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New Three Wave and Periodic Solutions for the Nonlinear (2+1)-Dimensional Burgers Equations

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Abstract: This research paper is about the new three wave, periodic wave and other analytical wave solutions of (2+1)-Dimensional Burgers equations by utilizing Hirota bilinear and extended sinh-Gordon equation expansion (EShGEE) schemes. Achieved solutions are verified and demonstrated by different plots with the use of Mathematica 11.01 software. Some of the achieved solutions are also described graphically by two-dimensional, three-dimensional and contour plots. The gained solutions are helpful for the future study of concerned models. Finally, these two schemes are simple, fruitful and reliable to handle the nonlinear PDEs.

Keywords: (2+1)-dimensional Burgers equations; Hirota bilinear scheme; extended sinh-Gordon equation expansion scheme; new three wave; new periodic wave; analytical wave solutions

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1. Introduction

Symmetry is used in real life to simplify calculations and solve problems more easily. Symmetry also offers human beings an additional extension to their capabilities. Applications of symmetry are determining the orbital overlap for molecular orbital, understanding the spectroscopic properties of molecules and identifying chiral molecular species, etc. Symmetry is a frequently recurring theme in mathematics, nature, science, etc. In mathematics, its most familiar manifestation appears in geometry [1,2].

The nonlinear partial differential equations (NLPDEs) arise in various types of physical problems such as fluid dynamics, plasma physics, quantum field theory, etc. The system of nonlinear partial differential Equations has been observed in chemical, biological, engineering and other areas of applied sciences. A lot of research has been performed in these areas to find the numerical and analytical results of NLPDEs. Various schemes have been developed for this purpose. For example, the auxiliary rational method [3], Kudryashov technique [4], two variable $(G'/G, 1/G)$-expansion technique [5], mapping method [6], generalized auxiliary equation method [7], modified F-expansion technique [8], unified method [9], modified extended tanh expansion method [10], modified simplest equation technique [11], extended Jacobi elliptic function scheme [12], He's semi-inverse and Riccati equation mapping schemes [13], the tanh-coth technique [14], $\exp(-\varphi(\mu))$-expansion scheme [15], etc.

Except for these schemes, there are two other simple, useful, and significant schemes: the Hirota bilinear scheme and the extended sinh-Gordon equation expansion scheme. The Hirota bilinear method can be used to search for new integrable evolution equations. Solutions obtained through the Hirota bilinear method have distinct structures, but all of them have emerged under the banner of the same scheme. This scheme solves solutions without setting solutions, and calculates transformations without making logarithmic transformations. The extended sinh-Gordon equation expansion scheme is a powerful scheme for solving non-linear partial differential equations. It is widely used in various areas of physics, engineering and mathematics. In the literature, there are many applications of these two schemes. For example, by using the Hirota bilinear method three-wave solutions of the (3+1)-dimensional Boiti–Leon–Manna–Pempinelli model have been gained [16], and new periodic wave solutions of the (2+1)-dimensional breaking soliton equation [17]. Some optical solitons of the new Hamiltonian Amplitude model have been achieved by applying the extended sinh-Gordon equation expansion scheme in [18]. New types of optical wave solutions of the Biswas–Arshed equation have been gained by the EShGEE scheme [19].

Our study model is a nonlinear (2+1)-dimensional Burgers equations given as [20]:

$$\begin{aligned} g_t = g g_y + a h g_x + b g_{yy} + a b g_{xx}. \\ g_x = h_y. \end{aligned} \tag{1}$$

where $g = g(x,y,t)$ and $h = h(x,y,t)$ denote the wave profiles, while a and b are the constants. Equation (1) has been studied by many researchers, e.g., Solitary, periodic and rational wave solutions have been gained by utilizing the new Riccati equation rational expansion method. Lump, rogue wave and interaction wave solutions have been gained by Hirota bilinear scheme [21]. Non-trivial wave solutions are achieved with by utilizing auxiliary equation method [22].

The main motivation of this research is to explore the new three wave, periodic wave and other analytical solitons of the non-linear (2+1)-dimensional Burgers equations based on the Hirota bilinear scheme and the extended sinh-Gordon equation expansion scheme. Our research model's Burgers equations are a fundamental partial differential equation from fluid mechanics. They occur in many fields of applied mathematics such as modeling of gas dynamics, traffic flow, etc. The schemes that are used in this research have never been applied to this model before, and the obtained results are newer than the existing results of this model in the literature. The diverse graphical analyses for the presented solutions show that the solutions are reliable for the further development in the model, and also in other areas of mathematical physics and engineering.

This paper is organized as follows. In Section 1; we brief the Hirota bilinear scheme, we apply the Hirota bilinear scheme to obtain the new periodic and three wave solutions to the (2+1)-dimensional Burgers equations. In Section 3; we give the details of the extended sinh-Gordon equation expansion scheme, and we apply the extended sinh-Gordon scheme to find out the exact solutions. In Section we apply the EShGEE scheme to gain the analytical wave solutions to the (2+1)-dimensional Burgers equations. In Section 4; we demonstrate some solutions with the help of different kinds of plots. In Section 5: we give the, conclusion.

2. Hirota Bilinear Scheme and Its Application

2.1. The Bilinear Form Polynomials

According to the [23], let us assume $\zeta = \zeta(x_1, x_2, \ldots, x_n)$ is a C^∞ function is shown below

$$Y_{n_1 x_1, \ldots, n_j x_j}(\zeta) \equiv Y_{n_1, \ldots, n_j}(\zeta_{s_1 x_1, \ldots, s_j x_j}) = e^{-\zeta} \partial_{x_1}^{n_1} \ldots \partial_{x_j}^{n_j} e^{\zeta}, \tag{2}$$

along binary Bell polynomials (BBPs) as below

$$\zeta_{s_1 x_1, \ldots, s_j x_j} = \partial_{x_1}^{s_1} \ldots \partial_{x_j}^{s_j} \zeta, \;\; \zeta_{0x_i} \equiv \zeta, \; s_1 = 0, \ldots, n_1; \ldots; s_j = 0, \ldots, n_j,$$

and we have

$$Y_1(\zeta) = \zeta_x, \;\; Y_2(\zeta) = \zeta_{2x} + \zeta_x^2, \;\; Y_3(\zeta) = \zeta_{3x} + 3\zeta_x \zeta_{2x} + \zeta_x^3, \ldots, \quad \zeta = \zeta(x,t),$$

$$Y_{x,t}(\zeta) = \zeta_{x,t} + \zeta_x \zeta_t, \;\; Y_{2x,t}(\zeta) = \zeta_{2x,t} + \zeta_{2x}\zeta_t + 2\zeta_{x,t}\zeta_x + \zeta_x^2 \zeta_t, \ldots. \tag{3}$$

The multi-D BBPs may be represented as in the below

$$\Sigma_{n_1x_1,\ldots,n_jx_j}(A,B) = \Upsilon_{n_1,\ldots,n_j}(\zeta)\Big|_{\zeta_{s_1x_1,\ldots,s_jx_j}=\begin{cases} A_{s_1x_1,\ldots,s_jx_j}, & s_1+s_2+\ldots+s_j, \text{ is odd} \\ B_{s_1x_1,\ldots,s_jx_j}, & s_1+s_2+\ldots+s_j, \text{ is even.}\end{cases}} \tag{4}$$

We have the conditions given as follows:

$$\Sigma_x(A) = A_x,\ \Sigma_{2x}(A,B) = B_{2x} + A_x^2,\ \Sigma_{x,t}(A,B) = B_{x,t} + A_xA_t, \ldots. \tag{5}$$

Proposition 1. *Consider*

$$A = \ln(\Theta/\Delta),\ \ B = \ln(\Theta\Delta), \tag{6}$$

then the connection between BBPs and Hirota D-operator can be written the below

$$\Sigma_{n_1x_1,\ldots,n_jx_j}(A,B)\Big|_{A=\ln(\Theta/\Delta),\ B=\ln(\Theta\Delta)} = (\Theta\Delta)^{-1}D_{x_1}^{n_1}\ldots D_{x_j}^{n_j}\Theta\Delta, \tag{7}$$

with Hirota formula

$$\prod_{i=1}^{j} D_{x_i}^{n_i} g.\,\eta = \prod_{i=1}^{j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'}\right)^{n_i}\Theta(x_1,\ldots,x_j)\Delta(x_1',\ldots,x_j')\Big|_{x_1=x_1',\ldots,x_j=x_j'}. \tag{8}$$

Proposition 2. *Take*

$$\Xi(A) = \sum_i \delta_i \mathfrak{P}_{s_1x_1,\ldots,s_jx_j} = 0, A = \ln(\Theta/\Delta),\ \ B = \ln(\Theta\Delta), \tag{9}$$

we have

$$\begin{cases} \sum_i \delta_{1i}\Upsilon_{n_1x_1,\ldots,n_jx_j}(A,B) = 0, \\ \sum_i \delta_{1i}\Upsilon_{s_1x_1,\ldots,s_jx_j}(A,B) = 0, \end{cases} \tag{10}$$

with below conditions

$$\mathfrak{R}(\aleph',\aleph) = \mathfrak{R}(\aleph') - \mathfrak{R}(\aleph) = \mathfrak{R}(B+A) - \mathfrak{R}(B-A) = 0. \tag{11}$$

The generalized Bell polynomials $\Upsilon_{n_1x_1,\ldots,n_jx_j}(\zeta)$ *is*

$$(\Theta\Delta)^{-1}D_{x_1}^{n_1}\ldots D_{x_j}^{n_j}\Theta\Delta = \Sigma_{n_1x_1,\ldots,n_jx_j}(A,B)\Big|_{A=\ln(\Theta/\Delta),\ B=\ln(\Theta\Delta)} \tag{12}$$

$$= \Sigma_{n_1x_1,\ldots,n_jx_j}(A,A+\gamma)\Big|_{A=\ln(\Theta/\Delta),\ \gamma=\ln(\Theta\Delta)}$$

$$= \sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{i=1}^{j}\begin{pmatrix} n_i \\ k_i \end{pmatrix}\mathfrak{P}_{k_1x_1,\ldots,k_jx_j}(\gamma)\Upsilon_{(n_1-k_1)x_1,\ldots,(n_j-k_j)x_j}(A).$$

The Cole-Hopf transformation becomes

$$\Upsilon_{k_1x_1,\ldots,k_jx_j}(A=\ln(f)) = \frac{f_{n_1x_1,\ldots,n_jx_j}}{f}, \tag{13}$$

$$(\Theta\Delta)^{-1}D_{x_1}^{n_1}\ldots D_{x_j}^{n_j}\Theta\Delta\Big|_{\Delta=\exp(\gamma/2),\ \Theta/\Delta=f} \tag{14}$$

$$= f^{-1}\sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{d=1}^{j}\begin{pmatrix} n_d \\ k_d \end{pmatrix}\mathfrak{P}_{k_1x_1,\ldots,k_dx_d}(\gamma)f_{(n_1-k_1)x_1,\ldots,(n_d-k_d)x_d},$$

with

$$\Upsilon_t(A) = \frac{f_t}{f}, \quad \Upsilon_{2x}(A, \beta) = \gamma_{2x} + \frac{f_{2x}}{f}, \quad \Upsilon_{2x,y}(A, B) = \frac{\gamma_{2x} f_y}{f} + \frac{2\gamma_{x,y} f_x}{f} + \frac{f_{2x,y}}{f}. \tag{15}$$

Let us assume the following transformations to obtain the Hirota bilinear form of Equation (1).

$$g(x,y,t) = 2b(\ln f)_y + g_0, \quad h(x,y,t) = 2b(\ln f)_x + h_0. \tag{16}$$

By using Equation (16) into Equations (1), we obtain

$$2ab^2 f_{xx} f_y - 2ab^2 f f_{xxy} + 2abh_0 f_x f_y - 2abfh_0 f_{xy} + 2b^2 f_y f_{yy} - 2b^2 f f_{yyy} + 2bg_0 f_y^2 - 2bfg_0 f_{yy} - 2bf_t f_y + 2bff_{yt} = 0. \tag{17}$$

2.2. New Three-Wave Solutions

We assume the transformation given as to gain the new three-wave results [16].

$$f(x,y,t) = \kappa_2 e^{a_1 x + b_1 y + d_1 t} + \kappa_3 \sin(a_3 x + b_3 y + d_3 t) + \kappa_1 \cos(a_2 x + b_2 y + d_2 t) + e^{-(a_1 x + b_1 y + d_1 t)} \tag{18}$$

Substituting Equation (18) into Equation (17), we gain a system of equations by summing up coefficients of every power of $e^{a_1 x + b_1 y + d_1 t}$, $e^{-(a_1 x + b_1 y + d_1 t)}$, $\sin(a_3 x + b_3 y + d_3 t)$ and $\cos(a_2 x + b_2 y + d_2 t)$ and putting them equal to 0. We gain the solution sets by solving the system given as follows:

Set 1:

$$\begin{aligned} \{a_2 = -\frac{}{\sqrt{a}}, a_3 = -\frac{R}{\sqrt{a}}, d_1 = aa_1 h_0 + b_1 g_0, d_2 = b_2 g_0 - \sqrt{a} h_0 \sqrt{-aa_1^2 - b_1^2 - b_2^2}, \\ d_3 = b_3 g_0 - \sqrt{a} h_0 R, R = \sqrt{-aa_1^2 - b_1^2 - b_3^2}, H = \sqrt{-aa_1^2 - b_1^2 - b_2^2}\} \end{aligned} \tag{19}$$

$$\begin{aligned} g(x,y,t) = 2b((b_1(-\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y)) + b_1 \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) \\ -b_2 \kappa_1 \sin(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y) + b_3 \kappa_3 \cos(t(b_3 g_0 - \sqrt{a} h_0 R) - \tfrac{xR}{\sqrt{a}} + b_3 y))/ \\ (\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y) + \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) + \kappa_3 \sin(t(b_3 g_0 - \sqrt{a} h_0 R) \\ -\tfrac{xR}{\sqrt{a}} + b_3 y) + \kappa_1 \cos(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y))) + g_0 \end{aligned} \tag{20}$$

$$\begin{aligned} h(x,y,t) = 2b((a_1(-\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y)) + a_1 \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) \\ +\frac{\kappa_1 H \sin(t(b_2 g_0 - \sqrt{a} h_0 H) - \frac{xH}{\sqrt{a}} + b_2 y)}{\sqrt{a}} - \frac{\kappa_3 R \cos(t(b_3 g_0 - \sqrt{a} h_0 R) - \frac{xR}{\sqrt{a}} + b_3 y)}{\sqrt{a}})/ \\ (\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y) + \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) \\ +\kappa_3 \sin(t(b_3 g_0 - \sqrt{a} h_0 R) - \tfrac{xR}{\sqrt{a}} + b_3 y) + \kappa_1 \cos(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y))) + h_0 \end{aligned} \tag{21}$$

Set 2:

$$\begin{aligned} \{a_2 = -\tfrac{xH}{\sqrt{a}}, a_3 = \tfrac{R}{\sqrt{a}}, d_1 = aa_1 h_0 + b_1 g_0, d_2 = b_2 g_0 - \sqrt{a} h_0 H, d_3 = \sqrt{a} h_0 R + b_3 g_0, \\ H = \sqrt{-aa_1^2 - b_1^2 - b_2^2}, R = \sqrt{-aa_1^2 - b_1^2 - b_3^2}\} \end{aligned} \tag{22}$$

$$\begin{aligned} g(x,y,t) = 2b((b_1(-\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y)) + b_1 \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) - b_2 \kappa_1 \\ \sin(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y) + b_3 \kappa_3 \cos(t(\sqrt{a} h_0 R + b_3 g_0) + \tfrac{xR}{\sqrt{a}} + b_3 y))/ \\ (\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y) + \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) + \kappa_3 \sin(t(\sqrt{a} h_0 R + b_3 g_0) \\ +\tfrac{xR}{\sqrt{a}} + b_3 y) + \kappa_1 \cos(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y))) + g_0 \end{aligned} \tag{23}$$

$$\begin{aligned} h(x,y,t) = 2b((a_1(-\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y)) + a_1 \kappa_2 \exp(t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y) + \\ \frac{\kappa_1 H \sin(t(b_2 g_0 - \sqrt{a} h_0 H) - \frac{xH}{\sqrt{a}} + b_2 y)}{\sqrt{a}} + \frac{\kappa_3 R \cos(t(\sqrt{a} h_0 R + b_3 g_0) + \frac{xR}{\sqrt{a}} + b_3 y)}{\sqrt{a}})/ \\ (\exp(-t(aa_1 h_0 + b_1 g_0) - a_1 x - b_1 y) + \kappa_2 e^{t(aa_1 h_0 + b_1 g_0) + a_1 x + b_1 y} + \kappa_3 \sin(t(\sqrt{a} h_0 R + b_3 g_0) + \\ \tfrac{xR}{\sqrt{a}} + b_3 y) + \kappa_1 \cos(t(b_2 g_0 - \sqrt{a} h_0 H) - \tfrac{xH}{\sqrt{a}} + b_2 y))) + h_0 \end{aligned} \tag{24}$$

Set 3:

$$\begin{aligned}\{a_2 = \tfrac{H}{\sqrt{a}}, a_3 = -\tfrac{R}{\sqrt{a}}, d_1 = aa_1h_0 + b_1g_0, d_2 = \sqrt{a}h_0H + b_2g_0,\\ d_3 = b_3g_0 - \sqrt{a}h_0R, H = \sqrt{-aa_1^2 - b_1^2 - b_2^2}, R = \sqrt{-aa_1^2 - b_1^2 - b_3^2}\}\end{aligned} \tag{25}$$

$$\begin{aligned}g(x,y,t) = 2b((b_1(-\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y)) + b_1\kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y) - b_2\kappa_1\\ \sin(t(\sqrt{a}h_0H + b_2g_0) + \tfrac{xH}{\sqrt{a}} + b_2y) + b_3\kappa_3\cos(t(b_3g_0 - \sqrt{a}h_0R) - \tfrac{xR}{\sqrt{a}} + b_3y))/\\ (\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y) + \kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y)\\ +\kappa_3\sin(t(b_3g_0 - \sqrt{a}h_0R) - \tfrac{xR}{\sqrt{a}} + b_3y) + \kappa_1\cos(t(\sqrt{a}h_0\sqrt{-aa_1^2 - b_1^2 - b_2^2} + b_2g_0)\\ + \tfrac{xH}{\sqrt{a}} + b_2y))) + g_0\end{aligned} \tag{26}$$

$$\begin{aligned}h(x,y,t) = 2b((a_1(-\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y)) + a_1\kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y)\\ -\frac{\kappa_1H\sin(t(\sqrt{a}h_0H + b_2g_0) + \frac{xH}{\sqrt{a}} + b_2y)}{\sqrt{a}} - \frac{\kappa_3R\cos(t(b_3g_0 - \sqrt{a}h_0R) - \frac{xR}{\sqrt{a}} + b_3y)}{\sqrt{a}})/\\ (\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y) + \kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y) + \kappa_3\sin(t(b_3g_0 - \sqrt{a}h_0R)\\ - \tfrac{xR}{\sqrt{a}} + b_3y) + \kappa_1\cos(t(\sqrt{a}h_0H + b_2g_0) + \tfrac{xH}{\sqrt{a}} + b_2y))) + h_0\end{aligned} \tag{27}$$

Set 4:

$$\begin{aligned}\{a_2 = \tfrac{}{\sqrt{a}}, a_3 = \tfrac{R}{\sqrt{a}}, d_1 = aa_1h_0 + b_1g_0, d_2 = \sqrt{a}h_0H + b_2g_0, d_3 = \sqrt{a}h_0R + b_3g_0,\\ H = \sqrt{-aa_1^2 - b_1^2 - b_2^2}, R = \sqrt{-aa_1^2 - b_1^2 - b_3^2}\}\end{aligned} \tag{28}$$

$$\begin{aligned}g(x,y,t) = 2b((b_1(-\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y)) + b_1\kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y) - b_2\kappa_1\\ \sin(t(\sqrt{a}h_0H + b_2g_0) + \tfrac{xH}{\sqrt{a}} + b_2y) + b_3\kappa_3\cos(t(\sqrt{a}h_0R + b_3g_0) + \tfrac{xR}{\sqrt{a}} + b_3y))/\\ (\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y) + \kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y) + \kappa_3\sin(t(\sqrt{a}h_0R + b_3g_0)\\ + \tfrac{xR}{\sqrt{a}} + b_3y) + \kappa_1\cos(t(\sqrt{a}h_0H + b_2g_0) + \tfrac{xH}{\sqrt{a}} + b_2y))) + g_0\end{aligned} \tag{29}$$

$$\begin{aligned}h(x,y,t) = 2b((a_1(-\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y)) + a_1\kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y)\\ -\frac{\kappa_1H\sin(t(\sqrt{a}h_0H + b_2g_0) + \frac{xH}{\sqrt{a}} + b_2y)}{\sqrt{a}} + \frac{\kappa_3R\cos(t(\sqrt{a}h_0R + b_3g_0) + \frac{xR}{\sqrt{a}} + b_3y)}{\sqrt{a}})/\\ (\exp(-t(aa_1h_0 + b_1g_0) - a_1x - b_1y) + \kappa_2\exp(t(aa_1h_0 + b_1g_0) + a_1x + b_1y) + \kappa_3\sin(t(\sqrt{a}h_0R + b_3g_0) +\\ \tfrac{xR}{\sqrt{a}} + b_3y) + \kappa_1\cos(t(\sqrt{a}h_0H + b_2g_0) + \tfrac{xH}{\sqrt{a}} + b_2y))) + h_0\end{aligned} \tag{30}$$

2.3. New Periodic Wave Solutions

Consider the following transformation [17]:

$$f(x,y,t) = \kappa_1e^{a_3t + a_1x + a_2y + a_4} + e^{-(a_3t + a_1x + a_2y + a_4)} + \kappa_2\cos(p(b_3t + b_1x + b_2y + b_4)) + \kappa_3\cosh(c_3t + c_1x + c_2y + c_4) + \kappa_4 \tag{31}$$

Inserting Equation (31) into Equation (17), we obtain a system of equations by summing up coefficients of every power of $e^{a_3t + a_1x + a_2y + a_4}$, $e^{-(a_3t + a_1x + a_2y + a_4)}$, cos $(p(b_3t + b_1x + b_2y + b_4))$, $\cosh(c_3t + c_1x + c_2y + c_4)$ and putting them equal to 0. We obtain the solution sets by solving the system given as follows:

Set 1:

$$\left\{a_1 = -\frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 - i\sqrt{a}a_2h_0, b_1 = -\frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 - i\sqrt{a}b_2h_0, c_1 = -\frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 - i\sqrt{a}c_2h_0\right\} \tag{32}$$

$$\begin{aligned}g(x,y,t) = 2b((-b_2\kappa_2p\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - ic_2\kappa_3\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y\\ +ic_4) + a_2\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2\\ \cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0)\\ - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0\end{aligned} \tag{33}$$

$$
\begin{aligned}
h(x,y,t) = 2b((&\tfrac{ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) \\
&- \frac{(ia_2\kappa_1)\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \frac{ia_2x}{\sqrt{a}} + a_2y + a_4)}{\sqrt{a}} + \frac{(ia_2)\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \frac{ia_2x}{\sqrt{a}} - a_2y - a_4)}{\sqrt{a}})/ \\
&(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 \\
&-i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0
\end{aligned} \tag{34}
$$

Set 2:

$$
\left\{a_1 = -\frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 - i\sqrt{a}a_2h_0, b_1 = -\frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 - i\sqrt{a}b_2h_0, c_1 = \frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 + i\sqrt{a}c_2h_0\right\} \tag{35}
$$

$$
\begin{aligned}
g(x,y,t) = 2b((&-b_2\kappa_2 p\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + ic_2\kappa_3\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} \\
&-ic_2y - ic_4) + a_2\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4))) \\
&/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 \\
&-i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
h(x,y,t) = 2b((&\tfrac{ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} \\
&-ic_2y - ic_4) - \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \tfrac{ia_2}{\sqrt{a}}\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)) \\
&/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 \\
&-i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0
\end{aligned} \tag{37}
$$

Set 3:

$$
\left\{a_1 = -\tfrac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 - i\sqrt{a}a_2h_0, b_1 = \tfrac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 + i\sqrt{a}b_2h_0, c_1 = -\tfrac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 - i\sqrt{a}c_2h_0\right\} \tag{38}
$$

$$
\begin{aligned}
g(x,y,t) = 2b((&-b_2\kappa_2 p\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - ic_2\kappa_3\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) \\
&+a_2\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2 \\
&\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) \\
&-\tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
h(x,y,t) = 2b((&\tfrac{-ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y \\
&+ic_4) - \frac{ia_2\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \frac{ia_2x}{\sqrt{a}} + a_2y + a_4)}{\sqrt{a}} + \frac{ia_2\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \frac{ia_2x}{\sqrt{a}} - a_2y - a_4)}{\sqrt{a}})/ \\
&(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 \\
&-i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0
\end{aligned} \tag{40}
$$

Set 4:

$$
\left\{a_1 = -\frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 - i\sqrt{a}a_2h_0, b_1 = \frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 + i\sqrt{a}b_2h_0, c_1 = \frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 + i\sqrt{a}c_2h_0\right\} \tag{41}
$$

$$
\begin{aligned}
g(x,y,t) = 2b((&-b_2\kappa_2 p\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + ic_2\kappa_3\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} \\
&-ic_2y - ic_4) + a_2\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} \\
&-a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) \\
&+\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
h(x,y,t) = 2b((&\tfrac{-ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} \\
&-ic_2y - ic_4) - \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \tfrac{ia_2}{\sqrt{a}}\exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y \\
&-a_4))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) \\
&+\kappa_1\exp(t(a_2g_0 - i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 - i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0
\end{aligned} \tag{43}
$$

Set 5:

$$
\left\{a_1 = \frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 + i\sqrt{a}a_2h_0, b_1 = -\frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 - i\sqrt{a}b_2h_0, c_1 = -\frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 - i\sqrt{a}c_2h_0\right\} \tag{44}
$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - ic_2\kappa_3\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + a_2\kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0 \quad (45)$$

$$h(x,y,t) = 2b((\tfrac{ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) - \tfrac{ia_2}{\sqrt{a}}\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4))/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0 \quad (46)$$

Set 6:

$$\left\{a_1 = \frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 + i\sqrt{a}a_2h_0, b_1 = -\frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 - i\sqrt{a}b_2h_0, c_1 = \frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 + i\sqrt{a}c_2h_0\right\} \quad (47)$$

$$g(x,y,t) = 2((-b_2\kappa_2 p\sin(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + ic_2\kappa_3\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + a_2\kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0 \quad (48)$$

$$h(x,y,t) = 2b((\tfrac{ib_2\kappa_2 p}{\sqrt{a}}\sin p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \tfrac{-ia_2}{\sqrt{a}}\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4))/(\kappa_2\cos(p(t(b_2g_0 - i\sqrt{a}b_2h_0) - \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0 \quad (49)$$

Set 7:

$$\left\{a_1 = \frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 + i\sqrt{a}a_2h_0, b_1 = \frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 + i\sqrt{a}b_2h_0, c_1 = -\frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 - i\sqrt{a}c_2h_0\right\} \quad (50)$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - ic_2\kappa_3\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + a_2\kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0 \quad (51)$$

$$h(x,y,t) = 2b((-\tfrac{ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) - \tfrac{ia_2}{\sqrt{a}}\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(it(c_2g_0 - i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} + ic_2y + ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0 \quad (52)$$

Set 8:

$$\left\{a_1 = \frac{ia_2}{\sqrt{a}}, a_3 = a_2g_0 + i\sqrt{a}a_2h_0, b_1 = \frac{ib_2}{\sqrt{a}}, b_3 = b_2g_0 + i\sqrt{a}b_2h_0, c_1 = \frac{ic_2}{\sqrt{a}}, c_3 = c_2g_0 + i\sqrt{a}c_2h_0\right\} \quad (53)$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + ic_2\kappa_3\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + a_2\kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + g_0 \quad (54)$$

$$h(x,y,t) = 2b((\tfrac{-ib_2\kappa_2 p}{\sqrt{a}}\sin(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) - \tfrac{c_2\kappa_3}{\sqrt{a}}\sin(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \tfrac{ia_2\kappa_1}{\sqrt{a}}\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) - \tfrac{ia_2}{\sqrt{a}}\exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4))/(\kappa_2\cos(p(t(b_2g_0 + i\sqrt{a}b_2h_0) + \tfrac{ib_2x}{\sqrt{a}} + b_2y + b_4)) + \kappa_3\cos(-it(c_2g_0 + i\sqrt{a}c_2h_0) + \tfrac{c_2x}{\sqrt{a}} - ic_2y - ic_4) + \kappa_1\exp(t(a_2g_0 + i\sqrt{a}a_2h_0) + \tfrac{ia_2x}{\sqrt{a}} + a_2y + a_4) + \exp(-t(a_2g_0 + i\sqrt{a}a_2h_0) - \tfrac{ia_2x}{\sqrt{a}} - a_2y - a_4) + \kappa_4)) + h_0 endarray \quad (55)$$

Set 9:

$$\{a_3 = a_2g_0 + aa_1h_0, b_1 = -\frac{A}{\sqrt{a}p}, b_3 = b_2g_0 - \frac{\sqrt{a}h_0A}{p}, c_1 = -\frac{B}{\sqrt{a}}, c_3 = c_2g_0 - \sqrt{a}h_0B,$$
$$\kappa_4 = 0, A = \sqrt{-aa_1^2 - a_2^2 - b_2^2p^2}, B = \sqrt{aa_1^2 + a_2^2 - c_2^2}\} \tag{56}$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) - \frac{xA}{\sqrt{a}p} + b_2y + b_4)) + c_2\kappa_3\sinh(t(c_2g_0 - \sqrt{a}h_0B)$$
$$-\frac{xB}{\sqrt{a}} + c_2y + c_4) + a_2\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + a_2(-\exp(-t(a_2g_0 + aa_1h_0)$$
$$+a_1(-x) - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) - \frac{xA}{\sqrt{a}p} + b_2y + b_4)) + \kappa_3\cosh(t(c_2g_0 - \sqrt{a}h_0B)$$
$$-\frac{xB}{\sqrt{a}} + c_2y + c_4) + \kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + \exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4))) + g_0 \tag{57}$$

$$h(x,y,t) = 2b((\frac{\kappa_2A\sin(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) - \frac{xA}{\sqrt{a}p} + b_2y + b_4))}{\sqrt{a}} - \frac{\kappa_3B\sinh(t(c_2g_0 - \sqrt{a}h_0B) - \frac{xB}{\sqrt{a}} + c_2y + c_4)}{\sqrt{a}}$$
$$+a_1\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + a_1(-\exp(-t(a_2g_0 + aa_1h_0) + a_1(-x) - a_2y - a_4)))/$$
$$(\kappa_2\cos(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) - \frac{xA}{\sqrt{a}p} + b_2y + b_4)) + \kappa_3\cosh(t(c_2g_0 - \sqrt{a}h_0B) - \frac{xB}{\sqrt{a}} + c_2y + c_4) +$$
$$\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + \exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4))) + h_0 \tag{58}$$

Set 10:

$$\{a_3 = a_2g_0 + aa_1h_0, b_1 = -\frac{A}{\sqrt{a}p}, b_3 = b_2g_0 - \frac{\sqrt{a}h_0A}{p}, c_1 = \frac{B}{\sqrt{a}}, c_3 = \sqrt{a}h_0B + c_2g_0,$$
$$\kappa_4 = 0, A = \sqrt{-aa_1^2 - a_2^2 - b_2^2p^2}, B = \sqrt{aa_1^2 + a_2^2 - c_2^2}\} \tag{59}$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) + b_1x + b_2y + b_4)) + c_2\kappa_3\sinh(t(\sqrt{a}h_0$$
$$B + c_2g_0) + \frac{xB}{\sqrt{a}} + c_2y + c_4) + a_2\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4)$$
$$+a_2(-\exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) + b_1x + b_2y$$
$$+b_4)) + \kappa_3\cosh(t(\sqrt{a}h_0B + c_2g_0) + \frac{xB}{\sqrt{a}} + c_2y + c_4) + \kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x$$
$$+a_2y + a_4) + \exp(-t(a_2g_0 + aa_1h_0) + a_1(-x) - a_2y - a_4))) + g_0 \tag{60}$$

$$h(x,y,t) = 2b((-b_1\kappa_2 p\sin(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) + b_1x + b_2y + b_4)) + \frac{\kappa_3B}{\sqrt{a}}$$
$$\sinh(t(\sqrt{a}h_0B + c_2g_0) + \frac{xB}{\sqrt{a}} + c_2y + c_4) + a_1\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4)$$
$$+a_1(-\exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4)))/(\kappa_2\cos(p(t(b_2g_0 - \frac{\sqrt{a}h_0A}{p}) + b_1x + b_2y$$
$$+b_4)) + \kappa_3\cosh(t(\sqrt{a}h_0B + c_2g_0) + \frac{xB}{\sqrt{a}} + c_2y + c_4) + \kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x$$
$$+a_2y + a_4) + \exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4))) + h_0 \tag{61}$$

Set 11:

$$\{a_3 = a_2g_0 + aa_1h_0, b_1 = \frac{A}{\sqrt{a}p}, b_3 = \frac{\sqrt{a}h_0A}{p} + b_2g_0,$$
$$c_1 = -\frac{B}{\sqrt{a}}, c_3 = c_2g_0 - \sqrt{a}h_0B, \kappa_4 = 0, A = \sqrt{-aa_1^2 - a_2^2 - b_2^2p^2}, B = \sqrt{aa_1^2 + a_2^2 - c_2^2}\} \tag{62}$$

$$g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(\frac{\sqrt{a}h_0A}{p} + b_2g_0) + b_1x + b_2y + b_4)) + c_2\kappa_3\sinh(t(c_2g_0$$
$$-\sqrt{a}h_0B) - \frac{xB}{\sqrt{a}} + c_2y + c_4) + a_2\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4)$$
$$+a_2(-\exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4)))/(\kappa_2\cos(p(t(\frac{\sqrt{a}h_0A}{p} + b_2g_0) + b_1x + b_2y + b_4))$$
$$+\kappa_3\cosh(t(c_2g_0 - \sqrt{a}h_0B) - \frac{xB}{\sqrt{a}} + c_2y + c_4) + \kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4)$$
$$+\exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4))) + g_0 \tag{63}$$

$$h(x,y,t) = 2b((-b_1\kappa_2 p\sin(p(t(\frac{\sqrt{a}h_0A}{p} + b_2g_0) + b_1x + b_2y + b_4)) - \frac{\kappa_3B}{\sqrt{a}}\sinh(t(c_2g_0 - \sqrt{a}h_0B)$$
$$-\frac{xB}{\sqrt{a}} + c_2y + c_4) + a_1\kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + a_1(-\exp(-t(a_2g_0 + aa_1h_0)$$
$$-a_1x - a_2y - a_4)))/(\kappa_2\cos(p(t(\frac{\sqrt{a}h_0A}{p} + b_2g_0) + b_1x + b_2y + b_4)) + \kappa_3\cosh(t(c_2g_0 - \sqrt{a}h_0B)$$
$$-\frac{xB}{\sqrt{a}} + c_2y + c_4) + \kappa_1\exp(t(a_2g_0 + aa_1h_0) + a_1x + a_2y + a_4) + \exp(-t(a_2g_0 + aa_1h_0) - a_1x - a_2y - a_4))) + h_0 \tag{64}$$

Set 12:

$$\{a_3 = a_2g_0 + aa_1h_0, b_1 = \frac{A}{\sqrt{a}p}, b_3 = \frac{\sqrt{a}h_0A}{p} + b_2g_0, c_1 = \frac{B}{\sqrt{a}},$$
$$c_3 = \sqrt{a}h_0B + c_2g_0, \kappa_4 = 0, A = \sqrt{-aa_1^2 - a_2^2 - b_2^2p^2}, B = \sqrt{aa_1^2 + a_2^2 - c_2^2}\} \tag{65}$$

$$\begin{aligned}g(x,y,t) = 2b((-b_2\kappa_2 p\sin(p(t(\tfrac{\sqrt{a}h_0A}{p}+b_2g_0)+\tfrac{xA}{\sqrt{ap}}+b_2y+b_4))\\ +c_2\kappa_3\sinh(t(\sqrt{a}h_0B+c_2g_0)+\tfrac{xB}{\sqrt{a}}+c_2y+c_4)+a_2\kappa_1\exp(t(a_2g_0+aa_1h_0)\\ +a_1x+a_2y+a_4)+a_2(-\exp(-t(a_2g_0+aa_1h_0)-a_1x-a_2y-a_4)))/(\kappa_2\cos(p(t(\tfrac{\sqrt{a}h_0A}{p}+b_2g_0)\\ +\tfrac{xA}{\sqrt{ap}}+b_2y+b_4))+\kappa_3\cosh(t(\sqrt{a}h_0B+c_2g_0)+\tfrac{xB}{\sqrt{a}}+c_2y+c_4)\\ +\kappa_1\exp(t(a_2g_0+aa_1h_0)+a_1x+a_2y+a_4)+\exp(-t(a_2g_0+aa_1h_0)-a_1x-a_2y-a_4)))+g_0\end{aligned} \tag{66}$$

$$\begin{aligned}h(x,y,t) = 2b((-\kappa_2A\sin(p(t(\tfrac{\sqrt{a}h_0A}{p}+b_2g_0)+\tfrac{xA}{\sqrt{ap}}+b_2y+b_4))+\kappa_3B\sinh(t(\sqrt{a}h_0B+c_2g_0)\\ +\tfrac{xB}{\sqrt{a}}+c_2y+c_4)+\sqrt{a}a_1\kappa_1\exp(t(a_2g_0+aa_1h_0)+a_1x+a_2y+a_4)+\sqrt{a}a_1(-\exp(-t(a_2g_0+aa_1h_0)\\ -a_1x-a_2y-a_4)))/(\sqrt{a}(\kappa_2\cos(p(t(\tfrac{\sqrt{a}h_0A}{p}+b_2g_0)+\tfrac{x\sqrt{-aa_1^2-a_2^2-b_2^2p^2}}{\sqrt{ap}}+b_2y+b_4))+\\ \kappa_3\cosh(t(\sqrt{a}h_0B+c_2g_0)+\tfrac{xB}{\sqrt{a}}+c_2y+c_4)+\kappa_1\exp(t(a_2g_0+aa_1h_0)+a_1x+a_2y+a_4)+\exp(-t(a_2g_0+aa_1h_0)\\ -a_1x-a_2y-a_4))))+h_0\end{aligned} \tag{67}$$

3. Analytical Solitons

Assume the wave transformation given below.

$$g(x,y,t) = G(\zeta); \quad h(x,y,t) = H(\zeta), \qquad \zeta = \tau(x+\Omega y+\lambda t). \tag{68}$$

By using Equation (68) into Equation (1), we obtain:

$$\begin{aligned}-\lambda G' + \Omega GG' + aHG' + b\tau\Omega^2G'' + ab\tau G'' = 0.\\ G' = \Omega H'.\end{aligned} \tag{69}$$

After integrating and taking integration constant zero, we obtain:

$$\begin{aligned}-2\Omega\lambda G + (\Omega^2+a)G^2 + 2(b\tau\Omega^3 + ab\Omega\tau)G' = 0.\\ \tfrac{G}{\Omega} = H.\end{aligned} \tag{70}$$

3.1. EShGEE Scheme

The main points of this scheme are given as:

Step 1:

Consider a non-linear PDE:

$$V(h,h^2,h^2h_x,h_y,h_{yy},h_{xx},h_{xy},h_{xt},\ldots) = 0, \tag{71}$$

where $h = h(x,y,t)$ denotes the wave function.

Assuming the traveling wave transformation:

$$h(x,y,t) = H(\xi), \;\; \xi = x - \nu y + \kappa t. \tag{72}$$

Putting Equation (72) into Equation (71), we gain the nonlinear ODE:

$$V(H(\xi), H^2(\xi)H^{'}(\xi), H^{''}(\xi), \ldots) = 0. \tag{73}$$

Step 2:

Assuming the results of Equation (73) in the series form:

$$F(p) = \alpha_0 + \sum_{i=1}^{m}(\beta_i\sinh(p) + \alpha_i\cosh(p))^i, \tag{74}$$

where α_0, α_i, β_i $(i = 1,2,3,\ldots,\ m)$ are unknowns. Consider a function p of ξ that satisfy the given equation:

$$\frac{dp}{d\zeta} = \sinh(p). \tag{75}$$

Natural number m can be attain with the use of homogeneous balance approach. Equation (75) is gained from sinh-Gordon equation shown as:

$$q_{xt} = \kappa\sinh(v). \tag{76}$$

By [24], we obtain the solutions of Equation (76) shown as:

$$\sinh p(\xi) = \pm \operatorname{csch}(\xi) \quad or \quad \cosh p(\xi) = \pm \coth(\xi), \tag{77}$$

and

$$\sinh p(\xi) = \pm\imath\, sech(\xi) \quad or \quad \cosh p(\xi) = \pm \tanh(\xi), \tag{78}$$
$$\imath^2 = -1.$$

Step 3:

Putting Equation (74), along with Equation (75), into Equation (73), we achieve the algebraic equations having $p'^k(\zeta) \sinh^l p(\zeta) \cosh^m p(\zeta)$ $(k = 0,1;\ l = 0,1;\ m = 0,1,2,\dots)$. Putting the every coefficient of $p'^k(\zeta) \sinh^l p(\zeta) \cosh^m p(\zeta)$ equal to 0, to achieve system of algebraic equations having $\nu, \kappa, \alpha_0, \alpha_i$ and $\beta_i (i = 1,2,3,\dots,m)$.

Step 4:

Solving the gained system of algebraic equations, we attain the value of $\nu, \kappa, \alpha_0, \alpha_i$ and β_i.

Step 5:

By achieved solutions for Equations (77) and (78), we obtain the wave solitons of Equation (71) shown as:

$$F(\xi) = \alpha_0 + \sum_{i=1}^{m} (\pm\beta_i \operatorname{csch}(\xi) \pm \alpha_i \coth(\xi))^i. \tag{79}$$

and

$$F(\xi) = \alpha_0 + \sum_{i=1}^{m} (\pm\imath\beta_i sech(\xi) \pm \alpha_i \tanh(\xi))^i. \tag{80}$$

By this technique, we can obtain the sech, csch, tanh and coth functions involving solutions.

3.2. Application of EShGEE Scheme

For $m = 1$, Equations (74), (79) and (80) change into:

$$G(\zeta) = \alpha_0 \pm \beta_1 \operatorname{csch}(\zeta) \pm \alpha_1 \coth(\zeta). \tag{81}$$

$$G(\zeta) = \alpha_0 \pm \imath\beta_1 sech(\zeta) \pm \alpha_1 \tanh(\zeta). \tag{82}$$

$$G(\zeta) = \alpha_0 + \beta_1 \sinh(p) + \alpha_1 \cosh(p). \tag{83}$$

where α_0, α_1 *and* β_1 are undetermined. Utilizing Equation (83) into Equation (70), we attain algebraic equations containing $\alpha_0, \alpha_1, \beta_1$ and other parameters. By using Mathematica software, we obtain sets:

Set 1:

$$\left\{\alpha_0 = -2b\tau\Omega, \alpha_1 = -2b\tau\Omega, \beta_1 = 0, \lambda = -2b\tau\left(a + \Omega^2\right)\right\} \tag{84}$$

$$g_1(x,y,t) = -2b\tau\Omega(1 \pm \coth(\tau(x + \Omega y - 2b\tau\left(a + \Omega^2\right)t))) \tag{85}$$

$$g_2(x,y,t) = -2b\tau\Omega(1 \pm \tanh(\tau(x + \Omega y - 2b\tau\left(a + \Omega^2\right)t))) \tag{86}$$

$$h1(x,y,t) = -2b\tau(1 \pm \coth(\tau(x + \Omega y - 2b\tau\left(a + \Omega^2\right)t))) \tag{87}$$

$$h_2(x,y,t) = -2b\tau(1 \pm \tanh(\tau(x + \Omega y - 2b\tau\left(a + \Omega^2\right)t))) \tag{88}$$

Set 2:

$$\left\{\alpha_0 = 2b\tau\Omega, \alpha_1 = -2b\tau\Omega, \beta_1 = 0, \lambda = 2b\tau\left(a + \Omega^2\right)\right\} \tag{89}$$

$$g_1(x,y,t) = 2b\tau\Omega(1 \mp \coth(\tau(x + \Omega y + 2b\tau\left(a + \Omega^2\right)t))) \tag{90}$$

$$g_2(x,y,t) = 2b\tau\Omega(1 \mp \tanh(\tau(x + \Omega y + 2b\tau\left(a + \Omega^2\right)t))) \tag{91}$$

$$h_1(x,y,t) = 2b\tau(1 \mp \coth(\tau(x + \Omega y + 2b\tau\left(a + \Omega^2\right)t))) \tag{92}$$

$$h_2(x,y,t) = 2b\tau(1 \mp \tanh(\tau(x + \Omega y + 2b\tau\left(a + \Omega^2\right)t))) \tag{93}$$

Set 3:

$$\left\{\alpha_0=-b\tau\Omega,\alpha_1=-b\tau\Omega,\beta_1=b\tau\Omega,\lambda=-b\tau\left(a+\Omega^2\right)\right\} \tag{94}$$

$$g_1(x,y,t)=-b\tau\Omega((1\pm\operatorname{csch}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t)))\pm\coth(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{95}$$

$$g_2(x,y,t)=-b\tau\Omega((1\pm\imath\operatorname{sech}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t)))\pm\tanh(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{96}$$

$$h_1(x,y,t)=-b\tau((1\pm\operatorname{csch}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t)))\pm\coth(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{97}$$

$$h_2(x,y,t)=-b\tau((1\pm\imath\operatorname{sech}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t)))\pm\tanh(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{98}$$

Set 4:

$$\left\{\alpha_0=b\tau\Omega,\alpha_1=-b\tau\Omega,\beta_1=-b\tau\Omega,\lambda=b\tau\left(a+\Omega^2\right)\right\} \tag{99}$$

$$g_1(x,y,t)=b\tau\Omega((1\mp\operatorname{csch}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\coth(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{100}$$

$$g_2(x,y,t)=b\tau\Omega((1\mp\imath\operatorname{sech}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\tanh(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{101}$$

$$h_1(x,y,t)=b\tau((1\mp\operatorname{csch}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\coth(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{102}$$

$$h_2(x,y,t)=b\tau((1\mp\imath\operatorname{sech}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\tanh(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{103}$$

Set 5:

$$\left\{\alpha_0=b\tau\Omega,\alpha_1=-b\tau\Omega,\beta_1=b\tau\Omega,\lambda=b\tau\left(a+\Omega^2\right)\right\} \tag{104}$$

$$g_1(x,y,t)=b\tau\Omega((1\pm\operatorname{csch}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\coth(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{105}$$

$$g_2(x,y,t)=b\tau\Omega((1\pm\imath\operatorname{sech}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\tanh(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{106}$$

$$h_1(x,y,t)=b\tau((1\pm\operatorname{csch}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\coth(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{107}$$

$$h_2(x,y,t)=b\tau((1\pm\imath\operatorname{sech}(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t)))\mp\tanh(\tau(x+\Omega y+b\tau\left(a+\Omega^2\right)t))) \tag{108}$$

Set 6:

$$\left\{\alpha_0=-b\tau\Omega,\alpha_1=-b\tau\Omega,\beta_1=-b\tau\Omega,\lambda=-b\tau\left(a+\Omega^2\right)\right\} \tag{109}$$

$$g_1(x,y,t)=-b\tau\Omega(1\pm\coth(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))\pm\operatorname{csch}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{110}$$

$$g_2(x,y,t)=-b\tau\Omega(1\pm\imath\operatorname{sech}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))\pm\tanh(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{111}$$

$$h_1(x,y,t)=-b\tau(1\pm\coth(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))\pm\operatorname{csch}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{112}$$

$$h_2(x,y,t)=-b\tau(1\pm\imath\operatorname{sech}(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))\pm\tanh(\tau(x+\Omega y-b\tau\left(a+\Omega^2\right)t))) \tag{113}$$

4. Graphical Representation of Solutions

In this section, some suitable numerical values of parameters selected for our analytical three wave solitons and periodic wave solutions are represented in the following Figures. In Figure 1, we gave graphs for new three wave solution (20), with suitable values of parameters, (a) in $y=-2$, (b) in $y=0$ and (c) in $y=2$ with $a=1,a_1=1,a_2=1,\kappa_3=-2,\kappa_1=1,b_1=1,b_2=1,b_3=1,$ $g_0=1,h_0=1,\kappa_2=1,b=1\ \ -10<t<10$ and $-10<x<10$. In Figure 2, we gave graphs for new three wave solution (20), with suitable values of parameters, (a) in $x=-2$, (b) in $x=0$ and (c) in $x=2$ with $a=1,a_1=1,a_2=1,\kappa_3=-2,\kappa_1=1,b_1=1,b_2=1,b_3=1,g_0=1,h_0=1,\kappa_2=1,b=1$

$-10 < t < 10$ and $-10 < Y < 10$. In Figure 3, we gave graphs for new three wave solution (20), with suitable values of parameters, (a) in $t = -2$, (b) in $t = 0$ and (c) in $t = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1$ $-10 < x < 10$ and $-10 < y < 10$. In Figure 4, we gave graphs for new three wave solution (21), with suitable values of parameters, (a) in $y = -2$, (b) in $y = 0$ and (c) in $y = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1$ $-10 < t < 10$ and $-10 < x < 10$. In Figure 5, we gave graphs for new three wave solution (21), with suitable values of parameters, (a) in $x = -2$, (b) in $x = 0$ and (c) in $x = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1$ $-10 < t < 10$ and $-10 < Y < 10$. In Figure 6, we gave graphs for new three wave solution (21), with suitable values of parameters, (a) in $t = -2$, (b) in $t = 0$ and (c) in $t = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1$ $-10 < x < 10$ and $-10 < y < 10$. In Figure 7, we gave graphs for new periodic solution (33), with suitable values of parameters, (a) in $y = -2$, (b) in $y = 0$ and (c) in $y = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1, \kappa_4 = 1, c_4 = 2, a_4 = 2, c_2 = 1, p = 1$ $-30 < t < 30$ and $-30 < x < 30$. In Figure 8, we gave graphs for new periodic solution (33), with suitable values of parameters, (a) in $x = -2$, (b) in $x = 0$ and (c) in $x = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1, \kappa_4 = 1, c_4 = 2, a_4 = 2, c_2 = 1, p = 1$ $-20 < t < 20$ and $-20 < Y < 20$. In Figure 9, we gave graphs for new periodic solution (33), with suitable values of parameters, (a) in $t = -2$, (b) in $t = 0$ and (c) in $t = 2$ with $a = 1, a_1 = 1, a_2 = 1, \kappa_3 = -2, \kappa_1 = 1, b_1 = 1, b_2 = 1, b_3 = 1, g_0 = 1, h_0 = 1, \kappa_2 = 1, b = 1, \kappa_4 = 1, c_4 = 2, a_4 = 2, c_2 = 1, p = 1$ $-10 < x < 10$ and $-10 < y < 10$. In Figure 10, we gave graphs for solution (84), with suitable values of parameters, a 3D graph shown in (a) with $\Omega = 1, b = 1, \tau = 0.1, a = 0.5, y = 0$ $-5 < x < 5$ and $-5 < t < 5$ and a 2D graph shown in (b) with $\Omega = 1, b = 1, \tau = 0.1, a = 0.5, y = 0$ $-10 < x < 10$ and $-1 < t < 1$. In Figure 11, we gave graphs for solution (95), with suitable values of parameters, a 3D graph shown in (a) with $\Omega = 2.3, b = 1.4, \tau = 0.1, a = 0.5, y = 0$ $-5 < x < 5$ and $-5 < t < 5$ and a 2D graph shown in (b) with $\Omega = 2.3, b = 1.4, \tau = 0.1, a = 0.5, y = 0$ $-10 < x < 10$ and $-1 < t < 1$.

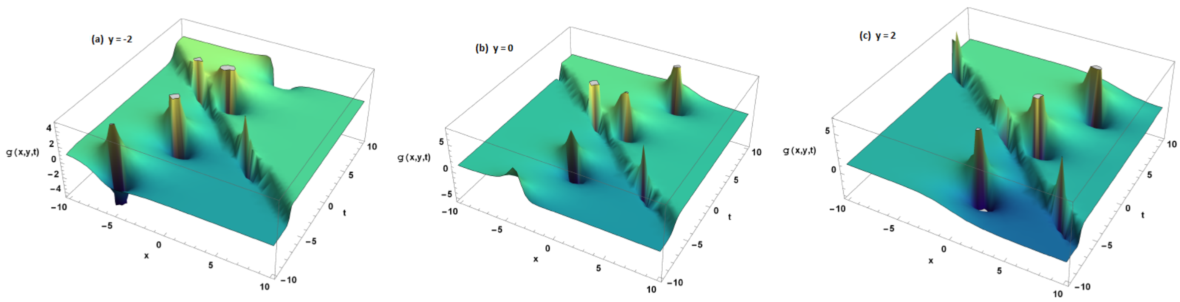

Figure 1. In this graph shows three wave behavior with different fix values of y in (**a**) $y = -2$, in (**b**) $y = 0$ and in (**c**) $y = 2$ contour surfaces representation of (20).

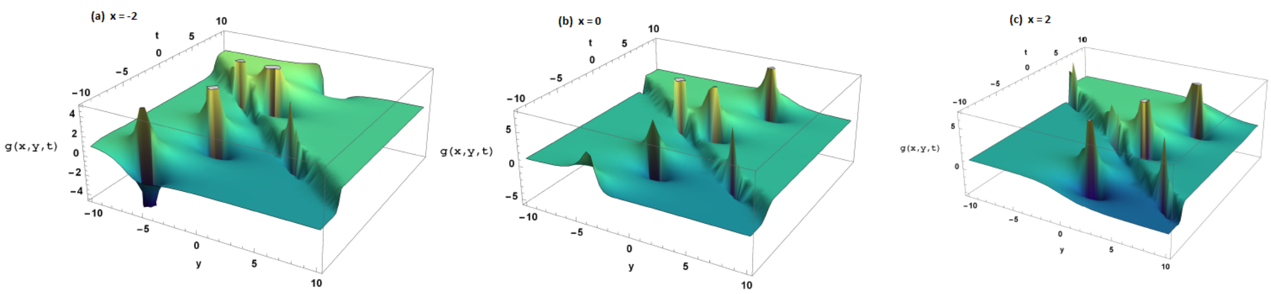

Figure 2. In this graph shows three wave behavior with different fix values of x in (**a**) $x = -2$, in (**b**) $x = 0$ and in (**c**) $x = 2$ contour surfaces representation of (20).

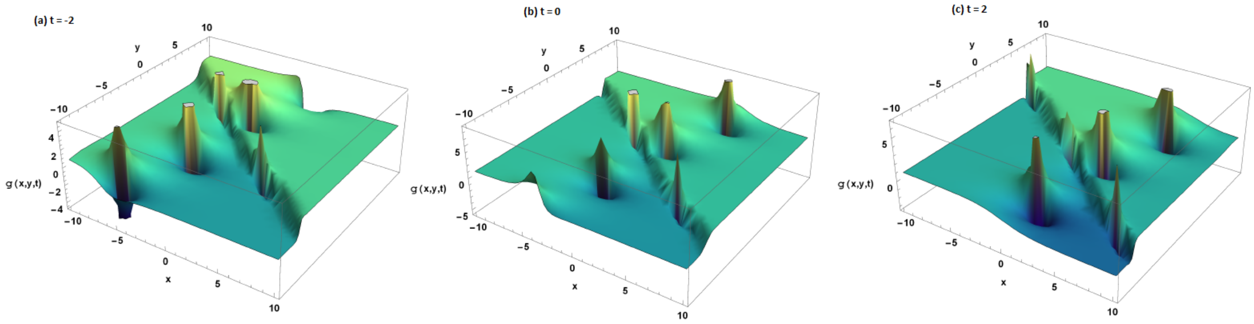

Figure 3. In this graph shows three wave behavior with different fix values of t in (**a**) $t = -2$, in (**b**) $t = 0$ and in (**c**) $t = 2$ contour surfaces representation of (20).

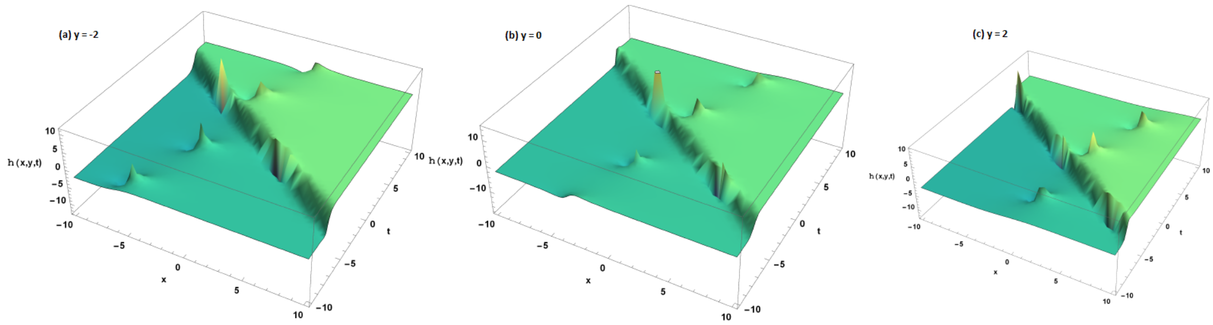

Figure 4. In this graph shows three wave behavior with different fix values of y in (**a**) $y = -2$, in (**b**) $y = 0$ and in (**c**) $y = 2$ contour surfaces representation of (21).

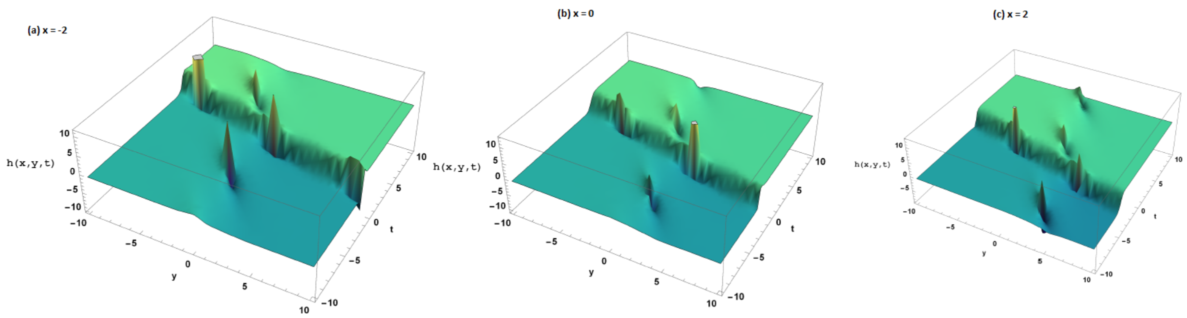

Figure 5. In this graph shows three wave behavior with different fix values of x in (**a**) $x = -2$, in (**b**) $x = 0$ and in (**c**) $x = 2$ contour surfaces representation of (21).

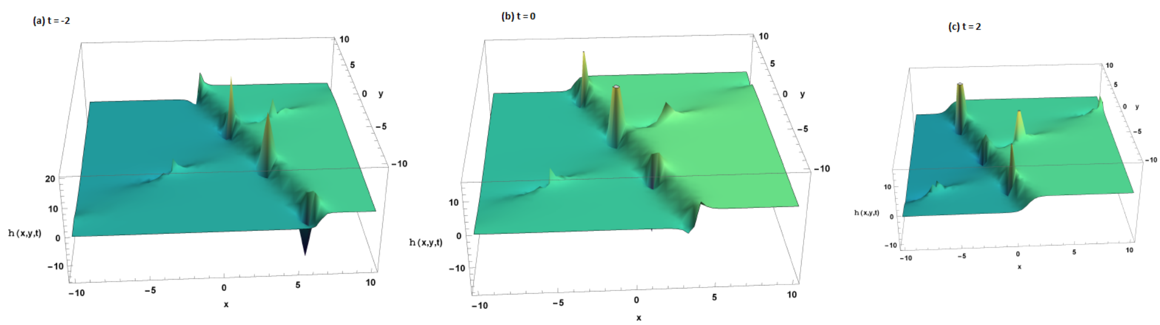

Figure 6. In this graph shows three wave behavior with different fix values of t in (**a**) $t = -2$, in (**b**) $t = 0$ and in (**c**) $t = 2$ contour surfaces representation of (21).

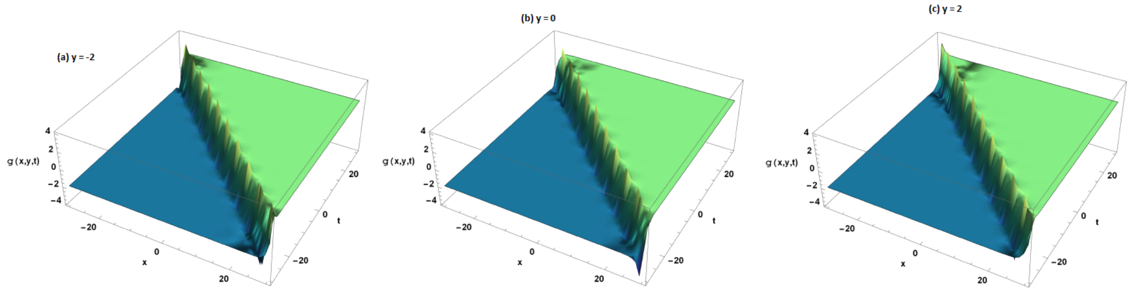

Figure 7. In this graph shows periodic wave behavior with different fix values of y in (**a**) $y = -2$, in (**b**) $y = 0$ and in (**c**) $y = 2$ contour surfaces representation of (33).

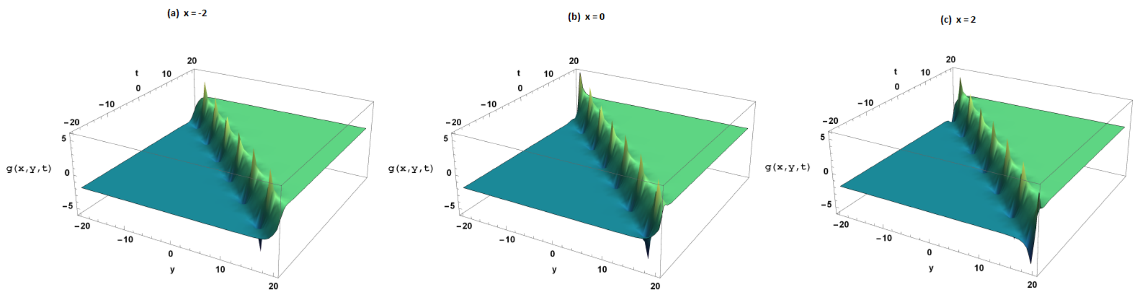

Figure 8. In this graph shows periodic wave behavior with different fix values of x in (**a**) $x = -2$, in (**b**) $x = 0$ and in (**c**) $x = 2$ contour surfaces representation of (33).

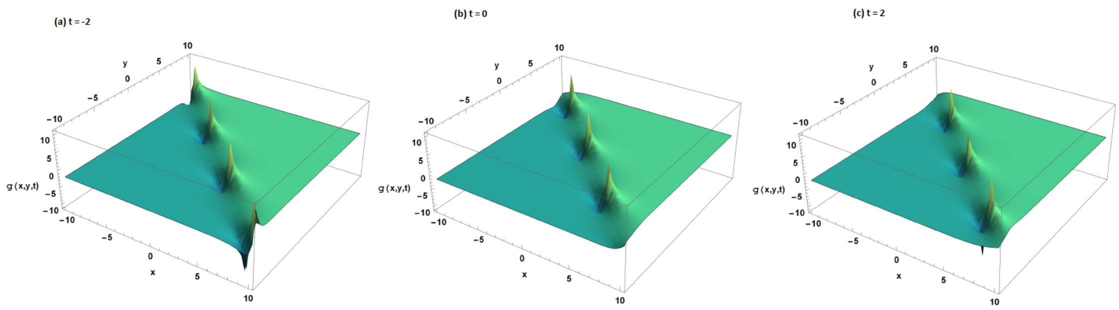

Figure 9. In this graph shows periodic wave behavior with different fix values of t in (**a**) $t = -2$, in (**b**) $t = 0$ and in (**c**) $t = 2$ contour surfaces representation of (33).

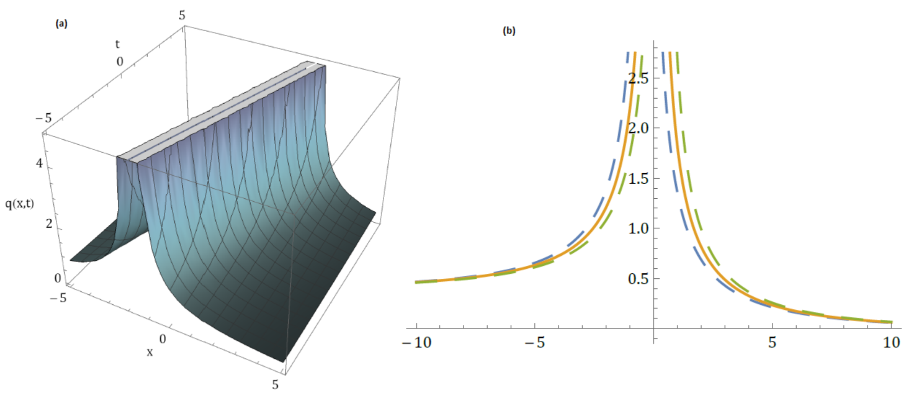

Figure 10. In this graph shows wave behavior 3D in (**a**) and 2D in (**b**) surfaces representation of (84).

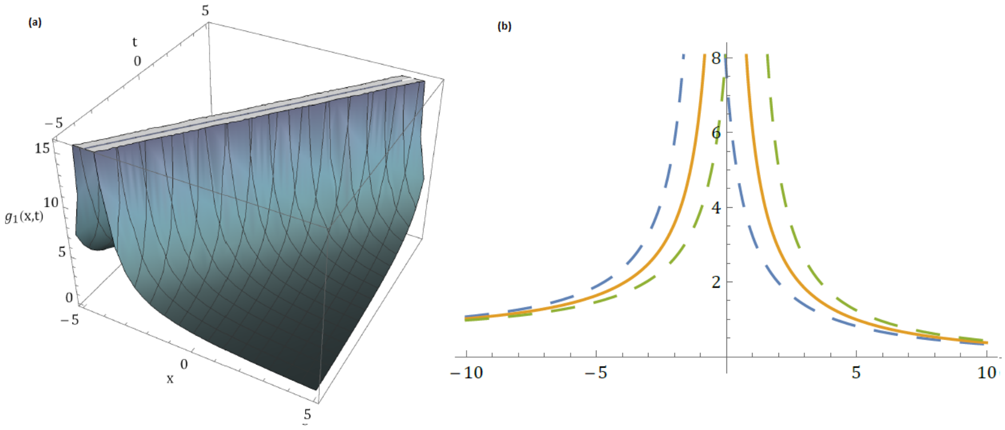

Figure 11. In this figure graph shows wave behavior 3D in (**a**) and 2D in (**b**) surfaces representation of (95).

5. Conclusions

In this paper, we succeed to obtain the new three wave, periodic wave and other exact wave solutions to the (2+1)-dimensional Burgers equations by utilizing the Hirota bilinear and EShGEE schemes. Achieved solutions are verified and demonstrated by different plots with the use of Mathematica software. Some of the achieved solutions are also described graphically by two-dimensional, three-dimensional and contour plots. Our research model Burgers equation is a fundamental partial differential equation from fluid mechanics. It occurs in many fields of applied mathematics such as modeling of gas dynamics, traffic flow, etc. The schemes that are used in this research have never been applied to this model before, and the obtained results are newer than the existing results of this model in the literature. The diverse graphical analyses for the presented solutions show that the solutions are reliable for the further development in the model, and also in other areas of mathematical physics and engineering; for example, in communication, optical computing, optical switching, etc. Finally, these two schemes are simple, fruitful and reliable to handle the nonlinear PDEs.

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