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Solution of Fractional Third-Order Dispersive Partial Differential Equations and Symmetric KdV via Sumudu–Generalized Laplace Transform Decomposition

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Abstract: This research work introduces a novel method called the Sumudu–generalized Laplace transform decomposition method (SGLDM) for solving linear and nonlinear non-homogeneous dispersive Korteweg–de Vries (KdV)-type equations. The SGLDM combines the Sumudu–generalized Laplace transform with the Adomian decomposition method, providing a powerful approach to tackle complex equations. To validate the efficacy of the method, several model problems of dispersive KdV-type equations are solved, and the resulting approximate solutions are expressed in series form. The findings demonstrate that the SGLDM is a reliable and robust method for addressing significant physical problems in various applications. Finally, we conclude that this transform is a symmetry to other symmetric transforms.

Keywords: one-dimensional fractional dispersive KdV equation; Sumudu–generalized Laplace transform; Adomian decomposition method; Sumudu–generalized Laplace transform decomposition

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1. Introduction

The dispersive wave phenomenon holds significant importance in the fields of plasma physics and quantum mechanics. Notably, the Korteweg–de Vries (KdV) equation, derived by Korteweg and de Vries, serves as a dimensional representation of these equations. However, obtaining accurate solutions for the KdV equation can often be challenging.

Laplace transformation is a mathematical gadget applied for resolving and converting differential equations. In some areas of science and engineering, the Laplace transformation methods are often used. In [1], the authors discussed the solution of fractional differential equations by applying Laplace transform. The Korteweg–de Vries (KdV) equation, first introduced in 1895, is a non-dimensionalized type of equation that has proven to be of immense importance in various scientific and technological fields. This mathematical model is widely employed to study and understand dispersive wave phenomena, with applications spanning disciplines such as plasma physics and quantum mechanics. Its versatility and ability to describe complex wave behaviors have made it a cornerstone in the exploration and analysis of diverse wave phenomena across different domains of science and technology [2].

Time-fractional third-order dispersive partial differential equations play a crucial role in mathematical sciences. Previous research has suggested combining the Laplace transform with the Adomian decomposition method to address the solution of such equations. These combined techniques have successfully solved four different types of KdV equations [3]. Additionally, researchers have explored numerical methods for solving the third- and fifth-order dispersive Korteweg-de Vries equations [4]. Several methodologies have been employed to investigate fractional partial differential equations of order three. These include the fractional-order variational iteration method [5], modified fractional-order differential transformation method [6], spline technique [7], and homotopy analysis transform technique [8]. Recent research has explored a fundamental transform coupled with Adomian's approach to tackling nonlinear growth equations endowed with non-integer derivatives [9]. In [10], the study focused on the n-th partial derivative of the G_α-transform for specific partial differential equations. The researchers in [11] examined the applicability range of the G_α-transform in solving ordinary differential equations with variable coefficients. The study conducted in [12] delves into the solutions of Abel's integral equations on distribution spaces using the distributional G_α-transform. Furthermore, the author of [13] analyzed the analytic solution of third-order dispersive partial differential equations.

The primary objective of this study is to introduce a novel definition for the Sumudu–generalized Laplace transform. Furthermore, we propose the application of this new transform to fractional partial derivatives. Finally, we leverage the Sumudu–generalized Laplace transform decomposition technique to successfully solve one- and two-dimensional fractional dispersive Korteweg–de Vries (KdV) equations. This approach opens up new avenues for solving complex fractional differential equations and sheds light on potential applications in various scientific and engineering domains.

2. Definitions and Ideas

Here, we introduce some fundamental requisite definitions and preliminary concepts related to fractional calculus and Sumudu–generalized Laplace transform decomposition, which are useful in this work. Generalized Laplace transform of the function $\psi(t)$ is given by G_α in the following definition.

Definition 1. *If $\psi(t)$ is an integrable function defined for all $t \geq 0$, its generalized Laplace transform G_α is the integral of $\psi(t)$ times $s^\alpha e^{-\frac{t}{s}}$ from $t = 0$ to ∞. It is a function of s, denoted by $\Psi(s)$, and is represented as $G_\alpha(\psi)$; thus,*

$$\Psi(s) = G_t(\psi) = s^\alpha \int_0^\infty \psi(t) e^{-\frac{t}{s}} dt,$$

where $s \in \mathbb{C}$ and $\alpha \in Z$,. For more details, see [14].

Definition 2 ([15])**.** *If $\psi(t) \in C([a,b])$ and $a < t < b$, then the Riemann–Liouville fractional integral of order σ is given by*

$$I^\sigma_{a+}\psi(t) = \frac{1}{\Gamma(\sigma)} \int_a^t (t-\iota)^{\sigma-1} \psi(\tau) d\tau \tag{1}$$

where $\sigma \in (-\infty, \infty)$ and I^σ_{a+} indicates the left side of the Riemann–Liouville fractional integral of order σ.

Definition 3 ([15])**.** *Whenever the integral exists, the Riemann–Liouville derivative of fractional order σ, where $n-1 < \sigma < n$, is defined by*

$$D^\sigma_{a+}\psi(t) = \frac{1}{\Gamma(n-\sigma)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\iota)^{n-\sigma-1} \psi(\tau) d\tau, \tag{2}$$

Here, D^σ_{a+} indicates the left Riemann–Liouvill derivative of fractional order σ.

Definition 4. *The Caputo time-fractional derivative operator of order $\sigma > 0$ is given by*

$$D^\sigma_t \psi(x,t) = \begin{cases} \frac{1}{\Gamma(m-\sigma)} \int_0^t (t-\tau)^{m-\sigma-1} \frac{\partial^m \psi(x,\tau)}{\partial \tau^m} d\tau, \\ \frac{\partial^m \psi(x,t)}{\partial t^m}, \text{ for } m=\sigma\in\mathbb{N} \end{cases} \quad m-1 < \sigma < m,$$

For more details, see [16–19].

In the next definition, we define the Sumudu–generalized Laplace transform:

Definition 5. *Let $\psi(x,t)$ be a function. The definition of the Sumudu–generalized Laplace transform of the function $\psi(x,t)$, $t, x \in \mathbb{R}^+$, is given by*

$$\Psi(\mu,s) = S_x G_t[\psi(x,t)] = \frac{s^\alpha}{\mu}\int_0^\infty\int_0^\infty e^{-(\frac{x}{\mu}+\frac{t}{s})}\psi(x,t)dxdt, \tag{3}$$

Here, S_xG_t indicates a Sumudu–generalized Laplace transform and the symbols μ and s indicate transforms of the variables x and t in Sumudu and generalized Laplace transforms, respectively.

Thus, an inverse Sumudu–generalized Laplace transform is denoted by

$$S_\mu^{-1}G_s^{-1}(S_xG_t[\psi(x,t)]) = \psi(x,t) = \frac{1}{(2\pi i)^2}\int_{\delta-i\infty}^{\delta-i\infty}\int_{\sigma-i\infty}^{\sigma-i\infty} e^{\frac{1}{\mu}x+\frac{1}{s}t} S_xG_t[\psi(x,t)]ds\,dp,$$

where the symbol $S_\mu^{-1}G_s^{-1}$ indicates an inverse Sumudu–generalized Laplace transform.

The Sumudu–generalized Laplace transform of the function $\psi(x,t)$ is offered by $S_xG_t[\psi(x,t)] = \Psi(\mu,s)$, so the Sumudu–generalized Laplace transform of , $\frac{\partial\psi(x,t)}{\partial t}$ and $\frac{\partial^2\psi(x,t)}{\partial t^2}$ is presented by

$$S_xG_t\left[\frac{\partial^\sigma\psi(x,t)}{\partial t^\sigma}\right] = \frac{\Psi(\mu,s)}{s^\sigma} - s^{\alpha-\sigma+1}S_x[\Psi(x,0)], \quad 0 < \sigma \le 1 \tag{4}$$

$$\begin{aligned} S_xG_t\left[\frac{\partial^{2\sigma}\psi(x,t)}{\partial t^{2\sigma}}\right] &= \frac{\Psi(\mu,s)}{s^{2\sigma}} - s^{\alpha-2\sigma+1}S_x[\psi(x,0)] - s^{\alpha-2\sigma+2}S_x[\psi_t(x,0)], \\ 0 &< \sigma \le 1 \end{aligned} \tag{5}$$

3. Main Results

The investigation of analytical solutions for third-order dispersive fractional partial differential equations has been explored by various authors through diverse methods. Notable approaches include the Laplace–Adomian decomposition method presented in [3], the Sumudu transform iterative method discussed in [20], and the homotopy analysis Sumudu transform method outlined in [8]. In this section, we aim to address the same problem but with a fresh perspective by employing the Sumudu–generalized Laplace transform decomposition (SGLDM) technique. This innovative approach holds the promise of providing valuable insights into solving such complex equations efficiently and effectively.

3.1. Sumudu–Generalized Laplace Transform Decomposition Method for Handling One-Dimentional KdV Equations

In this subsection, we harness the power of the Sumudu–generalized Laplace transform decomposition (SGLDM) method to tackle both linear and nonlinear one-dimensional KdV equations. These equations are expressed as follows:

$$\frac{\partial^\sigma\psi}{\partial t^\sigma} + a\frac{\partial\psi}{\partial x} + \frac{\partial^3\psi}{\partial x^3} = f(x,t), \quad t > 0, 0 < \sigma \le 1 \tag{6}$$

with the initial conditions

$$\psi(x,0) = f_1(x) \tag{7}$$

and

$$\frac{\partial^\alpha\psi}{\partial t^\alpha} + a\frac{\partial^3\psi}{\partial x^3} + b\psi\frac{\partial\psi}{\partial x} = f(x,t), \tag{8}$$

$$\psi(x,0) = f_1(x). \tag{9}$$

The SGLDM technique offers a promising approach to effectively handle the complexities of these KdV equations and to provide accurate solutions. Through this method, we aim to contribute to the understanding and analysis of dispersive wave phenomena in various fields of science and engineering.

3.1.1. Linear One-Dimensional Fractional KdV

In order to discuss the solution of Equation (6), the following steps are addressed:
Step 1: With the assistance of the Sumudu–generalized Laplace transform, Equation (6) becomes

$$S_x G_t\left[\frac{\partial^\sigma \psi}{\partial t^\sigma}\right] + S_x G_t\left[a\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3}\right] = S_x G_t[f(x,t)] \tag{10}$$

Step2: Applying Equation (4), we have

$$\frac{1}{s^\sigma}\Psi(\mu,s) - \frac{s^\alpha}{s^{\sigma-1}}F(\mu,0) = -S_x G_t\left[a\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3}\right] + F(\mu,s) \tag{11}$$

where $F(\mu,0)$ and $F(\mu,s)$ are the Sumudu– generalized Laplace transform for $f(x,0)$ and $f(x,t)$, respectively.
Step 3: Multiplying Equation (11) by s^σ, we have

$$\Psi(\mu,s) = s^{\alpha+1}F(\mu,0) - s^\sigma S_x G_t\left[a\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3}\right] + s^\sigma F(\mu,s) \tag{12}$$

Step 4: Taking an inverse Sumudu–generalized Laplace transform for Equation (12),

$$\psi(x,t) = S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] - S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[a\frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3}\right]\right] \tag{13}$$

Step 5: Using the ADM for Equation (13),

$$\sum_{n=0}^{\infty}\psi_n = S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] - S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[a\sum_{n=0}^{\infty}\frac{\partial \psi_n}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial^3 \psi_n}{\partial x^3}\right]\right] \tag{14}$$

where

$$\psi_0 = S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] \tag{15}$$

The other components are given by

$$\psi_{n+1} = -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[a\sum_{n=0}^{\infty}\frac{\partial \psi_n}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial^3 \psi_n}{\partial x^3}\right]\right] \tag{16}$$

The exact solution is given by

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots\ldots$$

Example 1 ([3])**.** *The fractional dispersive KdV equation is considered as follows:*

$$\frac{\partial^\sigma \psi}{\partial t^\sigma} + \frac{\partial^3 \psi}{\partial x^3} = -\sin(\pi x)\sin(t) - \pi^3\cos(\pi x)\cos(t), \quad x,t > 0, 0 < \sigma \leq 1 \tag{17}$$

subject to the initial condition

$$\psi(x,0) = \sin(\pi x). \tag{18}$$

Solution 1. *By employing the Sumudu–generalized Laplace transform of Equation (17) and using Equation (12), we have*

$$\Psi(\mu,s)=\frac{\pi\mu s^{\alpha+1}}{1+\mu^2\pi^2}-s^{\sigma}S_xG_t\left[\frac{\partial^3\psi}{\partial x^3}\right]+s^{\sigma}S_xG_t\Big[\sin(\pi x)\sin(t)-\pi^3\cos(\pi x)\cos(t)\Big]. \tag{19}$$

Applying the $\sin(t)$ *and* $\cos(t)$ *series in Equation (19), we obtain*

$$\begin{aligned}\Psi(\mu,s) &= \frac{\pi\mu s^{\alpha+1}}{1+\mu^2\pi^2}-s^{\sigma}S_xG_t\left[\frac{\partial^3\psi}{\partial x^3}\right]\\ &-s^{\sigma}S_xG_t\left[\sin(\pi x)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+...\right)\right]\\ &-s^{\sigma}S_xG_t\left[\pi^3\cos(\pi x)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+...\right)\right],\end{aligned} \tag{20}$$

$$\begin{aligned}\Psi(\mu,s) &= \frac{\pi\mu s^{\alpha+1}}{1+\mu^2\pi^2}-s^{\sigma}S_xG_t\left[\frac{\partial^3\psi}{\partial x^3}\right]\\ &-\frac{\pi\mu}{1+\mu^2\pi^2}\left(s^{\sigma+\alpha+2}-s^{\sigma+\alpha+4}+s^{\sigma+\alpha+6}-s^{\sigma+\alpha+8}+...\right)\\ &-\left[\frac{\pi^3}{1+\mu^2\pi^2}\left(s^{\sigma+\alpha+1}-s^{\sigma+\alpha+3}+s^{\sigma+\alpha+5}-s^{\sigma+\alpha+7}+...\right)\right].\end{aligned} \tag{21}$$

By involving an inverse Sumudu–generalized Laplace transform for Equation (22) and using ADM proceeding, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= \sin(\pi x)-\sin(\pi x)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)}-\frac{t^{\sigma+3}}{\Gamma(\sigma+4)}+\frac{t^{\sigma+5}}{\Gamma(\sigma+6)}-\frac{t^{\sigma+7}}{\Gamma(\sigma+8)}+...\right)\\ &-\pi^3\cos(\pi x)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\frac{t^{\sigma+2}}{\Gamma(\sigma+3)}+\frac{t^{\sigma+4}}{\Gamma(\sigma+5)}-\frac{t^{\sigma+6}}{\Gamma(\sigma+7)}+...\right)\\ &-S_{\mu}^{-1}G_s^{-1}\left[s^{\sigma}S_xG_t\left[\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial x^3}\right]\right]\end{aligned}$$

$$\begin{aligned}\psi_0(x,t) &= \sin(\pi x)-\sin(\pi x)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)}-\frac{t^{\sigma+3}}{\Gamma(\sigma+4)}+\frac{t^{\sigma+5}}{\Gamma(\sigma+6)}-\frac{t^{\sigma+7}}{\Gamma(\sigma+8)}+...\right)\\ &-\pi^3\cos(\pi x)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\frac{t^{\sigma+2}}{\Gamma(\sigma+3)}+\frac{t^{\sigma+4}}{\Gamma(\sigma+5)}-\frac{t^{\sigma+6}}{\Gamma(\sigma+7)}+...\right)\end{aligned}$$

and

$$\psi_{n+1}(x,t)=-S_{\mu}^{-1}G_s^{-1}\left[s^{\sigma}S_xG_t\left[\frac{\partial^3\psi_n}{\partial x^3}\right]\right] \tag{22}$$

where $n\geq 0$*, and the first terms are denoted by*

$$\begin{aligned}
\psi_1(x,t) &= -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\frac{\partial^3\psi_0}{\partial x^3}\right]\right] \\
&= S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^3\cos(\pi x)\right]\right] \\
&\quad -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^3\cos(\pi x)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)}-\frac{t^{\sigma+3}}{\Gamma(\sigma+4)}+\frac{t^{\sigma+5}}{\Gamma(\sigma+6)}-\frac{t^{\sigma+7}}{\Gamma(\sigma+8)}+\ldots\right)\right]\right] \\
&\quad +S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^6\sin(\pi x)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\frac{t^{\sigma+2}}{\Gamma(\sigma+3)}+\frac{t^{\sigma+4}}{\Gamma(\sigma+5)}-\frac{t^{\sigma+6}}{\Gamma(\sigma+7)}+\ldots\right)\right]\right] \\
\psi_1(x,t) &= \frac{\pi^3 t^\sigma}{\Gamma(\sigma+1)}\cos(\pi x)-\pi^3\cos(\pi x)\left[\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)}-\frac{t^{2\sigma+3}}{\Gamma(2\sigma+4)}+\frac{t^{2\sigma+5}}{\Gamma(2\sigma+6)}-\frac{t^{2\sigma+7}}{\Gamma(2\sigma+8)}+\ldots\right)\right] \\
&\quad +\pi^6\sin(\pi x)\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}-\frac{t^{2\sigma+2}}{\Gamma(2\sigma+3)}+\frac{t^{2\sigma+4}}{\Gamma(2\sigma+5)}-\frac{t^{2\sigma+6}}{\Gamma(2\sigma+7)}+\ldots\right)
\end{aligned}$$

$$\begin{aligned}
\psi_2(x,t) &= -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\frac{\partial^3\psi_1}{\partial x^3}\right]\right] \\
&= -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^6\frac{t^\sigma}{\Gamma(\sigma+1)}\sin(\pi x)\right]\right] \\
&\quad -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^6\sin(\pi x)\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)}-\frac{t^{2\sigma+3}}{\Gamma(2\sigma+4)}+\frac{t^{2\sigma+5}}{\Gamma(2\sigma+6)}-\frac{t^{2\sigma+7}}{\Gamma(2\sigma+8)}+\ldots\right)\right]\right] \\
&\quad -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\pi^9\cos(\pi x)\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}-\frac{t^{2\sigma+2}}{\Gamma(2\sigma+3)}+\frac{t^{2\sigma+4}}{\Gamma(2\sigma+5)}-\frac{t^{2\sigma+6}}{\Gamma(2\sigma+7)}+\ldots\right)\right]\right] \\
\psi_2(x,t) &= -\frac{\pi^6 t^{2\sigma}}{\Gamma(2\sigma+1)}\sin(\pi x)+\pi^6\sin(\pi x)\left[\left(\frac{t^{3\sigma+1}}{\Gamma(3\sigma+2)}-\frac{t^{3\sigma+3}}{\Gamma(3\sigma+4)}+\frac{t^{3\sigma+5}}{\Gamma(3\sigma+6)}-\frac{t^{3\sigma+7}}{\Gamma(3\sigma+8)}+\ldots\right)\right] \\
&\quad +\pi^9\cos(\pi x)\left(\frac{t^{3\sigma}}{\Gamma(3\sigma+1)}-\frac{t^{3\sigma+2}}{\Gamma(3\sigma+3)}+\frac{t^{3\sigma+4}}{\Gamma(3\sigma+5)}-\frac{t^{3\sigma+6}}{\Gamma(3\sigma+7)}+\ldots\right)
\end{aligned}$$

Eventually, the approximate solution of Equation (17) is given by

$$\begin{aligned}
\psi(x,t) &= \sin(\pi x)-\sin(\pi x)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)}-\frac{t^{\sigma+3}}{\Gamma(\sigma+4)}+\frac{t^{\sigma+5}}{\Gamma(\sigma+6)}-\frac{t^{\sigma+7}}{\Gamma(\sigma+8)}+\ldots\right) \\
&\quad -\pi^3\cos(\pi x)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\frac{t^{\sigma+2}}{\Gamma(\sigma+3)}+\frac{t^{\sigma+4}}{\Gamma(\sigma+5)}-\frac{t^{\sigma+6}}{\Gamma(\sigma+7)}+\ldots\right) \\
&\quad +\frac{\pi^3 t^\sigma}{\Gamma(\sigma+1)}\cos(\pi x)-\pi^3\cos(\pi x)\left[\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)}-\frac{t^{2\sigma+3}}{\Gamma(2\sigma+4)}+\frac{t^{2\sigma+5}}{\Gamma(2\sigma+6)}-\frac{t^{2\sigma+7}}{\Gamma(2\sigma+8)}+\ldots\right)\right] \\
&\quad +\pi^6\sin(\pi x)\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}-\frac{t^{2\sigma+2}}{\Gamma(2\sigma+3)}+\frac{t^{2\sigma+4}}{\Gamma(2\sigma+5)}-\frac{t^{2\sigma+6}}{\Gamma(2\sigma+7)}+\ldots\right) \\
&\quad -\frac{\pi^6 t^{2\sigma}}{\Gamma(2\sigma+1)}\sin(\pi x)+\pi^6\sin(\pi x)\left[\left(\frac{t^{3\sigma+1}}{\Gamma(3\sigma+2)}-\frac{t^{3\sigma+3}}{\Gamma(3\sigma+4)}+\frac{t^{3\sigma+5}}{\Gamma(3\sigma+6)}-\frac{t^{3\sigma+7}}{\Gamma(3\sigma+8)}+\ldots\right)\right] \\
&\quad +\pi^9\cos(\pi x)\left(\frac{t^{3\sigma}}{\Gamma(3\sigma+1)}-\frac{t^{3\sigma+2}}{\Gamma(3\sigma+3)}+\frac{t^{3\sigma+4}}{\Gamma(3\sigma+5)}-\frac{t^{3\sigma+6}}{\Gamma(3\sigma+7)}+\ldots\right)
\end{aligned}$$

Therefore, the exact solution at $\sigma = 1$ is presented by

$$
\begin{aligned}
\psi(x,t) \quad = \quad & \sin(\pi x) - \sin(\pi x)\left(\frac{t^2}{\Gamma(3)} - \frac{t^4}{\Gamma(5)} + \frac{t^6}{\Gamma(7)} - \frac{t^8}{\Gamma(7)} + ...\right) \\
& -\pi^3 \cos(\pi x)\left(\frac{t}{\Gamma(2)} - \frac{t^3}{\Gamma(4)} + \frac{t^{\sigma 5}}{\Gamma(6)} - \frac{t^7}{\Gamma(8)} + ...\right) \\
& +\frac{\pi^3 t}{\Gamma(2)}\cos(\pi x) - \pi^3\cos(\pi x)\left[\left(\frac{t^3}{\Gamma(4)} - \frac{t^5}{\Gamma(6)} + \frac{t^7}{\Gamma(8)} - \frac{t^9}{\Gamma(10)} + ...\right)\right] \\
& +\pi^6 \sin(\pi x)\left(\frac{t^2}{\Gamma(3)} - \frac{t^4}{\Gamma(5)} + \frac{t^6}{\Gamma(7)} - \frac{t^8}{\Gamma(9)} + ...\right) \\
& -\frac{\pi^6 t^2}{\Gamma(3)}\sin(\pi x) + \pi^6\sin(\pi x)\left[\left(\frac{t^4}{\Gamma(5)} - \frac{t^6}{\Gamma(7)} + \frac{t^8}{\Gamma(9)} - \frac{t^{10}}{\Gamma(11)} + ...\right)\right] \\
& +\pi^9 \cos(\pi x)\left(\frac{t^3}{\Gamma(4)} - \frac{t^5}{\Gamma(6)} + \frac{t^7}{\Gamma(8)} - \frac{t^9}{\Gamma(10)} + ...\right).
\end{aligned}
$$

By simplifying

$$
\begin{aligned}
\psi(x,t) \quad &= \quad \sin(\pi x) - \sin(\pi x)\left(\frac{t^2}{\Gamma(3)} - \frac{t^4}{\Gamma(5)} + \frac{t^6}{\Gamma(7)} - \frac{t^8}{\Gamma(7)} + ...\right) \\
&= \quad \sin(\pi x)\left(1 - \frac{t^2}{\Gamma(3)} + \frac{t^4}{\Gamma(5)} - \frac{t^6}{\Gamma(7)} + \frac{t^8}{\Gamma(7)} - ...\right)
\end{aligned}
$$

$$\psi(x,t) = \sin(\pi x)\cos(t)$$

3.1.2. Nonlinear One-Dimensional Fractional KdV

In this subsection, we delve into the nonlinear one-dimensional fractional Kortewegâ€"de Vries (KdV) equation and elucidate the Sumudu–generalized Laplace transform decomposition method (SGLDM) that we employ to find its solution.

Problem: Consider the nonlinear one-dimensional fractional dispersive KdV equation

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} + a\frac{\partial^3 \psi}{\partial x^3} + b\psi\frac{\partial \psi}{\partial x} = f(x,t), \tag{23}$$

$$\psi(x,0) = f_1(x). \tag{24}$$

where $f(x,t)$ and $f_1(x)$ are known functions and a and b are constants. For the ideal of obtaining the solution of Equation (23) by using the past examining method, the fundamental approximation is proposed via

$$\psi(x,t) = S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] - S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[a\frac{\partial^3\psi}{\partial x^3} + b\psi\frac{\partial\psi}{\partial x}\right]\right], \tag{25}$$

Therefore,

$$
\begin{aligned}
\sum_{n=0}^{\infty}\psi_n \quad = \quad & S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] \\
& -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[a\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial x^3} + b\sum_{n=0}^{\infty}\psi_n\frac{\partial\psi_n}{\partial x}\right]\right]
\end{aligned} \tag{26}
$$

where

$$\psi_0 = S_\mu^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,0) + s^\sigma F(\mu,s)\right] \tag{27}$$

The other components are given by

$$\psi_{n+1} = -S_{\mu}^{-1} G_s^{-1} \left[s^{\sigma} S_x G_t \left[a \frac{\partial^3 \psi_n}{\partial x^3} + b A_n \right] \right] \tag{28}$$

where the nonlinear term $A_n = \sum_{n=0}^{\infty} \psi_n \frac{\partial \psi_n}{\partial x}$ is determined by

$$\begin{aligned} A_0 &= \psi_0 \psi_{0x} \\ A_1 &= \psi_{0x} \psi_1 + \psi_0 \psi_{1x} \\ A_2 &= \psi_{0x} \psi_2 + \psi_0 \psi_{2x} + \psi_1 \psi_{1x} \\ A_3 &= \psi_{0x} \psi_3 + \psi_0 \psi_{3x} + \psi_{1x} \psi_2 + \psi_1 \psi_{2x}. \end{aligned} \tag{29}$$

Therefore, the approximate solution of Equation (23) is given by

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + ...$$

Assuming $a = 1$, $b = -2$, and $f(x,t) = 0$, in Equation (23), we have the following example:

Example 2. *Consider the following nonlinear one-dimensional fractional KdV equation*

$$\frac{\partial^{\alpha} \psi}{\partial t^{\alpha}} + \frac{\partial^3 \psi}{\partial x^3} - 2\psi \psi_x = 0, \tag{30}$$

subjected to the initial condition

$$\psi(x,0) = x. \tag{31}$$

Solution 2. *By utilizing Equations (27) and (28), we obtain*

$$\psi_0 = x, . \tag{32}$$

The additional components are provided by

$$\psi_{n+1} = -S_{\mu}^{-1} G_s^{-1} \left[s^{\sigma} S_x G_t \left[\frac{\partial^3 \psi_n}{\partial x^3} \right] \right] + S_{\mu}^{-1} G_s^{-1} [s^{\sigma} S_x G_t [2A_n]]. \tag{33}$$

By substituting $n = 0$ in Equation (33), we achieve

$$\begin{aligned} \psi_1 &= -S_{\mu}^{-1} G_s^{-1} \left[s^{\sigma} S_x G_t \left[\frac{\partial^3 \psi_0}{\partial x^3} \right] \right] + S_{\mu}^{-1} G_s^{-1} [s^{\sigma} S_x G_t [2A_0]] \\ &= -S_{\mu}^{-1} G_s^{-1} [s^{\sigma} S_x G_t [0]] + S_{\mu}^{-1} G_s^{-1} [s^{\sigma} S_x G_t [2\psi_0 \psi_{0x}]] \\ &= S_{\mu}^{-1} G_s^{-1} [s^{\sigma} S_x G_t [2x]] \\ &= S_{\mu}^{-1} G_s^{-1} S_2^{-1} \left[2\mu s^{\sigma+\alpha+1} \right] \\ \psi_1(x,t) &= 2x \frac{t^{\sigma}}{\Gamma(\sigma+1)}. \end{aligned}$$

At $n = 1$, we have

$$
\begin{aligned}
\psi_2 &= -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\frac{\partial^3\psi_1}{\partial x^3}\right]\right] + S_\mu^{-1}G_s^{-1}[s^\sigma S_x G_t[2A_1]] \\
&= 2S_\mu^{-1}G_s^{-1}[s^\sigma S_x G_t[\psi_0\psi_{1x} + \psi_1\psi_{0x}]] \\
&= S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[8x\frac{t^\sigma}{\Gamma(\sigma+1)}\right]\right] \\
&= S_\mu^{-1}G_s^{-1}\left[8\mu s^{2\sigma+\alpha+1}\right] \\
\psi_2(x,t) &= 8x\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}
\end{aligned}
$$

and at $n = 2$, we have

$$
\begin{aligned}
\psi_3 &= -S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[\frac{\partial^3\psi_1}{\partial x^3}\right]\right] + S_\mu^{-1}G_s^{-1}[s^\sigma S_x G_t[2A_2]] \\
&= S_\mu^{-1}G_s^{-1}\left[s^\sigma S_x G_t\left[32x\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + 8x\frac{t^{2\sigma}}{\Gamma(\sigma+1)\Gamma(\sigma+1)}\right]\right] \\
&= S_\mu^{-1}G_s^{-1}\left[32\mu s^{3\sigma+\alpha+1} + 8\mu s^{3\sigma+\alpha+1}\frac{\Gamma(2\sigma+1)}{\Gamma(\sigma+1)\Gamma(\sigma+1)}\right] \\
&= 32x\frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + 8x\frac{t^{3\sigma}\Gamma(2\sigma+1)}{\Gamma(\sigma+1)\Gamma(\sigma+1)\Gamma(3\sigma+1)}.
\end{aligned}
$$

Therefore, the approximation solution of Equation (30) is presented by

$$
\begin{aligned}
\psi &= x + 2x\frac{t^{\sigma\alpha}}{\Gamma(\sigma+1)} + 8x\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} \\
&\quad + 32x\frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + 8x\frac{t^{3\sigma}\Gamma(2\sigma+1)}{\Gamma(\sigma+1)\Gamma(\sigma+1)\Gamma(3\sigma+1)} + \ldots.
\end{aligned}
$$

In the particular case when $\sigma = 1$, we obtain

$$
\begin{aligned}
\psi &= x + 2xt + 4xt^2 + 8xt^3 + \ldots \\
&= x\left(1 + 2t + (2t)^2 + (2t)^3 +\right)\ldots \\
&= \frac{x}{1-2t}
\end{aligned}
$$

4. Sumudu–Generalized Laplace Transform Decomposition Method for Handling Two-Dimentional KdV Equations

Here, we present the details of the Sumudu–generalized Laplace transform decomposition method for solving the two-dimensional fractional KdV equation:

$$
\frac{\partial^\sigma\psi}{\partial t^\sigma} + a\frac{\partial^3\psi}{\partial x^3} + b\frac{\partial^3\psi}{\partial y^3} = f(x,y,t), \quad x,y,t > 0, 0 < \sigma \leq 1 \tag{34}
$$

with the initial condition

$$
\psi(x,0) = f_1(x,y) \tag{35}
$$

where a and b are constant. In order to obtain the solution of Equation (34), the following steps are proposed:

Step 1: Upon using double a Sumudu–generalized Laplace transform for Equation (34) and a double Sumudu transform for Equation (35), we obtain

$$
S_2G_t\left[\frac{\partial^\sigma\psi}{\partial t^\sigma}\right] + S_2G_t\left[a\frac{\partial^3\psi}{\partial x^3} + b\frac{\partial^3\psi}{\partial y^3}\right] = S_2G_t[f(x,,y,t)] \tag{36}
$$

where the symbol S_2 indicates a double Sumudu transform.
Step 2: Applying Equation (4), we obtain

$$\frac{1}{s^{\sigma}}\Psi(\mu,\lambda,s)-\frac{s^{\alpha}}{s^{\sigma-1}}F(\mu,\lambda,0)=-S_2G_t\left[a\frac{\partial^3\psi}{\partial x^3}+b\frac{\partial^3\psi}{\partial y^3}\right]+F(\mu,\lambda,s) \tag{37}$$

where $F(\mu,\lambda,0)$ and $F(\mu,\lambda,s)$ are the double Sumudu–generalized Laplace transform for $f(x,y,0)$ and $f(x,y,t)$, respectively.
Step 3: Multiplying Equation (37) by s^{σ}, we have

$$\Psi(\mu,\lambda,s)=s^{\alpha+1}F(\mu,\lambda,0)-s^{\sigma}S_2G_t\left[a\frac{\partial^3\psi}{\partial x^3}+b\frac{\partial^3\psi}{\partial y^3}\right]+s^{\sigma}F(\mu,\lambda,s) \tag{38}$$

Step 4: Employing the inverse double Sumudu–generalized Laplace transform for Equation (38),

$$\psi(x,y,t)=S_2^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,\lambda,0)+s^{\sigma}F(\mu,\lambda,s)\right]-S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[a\frac{\partial^3\psi}{\partial x^3}+b\frac{\partial^3\psi}{\partial y^3}\right]\right] \tag{39}$$

Step 5: Using the ADM for Equation (39),

$$\sum_{n=0}^{\infty}\psi_n=S_2^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,\lambda,0)+s^{\sigma}F(\mu,\lambda,s)\right]-S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[a\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial x^3}+b\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial y^3}\right]\right] \tag{40}$$

Then, we determine the repetition relations as

$$\psi_0=S_2^{-1}G_s^{-1}\left[s^{\alpha+1}F(\mu,\lambda,0)+s^{\sigma}F(\mu,\lambda,s)\right] \tag{41}$$

$$\psi_{n+1}=-S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[a\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial x^3}+b\sum_{n=0}^{\infty}\frac{\partial^3\psi_n}{\partial y^3}\right]\right] \tag{42}$$

The series solution is given by

$$\psi=\psi_0+\psi_1+\psi_2+\ldots\ldots$$

Example 3. *The non-homogeneous fractional dispersive KdV equation in two dimensions is considered as follows:*

$$\frac{\partial^{\sigma}\psi}{\partial t^{\sigma}}+\frac{\partial^3\psi}{\partial x^3}+\frac{\partial^3\psi}{\partial y^3}=\sin(x+y)\cos(t)-2\cos(x+y)\sin(t),\quad x,t>0,0<\sigma\leq 1 \tag{43}$$

subject to the initial condition

$$\psi(x,y,0)=0. \tag{44}$$

Solution 3. *With the help of Equations (41) and (42), we obtain*

$$\begin{aligned}\psi_0 \;=\;& S_2^{-1}G_s^{-1}\left[s^{\sigma}\left(\frac{\mu+\lambda}{(1+\mu^2)(1+\lambda^2)}\left(s^{\alpha+1}-s^{\alpha+3}+s^{\alpha+5}-s^{\alpha+7}+\ldots\right)\right)\right]\\ &-S_2^{-1}G_s^{-1}\left[s^{\sigma}\left(\frac{2(1-\mu\lambda)}{(1+\mu^2)(1+\lambda^2)}\left(s^{\alpha+2}-s^{\alpha+4}+s^{\alpha+6}-s^{\alpha+8}+\ldots\right)\right)\right]\end{aligned} \tag{45}$$

$$\psi_{n+1}=-S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[\frac{\partial\psi_n}{\partial x}+\frac{\partial^3\psi_n}{\partial x^3}\right]\right] \tag{46}$$

$$\begin{aligned}\psi_0 &= \sin(x+y)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)} - \frac{t^{\sigma+2}}{\Gamma(\sigma+3)} + \frac{t^{\sigma+4}}{\Gamma(\sigma+5)} - \frac{t^{\sigma+6}}{\Gamma(\sigma+7)} + ...\right) \\ &\quad -2\cos(x+y)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)} - \frac{t^{\sigma+3}}{\Gamma(\sigma+4)} + \frac{t^{\sigma+5}}{\Gamma(\sigma+6)} - \frac{t^{\sigma+7}}{\Gamma(\sigma+8)} + ...\right)\end{aligned}$$

where $n \geq 0$. The element of the solution is denoted by

$$\begin{aligned}\psi_1 &= -S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[\frac{\partial \psi_0}{\partial x} + \frac{\partial^3 \psi_0}{\partial x^3}\right]\right] \\ &= S_2^{-1}G_s^{-1}[s^{\sigma}S_2G_t[2\cos(x+y)[\Delta] + 4\sin(x+y)[\Pi]]]\end{aligned}$$

where

$$\begin{aligned}\Delta &= \left(\frac{t^{\sigma}}{\Gamma(\sigma+1)} - \frac{t^{\sigma+2}}{\Gamma(\sigma+3)} + \frac{t^{\sigma+4}}{\Gamma(\sigma+5)} - \frac{t^{\sigma+6}}{\Gamma(\sigma+7)} + ...\right) \\ \Pi &= \left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)} - \frac{t^{\sigma+3}}{\Gamma(\sigma+4)} + \frac{t^{\sigma+5}}{\Gamma(\sigma+6)} - \frac{t^{\sigma+7}}{\Gamma(\sigma+8)} + ...\right)\end{aligned}$$

The first three terms are given by

$$\begin{aligned}\psi_1 &= -S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[\frac{\partial \psi_0}{\partial x} + \frac{\partial^3 \psi_0}{\partial x^3}\right]\right] \\ &= 2\cos(x+y)\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} - \frac{t^{2\sigma+2}}{\Gamma(2\sigma+3)} + \frac{t^{2\sigma+4}}{\Gamma(2\sigma+5)} - \frac{t^{2\sigma+6}}{\Gamma(2\sigma+7)} + ...\right) \\ &\quad +4\sin(x+y)\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)} - \frac{t^{2\sigma+3}}{\Gamma(2\sigma+4)} + \frac{t^{2\sigma+5}}{\Gamma(2\sigma+6)} - \frac{t^{2\sigma+7}}{\Gamma(2\sigma+8)} + ...\right)\end{aligned}$$

$$\begin{aligned}\psi_2 &= -S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[\frac{\partial \psi_1}{\partial x} + \frac{\partial^3 \psi_1}{\partial x^3}\right]\right] \\ &= -4\sin(x+y)\left(\frac{t^{3\sigma}}{\Gamma(3\sigma+1)} - \frac{t^{3\sigma+2}}{\Gamma(3\sigma+3)} + \frac{t^{3\sigma+4}}{\Gamma(3\sigma+5)} - \frac{t^{3\sigma+6}}{\Gamma(3\sigma+7)} + ...\right) \\ &\quad +8\cos(x+y)\left(\frac{t^{3\sigma+1}}{\Gamma(3\sigma+2)} - \frac{t^{3\sigma+3}}{\Gamma(3\sigma+4)} + \frac{t^{3\sigma+5}}{\Gamma(3\sigma+6)} - \frac{t^{3\sigma+7}}{\Gamma(3\sigma+8)} + ...\right)\end{aligned}$$

$$\begin{aligned}\psi_3 &= -S_2^{-1}G_s^{-1}\left[s^{\sigma}S_2G_t\left[\frac{\partial \psi_2}{\partial x} + \frac{\partial^3 \psi_2}{\partial x^3}\right]\right] \\ &= -8\cos(x+y)\left(\frac{t^{4\sigma}}{\Gamma(4\sigma+1)} - \frac{t^{4\sigma+2}}{\Gamma(4\sigma+3)} + \frac{t^{4\sigma+4}}{\Gamma(4\sigma+5)} - \frac{t^{34\sigma+6}}{\Gamma(4\sigma+7)} + ...\right) \\ &\quad -16\sin(x+y)\left(\frac{t^{4\sigma+1}}{\Gamma(4\sigma+2)} - \frac{t^{4\sigma+3}}{\Gamma(4\sigma+4)} + \frac{t^{4\sigma+5}}{\Gamma(4\sigma+6)} - \frac{t^{4\sigma+7}}{\Gamma(4\sigma+8)} + ...\right)\end{aligned}$$

and so on. Therefore, upon adding up the above iterations, the solution is now denoted by

$$
\begin{aligned}
\psi(x,y,t) \quad = \quad & \sin(x+y)\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)} - \frac{t^{\sigma+2}}{\Gamma(\sigma+3)} + \frac{t^{\sigma+4}}{\Gamma(\sigma+5)} - \frac{t^{\sigma+6}}{\Gamma(\sigma+7)} + \ldots\right) \\
& -2\cos(x+y)\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)} - \frac{t^{\sigma+3}}{\Gamma(\sigma+4)} + \frac{t^{\sigma+5}}{\Gamma(\sigma+6)} - \frac{t^{\sigma+7}}{\Gamma(\sigma+8)} + \ldots\right) \\
& +2\cos(x+y)\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} - \frac{t^{2\sigma+2}}{\Gamma(2\sigma+3)} + \frac{t^{2\sigma+4}}{\Gamma(2\sigma+5)} - \frac{t^{2\sigma+6}}{\Gamma(2\sigma+7)} + \ldots\right) \\
& +4\sin(x+y)\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)} - \frac{t^{2\sigma+3}}{\Gamma(2\sigma+4)} + \frac{t^{2\sigma+5}}{\Gamma(2\sigma+6)} - \frac{t^{2\sigma+7}}{\Gamma(2\sigma+8)} + \ldots\right) \\
& -4\sin(x+y)\left(\frac{t^{3\sigma}}{\Gamma(3\sigma+1)} - \frac{t^{3\sigma+2}}{\Gamma(3\sigma+3)} + \frac{t^{3\sigma+4}}{\Gamma(3\sigma+5)} - \frac{t^{3\sigma+6}}{\Gamma(3\sigma+7)} + \ldots\right) \\
& +8\cos(x+y)\left(\frac{t^{3\sigma+1}}{\Gamma(3\sigma+2)} - \frac{t^{3\sigma+3}}{\Gamma(3\sigma+4)} + \frac{t^{3\sigma+5}}{\Gamma(3\sigma+6)} - \frac{t^{3\sigma+7}}{\Gamma(3\sigma+8)} + \ldots\right) \\
& -8\cos(x+y)\left(\frac{t^{4\sigma}}{\Gamma(4\sigma+1)} - \frac{t^{4\sigma+2}}{\Gamma(4\sigma+3)} + \frac{t^{4\sigma+4}}{\Gamma(4\sigma+5)} - \frac{t^{34\sigma+6}}{\Gamma(4\sigma+7)} + \ldots\right) \\
& -16\sin(x+y)\left(\frac{t^{4\sigma+1}}{\Gamma(4\sigma+2)} - \frac{t^{4\sigma+3}}{\Gamma(4\sigma+4)} + \frac{t^{4\sigma+5}}{\Gamma(4\sigma+6)} - \frac{t^{4\sigma+7}}{\Gamma(4\sigma+8)} + \ldots\right)
\end{aligned}
$$

In the particular when case $\sigma = 1$, the solution of becomes

$$
\begin{aligned}
\psi(x,y,t) \quad &= \quad \sin(x+y)\left(\frac{t}{\Gamma(2)} - \frac{t^3}{\Gamma(4)} + \frac{t^5}{\Gamma(6)} - \frac{t^7}{\Gamma(8)} + \ldots\right) \\
&= \quad \sin(x+y)\sin(t)
\end{aligned}
$$

5. Conclusions

Our research demonstrates the significance of employing the Sumudu–generalized Laplace transform decomposition to derive solutions for the one and two-dimensional fractional dispersive KdV equation. The method we employed proves to be straightforward in its fundamentals, and we have provided three illustrative examples to validate the accuracy and relevance of our approach. Building on these promising results, we anticipate exploring and solving various novel and intriguing scientific phenomena in the future, utilizing our technique to expand the horizons of modeling in our field.

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