cdots & \frac{(n-1)!}{(n-4)!}b_3 & \frac{(n-1)!}{(n-3)!}b_2 & \frac{(n-1)!}{(n-2)!}b_1 & \frac{(n-1)!}{(n-1)!}b_0 \\ n!a_n & \frac{n!}{0!}b_n & \frac{n!}{1!}b_{n-1} & \frac{n!}{2!}b_{n-2} & \cdots & \frac{n!}{(n-4)!}b_4 & \frac{n!}{(n-3)!}b_3 & \frac{n!}{(n-2)!}b_2 & \frac{n!}{(n-1)!}b_1 \end{vmatrix}$$

$$
= \frac{(-1)^n n!}{b_0^{n+1}} \begin{vmatrix}
a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_1 & b_1 & b_0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
a_2 & b_2 & b_1 & b_0 & \cdots & 0 & 0 & 0 & 0 \\
a_3 & b_3 & b_2 & b_1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{n-3} & b_{n-3} & b_{n-4} & b_{n-5} & \cdots & b_1 & b_0 & 0 & 0 \\
a_{n-2} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_2 & b_1 & b_0 & 0 \\
a_{n-1} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_3 & b_2 & b_1 & b_0 \\
a_n & b_n & b_{n-1} & b_{n-2} & \cdots & b_4 & b_3 & b_2 & b_1
\end{vmatrix} = n! c_n.
$$

The determinantal expression (23) is thus proved.

Remark 8. *In [1] (p. 55), one can find the Maclaurin power series expansion*

$$
\ln \cos z = -\sum_{\ell=1}^{\infty} \frac{2^{2\ell-1}(2^{2\ell}-1)}{\ell(2\ell)!} |B_{2\ell}| z^{2\ell} = -\frac{z^2}{2} - \frac{z^4}{12} - \frac{z^6}{45} - \frac{17z^8}{2520} - \cdots \tag{24}
$$

for $z^2 < \frac{\pi^2}{4}$. Taking the logarithm on both sides of the equality (17) and making use of the series expansion (24) reveal that

$$
\begin{aligned}
\ln J_{-1/2}(z) &= \frac{1}{2} \ln \frac{2}{\pi} - \frac{1}{2} \ln z + \ln \cos z \\
&= \frac{1}{2} \ln \frac{2}{\pi} - \frac{1}{2} \ln z - \sum_{\ell=1}^{\infty} \frac{2^{2\ell-1}(2^{2\ell}-1)}{\ell(2\ell)!} |B_{2\ell}| z^{2\ell}, \quad z^2 < \frac{\pi^2}{4}.
\end{aligned} \tag{25}
$$

This observation comes from one anonymous referee of this paper.

Taking $\nu = -\frac{1}{2}$ in (18) leads to

$$
\ln J_{-1/2}(z) = \frac{1}{2} \ln \frac{2}{\pi} - \frac{1}{2} \ln z - \sum_{\ell=1}^{\infty} \left[\sum_{n=1}^{\infty} \frac{1}{j_{-1/2,n}^{2\ell}} \right] \frac{z^{2\ell}}{\ell}. \tag{26}
$$

Comparing the series expansions in (25) and (26), employing the definition in (10), and simplifying result in an identity

$$
\zeta_{-1/2}(2\ell) = \sum_{n=1}^{\infty} \frac{1}{j_{-1/2,n}^{2\ell}} = \frac{2^{2\ell-1}(2^{2\ell}-1)}{(2\ell)!} |B_{2\ell}|, \quad \ell \ge 1.
$$

This is a simple recovery of the identity at the very ends of the equalities in (20).

Remark 9 (Absolute monotonicity). *From the Maclautin power series expansion (14), we see that the function $v(u) = \frac{e^{F(u)}}{3}$ is convex on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Strongly speaking, the function $v(u) = e^{f(u)}$ is absolutely monotonic on $(0, \frac{\pi}{2})$ and completely monotonic on $(-\frac{\pi}{2}, 0)$. On the other hand, the logarithmic function $\ln u$ is a Bernstein function on $(1, \infty)$.*

We guess that the function $F(u) = 3 \ln v(u)$ is absolutely monotonic on $(0, \frac{\pi}{2})$. In other words, we guess that the function $v(u)$ is logarithmically absolutely monotonic on $(0, \frac{\pi}{2})$, and then the sequence $D_{2\ell}$ is negative for all $\ell \in \mathbb{N}$. If this guess is true, then, from the Equation (19), the inequality

$$
\zeta_{-1/2}(2\ell) > \zeta_{3/2}(2\ell), \quad \ell \ge 1
$$

is true too. For the definitions, basic properties, and relations of the (logarithmically) absolutely (completely) monotonic functions and the Bernstein functions, please refer to the papers [26,27] and the monographs [28–30].

Remark 10 (Decreasing propoerty)**.** *Let $F(u)$ be defined by (7) and*

$$G(u)=\begin{cases}\ln\dfrac{\tan u}{u}, & 0<|u|<\dfrac{\pi}{2};\\ 0, & u=0.\end{cases}$$

On 5 April 2023, Chao-Ping Chen (Henan Polytechnic University, China) guessed that the ratio $\frac{F(u)}{G(u)}$ is decreasing on $(0,\frac{\pi}{2})$ and asked the second author for a proof.

On the night of 10 April 2023, Chao-Ping Chen, the initial proposer of the decreasing property of the function $\frac{F(u)}{G(u)}$, privately acknowledged the second author that he found out a proof of the decreasing property of the function $\frac{F(u)}{G(u)}$ and applied the property in a draft of his joint manuscript titled "Cusa-type inequalities for trigonometric and hyperbolic functions".

Iosif Pinelis (Michigan Technological University, USA) announced the second proof of the decreasing property of the function $\frac{F(u)}{G(u)}$ at the site https://mathoverflow.net/a/444526 (accessed on 10 April 2023) as an answer to his self-designed question at the site https://mathoverflow.net/q/444525 (accessed on 10 April 2023), which originates from the question at https://mathoverflow.net/q/444490 (accessed on 10 April 2023), Comment 1147661 at the site https://mathoverflow.net/questions/444490/ask-for-a-proof-of-an-inequality-involving-the-bernoulli-numbers#comment1147661_444490 (accessed on 10 April 2023), and Comment 1147671 at the site https://mathoverflow.net/questions/444490/ask-for-a-proof-of-an-inequality-involving-the-bernoulli-numbers#comment1147671_444490 (accessed on 10 April 2023).

The third proof of the decreasing property of the function $\frac{F(u)}{G(u)}$ was provided by River Li (https://stackexchange.com/users/14159649/river-li, accessed on 14 April 2023) at the site https://mathoverflow.net/a/444772 (accessed on 14 April 2023).

Remark 11 (Two guesses)**.** *In the paper [31], a monotonicity rule for the ratio of two Maclaurin power series was presented as follow.*

Let a_ℓ and b_ℓ for $\ell\in\{0\}\cup\mathbb{N}$ be real numbers and the power series

$$A(u)=\sum_{\ell=0}^{\infty}a_\ell u^\ell \quad \text{and} \quad B(u)=\sum_{\ell=0}^{\infty}b_\ell u^\ell$$

be convergent on $(-R,R)$ for some $R>0$. If $b_\ell>0$ and the ratio $\frac{a_\ell}{b_\ell}$ is (strictly) increasing for $\ell\ge0$, then the function $\frac{A(u)}{B(u)}$ is also (strictly) increasing on $(0,R)$.

In [32] (pp. 10–11, Theorem 1.25), a monotonicity rule for the ratio of two functions was established as follows.

For $a,b\in\mathbb{R}$ with $a<b$, let $p(u)$ and $q(u)$ be continuous on $[a,b]$, differentiable on (a,b), and $q'(u)\ne0$ on (a,b). If the ratio $\frac{p'(u)}{q'(u)}$ is increasing on (a,b), then both $\frac{p(u)-p(a)}{q(u)-q(a)}$ and $\frac{p(u)-p(b)}{q(u)-q(b)}$ are increasing in $u\in(a,b)$.

First guess:

Making use of the Maclaurin power series expansions (4) and (13), we can write

$$\frac{F(u)}{G(u)}=-\frac{\displaystyle\sum_{\ell=1}^{\infty}\frac{3^{2\ell}D_{2\ell}}{(2\ell)!}u^{2\ell}}{\displaystyle\sum_{\ell=1}^{\infty}\frac{2^{2\ell}(2^{2\ell-1}-1)}{\ell(2\ell)!}|B_{2\ell}|u^{2\ell}}=-\frac{\displaystyle\sum_{\ell=0}^{\infty}\frac{3^{2\ell+2}D_{2\ell+2}}{(2\ell+2)!}u^{2\ell}}{\displaystyle\sum_{\ell=0}^{\infty}\frac{2^{2\ell+2}(2^{2\ell+1}-1)}{(\ell+1)(2\ell+2)!}|B_{2\ell+2}|u^{2\ell}} \tag{27}$$

for $0<|u|<\frac{\pi}{2}$. By the above-mentioned monotonicity rule for the ratio of two Maclaurin power series, we see that if the ratio

$$\frac{\dfrac{3^{2\ell+2}D_{2\ell+2}}{(2\ell+2)!}}{\dfrac{2^{2\ell+2}(2^{2\ell+1}-1)}{(\ell+1)(2\ell+2)!}|B_{2\ell+2}|}=\frac{\ell+1}{2^{2\ell+1}-1}\left(\frac{3}{2}\right)^{2\ell+2}\frac{D_{2\ell+2}}{|B_{2\ell+2}|} \tag{28}$$

were increasing in $\ell\ge 0$, *then the function* $\frac{F(u)}{G(u)}$ *would be decreasing from the open interval* $(0,\frac{\pi}{2})$ *onto the open interval* $(1,\frac{6}{5})$.

We guess that the sequence in (28) *is increasing in* $\ell\ge 0$.

Second guess:

It is easy to see that we can write

$$\frac{F(u)}{G(u)}=\frac{F(u)-F(0)}{G(u)-G(0)},\quad u\in\left(0,\frac{\pi}{2}\right).$$

Let

$$T_\ell=\frac{2^{2\ell}}{(2\ell)!}|B_{2\ell}|,\quad \ell\ge 1.$$

Then, from the Maclaurin power series expansion (4) *and the relation* (8), *it follows that*

$$G'(u)=2u\sum_{\ell=0}^{\infty}(2^{2\ell+1}-1)T_{\ell+1}u^{2\ell},\quad F'(u)=\frac{2u\sum_{\ell=0}^{\infty}(\ell+1)(2^{2\ell+6}-1)T_{\ell+3}u^{2\ell}}{\sum_{\ell=0}^{\infty}(2^{2\ell+4}-1)T_{\ell+2}u^{2\ell}},$$

and

$$\begin{aligned}\frac{F'(u)}{G'(u)}&=\frac{\sum_{\ell=0}^{\infty}(\ell+1)(2^{2\ell+6}-1)T_{\ell+3}u^{2\ell}}{\left[\sum_{\ell=0}^{\infty}(2^{2\ell+1}-1)T_{\ell+1}u^{2\ell}\right]\left[\sum_{\ell=0}^{\infty}(2^{2\ell+4}-1)T_{\ell+2}u^{2\ell}\right]}\\&=\frac{\sum_{\ell=0}^{\infty}(\ell+1)(2^{2\ell+6}-1)T_{\ell+3}u^{2\ell}}{\sum_{\ell=0}^{\infty}\left[\sum_{j=0}^{\ell}(2^{2j+1}-1)(2^{2\ell-2j+4}-1)T_{j+1}T_{\ell-j+2}\right]u^{2\ell}}\end{aligned}$$

for $|u|<\frac{\pi}{2}$. *Denote*

$$S(\ell)=\frac{(\ell+1)(2^{2\ell+6}-1)T_{\ell+3}}{\sum_{j=0}^{\ell}(2^{2j+1}-1)(2^{2\ell-2j+4}-1)T_{j+1}T_{\ell-j+2}},\quad \ell\ge 0.$$

In the light of the above-mentioned monotonicity rules for the ratio of two functions and for the ratio of two Maclaurin power series in sequence, we see that if the sequence $S(\ell)$ *were decreasing in* $\ell\ge 0$, *that is, if the inequality*

$$\frac{\dfrac{1}{\ell+2}\sum_{j=0}^{\ell+1}[(2^{2j+1}-1)T_{j+1}][(2^{2\ell-2j+6}-1)T_{\ell-j+3}]}{\dfrac{1}{\ell+1}\sum_{j=0}^{\ell}[(2^{2j+1}-1)T_{j+1}][(2^{2\ell-2j+4}-1)T_{\ell-j+2}]}>\frac{(2^{2\ell+8}-1)T_{\ell+4}}{(2^{2\ell+6}-1)T_{\ell+3}} \tag{29}$$

were valid for $\ell \geq 0$, the function $\frac{F(u)}{G(u)}$ would be decreasing on $(0, \frac{\pi}{2})$.

At the site https://mathoverflow.net/q/444490 (accessed 10 April 2023), the second author asked for a proof of the inequality (29).

Remark 12. *We can formulate*

$$\frac{F'(u)}{G'(u)} = -\frac{3\sin^2 u\cos u - 2u\sin u\cos^2 u - u\sin u}{u\sin u + u\sin u\cos^2 u + \cos^3 u - u^2\cos u - \cos u}. \tag{30}$$

In [1] (p. 43) and [22] (Theorem 2.1), we find

$$\sin^3 u = \frac{1}{4}\sum_{\ell=1}^{\infty}(-1)^{\ell+1}\frac{3^{2\ell+1}-3}{(2\ell+1)!}u^{2\ell+1}, \quad |u| < \infty$$

and

$$\cos^3 u = \frac{1}{4}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{3^{2\ell}+3}{(2\ell)!}u^{2\ell}, \quad |u| < \infty.$$

Differentiating results in

$$\sin^2 u\cos u = \frac{1}{12}\sum_{\ell=1}^{\infty}(-1)^{\ell+1}\frac{3^{2\ell+1}-3}{(2\ell)!}u^{2\ell}, \quad |u| < \infty$$

and

$$\sin u\cos^2 u = \frac{1}{12}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{3^{2\ell+2}+3}{(2\ell+1)!}u^{2\ell+1}, \quad |u| < \infty.$$

Substituting the last three power series expansions into (30) and simplifying yield

$$\begin{aligned}\frac{F'(u)}{G'(u)} &= -\frac{\frac{1}{12}\sum_{\ell=3}^{\infty}(-1)^{\ell}\frac{3^{2\ell+2}-4\ell 3^{2\ell}-36\ell-9}{(2\ell)!}u^{2\ell}}{\frac{1}{12}\sum_{\ell=3}^{\infty}(-1)^{\ell}\frac{2\ell 3^{2\ell}-3^{2\ell+1}-48\ell^2+54\ell+3}{(2\ell)!}u^{2\ell}} \\ &= \frac{3\sum_{\ell=0}^{\infty}\frac{(4\ell+3)3^{2\ell+4}+4\ell+13}{(2\ell+6)!}(-u^2)^{\ell}}{\sum_{\ell=0}^{\infty}\frac{2(\ell+1)3^{2\ell+5}-(16\ell^2+78\ell+89)}{(2\ell+6)!}(-u^2)^{\ell}}.\end{aligned}$$

We now consider the sequence

$$T(\ell) = \frac{\frac{(4\ell+3)3^{2\ell+4}+4\ell+13}{(2\ell+6)!}}{\frac{2(\ell+1)3^{2\ell+5}-(16\ell^2+78\ell+89)}{(2\ell+6)!}} = \frac{(4\ell+3)3^{2\ell+4}+4\ell+13}{2(\ell+1)3^{2\ell+5}-(16\ell^2+78\ell+89)}$$

for $\ell \geq 0$. By induction, it follows that

$$2(\ell+1)3^{2\ell+5} - (16\ell^2+78\ell+89) > 0, \quad \ell \geq 0.$$

The increasing property of the sequence $T(\ell)$ is equivalent to

$$[3^{2\ell+7} - (256\ell^3+1760\ell^2+4032\ell+3180)]3^{2\ell+4} + 32\ell^2 + 240\ell + 433 > 0 \tag{31}$$

for $\ell \geq 0$. By induction, we can verify that

$$3^{2\ell+7} - (256\ell^3+1760\ell^2+4032\ell+3180) > 0, \quad \ell \geq 1$$

and that the inequality (31) is valid for $\ell \geq 1$, but not for $\ell = 0$. Hence, the sequence $T(\ell)$ is increasing for $\ell \geq 1$, but

$$T(0) = \frac{256}{397} = 0.645\cdots > T(1) = \frac{1024}{1713} = 0.598\cdots.$$

As a generalization of the monotonicity rule for the ratio of two Maclaurin power series, the following monotonicity rule was established in [33] (Theorem 2.1).

Let $f(u)$ and $g(u)$ be two differentiable functions on the finite or infinite interval (a,b) such that $g'(u) \neq 0$ on (a,b). Define

$$H_{f,g}(u) = \frac{f'(u)}{g'(u)}g(u) - f(u), \quad u \in (a,b).$$

Let

$$A(x) = \sum_{\ell=0}^{\infty} a_k x^\ell \quad \text{and} \quad B(x) = \sum_{\ell=0}^{\infty} b_k x^\ell$$

be two different real power series which are convergent on $(-r,r)$ and $b_\ell > 0$ for $\ell \geq 0$. Suppose that the sequence $\frac{a_\ell}{b_\ell}$ for $\ell \geq 0$ is decreasing for $0 \leq \ell \leq m$ and increasing for $\ell \geq m$, where $m \in \mathbb{N}$ is a fixed positive integer. Then the function $\frac{A(x)}{B(x)}$ is decreasing on $(0,r)$ if and only if $\lim_{x\to r^-} H_{A,B}(x) \leq 0$. If $\lim_{x\to r^-} H_{A,B}(x) > 0$, then there exists a number $x_0 \in (0,r)$ such that the function $\frac{A(x)}{B(x)}$ is decreasing on $(0,x_0)$ and increasing on (x_0,r).

By straightforward computation, we obtain

$$\begin{aligned}
\lim_{u\to(\pi/2)^-} H_{F',G'}(u) &= \frac{1}{4} \lim_{u\to(\pi/2)^-} \frac{\begin{bmatrix} 48u^3 + 32(u^3+u)\cos(2u) - 64u^2\sin(2u) \\ -40u^2\sin(4u) + 4(4u^2-11)u\cos(4u) \\ +12u - 19\sin(2u) + 8\sin(4u) + \sin(6u) \end{bmatrix}}{[8u^2\cos(2u) + \cos(4u) - 1](\sin u - u\cos u)^2} \\
&= \frac{4}{\pi} - \frac{\pi}{2} \\
&= -0.29755\ldots.
\end{aligned}$$

Accordingly, the function

$$\begin{aligned}
&\frac{\displaystyle\sum_{\ell=0}^{\infty} \frac{(4\ell+3)3^{2\ell+4} + 4\ell + 13}{(2\ell+6)!} x^\ell}{\displaystyle\sum_{\ell=0}^{\infty} \frac{2(\ell+1)3^{2\ell+5} - (16\ell^2 + 78\ell + 89)}{(2\ell+6)!} x^\ell} \\
&= \frac{\dfrac{2\sinh(\sqrt{x})(4\sqrt{x} - 3\sinh(2\sqrt{x}) + 2\sqrt{x}\cosh(2\sqrt{x}))}{3x^3}}{\dfrac{\begin{bmatrix} 120\sqrt{x}\sinh(\sqrt{x}) + 24\sqrt{x}\sinh(3\sqrt{x}) - 32\cosh(3\sqrt{x}) \\ +(24-96x)\cosh(\sqrt{x}) + 27x^2 + 36x + 8 \end{bmatrix}}{24x^3}} \\
&= \frac{16\sinh(\sqrt{x})[4\sqrt{x} - 3\sinh(2\sqrt{x}) + 2\sqrt{x}\cosh(2\sqrt{x})]}{\begin{bmatrix} 120\sqrt{x}\sinh(\sqrt{x}) + 24\sqrt{x}\sinh(3\sqrt{x}) - 32\cosh(3\sqrt{x}) \\ +(24-96x)\cosh(\sqrt{x}) + 27x^2 + 36x + 8 \end{bmatrix}}
\end{aligned}$$

is decreasing in $x > 0$.

Remark 13. *This paper is an important sibling of the articles [6,7].*

6. Conclusions

By virtue of the derivative Formula (9), we established the first form of the Maclaurin power series expansion (13) of the logarithmic expression $F(u)$ in term of the Hessenberg determinants $D_{2\ell}$ whose elements involve the Bernoulli numbers $B_{2\ell}$. On the other hand, we recited and revised the second form and its proof of the Maclaurin power series expansion (15) of the logarithmic expression $F(u)$ in terms of the Bessel zeta functions $\zeta_\lambda(2\ell)$ and the Bernoulli numbers $B_{2\ell}$ for $\ell \geq 0$.

In summary, our results in this paper are original. We believe that our results will be widely applied to other areas of mathematical sciences.

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