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Reliability Analysis of Kavya Manoharan Kumaraswamy Distribution under Generalized Progressive Hybrid Data

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Abstract: Generalized progressive hybrid censoring approaches have been developed to reduce test time and cost. This paper investigates the difficulties associated with estimating the unobserved model parameters and the reliability time functions of the Kavya Manoharan Kumaraswamy (KMKu) distribution based on generalized type-II progressive hybrid censoring using classical and Bayesian estimation techniques. The frequentist estimators' normal approximations are also used to construct the appropriate estimated confidence intervals for the unknown parameter model. Under symmetrical squared error loss, independent gamma conjugate priors are used to produce the Bayesian estimators. The Bayesian estimators and associated highest posterior density intervals cannot be derived analytically since the joint likelihood function is provided in a complicated form. However, they may be evaluated using Monte Carlo Markov chain (MCMC) techniques. Out of all the censoring choices, the best one is selected using four optimality criteria.

Keywords: Kavya Manoharan Kumaraswamy distribution; progressive hybrid generalized type-II censoring; Bayesian and classical estimators; Metropolis–Hastings algorithm; MCMC techniques; optimal plan for progressive censoring

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1. Introduction

The progressive type-II censoring (PCS-T2) method is the most popular scheme in reliability and survival analysis. Compared with the traditional type-II censoring method, it is better. Progressive censoring is advantageous in a variety of real-world applications, including business, medical research, and therapeutic settings. Up until the test's conclusion, it permits the removal of any remaining experimental units. Assume that n units are used in a life test and that it is not desirable to record every failure because of financial and time constraints. Consequently, only a portion of unit failures are seen. A sample like this is known as a censored sample. Assume that one of the units was accidentally damaged after the test started but before they all burned out. This unit needs to be taken out of the life test if the experiment is still going on. In this situation, a framework for analyzing this kind of data is provided by the progressive censoring scheme. A few examples of primary references are [1,2].

PCS-T2 has drawn a lot of attention in the literature as a very flexible censoring system (see [3] for further details). When testing n independent units at a time $T = 0$, the failure number to be noticed s and the progressive censored samples, $\underline{R} = (R_1, R_2, \ldots, R_s)$, where $n = \sum_{i=1}^{s} R_i + s$, are specified. When the initial failure is seen (suppose that $Y_{1:s:n}$), the other surviving units $n-1$ are chosen at random, and R_1 of those units is disqualified from the test. Similarly, at the moment of the second failure (suppose that $Y_{2:s:n}$), R_2 of $n - R_1 - 2$

are selected at random and deleted from the test, and so on. At the time of the $s-th$ failure (suppose that $Y_{s:s:n}$), every survivor unit still present $R_s = n - s - \sum_{j=1}^{s-1} R_j$ is removed from the experiment.

Whenever the test units are particularly reliable, the major drawback of this censoring is that it could take longer to finish the progressively type-II hybrid censored samples (PHCS-T2). The authors of [4] proposed a progressive type-I hybrid censored strategy (PHCS-T1) as a remedy for this issue. This method combines PCS-T2 with conventional type-I censoring. Under PHCS-T1, the trial period is stopped at T, maximum likelihood estimators (MLEs) were not always available due to the fact that relatively a few failures might occur before time T in PHCS-T1. To resolve this issue, [5] presented the PHCS-T2 scheme. At $T^* = max(Y_{s:s:n},\ T)$, the experiment comes to an end under PHCS-T2. It can take some time until such $s-th$ failures are really observed, despite the fact that PHCS-T2 promises a fixed number of failures.

It could take a while to gather the needed failures, even though the PHCS-T2 ensures an effective number of observable failures. Thus, [6] devised the generalized progressive type-II hybrid censoring (GPHC-T2). Assume that the thresholds $T_i, i = 1$, and 2, as well as the integer s, are preassigned in such a way that $0 < T_1 < T_2 < \infty$ and $1 < s < n$. c_1 and c_2 represent the overall number of failures up to periods T_1 and T_2, respectively. Then, at $Y_{1:s:n}$, R_1 of $n-1$ are arbitrarily excluded from the test, followed by R_2 of $n - R_1 - 2$, and so on.

The experiment is over, and all remaining units are deleted at $T^* = max(T_1, min(Y_{s:s:n},\ T_2))$. If $Y_{s:n} < T_1$, failures are observed without any further withdrawals up until time T_1 (Case-I); if $T_1 < Y_{s:s:n} < T_2$, the test is terminated at time $Y_{s:s:n}$ (Case-II); or, if not, the test is terminated at time T_2 (Case-III). Keep in mind that the GPHCS-T2 modifies the PHCS-T2 by guaranteeing that the test is completed at the scheduled time T_2. T_2 demonstrates the longest period of time the researcher is willing to let the experiment continue. As a result, one of the following three data types will be visible to the experimenter:

$$\left(\underset{-}{Y},\underset{-}{R}\right) = \begin{cases} \textbf{\textit{Case I}};\{[Y_{1:n},R_1],\ldots,[Y_{s-1:s:n},R_{s-1}],[Y_{s:s:n},0],\ldots,[Y_{c_1:n},0]\} \\ \textbf{\textit{Case II}};\{[Y_{1:s:n},R_1],\ldots,[Y_{c_1:n},R_{c_1}],[Y_{s-1:s:n},R_{s-1}],\ldots,[Y_{s:s:n},R_s]\} \\ \textbf{\textit{Case III}};\{[Y_{1:s:n},R_1],\ldots,[Y_{c_1:n},R_{c_1}],[Y_{c_2-1:n},R_{c_2-1}],\ldots,[Y_{c_2:n},R_{c_2}]\}. \end{cases}$$

Figure 1 indicates the cases of generalized type-II progressive hybrid sample as follows:

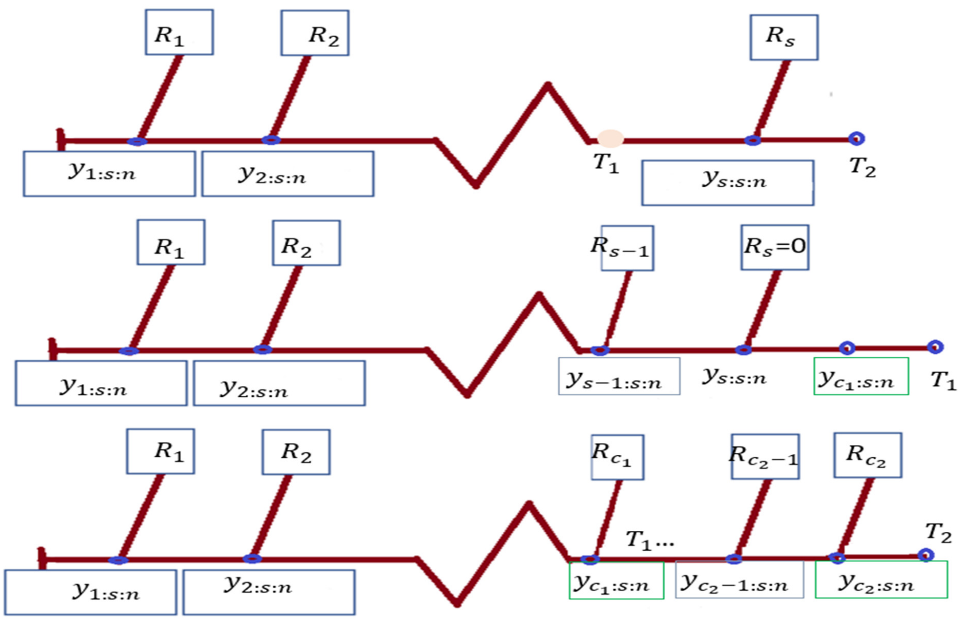

Figure 1. Generalized type-II progressive hybrid cases.

Assume that in a distribution with a cumulative distribution function (cdf) $F(.)$, and probability density function (pdf) $f(.)$, the variables $\underline{Y}$ and $\underline{R}$ represent the respective lifetimes. As a result, the GPHCS-T2 likelihood function is expressed as follows:

$$L_\varphi\left(\theta,\beta|\underline{y}\right) = C_\varphi \prod_{j=1}^{D_\varphi} f\left(Y_{j:s:n};\theta,\beta\right)\left[1-F\left(Y_{j:s:n};\theta,\beta\right)\right]^{R_j}\psi_\varphi(T_\tau;\theta,\beta), \tag{1}$$

where $\tau = 1,2,\ \varphi = 1,2,3$, stand in for Case-I, Case-II, and Case-III, respectively, and $\psi_\varphi(.)$ is a combination form of dependability functions. Table 1 displays the GPHCS-T2 notations from Equation (1). Many censoring techniques can also be inferred as particular examples from Equation (1), including

Table 1. The notations of the GPHCS-T2.

φ	C_φ	D_φ	$\psi_\varphi(T_\tau;\beta)$	$R^*_{c_\tau+1}$
1	$\prod_{j=1}^{c_1}\sum_{i=j}^{s}(R_i+1)$	c_1	$[1-F(T_1)]^{R^*_{c_1+1}}$	$n-c_1-\sum_{i=1}^{s-1}R_i$
2	$\prod_{j=1}^{s}\sum_{i=j}^{s}(R_i+1)$	s	1	0
3	$\prod_{j=1}^{c_2}\sum_{i=j}^{s}(R_i+1)$	c_2	$[1-F(T_2)]^{R^*_{c_2+1}}$	$n-c_2-\sum_{i=1}^{c_2}R_i$

1. With T_1 setting to 0, use PHCS-T1.
2. $T_2 \to \infty$. by setting PHCS-T2.
3. You may do hybrid type-I censoring by setting $T_1 \to 0,\ R_j = 0,\ j = 1,2,\ldots,s-1$, $R_s = n-s$.
4. $T_2 \to \infty,\ R_j = 0,\ j = 1,2,\ldots,s-1,\ R_s = n-s$ can be used to do hybrid type-II censoring.
5. To do type-I censoring, set $T_1 = 0,\ s = 1,\ R_j = 0,\ j = 1,\ 2,\ldots,\ s-1,\ R_s = n-s$.
6. A type-II censored sample is produced by setting $T_1 = 0,\ T_2 \to \infty,\ s = 1$, $R_j = 0,\ j = 1,\ 2,\ldots,\ s-1,\ R_s = n-s$.

On the basis of GPHCS-T2, more studies have been conducted. For instance, Ref. [7] investigated the prediction issue of forthcoming Burr-XII distribution failure rates. The authors of [8] created the Weibull distribution with little data with an objective Bayesian analysis. The authors of [9] addressed the competing risks from exponential data, and [10] more recently examined both the point and interval estimations of the Burr-XII parameters. Last but not least, [11] addressed the Fréchet distribution's optimality under generalized censoring schemes. In this paper, the KMKu model under generalized censoring samples is studied. Where the KMKu model was initially proposed by [11]. Also, they found that the Kumaraswamy model's and KMKu shape forms in the pdf for different parameter values are comparable. It may be asymmetric, unimodal, increasing, or decreasing. In addition, the bathtub, U-shape, J-shape, or increasing shapes of the hazard rate function (hrf) for the KMKu model are all possible. But suppose that Y is the lifespan random variable of a test item adheres to the KMKu distribution, denoted by the notation $KMKu(\theta,\beta)$, where $\theta > 0, \beta > 0$ are the shape parameters. Therefore, it is supplied by its pdf, cdf, reliability function (RF), $R(.)$, and hrf, all represented by the letters $f(.)$, $F(.)$, and $h(.)$ accordingly:

$$f(y;\theta,\beta) = \frac{\theta\beta y^{\theta-1}}{e-1}\left(1-y^\theta\right)^{\beta-1} e^{(1-y^\theta)^\beta}, 0 < y < 1; \theta\beta > 0, \tag{2}$$

$$F(y;\theta,\beta) = \frac{e}{e-1}\left(1-e^{-1}e^{(1-y^\theta)^\beta}\right), \tag{3}$$

$$R(y;\theta,\beta) = 1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-y^\theta)^\beta}\right), \tag{4}$$

and

$$h(y;\theta,\beta)=\frac{\frac{\theta\beta y^{\theta-1}}{e-1}\left(1-y^{\theta}\right)^{\beta-1}}{1-e^{(1-y^{\theta})^{\beta}}}. \tag{5}$$

Although the KMKu model has a lot of flexibility because of its different shapes of hrf and pdf, to our knowledge, no studies have yet been done under censorship. Particularly, the generalized type-II progressively hybrid censoring scheme has not produced any data for the new KMKu lifetime model's survival traits and model parameters. To fill this gap, the following are the objectives of this study: Firstly, the probability inference for any function of the unknown KMKu parameters, such as $R(t)$ or $h(t)$, is derived. The second objective is to derive independent gamma priors from the squared error (SE) loss and produce Bayes estimates for the same unknown parameters, employing the provided estimation procedures, such as classical and Bayesian approaches. The unknown parameters of the KMKu distribution are discovered using the approximation confidence intervals (ACIs) and highest posterior density (HPD) interval estimators. The acquired estimates are computed using the R programming language's "maxLik" and "coda" packages because the theoretical findings of θ and β obtained by the suggested estimation techniques cannot be represented in closed form. [12,13] offered these packages. Using four optimality criteria, the ultimate aim is to develop the most efficient progressively censored sample technique. The effectiveness of the different estimators is investigated using a Monte Carlo simulation with the entire sample size, which can be combined in a variety of ways, effective sample size, threshold timings, and progressively censored samples. We compare the average confidence lengths (ACLs), mean relative absolute biases (MRABs), and simulated root mean squared errors (RMSEs) of the derived estimators. The optimal censoring tactic should be chosen after evaluating how effectively the given techniques will function in practice. The remaining portions of this study are structured as follows: The maximum likelihood, Bayes inferences, and reliability functions of the unknown parameters are presented in Sections 2 and 3, respectively. The credible and asymptotic intervals are built into Section 4. Section 5 goes into depth about the results of the Monte Carlo simulation. The optimal methods for progressive censoring are discussed in Section 6. Two actual data sets are indicated in Section 7. Finally, the conclusion and discussion are given in Section 8.

2. Likelihood Estimation

Assume that the representation of a GPHCS-T2 sample of size c_2 taken from $KMKu(\theta,\beta)$ is $Y=((Y_{1:s:n},\ R_1),\ldots,(Y_{c_1:n},\ R_{c_1}),\ldots,(Y_{c_2:n},\ R_{c_2}))$. The probability function of GPHCS-T2 may be represented by substituting y_j for $y_{j:s:n}$ in Equation (1) and adding Equations (2) and (3); for more information, see [14].

$$L_{\varphi}(\theta,\beta|\underline{Y})\propto\prod_{j=1}^{D_{\varphi}}\frac{\theta\beta y_j^{\theta-1}}{e-1}\left(1-y_j^{\theta}\right)^{\beta-1}e^{(1-y_j^{\theta})^{\beta}}\left[1-\frac{e}{e-1}+e^{-1}e^{(1-y^{\theta})^{\beta}}\right]^{R_i}\psi_{\varphi}(T_{\tau};\theta,\beta), \tag{6}$$

where

$\psi_1(T_1;\theta,\beta)=\left(1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-T_1^{\theta})^{\beta}}\right)\right)^{R^*_{c_1+1}}$, $\psi_2(T_{\tau};\theta,\beta)=1$ and $\psi_3(T_2;\theta,\beta)=\left(1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-T_2^{\theta})^{\beta}}\right)\right)^{R^*_{c_2+1}}$.

The proper log-likelihood function for Equation (6) is $\ell_{\varphi}(.)\propto ln\ L_{\varphi}(.)$ as follows:

$$\begin{aligned}\ell_{\varphi}(\theta,\beta|\underline{Y})\propto{}& D_{\varphi}ln(\theta\beta)+(\theta-1)\textstyle\sum_{j=1}^{D_{\varphi}}ln(y_j)-D_{\varphi}ln(e-1)+(\beta-1)\sum_{j=1}^{D_{\varphi}}ln\left(1-y_j^{\theta}\right)+\beta\sum_{j=1}^{D_{\varphi}}(1-\\ & y_j^{\theta})+R_i\textstyle\sum_{j=1}^{D_{\varphi}}ln\left[1-\frac{e}{e-1}+e^{-1}e^{(1-y_j^{\theta})^{\beta}}\right]+\gamma_{\varphi}(T_{\tau};\theta,\beta),\end{aligned} \tag{7}$$

where

$\gamma_\varphi(T_1;\theta,\beta) = \left(R^*_{c_1+1}\right) ln\left[1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-T_1^\theta)^\beta}\right)\right], \gamma_2(T_\tau;\theta,\beta) = 1$, and $\gamma_3(T_2;\theta,\beta) = \left(R^*_{c_2+1}\right) ln\left[1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-T_2^\theta)^\beta}\right)\right]$.

By partially differentiating Equation (7) with reference to $\hat{\theta}$ and $\hat{\beta}$, the subsequent two findings are produced. After being equal to zero, likelihood equations must be simultaneously solved in order to create the MLEs.

$$\frac{\partial \ell_\varphi}{\partial \theta} = \frac{D_\varphi}{\theta} + \sum_{j=1}^{D_\varphi} ln(y_j) - (\beta-1)\sum_{j=1}^{D_\varphi} \frac{y_j^\theta ln(y_j)}{\left(1-y_j^\theta\right)} - \beta\sum_{j=1}^{D_\varphi} y_j^\theta ln(y_j) - R_i\sum_{j=1}^{D_\varphi} \frac{e^{(1-y_j^\theta)^\beta} y_j^\theta \beta\left(1-y_j^\theta\right)^{\beta-1} ln(y_j)}{e\left(1-\frac{e}{e-1}+e^{-1}e^{(1-y_j^\theta)^\beta}\right)} + \frac{\partial \gamma_\varphi(T_\tau;\theta,\beta)}{\partial \theta}, \tag{8}$$

and

$$\frac{\partial \ell_\varphi}{\partial \beta} = \frac{D_\varphi}{\beta} + 2\sum_{j=1}^{D_\varphi} ln\left(1-y_j^\theta\right) + R_i\sum_{j=1}^{D_\varphi} \frac{e^{(1-y_j^\theta)^\beta}\left(1-y_j^\theta\right)^\beta ln\left(1-y_j^\theta\right)}{e\left(1-\frac{e}{e-1}+e^{-1}e^{(1-y_j^\theta)^\beta}\right)} + \frac{\partial \gamma_\varphi(T_\tau;\theta,\beta)}{\partial \beta}, \tag{9}$$

where $\varphi = 1,\ 3$ and $\tau = 1,\ 2$, respectively, we have

$$\frac{\partial \gamma_\varphi(T_\tau;\theta,\beta)}{\partial \theta} = -\left(R^*_{c_\tau+1}\right)\frac{e^{(1-T_\tau^\theta)^\beta}\beta T_\tau^\theta ln(T_\tau)}{\left[1-\frac{e}{e-1}+e^{-1}e^{(1-T_\tau^\theta)^\beta}\right]},\ \frac{\partial \gamma_\varphi(T_\tau;\theta,\beta)}{\partial \beta} = -\left(R^*_{c_\tau+1}\right)\frac{e^{(1-T_\tau^\theta)^\beta}\beta ln\left(1-T_\tau^\theta\right)}{e\left[1-\frac{e}{e-1}+e^{-1}e^{(1-T_\tau^\theta)^\beta}\right]}.$$

According to Equations (8) and (9), it is necessary to simultaneously satisfy a system of two nonlinear equations in order to derive the MLEs of θ and β in the KMKu model. As a result, for θ and β, there is not, and cannot be computed, an analytical closed-form solution. Thus, it may be estimated for each specific GPHCS-T2 data set using numerical techniques like the Newton-Raphson iterative method. When the estimates of θ and β are derived by replacing them with $\hat{\theta}$ and $\hat{\beta}$, the MLEs $\hat{R}(t)$ and $\hat{h}(t)$, respectively, may be easily computed.

3. Bayes Estimator

The HPD intervals for the Bayes estimators of θ , β, $R(t)$, and $h(t)$ are developed using the SE loss function. To do this, it is assumed that the KMKu parameters θ and β, respectively, have independent gamma priors of the forms $\omega(\nu_1,\ \nu_2)$ and $\omega(\nu_3,\ \nu_4)$.

The normal distribution can be a standard choice for data if the domain of that distribution is from $-\infty$ to ∞, and the beta distribution can be a standard choice for data if the domain of that distribution is from 0 to 1. Similarly, the gamma distribution can be a standard choice for non-negative continuous data if the domain of the gamma distribution is from 0 to ∞. This is one of the most important reasons, but there are other reasons as follows:

- We believe the main motivation for the gamma prior is usually to constrain the random variables to positive values.
- The gamma distribution is considered one of the most important and well-known statistical distributions because it is compatible with many engineering, mathematical, statistical, and medical applications.
- The gamma distribution is one of the most famous distributions that is used in mathematical solutions (integrations), especially when the data are from 0 to ∞.
- In previous studies, the gamma distribution was the most popular prior distribution and was associated with the best statistical results.

Gamma priors should be considered for a variety of reasons, including the fact that they are (1) adjustable, (2) offer diverse shapes based on parameter values, and (3) fairly basic and brief and might not generate a solution to a challenging estimation problem. Then, the combined previous density of θ and β is determined; for more details on this topic, see [15,16].

$$\pi(\theta,\beta) \propto \theta^{\nu_1-1}\beta^{\nu_3-1}e^{-(\theta\nu_2+\beta\nu_4)} \tag{10}$$

If it is anticipated that for $i = 1,\ 2,\ 3, 4, \nu_i > 0$ are known. The joint posterior pdf of θ and β, Equations (6) and (10), when combined, results.

$$\pi_{\varphi}\left(\theta,\beta|\underline{y}\right) \propto \theta^{D_{\varphi}+\nu_1-1}\beta^{D_{\varphi}+\nu_3-1}e^{-(\theta\nu_2+\beta\nu_4)}\prod_{j=1}^{D_{\varphi}}\frac{\theta\beta y_j^{\theta-1}}{e-1}\left(1-y_j^{\theta}\right)^{\beta-1}e^{(1-y_j^{\theta})^{\beta}}\left[1-\frac{e}{e-1}+e^{-1}e^{(1-y^{\theta})^{\beta}}\right]^{R_i}\psi_{\varphi}(T_{\tau};\theta,\beta) \tag{11}$$

The Bayes estimate, $\widetilde{\eta}(\theta,\beta)$, of θ and β respectively, under SE loss, $\eta(\theta,\beta)$ is what is meant by the posterior expectation of Equation (11), which is given.

$$\widetilde{\eta}(\theta,\beta) = \int_0^{\infty}\int_0^{\infty}\eta(\theta,\beta)\pi_{\varphi}\left(\theta,\beta|\underline{y}\right)d\theta d\beta.$$

It is clear from Equation (11), that it is impossible to explicitly express the marginal pdfs of θ and β. In order to accomplish this, we recommend creating samples from Equation (11) utilizing Bayes MCMC methods to calculate the joint Bayes estimates and supplying their HPD intervals. The complete conditional pdfs of θ and β are provided for the MCMC sampler from Equation (11) to be performed as intended.

$$\pi_{\varphi}^{\theta}\left(\theta|\beta,\ \underline{y}\right) \propto \theta^{D_{\varphi}+\nu_1-1}e^{-\theta\nu_2}\prod_{j=1}^{D_{\varphi}}\frac{\theta\beta y_j^{\theta-1}}{e-1}\left(1-y_j^{\theta}\right)^{\beta-1}e^{(1-y_j^{\theta})^{\beta}}\left[1-\frac{e}{e-1}+e^{-1}e^{(1-y^{\theta})^{\beta}}\right]^{R_i}\psi_{\varphi}(T_{\tau};\theta,\beta), \tag{12}$$

and

$$\pi_{\varphi}^{\beta}\left(\beta|\theta,\ \underline{y}\right) \propto \beta^{D_{\varphi}+\nu_3-1}e^{-\beta\nu_4}\prod_{j=1}^{D_{\varphi}}\frac{\theta\beta y_j^{\theta-1}}{e-1}\left(1-y_j^{\theta}\right)^{\beta-1}e^{(1-y_j^{\theta})^{\beta}}\left[1-\frac{e}{e-1}+e^{-1}e^{(1-y^{\theta})^{\beta}}\right]^{R_i}\psi_{\varphi}(T_{\tau};\theta,\beta). \tag{13}$$

The Metropolis-Hastings (M-H) approach is considered to be the best solution to this problem because no analytical method exists to reduce the posterior pdfs of θ and β in Equations (12) and (13), respectively, to any known distribution (for further information, see [17,18]. The sampling method of the M-H algorithm is implemented according to:

First, establish the starting points, $\theta^{(0)} = \hat{\theta}$ and $\beta^{(0)} = \hat{\beta}$.

Set S = 1 after that.

Thirdly, from $N(\hat{\mu}_1,\hat{\sigma}_1)$ and $N(\hat{\mu}_2,\hat{\sigma}_2)$, respectively, create θ^* and β^*.

The fourth step: Obtaining $\varrho_{\theta} = min\left\{1, \frac{\pi_{\varphi}^{\theta}\left(\theta^*|\beta^{(s-1)};\underline{y}\right)}{\pi_{\varphi}^{\theta}\left(\theta^{(s-1)}|\beta^{(s-1)};\underline{y}\right)}\right\}$ and $\varrho_{\beta} = min\left\{1, \frac{\pi_{\varphi}^{\beta}\left(\beta^*|\theta^{(s)};\underline{y}\right)}{\pi_{\varphi}^{\beta}\left(\beta^{(s-1)}|\theta^{(s)};\underline{y}\right)}\right\}$.

Fifth, use the uniform $U(0,1)$ distribution to generate the samples u_1 and u_2.

Sixth: Set $\theta^{(S)} = \theta^*$ and $\beta^{(S)} = \beta^*$, respectively, if u_1 and u_2 are both smaller than ϱ_{θ} and ϱ_{β}, respectively. Set $\theta^{(S)} = \theta^{(S-1)}$ and $\beta^{(S)} = \beta^{(S-1)}$, correspondingly, if not.

Seventh: Establish that S equals S + 1.

Eighth: Repeating steps three through seven a number of times $\boldsymbol{B}$ will give you the values for $\theta^{(S)}$ and $\beta^{(S)}$ for $S = 1,\ 2,\ldots,\ \boldsymbol{B}$.

Ninth: To calculate the RF in Equation (4) and hrf in Equation (5), use $\theta^{(S)}$ and $\beta^{(S)}$ for $S = 1,\ 2,\ldots,\ \boldsymbol{B}$, respectively, for a given mission period $t > 0$.

$$R^{(S)}(t) = 1-\frac{e}{e-1}\left(1-e^{-1}e^{(1-y^{\theta^{(S)}})^{\beta^{(S)}}}\right), y > 0,$$

and

$$h^{(S)}(t) = \frac{\frac{\theta^{(S)}\beta^{(S)}y^{\theta^{(S)}-1}}{e-1}\left(1-y^{\theta^{(S)}}\right)^{\beta^{(S)}-1}}{1-e^{(1-y^{\theta^{(S)}})^{\theta^{(S)}}}}, y > 0.$$

The convergence of the MCMC sampler must be ensured, and starting, $\theta^{(0)}$ and $\beta^{(0)}$ values must be eliminated. The first simulated variants, let us say $\boldsymbol{B}_0$, are removed as burn-ins. Therefore, using the remaining $\boldsymbol{B}-\boldsymbol{B}_0$ samples of $\theta, \beta,\ R(t)$, or $h(t)$, (let us suppose η), the Bayesian estimates are computed. On the basis of the SE loss function, the Bayes MCMC estimates of η are shown.

$$\widetilde{\eta} = \frac{1}{\boldsymbol{B}-\boldsymbol{B}_0}\sum_{S=B_0+1}^{\mathrm{B}}\eta^{(S)}$$

4. Interval Estimators

The HPD interval estimators in this section are based on acquired MCMC-simulated variations, as opposed to the approximative confidence estimators of $\theta, \beta, R(t)$, or $h(t)$ that are based on observed Fisher information.

4.1. Asymptotic Intervals

To compute the ACIs for θ and β, the Fisher information matrix must first be inverted to produce the asymptotic variance-covariance (AVC) matrix. According to certain regularity criteria, $(\hat{\theta}, \hat{\beta})$ is nearly normal with a mean (θ, β) and variance $I^{-1}(\theta, \beta)$. In agreement with [19], we estimate $I^{-1}(\theta, \beta)$ by $I^{-1}(\hat{\theta}, \hat{\beta})$, replacing $\hat{\theta}$ and $\hat{\beta}$ for θ and β.

$$I^{-1}(\hat{\theta}, \hat{\beta}) = -\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \tag{14}$$

where

$$a_{11} = \frac{\partial^2 \ell_\varphi}{\partial \theta^2} = -\frac{D_\rho}{\theta^2} - (\beta - 1)\sum_{j=1}^{D_\varphi} \frac{y_j^\theta ln(y_j)\left(1 + ln(y_j)\left(1 - y_j^\theta\right)\right)}{\left(1 - y_j^\theta\right)^2} - \beta \sum_{j=1}^{D_\varphi} y_j^\theta \left[ln\left(y_j\right)\right]^2 + R_i \sum_{j=1}^{D_\varphi} \frac{\beta y_j^\theta (ln(y_j))^2 e^{(1-y_j^\theta)^\beta} \left(1 - y_j^\theta\right)^{\beta - 1}\left(1 + \beta y_j^\theta \left(1 - y_j^\theta\right)^\beta + \frac{y_j^\theta(\beta - 1)}{\left(1 - y_j^\theta\right)}\right)}{e\left(1 - \frac{e}{e-1} + e^{-1} e^{(1-y_j^\theta)^\beta}\right)^2} + \frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \theta^2} \tag{15}$$

$$a_{12} = a_{21} = \frac{\partial^2 \ell_\varphi}{\partial \theta \partial \beta} = -\sum_{j=1}^{D_\varphi} \frac{y_j^\theta ln(y_j)}{\left(1 - y_j^\theta\right)} - \sum_{j=1}^{D_\varphi} y_j^\theta ln\left(y_j\right) - R_i \sum_{j=1}^{D_\varphi} \frac{y_j^\theta ln(y_j) e^{(1-y_j^\theta)^\beta} \left(1 - y_j^\theta\right)^{\beta - 1}\left(1 + \beta\left(1 - y_j^\theta\right)^\beta \left(ln\left(1 - y_j^\theta\right)\right) + \frac{\beta ln\left(1 - y_j^\theta\right)}{\left(1 - y_j^\theta\right)}\right)}{e\left(1 - \frac{e}{e-1} + e^{-1} e^{(1-y_j^\theta)^\beta}\right)^2} + \frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \theta \partial \beta}, \tag{16}$$

$$a_{22} = \frac{\partial^2 \ell_\varphi}{\partial \beta^2} = -\frac{D_\varphi}{\beta^2} + R_i \sum_{j=1}^{D_\varphi} \frac{\left(ln\left(1 - y_j^\theta\right)\right)^2 e^{(1-y_j^\theta)^\beta}\left(\left(1 - y_j^\theta\right)^\beta + \left(1 - y_j^\theta\right)^{2\beta}\right)}{e\left(1 - \frac{e}{e-1} + e^{-1} e^{(1-y_j^\theta)^\beta}\right)^2} + \frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \beta^2}, \tag{17}$$

where

$$\frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \theta^2} = -\left(R^*_{c_\tau + 1}\right) \frac{\beta ln(T_\tau) T_\tau^\theta e^{(1-T_\tau^\theta)^\beta}\left(\left[1 - \frac{e}{e-1} + e^{-1} e^{(1-T_\tau^\theta)^\beta}\right]\left(1 - T_\tau^\theta\left(1 - T_\tau^\theta\right)^{\beta - 1}\right) + T_\tau^{2\theta} e^{-1} \beta \left(1 - T_\tau^\theta\right)^{\beta - 1}\right)}{\left[1 - \frac{e}{e-1} + e^{-1} e^{(1-T_\tau^\theta)^\beta}\right]^2},$$

$$\frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \theta \partial \beta} = \frac{\left(R^*_{c_1 + 1}\right) e^{-1} e^{(1-T_1^\theta)^\beta}\left(1 - T_\tau^\theta\right)^\beta ln\left(1 - T_\tau^\theta\right)}{\left[1 - \frac{e}{e-1}\left(1 - e^{-1} e^{(1-T_\tau^\theta)^\beta}\right)\right]},$$

and

$$\frac{\partial^2 \gamma_\varphi(T_\tau; \theta, \beta)}{\partial \beta^2} = \frac{-\left(R^*_{c_\tau + 1}\right) ln\left(1 - T_\tau^\theta\right) e^{(1-T_\tau^\theta)^\beta}}{e\left[1 - \frac{e}{e-1} + e^{-1} e^{(1-T_\tau^\theta)^\beta}\right]^2}$$

$$\left(\left[1 - \frac{e}{e-1} + e^{-1} e^{(1-T_\tau^\theta)^\beta}\right]\left(1 + \beta\left(1 - T_\tau^\theta\right)^\beta ln\left(1 - T_\tau^\theta\right)\right) - \beta e^{-1}\left(1 - T_\tau^\theta\right)^\beta ln\left(1 - T_\tau^\theta\right) e^{(1-T_\tau^\theta)^\beta}\right).$$

The two-sided $100(1 - \gamma)\%$ ACIs are therefore given by
$\hat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_1{}^2}$ and $\hat{\beta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_2{}^2}$, for θ and β, respectively, where $Z_{\frac{\gamma}{2}}$ stands for the top $\frac{\gamma}{2}$ percentage points of the standard normal distribution, $\hat{\sigma}_1{}^2$ and $\hat{\sigma}_2{}^2$ are the primary diagonal elements of

Equation (14). Furthermore, we employ the delta method to first establish the estimated variance of $\hat{R}(t)$ and $\hat{h}(t)$ (see [20]) before developing the ACIs of $R(t)$ and $h(t)$ as

$$\hat{\sigma}^2_{\hat{R}(t)} = \epsilon^T_{\hat{R}} I^{-1}(\hat{\varepsilon})\epsilon_{\hat{R}} \text{ and } \hat{\sigma}^2_{\hat{h}(t)} = \epsilon^T_{\hat{h}} I^{-1}(\hat{\varepsilon})\epsilon_{\hat{h}},$$

where $\epsilon^T_{\hat{R}} = \left[\frac{\partial \mathrm{R(t)}}{\partial \theta} \quad \frac{\partial \mathrm{R(t)}}{\partial \beta}\right]_{(\hat{\theta},\, \hat{\beta})}$, and $\epsilon^T_{\hat{h}} = \left[\frac{\partial \mathrm{h(t)}}{\partial \theta} \quad \frac{\partial \mathrm{h(t)}}{\partial \beta}\right]_{(\hat{\theta},\, \hat{\beta})}$

Then, $R(t)$ and $h(t)$ both have two-sided $100(1-\gamma)\%$ ACIs that are supplied by $\hat{R}(t) \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}_{\hat{R}(t)}}^2$ and $\hat{h}(t) \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}_{\hat{h}(t)}}^2$, respectively.

Adding bootstrapping techniques to improve estimators or create confidence intervals for θ, $\beta, R(t)$, or $h(t)$ is easy.

4.2. HPD Intervals

The method put forward by [21] is used to create $100(1-\gamma)\%$ HPD interval estimates of $\theta, \beta, R(t)$, or $h(t)$. First, we assign numerical values to the MCMC samples of $\varepsilon^{(j)}$ for $j = \boldsymbol{B}_0 + 1,\ \boldsymbol{B}_0 + 2, \ldots,\ \boldsymbol{B}$ *as* $\varepsilon_{(\boldsymbol{B}_0+1)}, \varepsilon_{(\boldsymbol{B}_0+2)}, \ldots, \varepsilon_{(\boldsymbol{B})}$ correspondingly. The discovery is that the $100(1-\gamma)\%$ two-sided HPD interval of ε is supplied by $\varepsilon_{(j^*)}, \varepsilon_{(j^*+(1-\epsilon)(\boldsymbol{B}-\boldsymbol{B}_0))}$, where $j^* = \boldsymbol{B}_0 + 1,\ \boldsymbol{B}_0 + 2, \ldots,\ \boldsymbol{B}$ is selected so that $\varepsilon_{(j^*+(1-\epsilon)(\boldsymbol{B}-\boldsymbol{B}_0))} - \varepsilon_{(j^*)} = \underset{1 \le j \le \gamma \le (\boldsymbol{B}-\boldsymbol{B}_0)}{min} \left\{\varepsilon_{(j+(1-\gamma)(\boldsymbol{B}-\boldsymbol{B}_0))} - \varepsilon_{(j)}\right\} \varepsilon_{(j^*)}, \varepsilon_{(j^*+(1-\gamma)(\boldsymbol{B}-\boldsymbol{B}_0))}$.

5. Optimal PCS-T2 Designs

The experimenter may want to pick the "best" censoring scheme out of a collection of all accessible censoring schemes in order to provide the most details about the unknown parameters under investigation, especially in the context of dependability. First, [1] examined the problem of deciding which censoring strategy is most appropriate under various circumstances. However, a number of optimality criteria, $R = (R_1, R_2, \ldots, R_s)$, where $\sum_{i=1}^{s} R_i$ have been proposed, and several assessments of the top censoring strategies have been made. The precise values of n (total test units), s (effective sample), and $T_i, i = 1, 2$ (ideal test thresholds) are picked in advance according to the accessibility of the units, the accessibility of the experimental settings, and cost factors (see [22]). A number of articles in the literature have addressed the topic of contrasting two (or more) different censoring techniques. For examples, see [23,24]. To help us choose the best censoring strategy, O_i, Table 2 offers a variety of widely used measures.

Table 2. Illustrations of numerous helpful censoring methods and best practices.

Criterion	Method
O_1	Maximize trace $[\boldsymbol{I}_{2\times2}(.)]$
O_2	Minimize trace $[\boldsymbol{I}_{2\times2}(.)]^{-1}$
O_3	Minimize det $[\boldsymbol{I}_{2\times2}(.)]^{-1}$
O_4	Minimize $\boldsymbol{Var}\left[\boldsymbol{log}\left(\hat{t}_p\right)\right]$, $\boldsymbol{0 < p < 1}$

It is advised that the observed Fisher information, $[\boldsymbol{I}_{2\times2}(.)]$ values for O_1, be maximized. For criterion O_2 and O_3, we also wish to reduce the determinant and trace of $[\boldsymbol{I}_{2\times2}(.)]^{-1}$. The best censoring strategy for multi-parameter distributions may be selected using scale-invariant criteria. While dealing with unknown multi-parameter distributions makes it more challenging to compare the two Fisher information matrices, dealing with single-parameter distributions allows for the use of scale-invariant criteria to compare a variety of criteria O_4. The logarithmic MLE of the $p-th$ quantile, $\boldsymbol{log}\left(\hat{t}_p\right)$, tends to have a variance that is minimized by the p-dependent criterion O_4. As a result, the logarithm of the KMKu distribution for time $\hat{t}_p$ may be calculated using

$$log\left(\hat{t}_p\right) = \left\{1 - \left[\ln\left(\mathrm{e}\left(1 - p\left(\frac{e-1}{e}\right)\right)\right)\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\theta}}, 0 < p < 1,$$

By using the delta technique to solve for Equation (4), the estimate of the variance for the $log(\hat{t}_p)$ of the KMKu distribution is given as

$$Var(log(\hat{t}_p)) = [\nabla log(\hat{t}_p)]^T I^{-1}_{2\times 2}(\hat{\theta},\ \hat{\beta})[\nabla log(\hat{t}_p)],$$

where

$$[\nabla log(\hat{t}_p)]^T = \left[\frac{\partial}{\partial\theta} log(\hat{t}_p), \frac{\partial}{\partial\beta} log(\hat{t}_p)\right]_{(\theta=\hat{\theta},\beta=\hat{\beta})}$$

$$P(R_1 = \mathfrak{K}_1) = \binom{n-s}{\mathfrak{K}_1} r^{\mathfrak{K}_1}(1-r)^{n-s-\mathfrak{K}_1}.$$

while $i = 2, 3, \ldots, s-1$. The maximum value of the O_1 criterion and the lowest value of $O_i,\ i = 2, 3, 4$, correspond to the best censoring. On the other hand, the greatest value of the O_1 criterion and the lowest value of the $O_i,\ i = 2, 3, 4$ criterion correspond to the best censoring.

6. Simulation

Using different combinations of $T_i; i = 1, 2$ (threshold points), n (sample size), s (size of censored sample), and R (censored removal), Monte-Carlo (MC) simulations were carried out to assess the true performance of the acquired point and interval estimators of $\theta, \beta,\ R(T)$, and $h(T)$. To establish this goal, for KMKu(1.4, 1.5), KMKu(1.4, 0.5), and KMKu(0.4, 0.5), we replicated the GPHCS-T2 mechanism 1000 times. Taking $(T_1, T_2) = (0.6,\ 0.85)$, two different choices of n and s were used as (n = 30, 50, 100), and the choices of s were used as (s = 20, 25) at n = 30, (s = 35, 45) at n = 50, and (s = 70, 90) at n = 100. At T_1 = 0.6, the true values of $R(T_1)$ and $h(T_1)$ were 0.4278 and 1.4899, respectively. At T_2 = 0.85, the true values of $R(T_2)$ and $h(T_2)$ were 0.2526 and 3.3106, respectively.

Additionally, by utilizing the binomial elimination distribution and taking into account different censoring schemes for each combination of s and n, the following is conducted: according to the following probability mass function, the number of units removed at each failure time is expected to follow a binomial distribution.

$$P(R_i = \mathfrak{K}_i | R_{i-1} = \mathfrak{K}_{i-1}, \ldots, R_1 = \mathfrak{K}_1) = \binom{n-s-\sum_{j=1}^{i-1}\mathfrak{K}_j}{\mathfrak{K}_i} r^{\mathfrak{K}_i}(1-r)^{n-s-\sum_{j=1}^{i}\mathfrak{K}_j}.$$

Additionally, assume that for any i, R_i is independent of X_i. In light of this, the likelihood function can be written as follows:

$$L(x_i, \beta,\ \theta,\ r) = L_1(x_i, \beta,\ \theta | R = \mathfrak{K}) P(R = \mathfrak{K}),$$

where

$$P(R = \mathfrak{K}) = P(R_1 = \mathfrak{K}_1, R_2 = \mathfrak{K}_2, \ldots, R_{s-1} = \mathfrak{K}_{s-1}) = P(R_{s-1} = \mathfrak{K}_{s-1} | R_{s-2} = \mathfrak{K}_{s-2}, \ldots, R_1 = \mathfrak{K}_1) \times$$
$$P(R_{s-2} = \mathfrak{K}_{s-2} | R_{s-3} = \mathfrak{K}_{s-3}, \ldots, R_1 = \mathfrak{K}_1) \ldots P(R_2 = \mathfrak{K}_2 | R_1 = \mathfrak{K}_1) P(R_1 = \mathfrak{K}_1).$$

That is,

$$P(R = \mathfrak{K}) = \frac{(n-s)!}{(n-s-\sum_{i=1}^{s-1}\mathfrak{K}_i)! \prod_{i=1}^{s-1}\mathfrak{K}_i} r^{\sum_{i=1}^{s-1}\mathfrak{K}_i}(1-r)^{(s-1)(n-s)-\sum_{i=1}^{s-1}(s-i)\mathfrak{K}_i},$$

where the GPHCS-T2-based KMKu distribution's parameters do not affect the binomial parameter r (Independent). We chose the binomial parameter r with varied values of 0.3 and 0.8.

The MLEs and 95% ACI estimates of $\theta, \beta,\ R(t)$, and $h(t)$ were assessed after 1000 GPHCS-T2 samples had been gathered using R 4.2.2 programming software and the "maxLik" library. We simulated 12,000 MCMC samples and omitted the first 2000 iterations as burn-in to obtain the Bayes point estimates along with their HPD interval estimates of the same unknown parameters using the "coda" library in the R 4.2.2 programming language. The estimates and their variances were equated with the Fisher information matrix of θ and β to produce the ML estimator, which it denoted as elective hyper-parameters, and this was contributed by [25]. This process allowed for the extraction of the hyper-parameters of the informative priors.

Some observations from Tables 3–5 include

Table 3. Bias, MSE, WCI, and CP for parameters and reliability measures: $\beta = 1.4$, $\theta = 1.5$.

						MLE					Bayesian	
n	**r**	**s**		**Bias**	**MSE**	**WACI**	**CP**	**Optimality**		**Bias**	**MSE**	**WCCI**
30	0.3	20	β	0.3647	0.96096	3.5687	95.8%	O_1	0.862132	0.0495	0.03658	0.9815
			θ	0.3083	0.30741	1.8074	95.3%	O_2	0.050431	0.0652	0.0178	0.6268
			$R(0.6)$	−1.0730	0.01021	3.5687	95.8%	O_3	24.39956	−1.0929	0.00159	0.9815
			$H(0.6)$	2.6011	2.15503	1.8074	95.3%	O_4	0.397259	2.2794	0.15897	0.6268
			$R(0.85)$	−1.3190	0.00311	3.5687	95.8%			−1.3269	0.00026	0.9815
			$H(0.85)$	9.8905	28.98923	1.8074	95.3%			8.2579	1.30669	0.6268
		25	β	0.3133	0.68518	3.0049	96.0%	O_1	0.603238	0.0348	0.01182	0.6606
			θ	0.1931	0.20528	1.6075	95.7%	O_2	0.028622	0.0196	0.00163	0.4069
			$R(0.6)$	−1.0962	0.00644	3.0049	96.0%	O_3	29.16632	−1.1016	0.00050	0.6606
			$H(0.6)$	2.6177	1.64301	1.6075	95.7%	O_4	0.637278	2.2792	0.05406	0.4069
			$R(0.85)$	−1.3278	0.00190	3.0049	96.0%			−1.3295	0.00009	0.6606
			$H(0.85)$	9.6683	20.97684	1.6075	95.7%			8.1852	0.42306	0.4069
	0.8	20	β	0.4478	1.11270	3.7458	95.6%	O_1	0.897504	0.0686	0.04368	1.0098
			θ	0.2881	0.29977	1.8260	95.3%	O_2	0.054545	0.0541	0.00853	0.6131
			$R(0.6)$	−1.0925	0.00827	3.7458	95.6%	O_3	23.77178	−1.0994	0.00147	1.0098
			$H(0.6)$	2.7704	2.38574	1.8260	95.3%	O_4	0.410967	2.3269	0.17886	0.6131
			$R(0.85)$	−1.3285	0.00217	3.7458	95.6%			−1.3295	0.00025	1.0098
			$H(0.85)$	10.3764	33.31616	1.8260	95.3%			8.3746	1.53529	0.6131
		25	β	0.3005	0.65995	2.9601	96.2%	O_1	0.588428	0.0312	0.00897	0.6494
			θ	0.1892	0.20103	1.5942	95.3%	O_2	0.02822	0.0199	0.00162	0.4063
			$R(0.6)$	−1.0954	0.00634	2.9601	96.2%	O_3	28.83162	−1.1009	0.00041	0.6494
			$H(0.6)$	2.5961	1.54340	1.5942	95.3%	O_4	0.290327	2.2713	0.04112	0.4063
			$R(0.85)$	−1.3273	0.00191	2.9601	96.2%			−1.3293	0.00008	0.6494
			$H(0.85)$	9.5939	20.02147	1.5942	95.3%			8.1634	0.31889	0.4063
50	0.3	35	β	0.2321	0.33855	2.0926	95.2%	O_1	0.3524	0.0219	0.00682	0.2936
			θ	0.2206	0.16385	1.3310	94.8%	O_2	0.0109	0.0389	0.00404	0.1817
			$R(0.6)$	0.0150	0.00491	0.2684	94.9%	O_3	40.4391	0.0044	0.00056	0.0984
			$H(0.6)$	0.2660	0.86775	3.5013	95.4%	O_4	0.4010	0.0230	0.03428	0.7479
			$R(0.85)$	0.0019	0.00151	0.1522	95.8%			0.0005	0.00010	0.0418
			$H(0.85)$	1.2233	10.31472	11.6464	95.4%			0.1173	0.24180	1.8061
		45	β	0.1009	0.17989	1.6157	95.3%	O_1	0.2308	0.0127	0.00185	0.1390
			θ	0.0688	0.07292	1.0241	95.5%	O_2	0.0051	0.0080	0.00043	0.0697
			$R(0.6)$	0.0038	0.00306	0.2163	95.2%	O_3	52.6779	−0.0009	0.00013	0.0468
			$H(0.6)$	0.1237	0.55947	2.8931	95.4%	O_4	0.3850	0.0228	0.00944	0.3526
			$R(0.85)$	0.0023	0.00099	0.1228	95.9%			−0.0008	0.00002	0.0202
			$H(0.85)$	0.5401	5.79962	9.2044	95.3%			0.0729	0.06640	0.8489
	0.8	35	β	0.2292	0.33120	2.0704	95.7%	O_1	0.3413	0.0302	0.00762	0.2866
			θ	0.1710	0.13353	1.2666	94.8%	O_2	0.0103	0.0289	0.00277	0.1549
			$R(0.6)$	0.0043	0.00413	0.2516	94.5%	O_3	40.8715	0.0003	0.00049	0.0905
			$H(0.6)$	0.2984	0.88222	3.4929	95.3%	O_4	0.5510	0.0472	0.03668	0.6994
			$R(0.85)$	−0.0013	0.00132	0.1424	95.8%			−0.0010	0.00009	0.0382
			$H(0.85)$	1.2328	10.27127	11.6023	95.7%			0.1697	0.26968	1.7342
		45	β	0.1024	0.17088	1.6527	95.8%	O_1	0.2313	0.0132	0.00183	0.1361
			θ	0.0614	0.07091	1.0767	95.4%	O_2	0.0052	0.0077	0.00047	0.0735
			$R(0.6)$	0.0020	0.00301	0.2196	95.0%	O_3	52.8830	−0.0011	0.00013	0.0450
			$H(0.6)$	0.1288	0.55691	2.9153	95.4%	O_4	0.3953	0.0242	0.00934	0.3352
			$R(0.85)$	0.0018	0.00100	0.1240	95.6%			−0.0009	0.00002	0.0195
			$H(0.85)$	0.5493	5.70139	9.3735	95.9%			0.0762	0.06541	0.8280

Table 3. *Cont.*

n	r	s		MLE Bias	MSE	WACI	CP	Optimality		Bayesian Bias	MSE	WCCI
100	0.3	70	β	0.1428	0.13884	1.3498	95.3%	O_1	0.1469	0.0129	0.00257	0.1755
			θ	0.1269	0.06502	0.8674	95.3%	O_2	0.0020	0.0189	0.00121	0.1107
			$R(0.6)$	0.0045	0.00218	0.1823	94.6%	O_3	80.6294	0.0016	0.00021	0.0575
			$H(0.6)$	0.1901	0.40464	2.3807	95.0%	O_4	0.6115	0.0162	0.01332	0.4378
			$R(0.85)$	−0.0017	0.00065	0.0999	96.1%			−0.0001	0.00004	0.0249
			$H(0.85)$	0.7729	4.38212	7.6300	95.3%			0.0702	0.09184	1.0646
		90	β	0.0668	0.07946	1.0740	95.7%	O_1	0.1065	0.0062	0.00054	0.0852
			θ	0.0497	0.03711	0.7299	95.5%	O_2	0.0011	0.0045	0.00020	0.0495
			$R(0.6)$	0.0013	0.00162	0.1577	95.0%	O_3	102.5854	−0.0003	0.00005	0.0275
			$H(0.6)$	0.0898	0.25612	1.9533	94.7%	O_4	0.8814	0.0107	0.00299	0.2093
			$R(0.85)$	0.0000	0.00050	0.0880	95.0%			−0.0004	0.00001	0.0120
			$H(0.85)$	0.3625	2.56087	6.1131	95.6%			0.0351	0.01967	0.5192
	0.8	70	β	0.1309	0.13318	1.3361	95.4%	O_1	0.1442	0.0146	0.00261	0.1649
			θ	0.1046	0.06010	0.8696	94.9%	O_2	0.0020	0.0157	0.00107	0.1047
			$R(0.6)$	0.0020	0.00220	0.1838	94.7%	O_3	80.5925	0.0005	0.00021	0.0568
			$H(0.6)$	0.1803	0.40288	2.3868	95.0%	O_4	0.6090	0.0220	0.01361	0.4150
			$R(0.85)$	−0.0018	0.00065	0.1001	95.9%			−0.0005	0.00004	0.0241
			$H(0.85)$	0.7125	4.25301	7.5901	95.2%			0.0816	0.09367	1.0046
		90	β	0.0505	0.06859	1.0078	95.4%	O_1	0.1039	0.0057	0.00047	0.0761
			θ	0.0318	0.03404	0.7127	94.5%	O_2	0.0011	0.0036	0.00019	0.0488
			$R(0.6)$	0.0004	0.00155	0.1544	95.2%	O_3	103.7077	−0.0004	0.00005	0.0271
			$H(0.6)$	0.0682	0.23049	1.8638	95.7%	O_4	0.7099	0.0103	0.00266	0.1933
			$R(0.85)$	0.0003	0.00047	0.0853	95.4%			−0.0004	0.00001	0.0113
			$H(0.85)$	0.2743	2.23659	5.7659	95.4%			0.0327	0.01719	0.4665

Table 4. Bias, MSE, WCI and CP for parameters and Reliability measures: $\beta = 1.4$, $\theta = 0.5$.

n	r	s		MLE Bias	MSE	WACI	CP	Optimality		Bayesian Bias	MSE	WCCI
30	0.3	20	β	0.2883	0.66402	2.9892	95.1%	O_1	0.5529	0.0373	0.0172	0.9061
			θ	0.0923	0.03017	0.5771	96.5%	O_2	0.0045	0.0191	0.0010	0.2005
			$R(0.6)$	−1.3142	0.00273	2.9892	95.1%	O_3	156.4684	−1.3223	0.0002	0.9061
			$H(0.6)$	4.4198	3.85075	0.5771	96.5%	O_4	3.6685	3.8534	0.1199	0.2005
			$R(0.85)$	−1.3780	0.00051	2.9892	95.1%			−1.3834	0.0000	0.9061
			$H(0.85)$	11.2462	28.57208	0.5771	96.5%			9.6102	0.7761	0.2005
		25	β	0.2330	0.41596	2.3587	95.6%	O_1	0.3694	0.0242	0.0044	0.6032
			θ	0.0565	0.01912	0.4950	95.3%	O_2	0.0024	0.0058	0.0001	0.1311
			$R(0.6)$	−1.3222	0.00166	2.3587	95.6%	O_3	187.5492	−1.3244	0.0001	0.6032
			$H(0.6)$	4.3140	2.50974	0.4950	95.3%	O_4	4.2533	3.8249	0.0314	0.1311
			$R(0.85)$	−1.3812	0.00029	2.3587	95.6%			−1.3839	0.0000	0.6032
			$H(0.85)$	10.8980	18.07372	0.4950	95.3%			9.5259	0.2003	0.1311
	0.8	20	β	0.3112	0.59476	2.7674	95.4%	O_1	0.5183	0.0446	0.0176	0.9133
			θ	0.0862	0.02909	0.5773	95.3%	O_2	0.0042	0.0168	0.0009	0.1972
			$R(0.6)$	−1.3207	0.00218	2.7674	95.4%	O_3	156.8018	−1.3236	0.0002	0.9133
			$H(0.6)$	4.4902	3.51118	0.5773	95.3%	O_4	1.9839	3.8741	0.1242	0.1972
			$R(0.85)$	−1.3808	0.00036	2.7674	95.4%			−1.3838	0.0000	0.9133
			$H(0.85)$	11.4035	25.72157	0.5773	95.3%			9.6601	0.7988	0.1972
		25	β	0.2760	0.46980	2.4606	94.4%	O_1	0.3856	0.0304	0.0058	0.6125
			θ	0.0560	0.01951	0.5019	95.5%	O_2	0.0025	0.0054	0.0001	0.1304
			$R(0.6)$	−1.3254	0.00173	2.4606	94.4%	O_3	188.0939	−1.3251	0.0001	0.6125
			$H(0.6)$	4.4269	2.85414	0.5019	95.5%	O_4	10.4186	3.8416	0.0413	0.1304
			$R(0.85)$	−1.3822	0.00027	2.4606	94.4%			−1.3842	0.0000	0.6125
			$H(0.85)$	11.1889	20.49050	0.5019	95.5%			9.5678	0.2637	0.1304

Table 4. *Cont.*

				MLE						Bayesian		
n	r	s		Bias	MSE	WACI	CP	Optimality		Bias	MSE	WCCI
50	0.3	35	β	0.1928	0.26587	1.8755	95.3%	O_1	0.2300	0.0209	0.00625	0.2453
			θ	0.0636	0.01464	0.4038	95.4%	O_2	0.0010	0.0111	0.00034	0.0531
			$R(0.6)$	0.0022	0.00138	0.1454	96.0%	O_3	266.3510	0.0003	0.00009	0.0379
			$H(0.6)$	0.4533	1.64429	4.7044	95.0%	O_4	4.7886	0.0504	0.04467	0.6665
			$R(0.85)$	0.0017	0.00022	0.0583	95.2%			−0.0001	0.00001	0.0118
			$H(0.85)$	1.2631	11.59297	12.4008	95.4%			0.1365	0.28360	1.6561
		45	β	0.1025	0.15357	1.4834	95.2%	O_1	0.1528	0.0111	0.00136	0.1177
			θ	0.0252	0.00856	0.3492	95.8%	O_2	0.0005	0.0027	0.00005	0.0238
			$R(0.6)$	0.0010	0.00090	0.1177	95.9%	O_3	332.8660	−0.0007	0.00002	0.0179
			$H(0.6)$	0.2438	0.97587	3.7545	95.7%	O_4	9.7083	0.0284	0.00981	0.3228
			$R(0.85)$	0.0015	0.00015	0.0473	95.2%			−0.0003	0.00000	0.0057
			$H(0.85)$	0.6769	6.72601	9.8189	95.4%			0.0737	0.06177	0.7925
	0.8	35	β	0.2216	0.30042	1.9661	95.3%	O_1	0.2360	0.0286	0.00733	0.2601
			θ	0.0591	0.01448	0.4111	95.5%	O_2	0.0010	0.0097	0.00031	0.0514
			$R(0.6)$	−0.0014	0.00128	0.1404	95.7%	O_3	264.3238	−0.0009	0.00009	0.0380
			$H(0.6)$	0.5296	1.84331	4.9029	95.3%	O_4	6.8233	0.0715	0.05206	0.7134
			$R(0.85)$	0.0005	0.00020	0.0555	94.9%			−0.0004	0.00001	0.0117
			$H(0.85)$	1.4571	13.09846	12.9931	95.3%			0.1886	0.33239	1.7468
		45	β	0.1251	0.17500	1.5656	94.3%	O_1	0.1585	0.0147	0.00189	0.1420
			θ	0.0244	0.00782	0.3334	95.3%	O_2	0.0005	0.0023	0.00004	0.0228
			$R(0.6)$	−0.0008	0.00097	0.1224	96.3%	O_3	331.8112	−0.0011	0.00003	0.0200
			$H(0.6)$	0.3053	1.12358	3.9811	94.4%	O_4	3.4415	0.0380	0.01355	0.3838
			$R(0.85)$	0.0010	0.00015	0.0485	95.8%			−0.0004	0.00000	0.0064
			$H(0.85)$	0.8313	7.71142	10.3916	94.4%			0.0977	0.08563	0.9557
100	0.3	70	β	0.1020	0.10312	1.1941	96.0%	O_1	0.0930	0.0071	0.00176	0.1407
			θ	0.0390	0.00642	0.2745	95.2%	O_2	0.0002	0.0063	0.00013	0.0334
			$R(0.6)$	0.0012	0.00065	0.0997	94.9%	O_3	526.4983	0.0006	0.00003	0.0220
			$H(0.6)$	0.2402	0.65129	3.0217	95.9%	O_4	3.0162	0.0158	0.01276	0.3863
			$R(0.85)$	0.0007	0.00010	0.0386	95.3%			0.0001	0.00002	0.0070
			$H(0.85)$	0.6667	4.49884	7.8970	96.1%			0.0451	0.07991	0.9471
		90	β	0.0661	0.06870	0.9948	95.5%	O_1	0.0688	0.0055	0.00046	0.0784
			θ	0.0164	0.00369	0.2292	95.4%	O_2	0.0001	0.0014	0.00002	0.0160
			$R(0.6)$	−0.0005	0.00048	0.0858	95.7%	O_3	654.6192	−0.0003	0.00001	0.0117
			$H(0.6)$	0.1605	0.44507	2.5397	95.2%	O_4	6.7361	0.0139	0.00338	0.2125
			$R(0.85)$	0.0003	0.00007	0.0332	95.6%			−0.0001	0.00001	0.0038
			$H(0.85)$	0.4370	3.01381	6.5894	95.3%			0.0361	0.02109	0.5289
	0.8	70	β	0.1036	0.10480	1.2028	95.8%	O_1	0.0929	0.0115	0.00204	0.1556
			θ	0.0320	0.00609	0.2791	95.3%	O_2	0.0002	0.0050	0.00011	0.0339
			$R(0.6)$	−0.0002	0.00064	0.0994	95.7%	O_3	525.8864	−0.0002	0.00004	0.0232
			$H(0.6)$	0.2484	0.66914	3.0567	95.8%	O_4	4.7902	0.0283	0.01472	0.4259
			$R(0.85)$	0.0003	0.00009	0.0381	96.1%			−0.0002	0.00002	0.0073
			$H(0.85)$	0.6812	4.58927	7.9658	95.9%			0.0754	0.09250	1.0493
		90	β	0.0652	0.07266	1.0258	95.7%	O_1	0.0689	0.0060	0.00052	0.0794
			θ	0.0161	0.00373	0.2310	95.1%	O_2	0.0001	0.0013	0.00002	0.0161
			$R(0.6)$	−0.0001	0.00052	0.0894	94.8%	O_3	650.3684	−0.0004	0.00001	0.0124
			$H(0.6)$	0.1583	0.47338	2.6260	95.6%	O_4	6.4073	0.0153	0.00379	0.2188
			$R(0.85)$	0.0005	0.00008	0.0345	94.8%			−0.0002	0.00001	0.0040
			$H(0.85)$	0.4317	3.19473	6.8025	95.6%			0.0396	0.02362	0.5400

Table 5. Bias, MSE, WCI and CP for parameters and Reliability measures: $\beta = 0.4$, $\theta = 0.5$.

				MLE						Bayesian		
n	***r***	***s***		**Bias**	**MSE**	**WACI**	**CP**	**Optimality**		**Bias**	**MSE**	**WCCI**
30	0.3	20	β	0.0477	0.03572	0.7173	96.0%	O_1	0.0822	0.0069	0.0009	0.2488
			θ	0.1693	0.10228	1.0642	95.4%	O_2	0.0011	0.0454	0.0066	0.3352
			$R(0.6)$	0.0653	0.01134	0.7173	96.0%	O_3	0.0822	0.0392	0.0014	0.2488
			$H(0.6)$	1.0736	0.26208	1.0642	95.4%	O_4	0.0011	1.0039	0.0089	0.3352
			$R(0.85)$	−0.1242	0.00977	0.7173	96.0%			−0.1418	0.0008	0.2488
			$H(0.85)$	3.1086	1.57995	1.0642	95.4%			2.8625	0.0409	0.3352
		25	β	0.0420	0.02739	0.6279	95.8%	O_1	0.0591	0.0051	0.0003	0.1640
			θ	0.0938	0.05590	0.8511	94.9%	O_2	0.0006	0.0139	0.0009	0.2075
			$R(0.6)$	0.0413	0.00854	0.6279	95.8%	O_3	170.1243	0.0291	0.0003	0.1640
			$H(0.6)$	1.0797	0.21444	0.8511	94.9%	O_4	5.0156	1.0038	0.0029	0.2075
			$R(0.85)$	−0.1395	0.00729	0.6279	95.8%			−0.1478	0.0002	0.1640
			$H(0.85)$	3.0747	1.22571	0.8511	94.9%			2.8470	0.0134	0.2075
	0.8	20	β	0.0747	0.04126	0.7408	96.2%	O_1	0.0818	0.0116	0.0012	0.2625
			θ	0.1633	0.09437	1.0204	96.1%	O_2	0.0011	0.0378	0.0051	0.3251
			$R(0.6)$	0.0443	0.01003	0.7408	96.2%	O_3	131.1160	0.0330	0.0012	0.2625
			$H(0.6)$	1.1578	0.29506	1.0204	96.1%	O_4	3.2292	1.0197	0.0110	0.3251
			$R(0.85)$	−0.1436	0.00864	0.7408	96.2%			−0.1467	0.0007	0.2625
			$H(0.85)$	3.2918	1.79718	1.0204	96.1%			2.8933	0.0527	0.3251
		25	β	0.0589	0.02782	0.6120	95.7%	O_1	0.0592	0.0065	0.0003	0.1698
			θ	0.0991	0.04971	0.7834	95.8%	O_2	0.0006	0.0125	0.0006	0.2058
			$R(0.6)$	0.0296	0.00743	0.6120	95.7%	O_3	159.3168	0.0275	0.0003	0.1698
			$H(0.6)$	1.1330	0.21505	0.7834	95.8%	O_4	2.2242	1.0084	0.0034	0.2058
			$R(0.85)$	−0.1517	0.00618	0.6120	95.7%			−0.1491	0.0002	0.1698
			$H(0.85)$	3.1924	1.23457	0.7834	95.8%			2.8564	0.0157	0.2058
50	0.3	35	β	0.0369	0.01839	0.5118	95.6%	O_1	0.0387	0.0044	0.00032	0.0627
			θ	0.1055	0.04398	0.7108	95.9%	O_2	0.0002	0.0229	0.00165	0.1006
			$R(0.6)$	0.0183	0.00581	0.2903	95.5%	O_3	230.4277	0.0052	0.00044	0.0790
			$H(0.6)$	0.0795	0.14627	1.4672	95.2%	O_4	2.9344	0.0102	0.00326	0.2111
			$R(0.85)$	0.0086	0.00523	0.2817	95.6%			0.0021	0.00026	0.0627
			$H(0.85)$	0.2401	0.84128	3.4719	95.8%			0.0328	0.01498	0.4164
		45	β	0.0292	0.01287	0.4300	95.8%	O_1	0.0284	0.0031	0.00009	0.0311
			θ	0.0523	0.02265	0.5534	95.8%	O_2	0.0001	0.0065	0.00019	0.0406
			$R(0.6)$	0.0030	0.00414	0.2520	94.7%	O_3	287.7103	0.0000	0.00010	0.0390
			$H(0.6)$	0.0702	0.10813	1.2599	95.7%	O_4	3.0968	0.0088	0.00092	0.1034
			$R(0.85)$	−0.0003	0.00374	0.2397	94.6%			−0.0008	0.00006	0.0317
			$H(0.85)$	0.1885	0.59457	2.9324	95.7%			0.0220	0.00423	0.2099
	0.8	35	β	0.0451	0.01827	0.4997	95.6%	O_1	0.0375	0.0065	0.00039	0.0608
			θ	0.0916	0.03438	0.6323	95.2%	O_2	0.0002	0.0181	0.00104	0.0866
			$R(0.6)$	0.0071	0.00524	0.2826	95.7%	O_3	223.9769	0.0018	0.00038	0.0752
			$H(0.6)$	0.1093	0.14684	1.4405	95.6%	O_4	1.6721	0.0177	0.00378	0.2016
			$R(0.85)$	−0.0004	0.00484	0.2727	95.5%			−0.0005	0.00024	0.0604
			$H(0.85)$	0.2965	0.83295	3.3853	95.6%			0.0468	0.01780	0.4059
		45	β	0.0295	0.01341	0.4392	95.7%	O_1	0.0280	0.0034	0.00010	0.0323
			θ	0.0448	0.02086	0.5386	96.2%	O_2	0.0001	0.0054	0.00016	0.0383
			$R(0.6)$	0.0003	0.00415	0.2525	93.7%	O_3	288.9783	−0.0006	0.00009	0.0395
			$H(0.6)$	0.0723	0.11132	1.2774	96.0%	O_4	3.5188	0.0098	0.00097	0.1069
			$R(0.85)$	−0.0018	0.00383	0.2426	94.8%			−0.0012	0.00006	0.0328
			$H(0.85)$	0.1896	0.61493	2.9842	95.8%			0.0236	0.00451	0.2164

Table 5. *Cont.*

n	r	s		MLE						Bayesian		
				Bias	MSE	WACI	CP	Optimality		Bias	MSE	WCCI
100	0.3	70	β	0.0267	0.00855	0.3473	95.7%	O_1	0.0171	0.0024	0.00012	0.0376
			θ	0.0690	0.01783	0.4484	95.0%	O_2	0.0000	0.0117	0.00043	0.0568
			$R(0.6)$	0.0090	0.00284	0.2061	95.8%	O_3	442.4535	0.0026	0.00016	0.0489
			$H(0.6)$	0.0645	0.07290	1.0283	95.1%	O_4	2.9947	0.0057	0.00125	0.1240
			$R(0.85)$	0.0020	0.00263	0.2008	95.7%			0.0010	0.00010	0.0387
			$H(0.85)$	0.1797	0.40180	2.3841	95.5%			0.0177	0.00568	0.2530
		90	β	0.0129	0.00536	0.2827	94.7%	O_1	0.0128	0.0013	0.00003	0.0181
			θ	0.0256	0.00907	0.3597	95.0%	O_2	0.0000	0.0029	0.00006	0.0233
			$R(0.6)$	0.0024	0.00205	0.1771	95.8%	O_3	572.9885	0.0000	0.00003	0.0222
			$H(0.6)$	0.0310	0.04803	0.8508	94.7%	O_4	2.3973	0.0038	0.00031	0.0595
			$R(0.85)$	0.0007	0.00187	0.1696	94.9%			−0.0003	0.00002	0.0179
			$H(0.85)$	0.0838	0.25437	1.9506	94.9%			0.0096	0.00137	0.1225
	0.8	70	β	0.0274	0.00795	0.3328	95.3%	O_1	0.0168	0.0034	0.00013	0.0392
			θ	0.0565	0.01500	0.4261	95.1%	O_2	0.0000	0.0098	0.00034	0.0558
			$R(0.6)$	0.0035	0.00280	0.2070	94.8%	O_3	441.2820	0.0011	0.00015	0.0484
			$H(0.6)$	0.0693	0.06869	0.9913	94.9%	O_4	3.2956	0.0092	0.00133	0.1304
			$R(0.85)$	−0.0016	0.00254	0.1976	95.5%			−0.0002	0.00010	0.0390
			$H(0.85)$	0.1834	0.37293	2.2845	94.9%			0.0243	0.00610	0.2614
		90	β	0.0109	0.00515	0.2782	95.1%	O_1	0.0127	0.0013	0.00003	0.0183
			θ	0.0212	0.00904	0.3635	95.5%	O_2	0.0000	0.0026	0.00005	0.0248
			$R(0.6)$	0.0022	0.00212	0.1802	95.2%	O_3	577.1218	0.0000	0.00003	0.0236
			$H(0.6)$	0.0253	0.04644	0.8393	94.9%	O_4	2.5652	0.0037	0.00028	0.0612
			$R(0.85)$	0.0011	0.00191	0.1712	94.9%			−0.0004	0.00002	0.0190
			$H(0.85)$	0.0693	0.24508	1.9224	94.9%			0.0092	0.00125	0.1231

- The key general finding is that the suggested values for $\theta, \beta,\ R(t)$, and $h(t)$ performed well.
- All estimations of $\theta, \beta, R(t)$, and $h(t)$ functioned satisfactorily as n(or s) grew.
- In most cases, the MSE, Bias, and WCI of all unknown parameters fell while their CPs grew as (T_1, T_2) increased.
- Due to the gamma information, the Bayes estimates of $\theta, \beta,\ R(t)$, and $h(t)$ behaved more predictably than the other estimates. Regarding credible HPD intervals, the same statement might be made.
- When the parameter of binomial r was increased, the proposed estimates of $\theta, \beta,\ R(t)$, and $h(t)$ performed better in most cases.

7. Application

The data set, which has been examined by [11], had 30 assessments of the tensile strength of polyester fibers. The following details are included in the data set: "0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926". For data on the strength of polyester fibers, where the Kolmogorov-Smirnov distance is 0.0569 with a p-value of 0.9999, [11] explores the MLE of this model using several measures of goodness-of-fit. The Kolmogorov-Smirnov test findings showed that the KMKu distribution fits the data on polyester fiber strength.

Two GPHCS-T2 samples with $s = 20$ and 25 were produced from the tensile strength of polyester fibers data in order to explain the proposed estimation methodology. The binomial removal has been used to obtain the GPHCS-T2 samples with different parameters of $p = 0.2$, 0.5, and 0.8. Table 6 lists the computed $R(t)$ and $h(t)$ at $t = 0.6$ and 0.85 by maximum likelihood estimates (MLE) and Bayesian estimation, respectively, along with their standard error (SE). By repeating the MCMC sampler 12,000 times and disregarding the first 2000 times as burn-in, the Bayes estimates (with their SE) were evaluated using incorrect gamma priors and are also provided in Table 4 because there was no prior knowledge about the unknown KMKu parameters θ, and β from the given data set. In order to estimate unknown hyperparameters for the computational logic, elective hyperparameters were employed. In terms of the minimum standard error and interval width values, it is evident from Table 6 that the MCMC estimates of $\theta, \beta, R(t)$, and $h(t)$performed better than the others.

Table 6. MLE and Bayesian estimation.

			MLE				Bayesian			
s	***p***		**Estimates**	**SE**	***R*(0.6)** / ***H*(0.6)**	***R*(0.85)** / ***H*(0.85)**	**Estimates**	**SE**	***R*(0.6)** / ***H*(0.6)**	***R*(0.85)** / ***H*(0.85)**
20	0.2	β	0.8884	0.5264	0.3253	0.1189	0.9884	0.3296	0.3047	0.1017
		θ	1.0033	0.2514	2.7480	6.4884	1.0556	0.2365	2.9811	7.1010
	0.5	β	1.5376	0.5760	0.2111	0.0436	1.3577	0.3243	0.2505	0.0608
		θ	1.2293	0.2692	4.1919	10.4235	1.2399	0.1761	3.7759	9.3192
	0.8	β	1.5404	0.5751	0.2091	0.0431	1.6921	0.3877	0.1800	0.0324
		θ	1.2231	0.2677	4.2022	10.4439	1.2090	0.1681	4.5521	11.3874
25	0.2	β	1.4928	0.4925	0.1858	0.0392	1.5520	0.3081	0.1795	0.0360
		θ	1.0776	0.2203	4.1819	10.2139	1.0966	0.1464	4.3074	10.5745
	0.5	β	1.5231	0.5001	0.1884	0.0389	1.5572	0.3157	0.1811	0.0362
		θ	1.1143	0.2294	4.2299	10.3859	1.1085	0.1507	4.3121	10.6013
	0.8	β	1.5221	0.4996	0.1883	0.0389	1.4511	0.3053	0.2044	0.0451
		θ	1.1129	0.2293	4.2285	10.3804	1.1252	0.1523	4.0571	9.9345

Figures 2–4 were created to examine the maximum values of the estimators by profile likelihood as well as the existence and uniqueness of the log-likelihood function by contour plot with regard to different d and q options based on GPHCS-T2 samples with s = 20 and distinct p = 0.2, 0.5, and 0.8, respectively. Figure 5 clearly shows that the MCMC technique converged favorably and that the recommended size of the burn-in sample was adequate to completely nullify the impact of the recommended beginning values. Figure 5 demonstrates that the estimated estimates of θ and β were roughly symmetrical for each sample when s = 20.

Figures 6–8 were created to examine the maximum values of the estimators by profile likelihood as well as the existence and uniqueness of the log-likelihood function by contour plot with regard to different d and q options based on GPHCS-T2 samples with s = 25 and distinct p = 0.2, 0.5, and 0.8, respectively. Figure 9 clearly shows that the MCMC technique converged favorably and that the recommended size of the burn-in sample was adequate to completely nullify the impact of the recommended beginning values. Figure 9 demonstrates that the estimated estimates of θ and β were roughly symmetrical for each sample when s = 25.

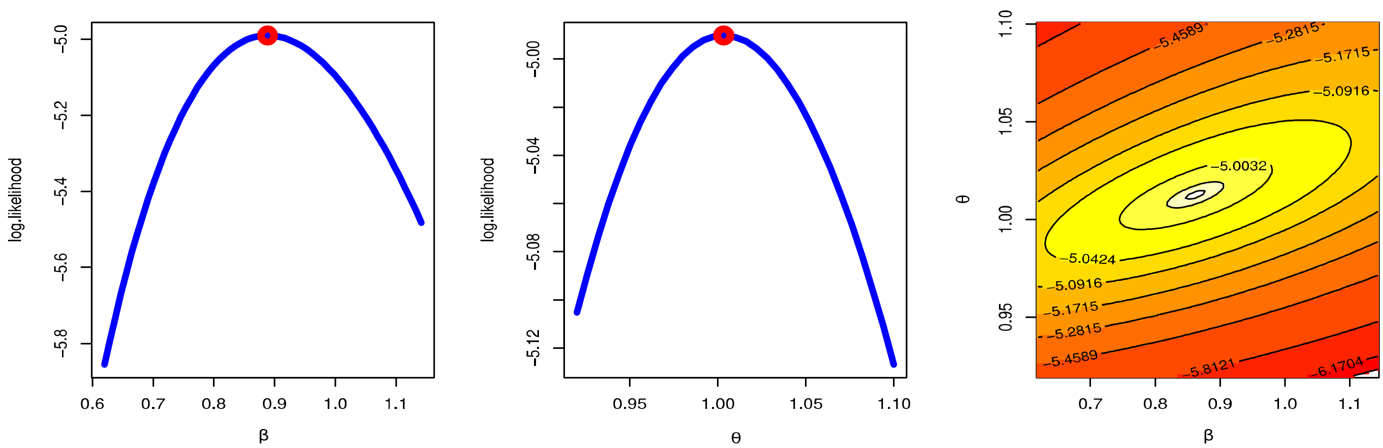

Figure 2. Profile likelihood and contour plot s = 20 p = 0.2.

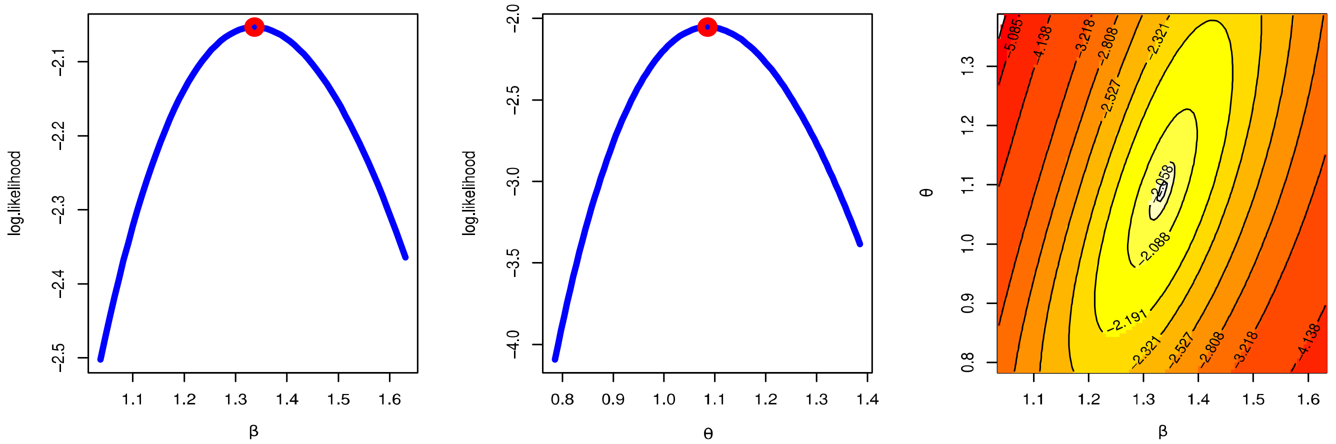

Figure 3. Profile likelihood and contour plot $s = 20$ $p = 0.5$.

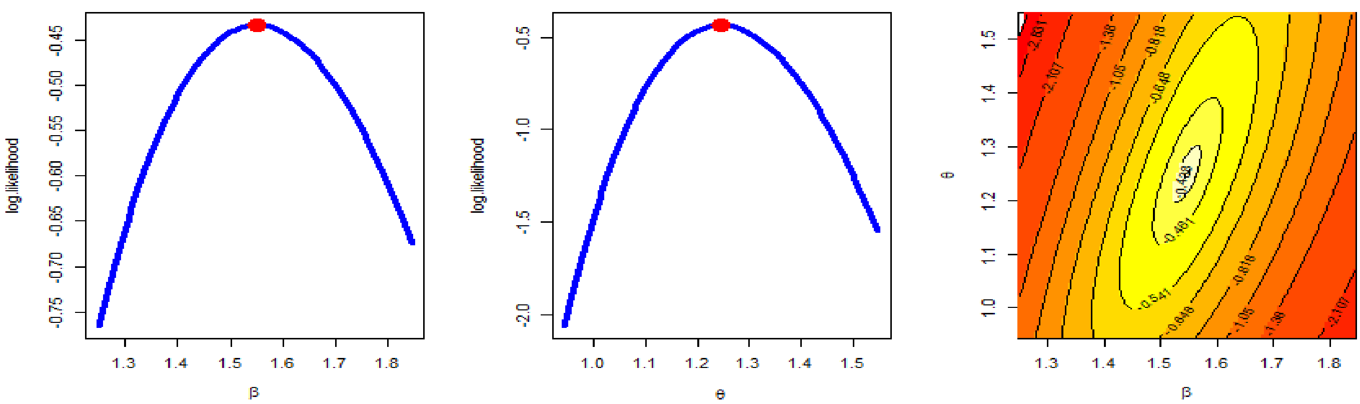

Figure 4. Profile likelihood and contour plot $s = 20$ $p = 0.8$.

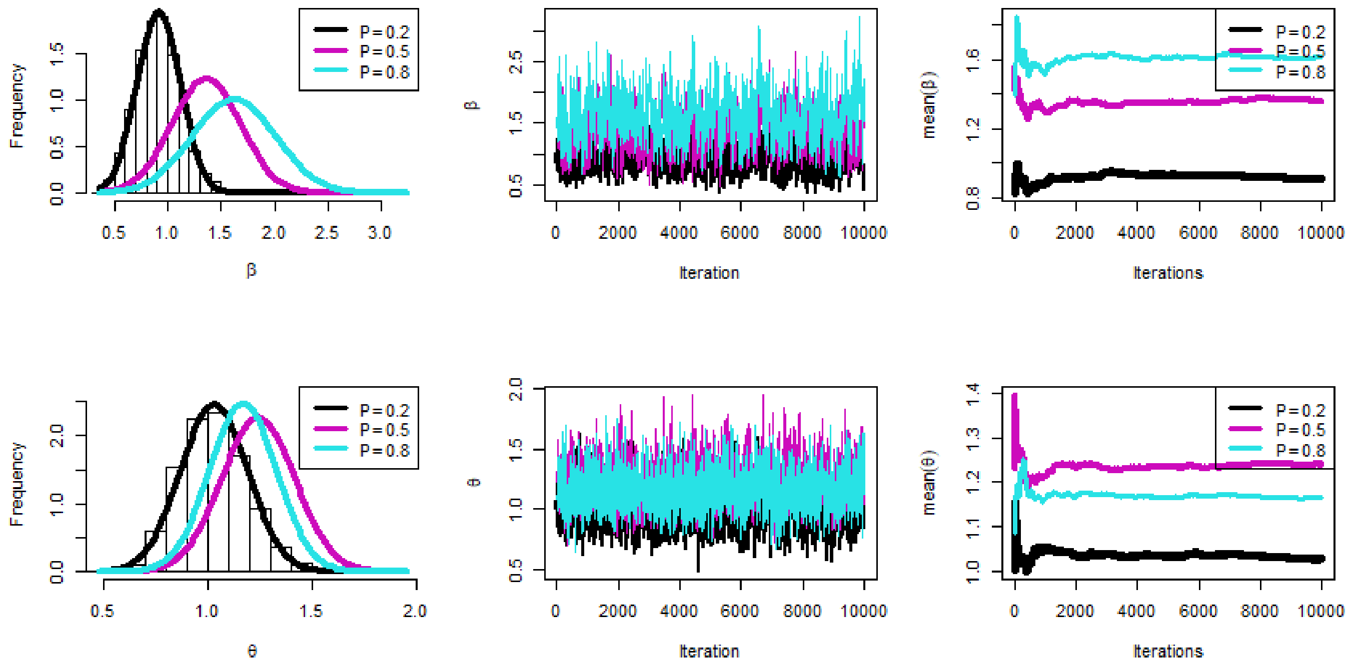

Figure 5. Bayesian checks when $s = 20$.

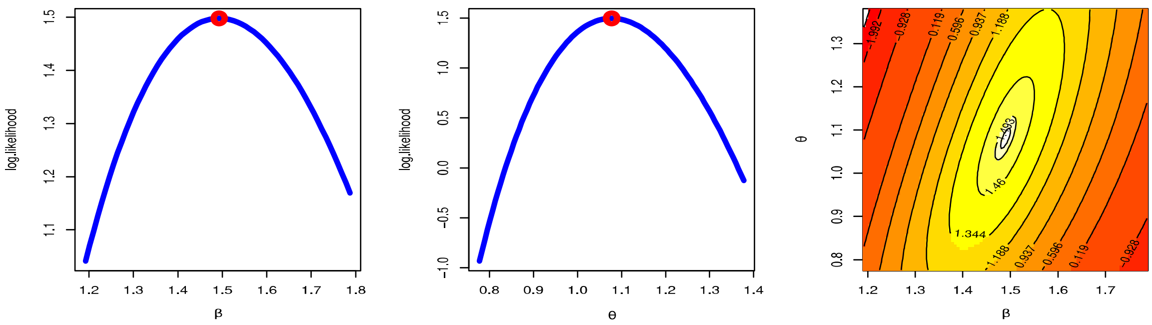

Figure 6. Profile likelihood and contour plot $s = 25$; $p = 0.2$.

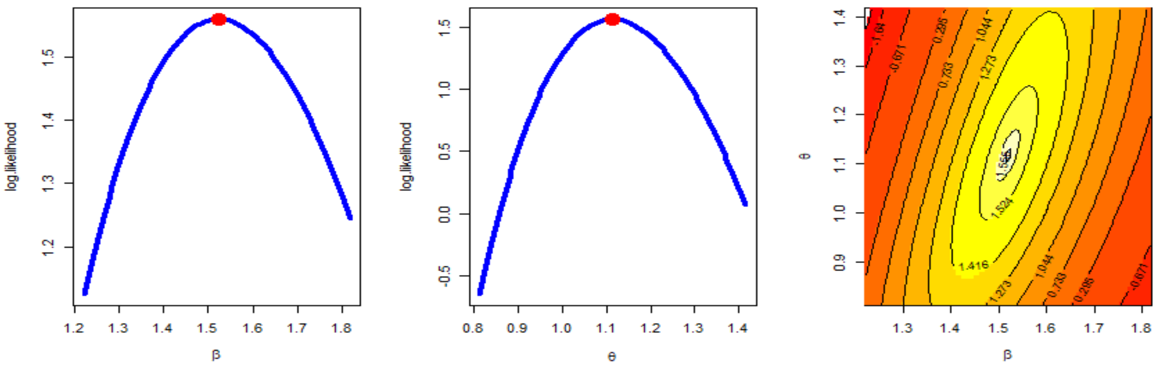

Figure 7. Profile likelihood and contour plot $s = 25$; $p = 0.5$.

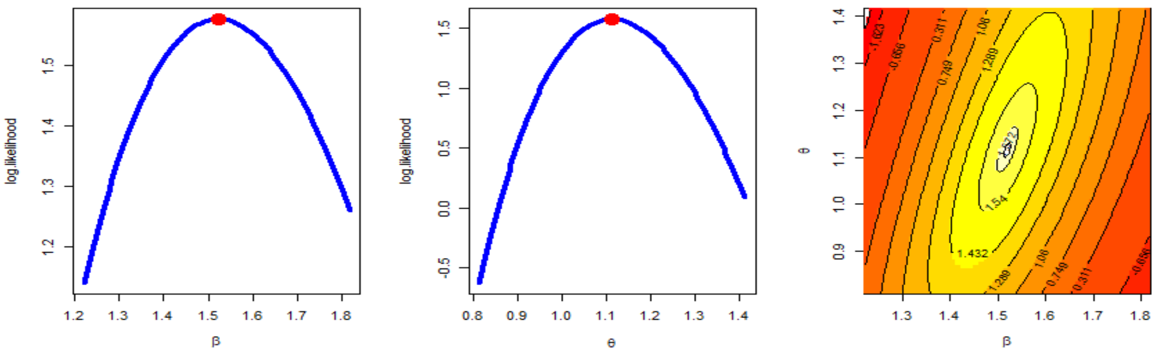

Figure 8. Profile likelihood and contour plot $s = 25$; $p = 0.8$.

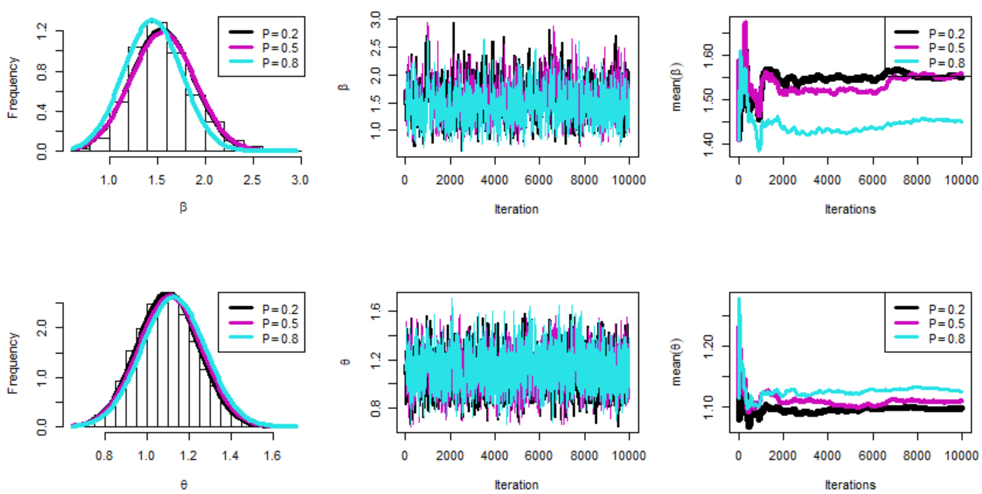

Figure 9. Bayesian checks when $s = 25$.

8. Conclusions and Discussion

This paper examines the reliability analysis of the unknown parameters, reliability, and hazard rate functions for the generalized type-II progressive hybrid censoring-based KMKu model. The "maxLik" package of the R programming language was used to compute the frequentist estimates with their asymptotic confidence intervals for the unknown parameters and any function of them. Since the likelihood function was produced in complex form, the posterior density function was obtained in nonlinear form. Consequently, the Bayesian estimates and the related HPD intervals were created using the Metropolis-Hastings technique and accounting for the squared error loss function. Numerous simulation experiments were run utilizing various total test unit choices, observed failure data, threshold times, and progressive censoring schemes in order to compare the behavior of the collected estimates. The outcomes demonstrated that the Bayes–MCMC strategy performed substantially better than the frequentist approach. Under generalized type-II progressive hybrid censoring, it was suggested to estimate the KMKu distribution's parameters, reliability, and hazard functions using the Bayesian MCMC paradigm. We believe that the technique and results described here will be helpful to reliability practitioners and that they will be used to inform future censoring tactics. The 30 assessments of the tensile strength of polyester fibers are used to demonstrate how the recommended strategies may be applied in real-world circumstances. The most important results can be summarized in the following points:

- The key general finding is that the suggested values for $\theta, \beta,\ R(t)$, and $h(t)$ performed well.
- All estimations of $\theta,\ \beta$, $R(t)$, and $h(t)$ functioned satisfactorily as n (or s) grew.
- In most cases, the MSE, Bias, and WCI of all unknown parameters fell while their CPs grew as (T_1, T_2) increased.
- Due to the gamma information, the Bayes estimates of $\theta, \beta, R(t)$, and $h(t)$ behaved more predictably than the other estimates. Regarding credible HPD intervals, the same statement might be made.
- In most cases, the proposed estimates of $\theta, \beta, R(t)$, and $h(t)$ performed better when the parameter of binomial r was increased.
- The MLE has a unique solution and a maximum value of log-likelihood.

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