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On Cyclic LA-Hypergroups

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Abstract: Symmetries in the context of hypergroups and their generalizations are closely related to the algebraic structures and transformations that preserve certain properties of hypergroup operations. Symmetric LA-hypergroups are indeed commutative hypergroups. This paper considers a category of cyclic hyperstructures called the cyclic LA-semihypergroup that is a conception of LA-semihypergroups and cyclic hypergroups. We inaugurate the idea of cyclic LA-hypergroups. The interconnected notions of single-power cyclic LA-hypergroups, non-single power cyclic LA-hypergroups and some of their properties are explored.

Keywords: cyclic LA-semihypergroups; cyclic LA-hypergroups; index of a generator; single-power cyclic LA-hypergroups; fundamental relations

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1. Introduction

An LA-semigroup is a groupoid $(S_{LA}, *)$ that satifies

$$(e * f) * g = (g * f) * e \text{ for all } e, f, g \in S_{LA}.$$

This notion was instituted by Kazim and Naseeruddin in the 1970s [1]. Afterward, several fundamental properties and new notions related to LA-semigroups were discussed in the literature [2–4]. In 1979, some advantageous outcomes were made by Mushtaq and Yusuf on LA-semigroups [2]. In this construction, congruences were explicated utilizing the powers of elements and quotient LA-semigroups were built.

Suppose that S_{LA} is an LA-semigroup. An equivalence relation ζ on S_{LA} becomes a congruence relation on S_{LA} if

$$(f, g) \in \zeta \text{ implies } (f * s, g * s) \in \zeta \text{ and } (s * f, s * g) \in \zeta$$

for all $s \in S_{LA}$ [5].

Marty [6] instituted the hypergroup theory in 1934, where he gave the definition of the hypergroup as a generality of the idea of a group. He demonstrated various applications of hpergroups to relational fractions, algebraic functions, and groups. Nowadays, hypergroups are studied both from the theoretical point of view and for their applications to many areas of pure and applied mathematics: graphs and hypergraphs, binary relations, geometry, cryptography and code theory, probability theory, topology, automata theory, theory of fuzzy and rough sets, economy, etc. [7–9]. A hypergroup is an algebraic structure similar to a group, but the composition of two elements is a non-empty set. Freni and De Salvo [10], Leoreanu [11], and Karimian and Davvaz [12] have studied cyclic (semi)hypergroups. In [13], Mousavi introduced, on a hypergroup, an equivalence relation that is strongly regular such that in a particular case the quotient is a cyclic group.

Tahan and Davvaz gave a large number of results on the class of cyclic hypergroups that are single-power in [14–16]. Ze introduced cyclic hypergroups with the idea of the index with respect to a generator. He proved in [17] that an element with index 2 generates a cyclic hypergroup that is a single power. When we applied the same definition of Ze in our structure (LA-hypergroups), we found some differences and some interesting observations. According to the Ze definition, our structure also become commutative and hence a cyclic hypergroup. This motivated us to apply the Ze definition, with some ammendments, to LA-hypergroups.

LA-semihypergroups are a generality of LA-semigroups and semihypergroups. The idea of the LA-semihypergroup was given by Hila and Dine in [18], where they have discussed some useful results about hyperideals and bi-hyperideals. The concept of LA-hypergroups was given in [19] by Rehman and Yaqoob, where they explored some interesting properties of LA-hyperrings. In [20], Massouros and Yaqoob investigated some very useful and fundamental characteristics of LA-hypergroups.

The organization of this paper is given in the following. Firstly, some fundamental definitions concerning LA-hyperstructures are given. Then, we launch the concepts of cyclic LA-hypergroups, single-power cyclic LA-hypergroups, non-single-power cyclic LA-hypergroups and discuss some of their characteristics and related outcomes in Sections 4–6.

2. Preliminaries and Basic Definitions

Definition 1. *A hypergroupoid $(\Upsilon, \tilde{o})$ is a nonempty set Υ embedded with a hyperoperation, which is a mapping $\tilde{o} : \Upsilon \times \Upsilon \to S^*(\Upsilon)$, and $S^*(\Upsilon)$ is the class of all nonempty subsets of Υ.*

If $x \in \Upsilon$ and P, Q are nonempty subsets of Υ, then

$$P\tilde{o}Q = U_{p\in P, q\in Q}\ p\tilde{o}q, \quad x\tilde{o}P = \{x\}\tilde{o}P \quad \text{and} \quad P\tilde{o}x = P\tilde{o}\{x\}.$$

The hypergroupoid $(\Upsilon, \tilde{o})$ is commutative if $p\tilde{o}q = q\tilde{o}p$, for all $p, q \in \Upsilon$.

Definition 2. *A semihypergroup is a hypergroupoid $(\Upsilon, \tilde{o})$ if for every $p, q, r \in \Upsilon$, we have $p\tilde{o}(q\tilde{o}r) = (p\tilde{o}q)\tilde{o}r$, and it is named a quasihypergroup if $p\tilde{o}\Upsilon = \Upsilon = \Upsilon\tilde{o}p$ for all $p \in \Upsilon$.*

Definition 3 ([6,7])**.** *A hypergroup $(\Upsilon, \tilde{o})$ is a hypergroupoid that is a semihypergroup as well as a quasihypergroup.*

We define $d^i = d\tilde{o}d\tilde{o}\ldots\tilde{o}d$, ($i$-times) for any $i \in \mathbb{N}$. A hypergroup $(\Upsilon, \tilde{o})$ is cyclic if there occurs $d \in \Upsilon$ such that $\Upsilon = d \cup d^2 \cup \cdots \cup d^i \cup \cdots$, where d is called a generator of Υ. If $\Upsilon = d \cup d^2 \cup \cdots \cup d^i$, then Υ is a cyclic hypergroup having a finite period. In other respects, Υ is named a cyclic hypergroup having an infinite period. Let $d \in \Upsilon$ such that $\Upsilon = d \cup d^2 \cup \cdots \cup d^i \cup \cdots$ and $d^{m-1} \subseteq d^m$ for all $m \in N_{\geq 2}$, where

$$N_{\geq 2} = \{2, 3, 4, 5, \ldots\},$$

then Υ is called a single-power cyclic hypergroup [14].

Assume that ϱ is a relation on a semihypergroup $(S, \tilde{o})$. For any nonempty subsets P and Q of S, we set

$$P\overline{\varrho}Q \Leftrightarrow \begin{pmatrix} p\varrho q & (\forall p \in P, \exists\, q \in Q) \\ p'\varrho q' & (\forall q' \in Q, \exists\, p' \in P) \end{pmatrix}$$

and

$$P\overline{\overline{\varrho}}Q \Leftrightarrow p\varrho q \quad (\forall p \in P, \forall q \in Q).$$

A relation of equivalence ϱ on S is named regular if

$$(\forall p, q, x \in S)\ p\varrho q \Longrightarrow p\tilde{o}x\overline{\varrho}q\tilde{o}x \text{ and } x\tilde{o}p\overline{\varrho}x\tilde{o}q,$$

and strongly regular if

$$(\forall p, q, x \in S)\ p\varrho q \Longrightarrow p\tilde{o}x\bar{\bar{\varrho}}q\tilde{o}x \text{ and } x\tilde{o}p\bar{\bar{\varrho}}x\tilde{o}q.$$

The fundamental relation [21] $\chi^* = \cup_{m\geq 1}\chi^m$ for a semihypergroup $(S, \tilde{o})$ is the transitive closure of $\chi = \cup_{n\geq 1}\chi^n$, where $\chi_1 = \{(s,s) \mid s \in S\}$ and for $n \in N_{\geq 2}, \chi_n$ is defined as follows:

$$p\chi_n q \Leftrightarrow (\exists (x_1, x_2, x_3, \ldots, x_n) \in S^n) \subseteq \prod_{i=1}^{n} x_i.$$

χ^* is the minimal relation on S which is strongly regular. Especially in case of a hypergroup, $\chi^* = \chi$ from [22], and in this case Υ/χ is said to be the fundamental group.

Definition 4 ([18])**.** *A hypergroupoid $(H_{LA}, \tilde{o})$ is an LA-semihypergroup if it satisfies*

$$(p\tilde{o}q)\tilde{o}r = (r\tilde{o}q)\tilde{o}p$$

for all $p, q, r \in H_{LA}$.

Definition 5 ([19,20])**.** *An LA-semihypergroup is an LA-hypergroup if*

$$p\tilde{o}H_{LA} = H_{LA} = H_{LA}\tilde{o}p,$$

for every $p \in H_{LA}$.

3. Regular Relations on LA-Hypergroups (LA-Semihypergroups)

In this segment, we discuss some results with respect to regular (strongly regular) relations on LA-hyperstructures and their related quotient structures.

Theorem 1. *Suppose that $(H_{LA}, \circ)$ is an LA-semihypergroup and ϱ is a relation of equivalence on it.*

1. In case that ϱ is regular, then H_{LA}/ϱ is an LA-semihypergroup, in regard to the following operation:
$$\bar{p} \circledcirc \bar{q} = \{\bar{w} \mid w \in p \circ q\}.$$
2. In case that the above operation is straightforward on H_{LA}/ϱ, then ϱ is regular.

Proof.

1. Initially, we examine that $\circledcirc$ is well defined on H_{LA}/ϱ. Let us examine $\bar{p} = \bar{p_1}$ and $\bar{q} = \bar{q_1}$. First, we look over $\bar{p} \circledcirc \bar{q} = \bar{p_1} \circledcirc \bar{q_1}$. We have $p\varrho p_1$ and $q\varrho q_1$. As ϱ is regular, it comes after that $(p \circ q)\varrho(p_1 \circ q)$ and $(p_1 \circ q)\varrho(p_1 \circ q_1)$; thus, $(p \circ q)\varrho(p_1 \circ q_1)$. Hence, for all $z \in p \circ q$ there exists $z_1 \in p_1 \circ q_1$ such that $z\varrho z_1$, which means that $\bar{z} = \bar{z_1}$. It follows that $\bar{p} \circledcirc \bar{q} \subseteq \bar{p_1} \circledcirc \bar{q_1}$. Similarly, $\bar{p_1} \circledcirc \bar{q_1} \subseteq \bar{p} \circledcirc \bar{q}$. Therefore, $\bar{p_1} \circledcirc \bar{q_1} = \bar{p} \circledcirc \bar{q}$. Now, we check the left inverter law for $\circledcirc$. Assume that $\bar{p}, \bar{q}, \bar{z}$ are arbitrary elements in H_{LA}/ϱ. Let $\bar{u} \in (\bar{p} \circledcirc \bar{q}) \circledcirc \bar{z}$. Thus, there exists $\bar{v} \in \bar{p} \circledcirc \bar{q}$ such that $\bar{u} \in \bar{v} \circledcirc \bar{z}$. We can also say that $v_1 \in p \circ q$ and $u_1 \in v \circ z$ exist such that $v\varrho v_1$ and $u\varrho u_1$. Because ϱ is a regular relation, so $u_2 \in v_1 \circ z \subseteq (p \circ q) \circ z = (z \circ q) \circ p$ such that $u_1\varrho u_2$. From here, we obtain that $u_3 \in z \circ q$ exist such that $u_2 \in u_3 \circ p$. We have $\bar{u} = \bar{u_1} = \bar{u_2} \in \bar{u_3} \circledcirc \bar{p} \subseteq (\bar{z} \circledcirc \bar{q}) \circledcirc \bar{p}$. It follows that $(\bar{p} \circledcirc \bar{q}) \circledcirc \bar{z} \subseteq (\bar{z} \circledcirc \bar{q}) \circledcirc \bar{p}$. Similarly, $(\bar{z} \circledcirc \bar{q}) \circledcirc \bar{p} \subseteq (\bar{p} \circledcirc \bar{q}) \circledcirc \bar{z}$. Hence, $(\bar{p} \circledcirc \bar{q}) \circledcirc \bar{z} = (\bar{z} \circledcirc \bar{q}) \circledcirc \bar{p}$. Thus, H_{LA}/ϱ is an LA-semihypergroup.
2. Let $c\varrho d$ and p be any member of H_{LA}. If $u \in c \circ p$, then $\bar{u} \in \bar{c} \circledcirc \bar{p} = \bar{d} \circledcirc \bar{p} = \{\bar{v} : v \in d \circ p\}$. Thus, an element $v \in d \circ p$ occurs such that $u\varrho v$; hence, $c \circ p\bar{\bar{\varrho}}d \circ p$, that is, ϱ is regular on the right. In parallel, ϱ is regular on the left. Hence, ϱ is regular.

□

Theorem 2. *Assume that ϱ is a relation of equivalence on an LA-hypergroup $(H_{LA}, \circ)$; then, ϱ is regular $\Leftrightarrow$ $(H_{LA}/\varrho, \circledcirc)$ is an LA-hypergroup.*

Proof. Let $(H_{LA}, \circ)$ is an LA-hypergroup, so for all p of H_{LA}, we have $H_{LA} = H_{LA} \circ p = p \circ H_{LA}$; hence, we obtain $H_{LA}/\varrho = \bar{p} \circledcirc H_{LA}/\varrho = H_{LA}/\varrho \circledcirc \bar{p}$. Thus, $(H_{LA}/\varrho, \circledcirc)$ is an LA-hypergroup, by Theorem 1. □

Theorem 3. *Suppose that $(H_{LA}, \circ)$ is an LA-semihypergroup and ϱ is a relation of equivalence on it.*

1. In the case where ϱ is strongly regular, then H_{LA}/ϱ is an LA-semigroup, in regard to the operation defined in the following:

$$\bar{p} \circledcirc \bar{q} = \bar{w} \text{ for every } w \in p \circ q;$$

2. In the case where the above operation is straightforward on H_{LA}/ϱ, then ϱ is strongly regular.

Proof. Proof is straightforward. □

Corollary 1. *Assume that ϱ is a relation of equivalence on an LA-hypergroup $(H_{LA}, \circ)$; then, ϱ is strongly regular $\Leftrightarrow$ $(H_{LA}/\varrho, \circledcirc)$ is an LA-group.*

Proof. The proof is evident. □

Proposition 1. *Let ϱ be a regular relation on a non-commutative LA-hypergroup $(H_{LA}, \circ)$. Then, $(H_{LA}/\varrho, \circledcirc)$ is a non-commutative LA-hypergroup.*

Proof. Assume that $\bar{p}, \bar{q}$ be any elements in H_{LA}/ϱ. It follows by Theorem 2 that $(H_{LA}/\varrho, \circledcirc)$ is an LA-hypergroup. Hence, $\bar{p} \circledcirc \bar{q} = \{\bar{w} \mid w \in p \circ q \neq q \circ p\} \neq \bar{q} \circledcirc \bar{p}$. □

Example 1. *Let $H_{LA} = \{1, 2, 3, 4\}$ with the hyperoperation $\circ$ defined as in Table 1.*

Table 1. Tabular form of the hyperoperation "$\circ$" defined in Example 1.

$\circ$	**1**	**2**	**3**	**4**
1	4	1	1	$\{2,3\}$
2	3	4	4	1
3	$\{2,3\}$	4	4	1
4	1	$\{2,3\}$	$\{2,3\}$	4

Then, $(H_{LA}, \circ)$ is an LA-semihypergroup. Consider

$$\varrho = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4)\}$$

as a strongly regular relation on H_{LA}. Then, $H_{LA}/\varrho = \{\{1\}, \{2,3\}, \{4\}\}$ is an LA-semigroup, and $\{4\}$ is the left identity.

Example 2. *Let $H_{LA} = \{1, 2, 3, 4\}$ be an LA-hypergroup explicated in Table 2.*

Table 2. Tabular form of the hyperoperation "$\circ$" defined in Example 2.

$\circ$	**1**	**2**	**3**	**4**
1	$\{1,2\}$	2	$\{3,4\}$	$\{3,4\}$
2	1	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$
3	4	$\{3,4\}$	$\{1,2\}$	$\{1,2\}$
4	$\{3,4\}$	$\{3,4\}$	$\{1,2\}$	$\{1,2\}$

And consider

$$\varrho = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$$

as a strongly regular relation on H_{LA}. Then, $H_{LA}/\varrho = \{\{1,2\},\{3,4\}\}$ is an LA-group.

4. Cyclic LA-Semihypergroups and LA-Hypergroups

In this section, we define a cyclic LA-hypergroup and present some properties of this new structure. We denote $per(c)$ for the period of c and $ind(c)$ for the index of c.

Definition 6. *An LA-hypergroup (LA-semihypergroup) $(H_{LA},\circ)$ is cyclic if there occurs $c \in H_{LA}$, such that*

$$H_{LA} = c \cup c^2 \cup \cdots \cup c^i \cup \cdots,$$

where

$$\begin{aligned} c^i &= c^{i-1} \circ c \\ &= (c^{i-2} \circ c) \circ c \\ &= ((c^{i-3} \circ c) \circ c) \circ c \\ &= (\ldots(((c \circ c) \circ c) \circ c)\ldots \circ c), \qquad i\text{-times}, \quad i \in \mathbb{N}. \\ \text{Also, } c^i c &= c^{i+1} \neq cc^i. \end{aligned}$$

c is named a generator of H_{LA}. If $H_{LA} = c \cup c^2 \cup \cdots \cup c^i$, then H_{LA} is named a cyclic LA-hypergroup (LA-semihypergroup) with a finite period. If i is the minimal number for which this relation holds, then we say that c has period i. Otherwise, H_{LA} is called a cyclic LA-hypergroup (LA-semihypergroup) with an infinite period.

Example 3. *Let $H_{LA} = \{1,2,3\}$ with the hyperoperation $\circ$ expounded in Table 3.*

Table 3. Cyclic LA-semihypergroup generated by 1.

$\circ$	**1**	**2**	**3**
1	2	2	2
2	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
3	2	2	2

Then, $(H_{LA},\circ)$ is a cyclic LA-semihypergroup generated by 1.

Example 4. *Consider the set $H_{LA} = \{1,2,3,4\}$. Then, $(H_{LA},\circ)$ is a cyclic LA-hypergroup generated by* 2*. See Table 4.*

Table 4. Cyclic LA-hypergroup generated by 2.

$\circ$	**1**	**2**	**3**	**4**
1	$\{1,3,4\}$	2	$\{3,4\}$	$\{3,4\}$
2	2	$\{1,3,4\}$	2	2
3	1	2	$\{1,3\}$	$\{1,4\}$
4	1	2	$\{1,4\}$	$\{1,3\}$

Example 5. *Let $H_{LA} = \{1,2,3,4\}$, with the hyperoperation $\circ$ defined in Table 5.*

Table 5. Cyclic LA-hypergroup generated by all of its elements.

$\circ$	**1**	**2**	**3**	**4**
1	2	$\{3,4\}$	$\{1,2\}$	$\{1,2\}$
2	$\{3,4\}$	$\{1,2\}$	$\{2,3,4\}$	$\{2,3,4\}$
3	$\{1,2\}$	$\{2,4\}$	$\{1,2,3\}$	$\{1,2,3\}$
4	$\{1,2\}$	$\{2,3,4\}$	$\{1,2,4\}$	$\{1,2,4\}$

Then, $(H_{LA}, \circ)$ is a cyclic LA-hypergroup generated by all of its elements.

Theorem 4. *Suppose that $(H_{LA}, \circ)$ is a cyclic LA-hypergroup and ϱ is a relation of equivalence on H_{LA}. Then, ϱ is strongly regular $\Leftrightarrow$ $(H_{LA}/\varrho, \circledcirc)$ is a fundamental LA-group. Here, the operation $\circledcirc$ is defined as*

$$(p)_\varrho \circledcirc (q)_\varrho = (w)_\varrho \ (\forall w \in p \circ q).$$

Proof. The proof is obvious. $\square$

Theorem 5. *Let $(H_{LA}, \circ)$ be a cyclic LA-hypergroup generated by c. Then, the fundamental relation is*

$$\chi^* = \{(x,y) \in H_{LA} \times H_{LA} \mid (\exists i \in N)\{x,y\} \subseteq c^i\}.$$

Proof. Let us denote

$$\varrho = \{(x,y) \in H_{LA} \times H_{LA} \mid (\exists i \in N)\{x,y\} \subseteq c^i\}.$$

For an LA-hypergroup $\chi^* = \chi$. Next, we just need to show that $\chi = \varrho$. If $(x,y) \in \varrho$, then $(x,y) \in \chi_i$ for some $i \in \mathbb{N}$. Thus, $(x,y) \in \chi$ so $\varrho \subseteq \chi$. Now, we show the reverse inclusion. Obviously, $\chi_1 \subseteq \varrho$. Assume that $(x,y) \in \chi_n$ for some $n \geq 2$. Then, $\{x,y\} \subseteq \prod_{i=1}^{n} x_i$ for some $x_1, x_2, x_3, \ldots, x_n \in H_{LA}$. Since $x_i \in c^{j_i}$for some $j_i \in \mathbb{N}$, we have $\{x,y\} \subseteq c^{\sum_{i=1}^{n} j_i}$. Thus, $(x,y) \in \varrho$. Hence, the proof is complete. $\square$

Example 6. *Let $H_{LA} = \{1,2,3,4,5\}$ be a cyclic LA-hypergroup (generated by 3 and 4) defined by Table 6.*

Table 6. Cyclic LA-hypergroup generated by 3 and 4.

$\circ$	**1**	**2**	**3**	**4**	**5**
1	1	$\{1,2,3\}$	$\{1,2,3\}$	$\{4,5\}$	$\{4,5\}$
2	$\{1,2,3\}$	$\{2,3\}$	$\{2,3\}$	$\{4,5\}$	$\{4,5\}$
3	$\{1,2,3\}$	2	2	$\{4,5\}$	$\{4,5\}$
4	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	$\{1,2,3\}$	$\{1,2,3\}$
5	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	$\{1,2,3\}$	$\{1,2,3\}$

And $\varrho = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$ be a fundamental relation on H_{LA}. Then, $H_{LA}/\varrho = \{\{1,2,3\},\{4,5\}\}$ is an LA-group.

Example 7. *Let $H_{LA} = \{1,2,3,4\}$ be a cyclic LA-hypergroup (generated by 1) defined by Table 7.*

Table 7. Cyclic LA-hypergroup generated by 1.

$\circ$	**1**	**2**	**3**	**4**
1	$\{2,3,4\}$	1	1	1
2	1	$\{2,3,4\}$	$\{3,4\}$	$\{3,4\}$
3	1	2	$\{2,3\}$	$\{2,4\}$
4	1	2	$\{2,4\}$	$\{2,3\}$

And consider

$$\varrho = \{(1,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3)(4,4)\}$$

as a fundamental relation on H_{LA}. Then, $H_{LA}/\varrho = \{\{1\},\{2,3,4\}\}$ is an LA-group.

5. Single-Power Cyclic LA-Hypergroups

In this segment, we study a few characteristics of single-power cyclic LA-hypergroups. We consider the idea of the index in a cyclic LA-hypergroup with respect to a generator and discuss some results on it with respect to the fundamental relation on a cyclic LA-hypergroup.

Definition 7. *A cyclic LA-hypergroup (LA-semihypergroup) $(H_{LA},\circ)$ is said to be a single-power cyclic LA-hypergroup (LA-semihypergroup) if $H_{LA} = c \cup c^2 \cup \cdots \cup c^i \cup \cdots$ and $c^{m-1} \subseteq c^m$ (for all $m \in N_{\geq 2}$) for some $c \in H_{LA}$.*

Lemma 1. *Suppose that $(H_{LA},\circ)$ is a cyclic LA-hypergroup generated by c. Then, $c \in c^i$ for some $i \in N_{\geq 2}$.*

Proof. Here, we examine the event that H_{LA} has an infinite period (the event that H_{LA} has a finite period follows correspondingly). So, in this case $H_{LA} = \cup_{i\geq 1} c^i = c \cup c^2 \cup \cdots \cup c^i \cup$. Since $H_{LA} = H_{LA} \circ c$, $H_{LA} = \cup_{i\geq 2} c^i$ (this result does not hold for $H_{LA} = c \circ H_{LA}$). Hence, we acquire our inference from $c \in H_{LA}$. The desired result follows from $c \in H_{LA}$. □

Remark 1.

1. *It is evident by Lemma 1 that the aforesaid i is not distinctive. The least number in the collection*

$$\{i \in N_{\geq 2} \mid c \in c^i\}$$

is named as the index of c and is represented by $ind(c)$. Specifically, if $ind(c) = 2$ then H_{LA} is called a single-power cyclic LA-hypergroup.

2. *Assume that H_{LA} is a single-power cyclic LA-hypergroup that has finite period p and is generated by c. If $per(c) > ind(c)$, then $per(c) = ind(c) + 1$. In fact, $ind(c) < per(c)$, $c \in \cup_{i=2}^{p} c^i$. Moreover,*

$$H_{LA} = H_{LA} \circ c = \cup_{i=2}^{p} c^i$$

(This result does not hold for $H_{LA} = c \circ H_{LA}$.). Thus, $c \in c^{p-1}$ and so $ind(c) = per(c) - 1$. Hence, $per(c) = ind(c) + 1$.

Example 8. *Let $H_{LA} = \{1,2,3\}$ with the hyperoperation $\circ$ defined in Table 8.*

Table 8. Single-power cyclic LA-hypergroup generated by 1.

$\circ$	**1**	**2**	**3**
1	$\{1,2\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,3\}$	3	$\{2,3\}$
3	$\{1,2,3\}$	$\{2,3\}$	$\{2,3\}$

Then, $(H_{LA}, \circ)$ *is a single-power cyclic LA-hypergroup generated by* 1*. Here, it is easy to see that* $per(1) = 3$ *and* $ind(1) = 2$*. This clearly shows that* $per(1) = ind(1) + 1$*.*

Example 9. *Let* $H = \{1, 2, 3\}$ *with the hyperoperation* $\circ$ *defined in Table 9.*

Table 9. Single-power cyclic LA-hypergroup generated by 1,2,3.

$\circ$	**1**	**2**	**3**
1	$\{1,3\}$	$\{1,3\}$	$\{1,2\}$
2	$\{2,3\}$	$\{2,3\}$	$\{1,3\}$
3	$\{1,2\}$	$\{1,2,3\}$	$\{2,3\}$

Then, $(H_{LA}, \circ)$ *is a single-power cyclic LA-hypergroup generated by* $1, 2, 3$*. Here, it is easy to see that for any* $h \in \{1, 2, 3\}$ *we have* $per(h) = 3$ *and* $ind(h) = 2$*. This clearly shows that* $per(h) = ind(h) + 1$*.*

Theorem 6. *Let* ϱ *be a relation of equivalence on a single power cyclic LA-hypergroup* $(H_{LA}, \circ)$*. Then,* ϱ *is regular* $\Leftrightarrow$ $(H_{LA}/\varrho, \circledcirc)$ *is an LA-hypergroup. The operation* $\circledcirc$ *is defined as*

$$(p)_\varrho \circledcirc (q)_\varrho = \{(w)_\varrho \mid w \in p \circ q\}.$$

Proof. The proof is obvious. □

Theorem 7. *Let* ϱ *be a relation of equivalence on a single power cyclic LA-hypergroup* $(H_{LA}, \circ)$*. Then,* ϱ *is strongly regular* $\Leftrightarrow$ $(H_{LA}/\varrho, \circledcirc)$ *is a fundamental LA-group. The operation* $\circledcirc$ *is defined as*

$$(p)_\varrho \circledcirc (q)_\varrho = (w)_\varrho \ (\forall w \in p \circ q).$$

Proof. The proof is obvious. □

Theorem 8. *Suppose that* $(H_{LA}, \circ)$ *is a cyclic LA-hypergroup generated by* c *which is also single power. Then, the fundamental relation*

$$\chi^* = \{(x, y) \in H \times H \mid (\exists i \in N)\{x, y\} \subseteq c^i\}.$$

Proof. It follows from the proof of Theorem 5. □

Proposition 2. *Let* $(H_{LA}, \circ)$ *be a cyclic LA-hypergroup generated by* c *that is also single power. Define a relation* ϱ *on* H_{LA} *by*

$$x\varrho y \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \{x, y\} \subseteq c^m.$$

Then, ϱ *is a strongly regular relation on* H_{LA}*.*

Proof. ϱ is a relation of equivalence on H_{LA}, which is evident. Since $(H_{LA}, \circ)$ is a cyclic LA-hypergroup that is also single-power, it follows that $H_{LA} = c \cup c^2 \cup \cdots \cup c^i \cup \cdots$ and $c \cup c^2 \cup \cdots \cup c^{i-1} \subseteq c^i$. It is obvious that for every $i \in \mathbb{N}, c \subset c^2 \subset \cdots \subset c^{i-1} \subset c^i \subset \ldots$. Suppose $x, y \in H_{LA}$. Then, $\{x, y\} \subseteq c^i \cup c^j \subseteq c^{i+j}$ for some $i, j \in \mathbb{N}$. The latter asserts that $x\varrho y$ for every $x, y \in H_{LA}$. We now obtain that the set H_{LA}/ϱ has only a single equivalence class, and therefore $(H_{LA}/\varrho, \circledcirc)$ is the trivial LA-group. The latter and Corollary 1 assert that ϱ is a strongly regular relation on H_{LA}. □

Theorem 9. *Let* $(H_{LA}, \circ)$ *be a cyclic LA-hypergroup generated by* c*. If* $c^i \cap c^j \neq \emptyset$ *for any* $i, j \in N_{\geq 2}$*, then the fundamental relation* $\chi^* = H_{LA} \times H_{LA}$*.*

Proof. Suppose $x, y \in H_{LA}$. We have from Lemma 1 and Remark 1 that $x \in c^i, y \in c^j$ for some $i, j \in N_{\geq 2}$. By hypothesis, $r \in c^i \cap c^j$ exists. Thus, $(x, r) \in \chi^*$ and $(r, y) \in \chi^*$. Since χ^* is transitive, so $(x, y) \in \chi^*$. □

Theorem 10. *Every cyclic LA-hypergroup that is single-power has a trivial fundamental LA-group.*

Proof. Let $(H_{LA}, \circ)$ be a cyclic LA-hypergroup generated by some $c \in H_{LA}$, which is also single-power. For every $p, q \in H_{LA}$, there occurs $i, j \in N_{\geq 2}$ s.t. $p \in c^i, q \in c^j$. It is obvious that $\{p, q\} \subseteq c^k$, where $k = i + j$. It follows from the latter that $p\chi_k q$ for all $p, q \in H_{LA}$. Finally, we attain that the quotient set $H_{LA}/\chi = H_{LA}/\chi^*$ contains just a single equivalence class. Accordingly, H_{LA} has a trivial fundamental LA-group. □

Theorem 11. *Let ϱ be a regular relation on a cyclic LA-hypergroup $(H_{LA}, \circ)$ generated by c. Then, $(H_{LA}/\varrho, \circledcirc)$ is a cyclic LA-hypergroup generated by $(c)_\varrho$ and $ind(c) = ind((c)_\varrho)$.*

Proof. From Theorem 6, we obtain that H_{LA}/ϱ is an LA-hypergroup. We examine that case H_{LA} has an infinite period (likewise we can see the case that H_{LA} has a finite period). Initially, we use induction to show that $(c)^i_\varrho = \{ (t)_\varrho : t \in c^i\}$ for all $i \geq 2$. For $i = 2$, we have $(c)^2_\varrho = (c)_\varrho \circledcirc (c)_\varrho = \{(t)_\varrho : t \in c \circ c\} = \{ (t)_\varrho : t \in c^2\}$. We assume that $(c)^{i-1}_\varrho = \{ (t)_\varrho : t \in c^{i-1}\}$. We have $(c)^i_\varrho = (c)^{i-1}_\varrho \circledcirc (c)_\varrho$ (left invertive law). Using our assumption

$$\begin{aligned}
(c)^i_\varrho &= (c)^{i-1}_\varrho \circledcirc (c)_\varrho \\
&= \bigcup_{(x)_\varrho \in (c)^{i-1}_\varrho} (x)_\varrho \circledcirc (c)_\varrho \\
&= \bigcup_{x \in c^{i-1}} \{(t)_\varrho : t \in x \circ c\} \\
&= \{(t)_\varrho : t \in c^{i-1} \circ c\} \\
&= \{(t)_\varrho : t \in c^i\}.
\end{aligned}$$

Since $H_{LA} = \cup_{i \in \mathbb{N}} c^i$, we have

$$\cup_{i \in \mathbb{N}} (c)^i_\varrho = \{(t)_\varrho : t \in H\} = H/\varrho.$$

That is, H_{LA}/ϱ is a cyclic LA-hypergroup generated by $(c)_\varrho$. Furthermore, it can be simply seen that $ind(c) = ind((c)_\varrho)$. □

Theorem 12. *Let $(H_{LA}, \circ)$ be a cyclic LA-hypergroup that is also single-power and ϱ be a regular equivalence relation on H_{LA}. Then, $(H_{LA}/\varrho, \circledcirc)$ is a single-power cyclic LA-hypergroup.*

Proof. We can see from Theorem 6 that $(H_{LA}/\varrho, \circledcirc)$ is an LA-hypergroup. As $(H_{LA}, \circ)$ is a single power cyclic LA-hypergroup, it follows that $H_{LA} = c \cup c^2 \cup c^3 \cup \cdots$ and

$$c^{u-1} \subset \begin{cases} c^u, & \text{for all } u \geq 2 \text{ if } H_{LA} \text{ has infinite period} \\ c^u, & \text{for all } 2 \leq u \leq o \text{ if } H_{LA} \text{ has finite period } o \end{cases},$$

for some $c \in H_{LA}$. We examine the case when H_{LA} has an infinite period (likewise we can see the case that H_{LA} has a finite period). Firstly, we use mathematical induction to show that $\bar{c}^m = \{\bar{w} : w \in c^m\}$ for all $m \geq 2$. For $m = 2$,

$$\bar{c}^2 = \bar{c} \circledcirc \bar{c} = \{\bar{w} : w \in h \circ h\} = \{\bar{w} : w \in h^2\}.$$

We assume that $\bar{c}^{m-1} = \{\bar{w} : w \in c^{m-1}\}$. We have that $\bar{c}^m = \bar{c}^{m-1} \circ \bar{c} = \underset{\bar{w}\in\bar{c}^{m-1}}{\cup} \bar{w} \circ \bar{c}$ (left invertive law). Using our assumption, we obtain that

$$\bar{c}^m = \underset{\bar{w}\in\bar{c}^{m-1}}{\cup} \bar{w} \circ \bar{c} = \underset{\bar{w}\in\bar{c}^{m-1}}{\cup} \{\bar{x} : x \in w \circ c\}.$$

Since $c \subset c^2 \subset c^3 \subset \dots$, it follows that

$$\bar{h}^m = \{\bar{x} : x \in h^{m-1} \circ h\} = \{\bar{w} : w \in h^m\}.$$

Since H_{LA} is generated by c, it follows that

$$\underset{m\in\mathbb{N}}{\cup} \bar{h}^m = \{\bar{w} : w \in H\} = H/\varrho.$$

Thus, H_{LA}/ϱ is a cyclic LA-hypergroup generated by $\bar{c}$ with $\bar{c}^{m-1} \subset \bar{c}^m$ for all $m \geq 2$, which suggests that

$$\bar{c}^{m-1} = \{\bar{w} : w \in c^{m-1}\} \subset \{\bar{w} : w \in c^m\} = \bar{c}^m.$$

Therefore, $(H_{LA}/\varrho, \circledcirc)$ is a single-power cyclic LA-hypergroup. □

Theorem 13. *Assume that $(H_{LA}, \circ)$ is a single-power cyclic LA-hypergroup generated by c and ϱ is a strongly regular relation on H_{LA}. Then, $(H_{LA}/\varrho, \circledcirc)$ is a cyclic LA-group generated by $(c)_\varrho$ and $ord(H_{LA}/\varrho) \leq ind(c) - 1$.*

Proof. From Theorem 7, we see that H_{LA}/ϱ is an LA-group. We examine the case that H_{LA} has a infinite period (likewise, we can see the case that H_{LA} has a finite period). We first prove that $(c)^i_\varrho = (p)_\varrho (\forall p \in c^i)$ for all $i \geq 2$. ϱ is a strongly regular relation, so ϱ is a regular relation. Hence, $(c)^i_\varrho = \{(p)_\varrho : p \in c^i\}$ by Theorem 11. The fundamental relation $\chi^* \subseteq \varrho$, so $(p)_\varrho = (q)_\varrho$ for any $p, q \in c^i$ by Theorem 8. Thus, $(c)^i_\varrho = (p)_\varrho (\forall p \in c^i)$. Let $p \in H_{LA}$; then, $p \in h^i$ for some $i \in \mathbb{N}$. Thus, $(p)_\varrho = (c)^i_\varrho$. It comes after that H_{LA}/ϱ is a cyclic LA-group generated by $(c)_\varrho$. Furthermore, $c \in c^{ind(c)}$. Thus, $(h)_\varrho = (c)^{ind(c)}_\varrho$; therefore, $ord(H_{LA}/\varrho) \leq ind(c) - 1$. □

6. Non-Single-Power Cyclic LA-Hypergroups

In this segment, we see some examples and discuss some characteristics of non-single-power cyclic LA-hypergroups.

Definition 8. *A cyclic LA-hypergroup (LA-semihypergroup) $(H_{LA}, \circ)$ is said to be a non-single-power cyclic LA-hypergroup (LA- semihypergroup) if $c \in H_{LA}$ exists such that*

$$H_{LA} = c \cup c^2 \cup \dots \cup c^i \cup \dots$$

and $c^{m-1} \subseteq c^m$ for all $m \in N_{\geq 3}$,

$$N_{\geq 3} = \{3, 4, 5, 6, \dots\}$$

but $c \notin c^2$. In particular, if $ind(h) \neq 2$, then H is a non-single-power cyclic LA-hypergroup.

Remark 2. *Let H_{LA} be a non-single-power cyclic LA-hypergroup generated by c with finite period p. If $per(c) < ind(c)$, then $ind(c) = per(c) + 1$. Indeed, $ind(c) > per(c), c \notin \cup_{i=2}^{p} c^i$. Moreover, $H_{LA} = H_{LA} \circ c = \cup_{i=2}^{p} h^i$ (this result does not hold for $H_{LA} = c \circ H_{LA}$.). Thus, $c \in h^{p+1}$ and so $ind(c) = per(c) + 1$.*

Example 10. *Let $H_{LA} = \{1, 2, 3\}$ with the hyperoperation $\circ$ defined in Table 10.*

Table 10. Non-single-power cyclic LA-hypergroup generated by 1, 2, 3.

$\circ$	**1**	**2**	**3**
1	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,3\}$	$\{1,3\}$	$\{2,3\}$
3	$\{1,2\}$	$\{2,3\}$	$\{1,2\}$

Then, $(H_{LA}, \circ)$ is a non-single-power cyclic LA-hypergroup generated by $1,2,3$. Here, it is easy to see that for any $c \in \{1,2,3\}$ we have $per(c) = 2$ and $ind(c) = 3$. This clearly shows that $ind(c) = per(c) + 1$.

Example 11. *Let $H_{LA} = \{1,2,3,4\}$ with the hyperoperation $\circ$ defined in Table 11.*

Table 11. Non-single-power cyclic LA-hypergroup generated by 1.

$\circ$	**1**	**2**	**3**	**4**
1	$\{2,3,4\}$	1	1	1
2	1	$\{2,3,4\}$	$\{2,3,4\}$	4
3	1	$\{3,4\}$	$\{2,4\}$	4
4	1	$\{2,3\}$	$\{2,3\}$	$\{2,3,4\}$

Then, $(H_{LA}, \circ)$ is a non-single-power cyclic LA-hypergroup generated by 1. Here, it is easy to see that $per(1) = 2$ and $ind(1) = 3$. This clearly shows that $ind(1) = per(1) + 1$.

7. Conclusions

In this paper, we have presented a new class of cyclic hyperstructures named cyclic LA-hypergroups. Then, we have introduced the notions of single-power cyclic LA-hypergroups and non-single-power cyclic LA-hypergroups and have explored some interesting results and characteristics about these LA-hyperstructures. We have also identified some results for the quotient structures induced by the factorization of the cyclic LA-hypergroups by regular and strongly regular relations. In future work, we will explore the applications of our main results by incorporating ideas from singularity theory, submanifold theory, and other related fields [23–29]. This will likely lead to the discovery of many new insights and push the boundaries of our understanding even further. Furthermore, the following topics may be considered for further research:

1. Enumerations of (cyclic) LA-hypergroups of order 3 and 4.
2. Combine LA-hypergroups with singularity theory and submanifolds.

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