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Results of Hyperbolic Ricci Solitons

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Abstract: We obtain some properties of a hyperbolic Ricci soliton with certain types of potential vector fields, and we point out some conditions when it reduces to a trivial Ricci soliton. We also study those soliton submanifolds whose vector fields are the tangential components of a concurrent vector field on the ambient manifold, and in particular, we show that a totally umbilical hyperbolic Ricci soliton is an Einstein manifold. We prove that if the hyperbolic Ricci soliton hypersurface of a Riemannian manifold of constant curvature and endowed with a concurrent vector field has a parallel shape operator, then it is a metallic-shaped hypersurface, and we determine some conditions for it to be minimal. Moreover, we show that it is also a pseudosymmetric hypersurface.

Keywords: hyperbolic Ricci soliton; concurrent vector field; submanifold; hypersurface

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1. Introduction

Stationary solutions to the hyperbolic Ricci flow [1–3], hyperbolic Ricci solitons, have been very recently considered in [4], where the authors studied the three-dimensional homogeneous case of manifolds with Riemannian and Lorentzian metric tensors. When the potential vector field of a soliton is of a certain type, more information about the geometry of the manifold can be obtained.

Nevertheless, smooth manifolds can be sometimes characterized by the existence of certain distinguished vector fields on them. Concircular vector fields are of special interest, having interesting applications in physics. For instance, in [5], B.-Y. Chen proved that a Lorentzian manifold is a generalized Robertson–Walker spacetime if and only if it admits a time-like concircular vector field. Concurrent vector fields are particular cases of concircular vector fields, such an example being provided by the position vector field in the Euclidean space. More generally, any warped product $I \times_s M$ of an open real interval I, with s as its arclength, and an arbitrary Riemannian manifold (M, g) endowed with the metric given by $h = ds^2 + s^2 g$, admits a concurrent vector field, namely $s\frac{\partial}{\partial s}$.

Having these in mind, the aim of this paper is to present some properties of hyperbolic Ricci solitons with a torse-forming potential vector field, or with particular cases of it. We also investigate the geometry of a submanifold, which is a hyperbolic Ricci soliton that is isometrically immersed into a Riemannian manifold and whose potential vector field is the tangential component of a concurrent vector field on the ambient space. In particular, we show that a totally umbilical hyperbolic Ricci soliton is an Einstein manifold. We also prove that if a hyperbolic Ricci soliton hypersurface of a Riemannian manifold of constant curvature and endowed with a concurrent vector field has parallel shape operator, then it is a metallic-shaped hypersurface, and we determine some conditions for it to be minimal. Moreover, we show that it is also a pseudosymmetric hypersurface.

2. Hyperbolic Ricci Solitons

Let (M, g) be a Riemannian manifold. We shall denote by $\chi(M)$ the set of smooth sections on M and by ∇ the Levi-Civita connection of g.

We recall that $(M, g, \xi, \lambda, \mu)$ is called a *hyperbolic Ricci soliton* [4] if the vector field ξ and the scalars $\lambda, \mu \in \mathbb{R}$ satisfy

$$\pounds_\xi \pounds_\xi g + \lambda \pounds_\xi g + \mathrm{Ric} = \mu g, \tag{1}$$

where Ric is the Ricci curvature of (M, g).

We remark that if ξ is a Killing vector field, i.e., $\pounds_\xi g = 0$, then the manifold is an Einstein manifold, and if ξ is a 2-Killing vector field [6], i.e., $\pounds_\xi \pounds_\xi g = 0$ and $\lambda = \frac{1}{2}$, then the manifold is a Ricci soliton.

Considering particular cases of vector fields, we recover the mixed generalized quasi-Einstein, generalized quasi-Einstein and Einstein manifolds, whose definitions we shall briefly recall.

Definition 1. *Let (M, g) be a nonflat Riemannian manifold of dimension $n \geq 3$. Then, M is said to be a mixed generalized quasi-Einstein manifold [7] if its Ricci tensor field* Ric *is not identically zero and verifies*

$$\mathrm{Ric} = \alpha g + \beta A \otimes A + \gamma B \otimes B + \delta(A \otimes B + B \otimes A),$$

where α, β, γ and δ are smooth functions and A, B are 1-forms on M. In particular, the manifold M is called

1. *generalized quasi-Einstein [8] if $\beta = 0$ or $\gamma = 0$;*
2. *quasi-Einstein [9] if $\beta = \delta = 0$ or $\gamma = \delta = 0$;*
3. *Einstein [10] if $\beta = \gamma = \delta = 0$.*

2.1. The Trivial Case

It would be interesting to find more relaxed, nontrivial conditions under which a hyperbolic Ricci soliton reduces to an Einstein manifold. For the compact case, we prove the following results:

Proposition 1. *Let $(M, g, \xi, \lambda, \mu)$ be a compact and connected hyperbolic Ricci soliton with $\lambda \neq 0$. If the second Lie derivative of g in the direction of ξ is divergence-free and the scalar curvature r is constant on the integral curves of ξ, then ξ is a parallel vector field, i.e., $\nabla \xi = 0$, and M is a Ricci-flat manifold.*

Proof. Since $\mathrm{div}(\pounds_\xi \pounds_\xi g) = 0$, then $\mathrm{tr}(\pounds_\xi \pounds_\xi g)$ is a constant. By taking the divergence into the soliton Equation (1) and taking into account that $\mathrm{div}(\mathrm{Ric}) = \frac{dr}{2}$, we infer

$$\lambda\, \mathrm{div}(\pounds_\xi g) = -\frac{dr}{2}.$$

We know [11] that

$$(\mathrm{div}(\pounds_\xi g))(\xi) = 2[\xi(\mathrm{div}(\xi)) + \mathrm{Ric}(\xi, \xi)]$$

and we obtain

$$-4\lambda[\xi(\mathrm{div}(\xi)) + \mathrm{Ric}(\xi, \xi)] = \xi(r)(= 0).$$

By means of Bochner's formula,

$$\frac{1}{2}\Delta(\|\xi\|^2) = \|\nabla \xi\|^2 + \xi(\mathrm{div}(\xi)) + \mathrm{Ric}(\xi, \xi),$$

we obtain
$$\lambda\left(\frac{1}{2}\Delta(\|\xi\|^2) - \|\nabla\xi\|^2\right) = 0,$$
which, since $\lambda \neq 0$, by integration with respect to the canonical measure, gives
$$\int_M \|\nabla\xi\|^2 = 0,$$
hence, $\nabla\xi = 0$. Then, $\operatorname{div}(\xi) = 0$, $\pounds_\xi g = 0$, $\mathrm{Ric} = \mu g$ and since $\|\xi\|^2$ is a constant, from Bochner's formula, we find
$$2\mu\|\xi\|^2 = \Delta(\|\xi\|^2)(= 0),$$
hence the conclusion. □

A compact and connected hyperbolic Ricci soliton reduces to a Ricci-flat manifold in the following case too:

Proposition 2. *Let $(M^n, g, \xi, \lambda, \mu)$ be a compact and connected hyperbolic Ricci soliton with $\lambda \neq 0$. If the second Lie derivative of g in the direction of ξ has a constant trace, the scalar curvature r is constant on the integral curves of ξ and $\mathrm{Ric}(\xi, \xi) \geq 0$, then ξ is a parallel vector field and M is a Ricci-flat manifold.*

Proof. By taking the trace into the soliton Equation (1), we infer
$$\lambda \operatorname{div}(\xi) = \frac{n\mu - \operatorname{tr}(\pounds_\xi \pounds_\xi g)}{2} - \frac{r}{2},$$
which implies
$$\lambda\xi(\operatorname{div}(\xi)) = -\frac{\xi(r)}{2}(= 0).$$

Then, Bochner's formula becomes
$$\frac{1}{2}\Delta(\|\xi\|^2) = \|\nabla\xi\|^2 + \mathrm{Ric}(\xi, \xi),$$
which, by integration with respect to the canonical measure, gives
$$\int_M (\|\nabla\xi\|^2 + \mathrm{Ric}(\xi, \xi)) = 0,$$
hence, $\nabla\xi = 0$ and $\mathrm{Ric}(\xi, \xi) = 0$. Then, $\operatorname{div}(\xi) = 0$, $\pounds_\xi g = 0$, $\mu = 0$ and $\mathrm{Ric} = 0$, hence the conclusion. □

We end this section by bringing into light the connection between the hyperbolic Ricci solitons that satisfy $\operatorname{div}(\pounds_\xi \pounds_\xi g) = 0$ and the Schrödinger–Ricci equation [12]. The nonlinear Schrödinger equation mathematically describes the wave propagation, modeling the nonlinear effects, e.g., the like formation of solitons. Closely related to Ricci-type solitons, the Schrödinger–Ricci equation has, under certain assumptions, the dual 1-form of the potential vector field ξ as a solution.

Proposition 3. *Let $(M^n, g, \xi, \lambda, \mu)$ be a connected hyperbolic Ricci soliton with $\lambda \neq 0$. If the second Lie derivative of g in the direction of ξ is divergence-free, then the dual 1-form η of ξ is a solution of the Schrödinger–Ricci equation.*

Proof. Since $\operatorname{div}(\pounds_\xi \pounds_\xi g) = 0$, $\operatorname{tr}(\pounds_\xi \pounds_\xi g)$ is a constant. By taking the divergence into the soliton Equation (1) and taking into account that $\operatorname{div}(\mathrm{Ric}) = \frac{dr}{2}$, we infer
$$\operatorname{div}(\pounds_\xi g) = -\frac{dr}{2\lambda}. \tag{2}$$

Now, taking the trace into the soliton Equation (1), we obtain

$$\mathrm{tr}(\pounds_\xi \pounds_\xi g) + 2\lambda\, \mathrm{div}(\xi) + r = n\mu,$$

which, by differentiating, gives

$$2\lambda d(\mathrm{div}(\xi)) + dr = 0. \tag{3}$$

We know (see, e.g., [13]) that

$$\mathrm{div}(\pounds_\xi g) = (\Delta + \mathrm{Ric}_\sharp)(\eta) + d(\mathrm{div}(\xi)),$$

where Δ is the Laplace–Hodge operator on differential forms with respect to the metric g and $\mathrm{Ric}_\sharp(\eta)(X) := \mathrm{Ric}(\xi, X)$ for $X \in \chi(M)$; from (2) and (3), we obtain

$$(\Delta + \mathrm{Ric}_\sharp)(\eta) = 0,$$

that is, η is a solution of the Schrödinger–Ricci equation. □

2.2. Solitons with Torse-Forming Vector Field

Definition 2 ([14]). *Let (M, g) be a Riemannian manifold. A smooth vector field ξ on M is said to be torse-forming if $\nabla\xi = aI + \psi \otimes \xi$, where a is a smooth function and ψ is a 1-form; concircular if $\nabla\xi = aI$; and concurrent if $\nabla\xi = I$, where ∇ is the Levi-Civita connection of g and I is the identity endomorphism.*

Next we will prove the following results concerning torse-forming vector fields:

Proposition 4. *Let ξ be a torse-forming vector field satisfying*

$$\nabla\xi = aI + \psi \otimes \xi,$$

where a is a smooth function and ψ is a 1-form on the Riemannian manifold (M, g). Then,

$$\begin{aligned} a &= \frac{\xi(\|\xi\|^2)}{2\|\xi\|^2} - \psi(\xi), \\ \pounds_\xi \psi &= i_{(\nabla_\xi \zeta + (a+\psi(\xi))\zeta)} g, \\ \pounds_\xi g &= 2ag + \psi \otimes \eta + \eta \otimes \psi, \\ \pounds_\xi \pounds_\xi g &= 2\big(2a^2 + \xi(a)\big)g + 2\|\xi\|^2 \psi \otimes \psi + \big(4a + \psi(\xi)\big)(\psi \otimes \eta + \eta \otimes \psi) \\ &\quad + (\pounds_\xi \psi) \otimes \eta + \eta \otimes (\pounds_\xi \psi), \end{aligned}$$

where ζ is the dual vector field of ψ and η is the dual 1-form of ξ.

Proof. We have

$$\xi(\|\xi\|^2) = 2g(\nabla_\xi \xi, \xi) = 2\big(a + \psi(\xi)\big)\|\xi\|^2,$$

hence,

$$a = \frac{\xi(\|\xi\|^2)}{2\|\xi\|^2} - \psi(\xi).$$

We also have

$$\begin{aligned} (\pounds_\xi \psi)X :&= \xi(\psi(X)) - \psi([\xi, X]) \\ &= \xi(g(\zeta, X)) - g(\zeta, \nabla_\xi X) + g(\zeta, \nabla_X \xi) \\ &= g(\nabla_\xi \zeta, X) + ag(\zeta, X) + \psi(X)g(\zeta, \xi) \\ &= g(\nabla_\xi \zeta, X) + ag(\zeta, X) + g(\zeta, X)\psi(\xi) \end{aligned}$$

for any $X \in \chi(M)$. Also,

$$\begin{aligned}(\pounds_\xi g)(X,Y) :&= \xi(g(X,Y)) - g([\xi,X],Y) - g(X,[\xi,Y]) \\ &= g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi) \\ &= 2ag(X,Y) + \psi(X)g(\xi,Y) + \psi(Y)g(X,\xi)\end{aligned}$$

for any $X, Y \in \chi(M)$. Finally, we obtain the expression of the second Lie derivative of g in the direction of ξ:

$$\begin{aligned}(\pounds_\xi\pounds_\xi g)(X,Y) :&= \xi((\pounds_\xi g)(X,Y)) - (\pounds_\xi g)([\xi,X],Y) - (\pounds_\xi g)(X,[\xi,Y]) \\ &= \xi(2ag(X,Y) + \psi(X)\eta(Y) + \psi(Y)\eta(X)) \\ &\quad - 2ag([\xi,X],Y) - \psi([\xi,X])\eta(Y) - \psi(Y)\eta([\xi,X]) \\ &\quad - 2ag(X,[\xi,Y]) - \psi(X)\eta([\xi,Y]) - \psi([\xi,Y])\eta(X) \\ &= 2\xi(a)g(X,Y) + \psi(X)(\xi(g(Y,\xi)) - g([\xi,Y],\xi)) \\ &\quad + \psi(Y)(\xi(g(X,\xi)) - g([\xi,X],\xi)) \\ &\quad + \eta(Y)(\xi(\psi(X)) - \psi([\xi,X])) + \eta(X)(\xi(\psi(Y)) - \psi([\xi,Y])) \\ &\quad + 2a(g(aX + \psi(X)\xi,Y) + g(aY + \psi(Y)\xi,X)) \\ &= 2\xi(a)g(X,Y) + \psi(X)(g(Y,a\xi + \psi(\xi)\xi) + g(aY + \psi(Y)\xi,\xi)) \\ &\quad + \psi(Y)(g(X,a\xi + \psi(\xi)\xi) + g(aX + \psi(X)\xi,\xi)) \\ &\quad + \eta(Y)(\pounds_\xi\psi)X + \eta(X)(\pounds_\xi\psi)Y + 4a^2g(X,Y) \\ &\quad + 2a(\psi(X)\eta(Y) + \psi(Y)\eta(X)) \\ &= 2\Big(2a^2 + \xi(a)\Big)g(X,Y) + (4a + \psi(\xi))(\psi(X)\eta(Y) + \psi(Y)\eta(X)) \\ &\quad + 2\psi(X)\psi(Y)\|\xi\|^2 + \eta(Y)(\pounds_\xi\psi)X + \eta(X)(\pounds_\xi\psi)Y\end{aligned}$$

for any $X, Y \in \chi(M)$, and we obtain the conclusion. □

From the above proposition, we deduce the following:

Theorem 1. *If $(M, g, \xi, \lambda, \mu)$ is a hyperbolic Ricci soliton with a torse-forming potential vector field ξ, then*

$$\begin{aligned}\mathrm{Ric} &= \big(\mu - 4a^2 - 2\xi(a) - 2a\lambda\big)g - 2\|\xi\|^2\psi\otimes\psi \\ &\quad - \big(4a + \psi(\xi) + \lambda\big)(\psi\otimes\eta + \eta\otimes\psi) \\ &\quad - \big(\iota_{(\nabla_\xi\zeta + (a+\psi(\xi))\zeta)}g\big)\otimes\eta - \eta\otimes\big(\iota_{(\nabla_\xi\zeta + (a+\psi(\xi))\zeta)}g\big).\end{aligned} \tag{4}$$

From (4), we conclude the following:

Proposition 5. *(i) If ξ is concircular (i.e., $\psi = 0$; hence, $\zeta = 0$), then*

$$\mathrm{Ric} = \big(\mu - 4a^2 - 2\xi(a) - 2a\lambda\big)g,$$

hence, (M, g) is an Einstein manifold provided that $\dim(M) > 2$.

Even more generally, if ξ is a conformal vector field, $\pounds_\xi g = 2ag$, for a a smooth function on M, then $\pounds_\xi\pounds_\xi g = 2\big(2a^2 + \xi(a)\big)g$, and we obtain the same conclusion as before.

(ii) If $\eta = \psi$, and hence $\zeta = \xi$, then

$$\mathrm{Ric} = \big(\mu - 4a^2 - 2\xi(a) - 2a\lambda\big)g - 2\Big(4\|\xi\|^2 + 6a + \lambda\Big)\eta\otimes\eta,$$

hence, (M, g) is a quasi-Einstein manifold.

(iii) If $\pounds_\xi \psi = k\psi$ *with k a smooth function on M, then*

$$\begin{aligned}\mathrm{Ric} = &\left(\mu - 4a^2 - 2\xi(a) - 2a\lambda\right)g - 2\|\xi\|^2\psi \otimes \psi \\ &- \left(4a + \psi(\xi) + \lambda + k\right)(\psi \otimes \eta + \eta \otimes \psi),\end{aligned}$$

hence, (M, g) *is a generalized quasi-Einstein manifold.*

(iv) If $\pounds_\xi \psi = h\eta + k\psi$ *with h and k smooth functions on M, then*

$$\begin{aligned}\mathrm{Ric} = &\left(\mu - 4a^2 - 2\xi(a) - 2a\lambda\right)g - 2\|\xi\|^2\psi \otimes \psi - 2h\eta \otimes \eta \\ &- \left(4a + \psi(\xi) + \lambda + k\right)(\psi \otimes \eta + \eta \otimes \psi),\end{aligned}$$

hence, (M, g) *is a mixed generalized quasi-Einstein manifold.*

Lemma 1. *If the second Lie derivative of g in the direction of a torse-forming vector field* ξ *is traceless;* ξ *is torqued, i.e.,* $\psi(\xi) = 0$*; and a is constant on the integral curves of* ξ*, then* ξ *is a parallel vector field.*

Proof. By taking the trace of $\pounds_\xi \pounds_\xi g$, we obtain

$$\begin{aligned}\mathrm{tr}(\pounds_\xi \pounds_\xi g) &= 2n\left(2a^2 + \xi(a)\right) + 2\|\xi\|^2\|\zeta\|^2 + 2(2a - \psi(\xi))\psi(\xi) + 2\xi(\psi(\xi)) \\ &= 4na^2 + 2\|\xi\|^2\|\zeta\|^2,\end{aligned}$$

hence, $a = 0$ and $\zeta = 0$, so $\nabla\xi = 0$. □

Using the above lemma, we can state the following:

Proposition 6. *Let* $(M, g, \xi, \lambda, \mu)$ *be a hyperbolic Ricci soliton with a torqued potential vector field* ξ*. If the second Lie derivative of g in the direction of* ξ *is traceless and a is constant on the integral curves of* ξ*, then* ξ *is a parallel vector field and M is a Ricci-flat manifold.*

Proof. We get $\nabla\xi = 0$, which implies $\mathrm{div}(\xi) = 0$, $\pounds_\xi g = 0$ and $\mathrm{Ric} = \mu g$, and since $\|\xi\|^2$ is a constant, from Bochner's formula, we find

$$2\mu\|\xi\|^2 = \Delta(\|\xi\|^2)(= 0),$$

hence the conclusion. □

Now we shall provide a characterization for the Einstein case.

Proposition 7. *Let* $(M, g, \xi, \lambda, \mu)$ *be a hyperbolic Ricci soliton with a torse-forming potential vector field* ξ *satisfying* $\nabla\xi = I \pm \eta \otimes \xi$ *such that* $\|\xi\|^2 \neq n$*, where* $n = \dim(M)$*. Then, M is an Einstein manifold if and only if the scalar curvature r of M is* $r = \pm n(n-1)$*.*

Proof. In this case, $a = 1$, $\psi = \pm\eta$, $\zeta = \pm\xi$ and we have

$$\pounds_\xi g = 2(g - \eta \otimes \eta), \;\; \pounds_\xi \pounds_\xi g = 4(g - \eta \otimes \eta), \;\; \pounds_\xi \eta = 0.$$

Then, (4) becomes

$$\mathrm{Ric} = \left(\mu - 2\lambda - 4\right)g + 2(\lambda + 2)\eta \otimes \eta, \tag{5}$$

hence, (M, g) is a quasi-Einstein manifold. Also, by a direct computation, we obtain

$$\begin{aligned}&R(X, Y)\xi = \pm\left(\eta(Y)X - \eta(X)Y\right), \\ &\mathrm{Ric}(Y, \xi) = \pm(n-1)\eta(Y),\end{aligned}$$

for any $X, Y \in \chi(M)$, hence,

$$\mathrm{Ric}(\xi, \xi) = \pm(n-1)\|\xi\|^2.$$

But from the soliton equation, we have

$$\mathrm{Ric}(\xi, \xi) = \mu\|\xi\|^2,$$

hence, $\mu = \pm(n-1)$. By taking the trace in (5), we find

$$\lambda = \frac{r}{2(\|\xi\|^2 - n)} - \frac{4\|\xi\|^2 - n(4 \mp (n-1))}{2(\|\xi\|^2 - n)}.$$

Replacing λ and μ in (5), we infer

$$\mathrm{Ric} = \frac{r \mp (n-1)\|\xi\|^2}{n - \|\xi\|^2} g - \frac{r \mp n(n-1)}{n - \|\xi\|^2} \eta \otimes \eta,$$

and we obtain the conclusion. $\square$

A particular case of torse-forming vector field is the Reeb vector field of a Kenmotsu manifold.

Example 1. *If $(M^n, \phi, \xi, \eta, g, \lambda, \mu)$ (for $n \geq 3$ odd) is a Kenmotsu hyperbolic Ricci soliton with the Reeb potential vector field ξ, since $\|\xi\|^2 = 1$, we obtain $\lambda = -2$ and $\mu = -(n-1)$.*

Based on Theorem 3.1. from [4], we shall give another two examples of nontrivial hyperbolic Ricci solitons.

Example 2. *Let $\mathcal{G}_1$ be the three-dimensional unimodular Lorentzian Lie algebra described by*

$$[e_1, e_2] = e_1, \quad [e_2, e_3] = e_2 + e_3, \quad [e_3, e_1] = e_1,$$

where $\{e_1, e_2, e_3\}$ is a suitable pseudo-orthonormal basis with e_3 a time-like vector field on the Minkowski space. Then, $(\xi, 1, 0)$ defines a hyperbolic Ricci soliton for $\xi = e_2 + e_3$.

Example 3. *Let $\mathcal{G}_1$ be the three-dimensional unimodular Lorentzian Lie algebra described by*

$$[e_1, e_2] = e_1 - \sqrt{2}e_3, \quad [e_2, e_3] = \sqrt{2}e_1 + e_2 + e_3, \quad [e_3, e_1] = e_1 + \sqrt{2}e_2,$$

where $\{e_1, e_2, e_3\}$ is a suitable pseudo-orthonormal basis with e_3 a time-like vector field on the Minkowski space. Then, $(\xi, 0, -1)$ defines a hyperbolic Ricci soliton for $\xi = \sqrt{2}e_1$.

3. Submanifolds as Hyperbolic Ricci Solitons

Let M be an isometrically immersed submanifold of a Riemannian manifold $(\bar{M}, \bar{g})$ and denote the induced metric on M with g. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections on (M, g) and $(\bar{M}, \bar{g})$, respectively. Then, the Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

where h is the second fundamental form and A_N is the shape operator in the direction of the normal vector field N, defined by $g(A_N X, Y) = \bar{g}(h(X, Y), N)$ for $X, Y \in \chi(M)$ [15].

We assume now that $\xi \in \chi(\bar{M})$ is a concurrent vector field, i.e., $\bar{\nabla}\xi = I$. Then, for any $X \in \chi(M)$, we have:

$$\nabla_X \xi^{\top} + h(X, \xi^{\top}) - A_{\xi^{\perp}} X + \nabla^{\perp}_X \xi^{\perp} = X,$$

hence,
$$\nabla_X \xi^\top = X + A_{\xi^\perp} X.$$

We consequently obtain
$$\begin{aligned}
(\pounds_{\xi^\top} g)(X,Y) &= g(\nabla_X \xi^\top, Y) + g(X, \nabla_Y \xi^\top) \\
&= 2g(X,Y) + 2g(A_{\xi^\perp} X, Y), \\
\xi^\top((\pounds_{\xi^\top} g)(X,Y)) &= 2\xi^\top(g(X,Y)) + 2\xi^\top(g(A_{\xi^\perp} X, Y)) \\
&= 2g(\nabla_{\xi^\top} X, Y) + 2g(X, \nabla_{\xi^\top} Y) \\
&\quad + 2g(\nabla_{\xi^\top}(A_{\xi^\perp} X), Y) + 2g(A_{\xi^\perp} X, \nabla_{\xi^\top} Y), \\
(\pounds_{\xi^\top} g)([\xi^\top, X], Y) &= 2g([\xi^\top, X], Y) + 2g(A_{\xi^\perp}[\xi^\top, X], Y) \\
&= 2g(\nabla_{\xi^\top} X, Y) - 2g(\nabla_X \xi^\top, Y) \\
&\quad + 2g(A_{\xi^\perp}(\nabla_{\xi^\top} X), Y) - 2g(A_{\xi^\perp}(\nabla_X \xi^\top), Y) \\
&= 2g(\nabla_{\xi^\top} X, Y) - 2g(X,Y) - 2g(A_{\xi^\perp} X, Y) \\
&\quad + 2g(A_{\xi^\perp} Y, \nabla_{\xi^\top} X) - 2g(A_{\xi^\perp} X, Y) - 2g(A^2_{\xi^\perp} X, Y), \\
(\pounds_{\xi^\top} g)(X, [\xi^\top, Y]) &= 2g(\nabla_{\xi^\top} Y, X) - 2g(X,Y) - 2g(A_{\xi^\perp} X, Y) \\
&\quad + 2g(A_{\xi^\perp} X, \nabla_{\xi^\top} Y) - 2g(A_{\xi^\perp} X, Y) - 2g(A^2_{\xi^\perp} X, Y)
\end{aligned}$$

for any $X, Y \in \chi(M)$. Then,
$$\begin{aligned}
(\pounds_{\xi^\top} \pounds_{\xi^\top} g)(X,Y) &= 4g(X,Y) + 4g(A^2_{\xi^\perp} X, Y) + 8g(A_{\xi^\perp} X, Y) \\
&\quad + 2g((\nabla_{\xi^\top} A_{\xi^\perp}) X, Y)
\end{aligned} \tag{6}$$

for any $X, Y \in \chi(M)$.

From the above computations, we obtain the following:

Theorem 2. *Let M be a submanifold that is isometrically immersed into a Riemannian manifold $(\bar{M}, \bar{g})$, and let ξ be a concurrent vector field on $\bar{M}$. Then, $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton if and only if the Ricci tensor field of M satisfies*
$$\begin{aligned}
\mathrm{Ric}_M(X,Y) &= (\mu - 4 - 2\lambda) g(X,Y) - 4g(A^2_{\xi^\perp} X, Y) - 2(4+\lambda) g(A_{\xi^\perp} X, Y) \\
&\quad - 2g((\nabla_{\xi^\top} A_{\xi^\perp}) X, Y)
\end{aligned} \tag{7}$$

for any $X, Y \in \chi(M)$.

Obviously, if $A_{\xi^\perp} = 0$ (in particular, if M is totally geodesic, i.e., $A_N = 0$ for any normal vector field N), then it is an Einstein manifold provided that $n = \dim(M) > 2$, of scalar curvature $r_M = n(\mu - 4 - 2\lambda)$.

If M is totally umbilical, then $A_{\xi^\perp} = fI$, where f is a smooth function on M and I is the identity map. So,
$$\begin{aligned}
g(A^2_{\xi^\perp} X, Y) &= f^2 g(X,Y), \\
g(\nabla_{\xi^\top}(A_{\xi^\perp} X), Y) &= \xi^\top(f) g(X,Y) + f g(\nabla_{\xi^\top} X, Y), \\
g(A_{\xi^\perp} X, Y) &= f g(X,Y), \\
g(A_{\xi^\perp} Y, \nabla_{\xi^\top} X) &= f g(\nabla_{\xi^\top} X, Y),
\end{aligned}$$

and we have
$$\mathrm{Ric}_M = \Big(\mu - 4 - 2\lambda - 4f^2 - 2f(4+\lambda) - 2\xi^\top(f)\Big) g.$$

Then, we conclude the following:

Proposition 8. *A totally umbilical hyperbolic Ricci soliton $(M, g, \xi^\top, \lambda, \mu)$ that is isometrically immersed into a Riemannian manifold $(\bar{M}, \bar{g})$ with a concurrent vector field ξ is an Einstein manifold provided that* $\dim(M) > 2$.

From (6), we notice that if $\xi^\top$ is a 2-Killing vector field and $\nabla A_{\xi^\perp} = 0$, then

$$g(X, Y) + g(A^2_{\xi^\perp} X, Y) + 2g(A_{\xi^\perp} X, Y) = 0$$

for any $X, Y \in \chi(M)$. By taking the trace, we find

$$n + \mathrm{tr}\left(A^2_{\xi^\perp}\right) + 2\,\mathrm{tr}\left(A_{\xi^\perp}\right) = 0,$$

and we can state the following:

Proposition 9. *There does not exist a* 2*-Killing vector field $\xi^\top$ on a minimal submanifold with parallel shape operator that is isometrically immersed into a Riemannian manifold endowed with a concurrent vector field ξ.*

3.1. Submanifolds of Manifolds with Constant Curvature

We shall further consider hyperbolic Ricci solitons M^n that are isometrically immersed into a Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c. From (7), since

$$\mathrm{Ric}_M(X, Y) = (n-1)cg(X, Y) + n\bar{g}(h(X, Y), H) - \sum_{i=1}^{n} \bar{g}(h(X, e_i), h(Y, e_i))$$

for any $X, Y \in \chi(M)$, where $\{e_i\}_{1 \leq i \leq n}$ is a local orthonormal frame field on M and H is the mean curvature vector field defined by $H := \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$, we obtain the following:

Proposition 10. *If M^n is a submanifold that is isometrically immersed into a Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, and ξ is a concurrent vector field on $\bar{M}$, then $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton if and only if*

$$\begin{aligned} n\bar{g}(h(X, Y), H) - \sum_{i=1}^{n} \bar{g}(h(X, e_i), h(Y, e_i)) &= (\mu - 4 - 2\lambda - (n-1)c)g(X, Y) \\ &\quad - 4g(A^2_{\xi^\perp} X, Y) - 2(4 + \lambda)g(A_{\xi^\perp} X, Y) - 2g((\nabla_{\xi^\top} A_{\xi^\perp})X, Y) \end{aligned}$$

for any $X, Y \in \chi(M)$.

Then, we have the following:

Corollary 1. *Let M^n be a totally geodesic submanifold that is isometrically immersed into a Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, and let ξ be a concurrent vector field on $\bar{M}$. Then, $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton if and only if*

$$\mu - 2\lambda = 4 + (n-1)c.$$

Now we will assume that M is a hypersurface with parallel shape operator A that is isometrically immersed into a Riemannian manifold $(\bar{M}^{n+1}, \bar{g})$, and we consider ξ a concurrent vector field on $\bar{M}$. If M has parallel shape operator, i.e., $\nabla A_{\xi^\perp} = 0$, then Equation (7) can be written as

$$\mathrm{Ric}_M(X, Y) = (\mu - 4 - 2\lambda)g(X, Y) - 4g(A^2_{\xi^\perp} X, Y) - 2(4 + \lambda)g(A_{\xi^\perp} X, Y)$$

for any $X, Y \in \chi(M)$, and we can state the following:

Proposition 11. *Let M be a hypersurface that is isometrically immersed into a Riemannian manifold $(\bar{M}^{n+1}(c), \bar{g})$ of constant curvature c, and let ξ be a concurrent vector field on $\bar{M}$. If M has parallel shape operator, then*

$$A^2_{\xi^\perp} = \frac{\left(-2(4+\lambda)\rho^2 - \operatorname{tr}\left(A_{\xi^\perp}\right)\right)}{4\rho^2 - 1} A_{\xi^\perp} + \frac{(\mu - 4 - 2\lambda - (n-1)c)\rho^2}{4\rho^2 - 1} I \tag{8}$$

provided that $\rho(x) \neq \frac{1}{2}$ for any $x \in M$, where $\rho := \|\xi^\perp\|$.

If $\rho = \frac{1}{2}$, then $A_{\xi^\perp} = \dfrac{(\mu - 4 - 2\lambda - (n-1)c)}{4\left(4 + \lambda + 2\operatorname{tr}\left(A_{\xi^\perp}\right)\right)} I$ provided that $\lambda \neq -2\left(\operatorname{tr}\left(A_{\xi^\perp}\right) + 2\right)$ or $\lambda = -2\left(\operatorname{tr}\left(A_{\xi^\perp}\right) + 2\right)$ and $\mu = -4\left(\operatorname{tr}\left(A_{\xi^\perp}\right) + 1\right) + (n-1)c$.

3.2. Metallic-Shaped Hypersurfaces

We recall that a hypersurface M is called a *metallic-shaped hypersurface* (see [16]) if its shape operator A satisfies

$$A^2 = pA + qI, \tag{9}$$

where $p, q \in \mathbb{R}$. For certain values of p and q, they are called *golden shaped* ($p = q = 1$) (see [17]), *silver shaped* ($p = 2$ and $q = 1$), *bronze shaped* ($p = 3$ and $q = 1$), *copper shaped* ($p = 1$ and $q = 2$) and *nickel shaped* ($p = 1$ and $q = 3$) (see [16]).

By using (8) and (9), we have the following:

Proposition 12. *Let M be a connected hypersurface that has parallel shape operator that is isometrically immersed into a Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, and let ξ be a concurrent vector field on $\bar{M}$ such that $\|\xi^\perp\|$ is constant. If $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton, then it is a metallic-shaped hypersurface.*

For particular cases, we deduce the following:

Corollary 2. *Let M be a connected hypersurface that has parallel shape operator that is isometrically immersed into a Riemannian manifold of constant curvature $(\bar{M}^{n+1}(c), \bar{g})$, and let ξ be a concurrent vector field on $\bar{M}$ such that $\|\xi^\perp\|$ is constant. If $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton, and if one of the following cases occurs:*

1. *M is golden shaped and $\mu = (n-1)c - 4$;*
2. *M is silver shaped and $\mu = (n-1)c + \dfrac{1 - 8\rho^2}{\rho^2}$;*
3. *M is bronze shaped and $\mu = (n-1)c + \dfrac{2 - 12\rho^2}{\rho^2}$;*
4. *M is copper shaped and $\mu = (n-1)c - \dfrac{1}{\rho^2}$;*
5. *M is nickel shaped and $\mu = (n-1)c - \dfrac{2 - 4\rho^2}{\rho^2}$;*

then M is minimal, i.e., $H = 0$.

Remark 1. *In the above cases, we obtain the following values for λ: $\dfrac{1 - 12\rho^2}{2\rho^2}$ for the golden-, copper- and nickel-shaped hypersurfaces; $\dfrac{1 - 8\rho^2}{\rho^2}$ for the silver-shaped hypersurface; and $\dfrac{3 - 20\rho^2}{2\rho^2}$ for the bronze-shaped hypersurface.*

3.3. Pseudosymmetric Hypersurfaces

Let (M^n, g), $n > 2$, be a Riemannian manifold and R be its Riemannian curvature tensor field. Let $R \cdot R$ and $Q(g, R)$ be the tensor fields defined by

$$\begin{aligned}(R(X,Y)\cdot R)(U_1,U_2,U_3,U_4) := &-R(R(X,Y)U_1,U_2,U_3,U_4) - \dots \\ &-R(U_1,U_2,U_3,R(X,Y)U_4)\end{aligned}$$

and

$$\begin{aligned}Q(g,R)(U_1,U_2,U_3,U_4;X,Y) := &-R((X\wedge Y)U_1,U_2,U_3,U_4) - \dots \\ &-R(U_1,U_2,U_3,(X\wedge Y)U_4),\end{aligned}$$

respectively, where $X \wedge Y$ is given by

$$(X \wedge Y)U_i := g(Y, U_i)X - g(X, U_i)Y$$

for $X, Y, U_i \in \chi(M)$, $1 \leq i \leq 4$ (see [18]). Then, (M, g) is called *pseudosymmetric* if the condition

$$R \cdot R = f_R Q(g, R)$$

is satisfied on the set

$$\mathcal{V}_R = \left\{ x \in M : \left(R - \frac{r}{n(n-1)} G \right) \neq 0 \text{ at } x \right\},$$

where f_R is some function on $\mathcal{V}_R$, the $(0, 4)$-tensor field G is defined by

$$G(U_1, U_2, U_3, U_4) := g((U_1 \wedge U_2)U_3, U_4),$$

r is the scalar curvature and $U_i \in \chi(M)$, $1 \leq i \leq 4$ [18]. If $R \cdot R = 0$, then (M, g) is called *semisymmetric* [19].

R. Deszcz, L. Verstraelen and Ş. Yaprak proved the following result:

Lemma 2 ([20]). *Let (M, g) be a hypersurface that is isometrically immersed into a Riemannian space of constant curvature $(\bar{M}^{n+1}(c), \bar{g})$, $n \geq 3$. If at a point $x \in \mathcal{V}_R \subset M$, the shape operator A of M satisfies the condition*

$$A^2 = pA + qI$$

where $p, q \in \mathbb{R}$, then the relation

$$R \cdot R = (c - q)Q(g, R)$$

holds at x.

Using the above lemma, we can state the following:

Proposition 13. *Let M be a hypersurface with parallel shape operator that is isometrically immersed into a Riemannian manifold of constant curvature $(\bar{M}^{n+1}(c), \bar{g})$, and let ξ be a concurrent vector field on $\bar{M}$. If $(M, g, \xi^\top, \lambda, \mu)$ is a hyperbolic Ricci soliton, then it is pseudosymmetric.*

In particular, if $\mu - 2\lambda = 4 + \dfrac{(n+3)\rho^2 - 1}{\rho^2} c$, then M is semisymmetric.

4. Conclusions

The theory of solitons, which are stationary solutions to geometric flows on Riemannian manifolds, has been continuously developed in the last decades. Still insufficiently investigated, hyperbolic Ricci solitons, corresponding to the hyperbolic Ricci flow, have been very recently considered [4]. The aim of the present study is to point out some properties of hyperbolic Ricci soliton submanifolds whose potential vector fields are the tangential components of a concurrent vector field on the ambient space. Since concircular (in particular, concurrent) vector fields play an important role in general relativity, this paper, being among the first ones in this direction, could generate further studies of large interest, with possible applications in physics.

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