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# Hankel Determinant for a Subclass of Starlike Functions with Respect to Symmetric Points Subordinate to the Exponential Function

**Zongtao Li [1,*], Dong Guo [2] and Jinrong Liang [3]**

[1] Faculty of Humanities and Social Sciences, Guangzhou Civil Aviation College, Guangzhou 510403, China
[2] School of Mathematical Sciences, Yangzhou Polytechnic College, Yangzhou 225009, China; 102241@yzpc.edu.cn
[3] Department of Basic Disciplines, Chuzhou Polytechnic College, Chuzhou 239000, China; liangjinrong@chzc.edu.cn
* Correspondence: lizt2046@163.com

**Abstract:** Let $S_s^*(e^z)$ denote the class of starlike functions with respect to symmetric points subordinate to the exponential function, i.e., the functions which satisfy in the unit disk $\mathbb{U}$ the condition $\frac{2zf'(z)}{f(z)-f(-z)} \prec e^z (z \in \mathbb{U})$. We obtained the sharp estimate of the second-order Hankel determinants $H_{2,3}(f)$ and improved the estimate of the third-order $H_{3,1}(f)$ for this functions class $S_s^*(e^z)$.

**Keywords:** analytic function; starlike functions with respect to symmetric points; Hankel determinant; exponential function

**MSC:** 30C45

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## 1. Introduction and Definitions

Let the notation $\mathcal{A}$ denote the class of all analytic functions $f(z)$ in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which are normalized so that the Taylor expansion has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

If $f(z)$ is univalent and has the form (1), it is referred to as a normalized univalent function. The class of all $f(z)$ is denoted as $S$.

In 1916, Bieberbach [1] studied the coefficients of class $S$ and proposed the well-known coefficient conjecture, known as the "Bieberbach conjecture", that if $f(z) \in S$, then $|a_n| \leq n$ for all $n \geq 2$. Although de Branges [2] proved this conjecture in 1985, research on the coefficient estimation of univalent analytic functions continues to increase. In 1959, Sakaguchi [3] introduced the class $S_s^*$ of starlike functions with respect to symmetric points, which consist of functions $f(z) \in S$, satisfying the condition $Re\frac{2zf'(z)}{f(z)-f(-z)} > 0$, and obtained coefficient estimates $|a_n| \leq 1$ for all $n \geq 2$. Subsequently, the class $S_s^*$ has been studied by several authors [4–6]. In 1991, Goodman [6] proved that if $f(z) \in S_s^*$, then for $n \geq 2$, $|a_n| \leq \frac{2}{n}$.

Let $B_0$ denote the class of Schwarz functions in the unit disk $\mathbb{U} = \{z : |z| < 1\}$, i.e., $w(z) \in B_0$ if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ is analytic and $|w(z)| < 1$ on $\mathbb{U}$. Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U}$. If there exists a Schwarz function $w(z)$ in the unit disk that satisfies $w(0) = 0, |w(z)| < 1$ and $f(z) = g(w(z))$, then $f(z)$ is said to be subordinate to $g(z)$ and denoted by $f(z) \prec g(z)$.

Ravichandran [7] introduced a class $S_s^*(\phi)$ of symmetric starlike functions, which is given by

$$S_s^*(\phi) = \{f(z) \in S, \frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z), z \in \mathbb{U}\},$$

where $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ is a univalent starlike function with respect to 1, $B_1 > 0$ and $Re\phi(z) > 0$.

Taking different parameters for the coefficients of the function $\phi(z)$, we can obtain the following subclasses:

(i) $S_s^*(\frac{1+Az}{1+Bz}) = S_s^*(A,B), (-1 \le B < A < 1)$ (see [8–13]; for $S_s^*(\frac{1+z}{1-z}) = S_s^*$, see [3]);

(ii) $S_s^*(\frac{1+(1-2\alpha)z}{1-z}) = S_s^*(\alpha), (0 \le \alpha < \frac{1}{2})$ (see [9,14]);

(iii) $S_s^*(\frac{1+\beta z}{1-\alpha\beta z}) = S_s^*(\alpha,\beta)$ (see [10,15]);

(iv) $S_s^*((1-\beta)[p(z)]^\alpha + \beta) = \bar{S}_{s,\beta}^*(\alpha)$, where $p \in P$, the class of all functions $p(z)$ analyzed in $\mathbb{U}$, for which $Rep(z) > 0$ and $p(z) = 1 + c_1 z + c_2 z^2 + \cdots, 0 < \alpha \le 1, 0 \le \beta < 1$ (see [16]);

(v) $S_s^*(\frac{2}{1+e^{-z}}) = SS_{sG}^*$ (see [17]).

Pommerenke [18] introduced the $q$-th order Hankel determinant of $f(z)$ of the form given by (1)

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

When certain special parameters are chosen for q and n, different Hankel determinants will be obtained. For example,

$$H_{2,3}(f) = a_3 a_5 - a_4^2 \tag{2}$$

and

$$H_{3,1}(f) = a_1 a_3 a_5 + 2a_2 a_3 a_4 - a_3^3 - a_2^2 a_5 - a_1 a_4^2. \tag{3}$$

The Hankel determinant plays a vital role in studies of power series with integral coefficients. Initially, the determinant was studied by researchers such as Hyman [19], Noonan [20] and Ehrenborg [21]. These studies explore the properties and behavior of power series, particularly their coefficients. It was difficult for scholars to determine the exact bounds of the Hankel determinants for a particular class of functions. After that, many papers focused on the upper-bound estimates of the second-order Hankel determinants $H_{2,1}$ and $H_{2,2}$ [22–24]. Recently, many scholars [25–30] have investigated the third-order Hankel determinant. Some sharp upper-bound estimates of the Hankel determinant for a subclass of univalent analytic functions were obtained via complex calculations.

In 2020, Ganesh et al. introduced and investigated a class $S_s^*(e^z)$ of analytic functions in [31], defined as follows:

$$S_s^*(e^z) = \{f \in A : \frac{2zf'(z)}{f(z)-f(-z)} \prec e^z, z \in \mathbb{U}\}.$$

They expressed the coefficients of functions in class $S_S^*(e^z)$ with the coefficients of the corresponding functions with positive real part, and these lemmas on the coefficients of the positive real part were obtained by Libera and Zlotkiewicz [32]. Using this method, the following results were obtained:

**Theorem A.** If $f \in S_s^*(e^z)$, then $|a_2| \le \frac{1}{2}, |a_3| \le \frac{1}{2}, |a_4| \le \frac{19}{48}, |a_5| \le \frac{13}{24}$.

**Theorem B.** If $f \in S_s^*(e^z)$, then $H_{2,1} = |a_3 - a_2^2| \le \frac{1}{2}$.

**Theorem C.** If $f \in S_s^*(e^z)$, then $|a_4 - a_2 a_3| \le \frac{765+59\sqrt{118}}{3468}$.

**Theorem D.** If $f \in S_s^*(e^z)$, then $H_{2,2} = 2|a_2a_4 - a_3^2| \le \frac{3}{8}$.

By applying the above theorems together, the estimate of the third-order Hankel determinant is derived as follows:

**Theorem E.** If $f \in S_s^*(e^z)$, then $H_{3,1} = \frac{90831+1121\sqrt{118}}{166464} = 0.618$.

In 2022, Zaprawa [33] researched the class $S_s^*(e^z)$ using a new approach, which is based on relating the coefficients of functions in a given class with the coefficients of corresponding Schwarz functions. Some coefficient estimates of the function class $S_s^*(e^z)$ were obtained, including logarithmic coefficient estimates, Zalcman functional estimates and upper bounds for the second- and third-order Hankel determinants, etc. The main results are as follows:

**Theorem A′.** If $f \in S_s^*(e^z)$, then $|a_4| \le \frac{1}{4}, |a_5| \le \frac{1}{4}$.

**Theorem B′.** If $f \in S_s^*(e^z)$, then $H_{2,2} = 2|a_2a_4 - a_3^2| \le \frac{1}{4}$.

**Theorem C′.** If $f \in S_s^*(e^z)$, then $|a_4 - a_2a_3| \le \frac{1}{4}, |a_5 - a_3^2| \le \frac{1}{4}$.

**Theorem D′.** If $f \in S_s^*(e^z)$, then $H_{2,3} \le \frac{1}{8}$.

**Theorem E′.** If $f \in S_s^*(e^z)$, then $H_{3,1} = \frac{13}{128}$.

Comparing references [31,33], it is evident that Zaprawa's research results improved upon some of Ganesh's conclusions.

In this paper, we re-examined the upper bounds of Hankel determinants $H_{2,3}$ and $H_{3,1}$ of the class $S_s^*(e^z)$ and presented a complete proof. We proposed a proof method which differed from that of Zaprawa. This method is based on the use of a lemma derived by Carlson [34] about the coefficients of the Schwarz function. In particular, for the Hankel determinant $H_{3,1}$, we obtained a more precise upper-bound estimate than in [33].

These proofs require the following lemma.

**Lemma 1.** *[34] Let $w(z) \in B_0$. Then*

$$|c_2| \le 1 - |c_1|^2, |c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}, |c_4| \le 1 - |c_1|^2 - |c_2|^2.$$

## 2. Main Results and Proofs

**Theorem 1.** *If $f \in S_s^*(e^z)$ , then $|H_{2,3}(f)| \le \frac{1}{8}$. The equality is sharp.*

**Proof.** Since $f \in S_s^*(e^z)$ , we have

$$\frac{2zf'(z)}{f(z) - f(-z)} = e^{w(z)}$$

where $w(z) \in B_0$.

Comparing the coefficients on both sides yields

$$\begin{cases} a_2 = \frac{c_1}{2}, \\ a_3 = \frac{c_2}{2} + \frac{c_1^2}{4}, \\ a_4 = \frac{c_3}{4} + \frac{3}{8}c_1c_2 + \frac{5}{48}c_1^3, \\ a_5 = \frac{c_4}{4} + \frac{1}{4}c_1c_3 + \frac{1}{4}c_2^2 + \frac{1}{4}c_1^2c_2 + \frac{1}{24}c_1^4. \end{cases} \tag{4}$$

Based on (2) and (4), we have

$$|H_{2,3}(f)| = \frac{1}{2304}|-c_1^6 + 12c_1^4c_2 + 108c_1^2c_2^2 + 288c_2c_4 + 144c_1^2c_4 + 288c_2^3 + 24c_1^3c_3 - 144c_1c_2c_3 - 144c_3^2|. \tag{5}$$

Applying the triangle inequality and Lemma 1 to (5) yields

$$
\begin{aligned}
|H_{2,3}(f)| \leq & \frac{1}{2304}[|c_1|^6+12|c_1|^4|c_2|+108|c_1|^2|c_2|^2+288|c_2|(1-|c_1|^2-|c_2|^2)+144|c_1|^2(1-|c_1|^2-|c_2|^2) \\
& +288|c_2|^3+24|c_1|^3((1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})+144|c_1||c_2|(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})+144(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})^2] \\
= & \frac{1}{2304}[|c_1|^6-24|c_1|^5+12|c_2||c_1|^4+144|c_1|^4-144|c_2||c_1|^3-120|c_1|^3-36|c_2|^2|c_1|^2-288|c_2||c_1|^2 \\
& -144|c_1|^2+288|c_2|^2|c_1|+144|c_2||c_1|-288|c_2|^2+288|c_2|+144+\frac{144|c_2|^4}{(1+|c_1|)^2}-\frac{144|c_1||c_2|^3}{1+|c_1|}-\frac{24|c_1|^3|c_2|^2}{1+|c_1|}].
\end{aligned}
$$

Let $x=|c_1|\in[0,1], y=|c_2|\in[0,1-x^2]$. Then,

$$
\begin{aligned}
2304|H_{2,3}(f)| \leq & x^6-24x^5+(12y+144)x^4+(-144y-120)x^3+(-36y^2-288y-144)x^2 \\
& +(288y^2+144y)x-288y^2+288y+144+\frac{144y^4}{(1+x)^2}-\frac{144xy^3}{1+x}-\frac{24x^3y^2}{1+x}=T(x,y).
\end{aligned}
$$

We need to find the maximum value 288 of $T(x,y)$ on $\Omega=\{(x,y):0\leq x\leq 1, 0\leq y\leq 1-x^2\}$. First, we find the possible extreme points of $T(x,y)$ inside $\Omega$. Let

$$
\begin{cases}
\frac{\partial T}{\partial x}=6x^5-120x^4+(48y+576)x^3+(-432y-360)x^2+(-72y^2-576y-288)x+288y^2+144y \\
\qquad -\frac{288y^4}{(1+x)^3}-\frac{144y^3}{(1+x)^2}-\frac{24y^2(3x^2+2x^3)}{(1+x)^2}=0, \\
\frac{\partial T}{\partial y}=12x^4-144x^3+(-72y-288)x^2+(576y+144)x-576y+288 \\
\qquad +\frac{576y^3}{(1+x)^2}-\frac{432xy^2}{1+x}-\frac{48x^3y}{1+x}=0.
\end{cases}
$$

In this case, it is equivalent to

$$
\begin{cases}
x^8-17x^7+(8y+39)x^6+(-48y+169)x^5+(-20y^2-288y+40)x^4+(-8y^2-472y-180)x^3 \\
\qquad +(96y^2-288y-156)x^2+(-24y^3+132y^2-24y-48)x-48y^4-24y^3+48y^2+24y=0, \\
x^6-10x^5+(-10y-47)x^4+(32y-48)x^3+(-36y^2+42y+24)x^2 \\
\qquad +(36y^2-48y+60)x+48y^3-48y+24=0.
\end{cases}
$$

By solving the system of binary nonlinear equations, we find that

$$
\begin{cases} x_1=-1 \\ y_1=0, \end{cases}
\begin{cases} x_2=0.1126\cdots \\ y_2=-1.3016\cdots, \end{cases}
\begin{cases} x_3=1.0265\cdots \\ y_3=-0.1197\cdots, \end{cases}
\begin{cases} x_4=13.2848\cdots \\ y_4=-0.9468\cdots, \end{cases}
\begin{cases} x_5=-0.3053\cdots \\ y_5=0.8011\cdots, \end{cases}
$$

$$
\begin{cases} x_6=-0.4422\cdots \\ y_6=-0.5837\cdots, \end{cases}
\begin{cases} x_7=-1.4723\cdots \\ y_8=-0.0182\cdots, \end{cases}
\begin{cases} x_8=-1.4765\cdots \\ y_8=-0.2346\cdots, \end{cases}
\begin{cases} x_9=-4.4675\cdots \\ y_9=-6.4232\cdots. \end{cases}
$$

Obviously, there are no extreme points of $T$ inside $\Omega$.

It is assumed that the extreme points lie on the edges of $\Omega$.

(a) When $x=0, T(0,y)=144y^4-288y^2+288y+144$.

Since $T'(0,y)=576y^3-576y+288>0$, $T(0,y)\leq T(0,1)=288$.

(b) When $y=0$, $T(x,0)=x^6-24x^5+144x^4-120x^3-144x^2+144$.

Let $T'(x,0)=6x^5-120x^4+576x^3-360x^2-288x=0$. A simple calculation provides the stationary point $x=0$.

From $T''(x,0)=30x^4-488x^3+1728x^2-720x-288$, we obtain $T''(0,0)=-288<0$. This implies $T(x,0)\leq T(0,0)=1448$.

Therefore, we have $|H_{2,3}(f)|\leq\frac{288}{2304}=\frac{1}{8}$. Considering $f(z)=\frac{z^3}{2-z^2}=\frac{1}{2}z^3+\frac{1}{4}z^5+\cdots$, the equality condition in the theorem's statement is true. □

**Theorem 2.** *Let $f(z)\in S_s^*(e^z)$. Then, $|H_{3,1}(f)|\leq\frac{203.5083\cdots}{2304}=0.08832\cdots$.*

**Proof.** From (3) and (4), we have

$$|H_{3,1}(f)| = \frac{1}{2304}|-c_1^6 - 12c_1^4c_2 - 36c_1^2c_2^2 + 288c_2c_4 + 24c_1^3c_3 + 144c_1c_2c_3 - 144c_3^2|. \quad (6)$$

Applying the triangle inequality and Lemma 1 to Equation (6), we obtain

$$\begin{aligned}
|H_{3,1}(f)| \le& \frac{1}{2304}[|c_1|^6 + 12|c_1|^4|c_2|| + 36|c_1|^2|c_2|^2 + 288|c_2|(1-|c_1|^2-|c_2|^2) + 24|c_1|^3(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}) \\
&+ 144|c_1||c_2|(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}) + 144(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})^2] \\
=& \frac{1}{2304}[|c_1|^6 - 24|c_1|^5 + 12|c_2||c_1|^5 + 144|c_1|^4 - 144|c_2||c_1|^3 + 24|c_1|^3 + 36|c_2|^2|c_1|^2 \\
&- 288|c_2||c_1|^2 - 288|c_1|^2 + 288|c_2|^2|c_1| + 144|c_2||c_1| - 288|c_2|^3 - 288|c_2|^2 + 288|c_2| \\
&+ 144 + \frac{144|c_2|^4}{(1+|c_1|)^2} - \frac{144|c_1||c_2|^3}{1+|c_1|} - \frac{24|c_1|^3|c_2|^2}{1+|c_1|}.
\end{aligned}$$

Setting $x = |c_1| \in [0,1]$ and $y = |c_2| \in [0, 1-x^2]$, we obtain

$$\begin{aligned}
2304|H_{3,1}(f)| \le& x^6 - 24x^5 + (12y+144)x^4 + (-144y+24)x^3 + (36y^2-288y-288)x^2 + (288y^2+144y)x \\
&- 288y^3 - 288y^2 + 288y + 144 + \frac{144y^4}{(1+x)^2} - \frac{144xy^3}{1+x} - \frac{24x^3y^2}{1+x} = \Psi(x,y).
\end{aligned}$$

Next, we will seek the maximum value of $\Psi$ on $\Omega = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1-x^2\}$. First, we need to find the possible extreme points inside $\Omega$ for $\Psi$. Let

$$\begin{cases}
\frac{\partial\Psi}{\partial x} = 6x^5 - 120x^4 + (48y+576)x^3 + (-432y+72)x^2 + (72y^2-576y-576)x + 288y^2 + 144y \\
\qquad - \frac{288y^4}{(1+x)^3} - \frac{144y^3}{(1+x)^2} - \frac{24y^2(3x^2+2x^3)}{(1+x)^2} = 0, \\
\frac{\partial\Psi}{\partial y} = 12x^4 - 144x^3 + (72y-288)x^2 + (576y+144)x - 864y^2 - 576y + 288 \\
\qquad + \frac{576y^3}{(1+x)^2} - \frac{432xy^2}{1+x} - \frac{48x^3y}{1+x} = 0.
\end{cases}$$

Then, it is equivalent to

$$\begin{cases}
x^8 - 17x^7 + (8y+39)x^6 + (-48y+241)x^5 + (4y^2-288y+208)x^4 + (64y^2-472y-156)x^3 \\
\qquad + (168y^2-288y-276)x^2 + (-24y^3+156y^2-24y-96)x - 48y^4 - 24y^3 + 48y^2 + 24y = 0, \\
x^6 - 10x^5 + (-2y-47)x^4 + (56y-48)x^3 + (-108y^2+54y+24)x^2 \\
\qquad + (-180y^2-48y+60)x + 48y^3 - 72y^2 - 48y + 24 = 0.
\end{cases}$$

Solving this nonlinear system of equations gives

$$\begin{cases} x_0 = -1 \\ y_0 = 0, \end{cases} \begin{cases} x_1 = 0.1084\cdots \\ y_1 = 0.3801\cdots, \end{cases} \begin{cases} x_2 = 1.0099\cdots \\ y_2 = -0.0399\cdots, \end{cases} \begin{cases} x_3 = -0.2586\cdots \\ y_3 = -0.7152\cdots, \end{cases} \begin{cases} x_4 = -0.4558\cdots \\ y_4 = 0.6786\cdots. \end{cases}$$

Thus, the extreme point inside $\Omega$ is $\begin{cases} x_1 = 0.1084\cdots \\ y_1 = 0.3801\cdots \end{cases}$ and accordingly,

$$\Psi(0.1084\cdots, 0.3801\cdots) = 203.5083\cdots.$$

Second, we discuss the extreme values of $\Psi$ on the boundary of $\Omega$ separately.
(a) When $x = 0$, $\Psi(0,y) = 144y^4 - 288y^3 - 288y^2 + 288y + 144 = \phi_1(y)$.
Let $\phi_1'(y) = 576y^3 - 864y^2 - 576y + 288 = 0$. Then, $y = 0.3554$.
Since $\phi_1''(y) = 1728y^2 - 1728y - 576$, $\phi_1''(0.3554) = -971.8690 < 0$. Thus, $\phi_1(y) \le \phi_1(0.3554) = 199.3471$.

(b) When $y = 0$, $\Psi(x,0) = x^6 - 24x^5 + 144x^4 + 24x^3 - 288x^2 + 144 = \phi_2(x)$.
Let $\phi_2'(x) = 6x^5 - 120x^4 + 576x^3 + 72x^2 - 576x = 0$. Then, $x = 0$.
Since $\phi_2''(x) = 30x^4 - 480x^3 + 1728x^2 + 144x - 576$, $\phi_2''(0) = -576 < 0$. Thus, $\phi_2(x) \le \phi_2(0) = 144$.
(c) When $y = 1 - x^2$, $\Psi(x, 1-x^2) = 577x^6 - 1188x^4 + 612x^2 = \phi_3(x)$.
Let $\phi_3'(x) = 3462x^5 - 4752x^3 + 1224x = 0$. Then, $x = \frac{\sqrt{228492 - 1154\sqrt{9777}}}{577} = 0.5862\cdots$.
Since $\phi_3''(x) = 17310x^4 - 14256x^2 + 1224$, $\phi_3''(0.5862\cdots) = -1630.8\cdots < 0$. Thus, $\phi_3(x) \le \phi_3(0.5862) = 93.4332\cdots$.
Therefore,
$$|H_{3,1}(f)| \le \frac{203.5083\cdots}{2304} = 0.08832\cdots.$$
This completes the proof of Theorem 2. □

## 3. Conclusions

In this paper, upper-bound estimates for the Hankel determinant of the class $S_S^*(e^z)$ are studied using a new Lemma and method. Theorem 5 in reference [33] is proved once again, and the extremal function is obtained. The upper-bound estimate of $|H_{3,1}(f)|$ for the functions class $S_s^*(e^z)$ is more accurate than Theorem 6 in reference [33]. The proof of Theorem 2 indicates that the exact upper-bound estimate of $|H_{3,1}(f)|$ cannot be obtained at the boundary of $\Omega$.

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