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# Representations of Flat Virtual Braids by Automorphisms of Free Group

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**Abstract:** Representations of braid group $B_n$ on $n \geq 2$ strands by automorphisms of a free group of rank $n$ go back to Artin. In 1991, Kauffman introduced a theory of virtual braids, virtual knots, and links. The virtual braid group $VB_n$ on $n \geq 2$ strands is an extension of the classical braid group $B_n$ by the symmetric group $S_n$. In this paper, we consider flat virtual braid groups $FVB_n$ on $n \geq 2$ strands and construct a family of representations of $FVB_n$ by automorphisms of free groups of rank $2n$. It has been established that these representations do not preserve the forbidden relations between classical and virtual generators. We investigated some algebraic properties of the constructed representations. In particular, we established conditions of faithfulness in case $n = 2$ and proved that the kernel contains a free group of rank two for $n \geq 3$.

**Keywords:** braid; virtual braid; flat virtual braid group; automorphism of free group



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## 1. Introduction

The foundations of the braid groups theory were laid down in the works of E. Artin in the 1920s. In [1] he defined the *braid group* $B_n$ on $n \geq 2$ strands as a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| \geq 2, \tag{1}$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \leq i \leq n-2. \tag{2}$$

A set $\{\sigma_1, \ldots, \sigma_{n-1}\}$ is called the standard generators, or the Artin generators of the braid group $B_n$. The generator $\sigma_i \in B_n$ and its inverse $\sigma_i^{-1}$ are presented geometrically in Figure 1.

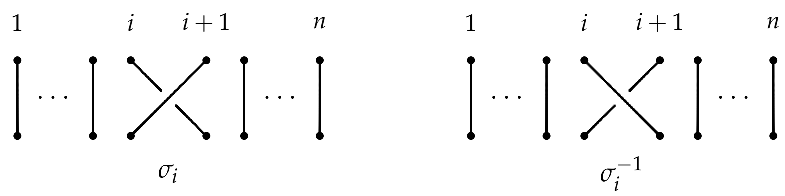


**Figure 1.** Generator $\sigma_i \in B_n$ and its inverse $\sigma_i^{-1}$.

There is a useful relation between braid groups and knot theory, see for example, [2–7]. This relation is based on Alexander's theorem [8], that states that every knot or link in $S^3$ is ambient isotopic to a closed braid, and on Markov's theorem [9], that describes the elementary operations generating the equivalence relations on braids given by the equivalence of their closures. These operations are said to be Markov moves. Invariants, arising from representations of braid groups, play an important role in classical knot theory and its generalizations.

Artin discovered faithful representation $\varphi_n \colon B_n \to \mathrm{Aut}(\mathbb{F}_n)$, where $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$ is the free group of rank $n$ for $n \geq 2$. Homomorphism $\varphi_n$, known as the Artin representation, maps generator $\sigma_i \in B_n$ to the following automorphism $\varphi_n(\sigma_i)$:

$$\varphi_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \qquad j \neq i, i+1. \end{cases}$$

Note that for each $i$, $1 \leq i \leq n-1$, one has $\varphi_n(\sigma_i)(x_1 \cdots x_n) = x_1 \cdots x_n$. Therefore $\varphi_n(\beta)(x_1 \cdots x_n) = (x_1 \cdots x_n)$ for every $\beta \in B_n$. Moreover, it is shown by Artin [1,10] that an automorphism $g \in \mathrm{Aut}(\mathbb{F}_n)$ is equal to $\varphi_n(\beta)$ for some $\beta \in B_n$ if and only if the following two conditions are satisfied: (i) $f(x_i)$ is conjugate of some $x_j$ for $i = 1, \ldots, n-1$; and (ii) $f(x_1 \cdots x_n) = x_1 \cdots x_n$.

The braid group $B_n$ can be naturally identified with $MCG(D_n, \partial D_n)$, the relative mapping class group of the $n$-punctured disc $D_n$, and the Artin representation is induced by the action of $B_n$ on the group $\pi_1(D_n) = \mathbb{F}_n$, where $x_i$ is a loop represented by the boundary of the $i$-th puncture. Moreover, the Artin representation has the following useful property. Let $L$ be a link in $S^3$. Suppose $L$ is obtained by closure from an $n$-strand braid $\beta \in B_n$, i.e., $L = \hat{\beta}$. Then the link group $\pi_1(S^3 \setminus L)$ is isomorphic to a group $G_\beta$ defined by the following presentation

$$G_\beta = \langle x_1, x_2, \ldots, x_n \quad | \quad \varphi_n(\beta)(x_i) = x_i, \quad i = 1, \ldots, n \rangle.$$

In [11] Wada introduced some other representations $\{\psi_n\}_{n=2}^{\infty}$ of braid groups $B_n$ by automorphisms of free groups which, in the same way as above, give groups invariants of links

$$G_\beta(\psi_n) = \langle x_1, x_2, \ldots, x_n \quad | \quad \psi_n(\beta)(x_i) = x_i, \quad i = 1, \ldots, n \rangle. \tag{3}$$

It is evident that the family of representations $\{\psi_n\}_{n=2}^{\infty}$ should be such that the group $G_\beta(\psi_n)$ exhibits the property of invariance with respect to Markov moves.

For this purpose, the so-called Wada-type representations or local homogeneous representations have proven to be a useful tool. Recall that a representation is local whenever the image of $\sigma_i$, for $i = 1, \ldots, n-1$, acts non-trivially on the pair of adjacent generators $x_i$ and $x_{i+1}$, and the image of $x_i$ is the word $u(x_i, x_{i+1})$, while the image of $x_{i+1}$ is a word $v(x_i, x_{i+1})$, where $u$ and $v$ are reduced words in the group generated by $x_i$ and $x_{i+1}$. In [11], seven types of such representations were discovered, and a hypothesis was formulated regarding the existence of other local homogeneous representations. Four of these seven types are faithful. The classification of such representations was provided in [12].

In [13], new families of representations $\{\psi_n\}_{n=2}^{\infty}$ are considered, for which a similar group invariant for links can be defined, analogous to (3). By deviating from the requirement of local homogeneous representation (see [13]), it is possible to expand the list of representations of braid groups by automorphisms of free groups. However, in terms of group invariants, as demonstrated by Ito [13] (Theorem 4.1), no new additions are made.

In what follows, if any automorphism acts on a generator identically, we will not write this action. We write the composition of automorphisms in the order of their application from left to right, namely, $\varphi\psi(f) = \psi(\varphi(f))$.

Virtual braids were introduced by Kauffman in his founding paper [14] together with virtual knots and links. See [15–18] for more information about virtual knots and links, and [19,20] for their applications to study of proteins. In the same paper, Kauffman defined the virtual braid group $VB_n$ on $n \geq 2$ strands, generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$

similarly to the classical braid group and $\rho_1, \ldots \rho_{n-1}$ that satisfy braid relations (1)–(2), symmetric group relations (4)–(6) and mixed relations (7)–(8):

$$\rho_i^2 = 1, \qquad 1 \le i \le n-1, \tag{4}$$
$$\rho_i \rho_j = \rho_j \rho_i, \qquad |i-j| \ge 2, \tag{5}$$
$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \qquad 1 \le i \le n-2, \tag{6}$$
$$\rho_i \sigma_j = \sigma_j \rho_i, \qquad |i-j| \ge 2, \tag{7}$$
$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \qquad 1 \le i \le n-2. \tag{8}$$

Generator $\rho_i \in VB_n$ is presented geometrically in Figure 2.

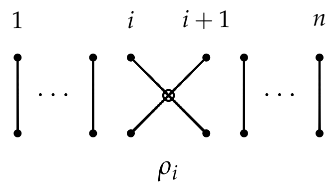


**Figure 2.** Generator $\rho_i \in VB_n$.

Geometric braids corresponding to the mixed relation (8) are presented in Figure 3.

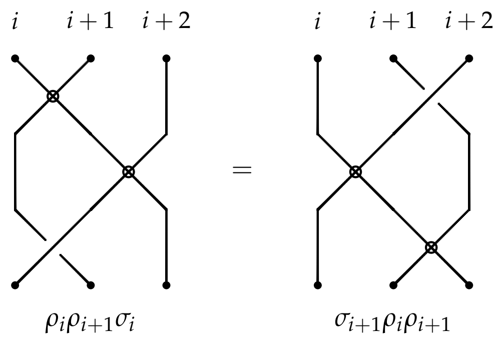


**Figure 3.** The mixed relation $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$.

Kamada [21] established that the following Alexander theorem for virtual braids holds: If $L$ is a virtual link, then for some $n$ there exists a virtual braid $\beta \in VB_n$ such that $L = \hat{\beta}$ is the closure of $\beta$.

It is shown in [22] that relations

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \qquad 1 \le i \le n-2, \tag{9}$$
$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \qquad 1 \le i \le n-2. \tag{10}$$

do not hold in the $VB_n$ group. These relations (9) and (10) are called forbidden relations, see Figures 4 and 5. The group $WB_n$ is obtained from $VB_n$ by adding the relation (9) and is called welded braid group [23]. The same group $WB_n$ is obtained by adding the relation (10) to the group $VB_n$. Adding both relations (9) and (10) to $VB_n$ leads to unknotting transformations for virtual knots and links [21,22,24]. Other unknotting operations for links, virtual links and welded links are given, for example, in [25–27]. Note that the representations $VB_n \to \mathrm{Aut}(G_n)$ were constructed, for example, for groups $G_n$ of the following form: $G_n = \mathbb{F}_n * \mathbb{Z}^{n+1}$ in [28], $G_n = \mathbb{F}_n * \mathbb{Z}^2$ in [29], $G_n = \mathbb{F}_n * \mathbb{Z}^{2n+1}$ and $G_n = \mathbb{F}_n * \mathbb{Z}^n$ in [30]. For structural properties and other representations of the virtual braid groups see [31,32].

In the last decade many polynomial invariants of virtual knots and links have been introduced. Among them are affine index polynomial by Kauffman [33], writhe polynomial by Cheng and Cao [34], wriggle polynomial by Folwaczny and Kauffman [35], arrow polynomial by Dye and Kauffman [36], extended bracket polynomial by Kauffman [37], index polynomial by Im, Lee and Lee [38], zero polynomial by Jeong [39], sequences of $L$-polynomials and $F$-polynomials by Kaur, Prabhakar, and Vesnin [40] and recurrent generalizations of $F$-polynomials [41].

In [42], Mihalchishina constructs an extension of Wada representation to the virtual and welded braid groups. By utilizing the generated representations, she established the construction of virtual link groups and demonstrated their invariance under link transformations. The conditions on the representations here for group invariance appear to be more complex than in the classical case.

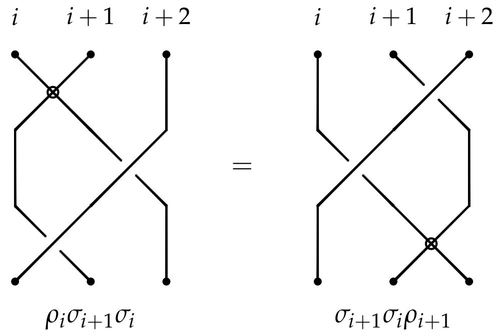


**Figure 4.** The forbidden relation $\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}$.

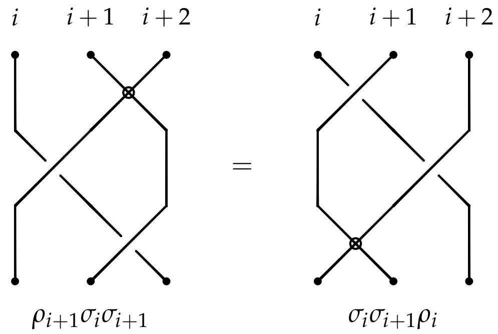


**Figure 5.** The forbidden relation $\rho_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\rho_i$.

In [43] the group of flat virtual braids $FVB_n$ on $n$ strands was introduced as a result of adding the relations (11) to the group $VB_n$:

$$\sigma_i^2 = 1, \qquad 1 \le i \le n-1. \tag{11}$$

We summarize the above discussions in the following definition.

**Definition 1.** *For $n \geq 2$ a group with generators $\sigma_1, \dots, \sigma_{n-1}$, $\rho_1, \dots, \rho_{n-1}$ and the following defining relations:*

$$\begin{array}{ccl} \sigma_i^2 = 1, & \rho_i^2 = 1, & 1 \le i \le n-1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, & 1 \le i \le n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i, & \rho_i \rho_j = \rho_j \rho_i, & |i-j| \ge 2 \end{array}$$

*and*

$$\begin{array}{cl} \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, & 1 \le i \le n-2, \\ \rho_i \sigma_j = \sigma_j \rho_i, & |i-j| \ge 2. \end{array}$$

*is called the flat virtual braid group $FVB_n$ on $n$ strands.*

Generator $\sigma_i \in FVB_n$ is presented geometrically in Figure 6 and generator $\rho \in FVB_n$ is presented geometrically in Figure 2. Flat virtual knots and links arise naturally as closures of flat virtual braids. In [44] Im, Lee, and Son demonstrate a construction of a polynomial invariant for flat virtual knots induced from an index polynomial invariant of virtual knots in [38].

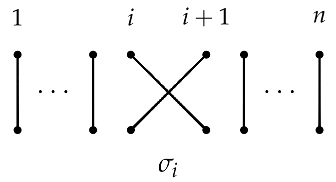


**Figure 6.** Generator $\sigma_i \in FVB_n$.

In [45] the following problem was formulated: Does it exist a representation of the $FVB_n$ group by automorphisms of some group for which the forbidden relations would not hold? In [46], such representation $\eta_n \colon FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ was constructed, here $\mathbb{F}_{2n} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ is a free group of rank $2n$. The homomorphism $\eta_n$ maps generators $\sigma_i, \rho_i \in FVB_n$, where $i = 1, \ldots, n-1$, to the following automorphisms:

$$\eta_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}y_{i+1}, \\ x_{i+1} \mapsto x_i y_{i+1}^{-1}, \end{cases} \qquad \eta_n(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i. \end{cases} \tag{12}$$

It was shown in [46] that the representation $\eta_2 : FVB_2 \to \mathrm{Aut}(\mathbb{F}_4)$ is faithful.

In this paper we construct a family of representations of the group $FVB_n$ by automorphisms of the free group $\mathbb{F}_{2n} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$, which generalize the representation (12). Namely, we consider a family of homomorphisms $\Theta_n : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$, which are given by mapping generators $\sigma_i, \rho_i \in FVB_n$, where $i = 1, \ldots, n-1$, to the following automorphisms:

$$\Theta_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}\, a_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, b_i(y_i, y_{i+1}), \end{cases} \qquad \Theta_n(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}\, c_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, d_i(y_i, y_{i+1}), \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases} \tag{13}$$

where the elements $a_i(y_i, y_{i+1})$, $b_i(y_i, y_{i+1})$, $c_i(y_i, y_{i+1})$ and $d_i(y_i, y_{i+1})$ are words in a free group of rank two with generators $\{y_i, y_{i+1}\}$ for each $i = 1, \ldots, n-1$. Thus, the homomorphisms $\Theta_n$ depend only on the choice of the words $a_i, b_i, c_i, d_i$, which define the locally nontrivial action of the automorphism corresponding to the generator of the group $FVB_n$, and in this sense the homomorphisms $\Theta_n$ are local homomorphisms.

The article has the following structure. In Section 2, the existence of local representations is discussed. Namely, in Theorem 1, we establish for which $a_i$, $b_i$, $c_i$, and $d_i$ there exists a local homomorphism $\Theta_n$ of the group $FVB_n$ into the automorphism group of the free group $\mathbb{F}_{2n}$. In Sections 3 and 4, we obtain results about the structure of the kernel of the homomorphism $\Theta_n$, in particular, in Theorem 3, we describe the kernel of this homomorphism for $n = 2$. In Theorem 4, it will be established that for $n \geq 3$ the kernel of the homomorphism $\Theta_n$ contains a free group of rank 2. We note it was shown earlier in [46] that for $n \geq 3$ the kernel of the homomorphism $\eta_n$, which is a special case of $\Theta_n$, contains an infinite cyclic group. In Section 5, we present a family of local non-homogeneous representations, see Theorem 5.

## 2. Existence of Local Representations

Let $\mathbb{F}_{2n}$ be a free group of rank $2n$ with free generators $x_1, \ldots, x_n, y_1, \ldots, y_n$.

**Theorem 1.** *Let $n \geq 2$ and $a_i(y_i, y_{i+1})$, $b_i(y_i, y_{i+1})$, $c_i(y_i, y_{i+1})$, $d_i(y_i, y_{i+1})$ be words in a free group of rank two with generators $\{y_i, y_{i+1}\}$, where $1 \leq i \leq n-1$. Define the map $\Theta_n : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ by mapping $\sigma_i$ and $\rho_i$ to automorphisms:*

$$\Theta_n(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}\, a_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, b_i(y_i, y_{i+1}), \end{cases} \qquad \Theta_n(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}\, c_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, d_i(y_i, y_{i+1}), \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i. \end{cases} \tag{14}$$

*Then, the following properties hold.*

*(1) The map $\Theta_n$ is homomorphism iff*

$$b_i(y_i, y_{i+1}) = a_i^{-1}(y_i, y_{i+1}), \qquad c_i(y_i, y_{i+1}) = y_{i+1}^{m_i}, \qquad d_i(y_i, y_{i+1}) = y_i^{-m_i},$$

*where $m_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$, and*

$$a_j(y_j, y_{j+1}) = y_{j+1}^{m_j}\, a_{j-1}(y_j, y_{j+1})\, y_j^{-m_{j-1}}, \quad 2 \leq j \leq n-1,$$

*with $n \geq 3$, where $a_1 = w(y_1, y_2)$ for some word $w(A, B) \in \mathbb{F}_2 = \langle A, B \rangle$.*

*(2) The map $\Theta_n$ does not preserve the forbidden relations.*

**Proof.** (1) Let us verify that the relations (1)–(8) and the relation (11) are preserved under the map $\Theta_n$. Denote $s_i = \Theta_n(\sigma_i) \in \mathrm{Aut}(\mathbb{F}_{2n})$ and $r_i = \Theta_n(\rho_i) \in \mathrm{Aut}(\mathbb{F}_{2n})$.

The relations (1), (5), and (7) are preserved because $s_i$ acts non-trivially only on $x_i$ and $x_{i+1}$, while $r_i$ acts non-trivially only on $x_i$, $x_{i+1}$, $y_i$ and $y_{i+1}$.

Since

$$s_i^2 : \begin{cases} x_i \mapsto x_{i+1}\, a_i(y_i, y_{i+1}) \mapsto x_i\, b_i(y_i, y_{i+1})\, a_i(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i\, b_i(y_i, y_{i+1}) \mapsto x_{i+1}\, a_i(y_i, y_{i+1})\, b_i(y_i, y_{i+1}), \end{cases}$$

the relation (11) is preserved if and only if $b_i(y_i, y_{i+1}) = a_i^{-1}(y_i, y_{i+1})$ for all $1 \leq i \leq n-1$. Further,

$$r_i^2 : \begin{cases} x_i \mapsto x_{i+1}\, c_i(y_i, y_{i+1}) \mapsto x_i\, d_i(y_i, y_{i+1})\, c_i(y_{i+1}, y_i), \\ x_{i+1} \mapsto x_i\, d_i(y_i, y_{i+1}) \mapsto x_{i+1}\, c_i(y_i, y_{i+1})\, d_i(y_{i+1}, y_i), \end{cases}$$

and relation (4) is preserved iff

$$d_i(y_i, y_{i+1}) = c_i^{-1}(y_{i+1}, y_i), \qquad 1 \leq i \leq n-1. \tag{15}$$

Consider the actions of automorphisms $r_i r_{i+1} s_i$ and $s_{i+1} r_i r_{i+1}$. We have

$$r_i r_{i+1} s_i : \begin{cases} x_i \mapsto x_{i+1} c_i(y_i, y_{i+1}) \mapsto x_{i+2} c_{i+1}(y_{i+1}, y_{i+2}) c_i(y_i, y_{i+2}), \\ x_{i+1} \mapsto x_i c_i^{-1}(y_{i+1}, y_i) \mapsto x_i c_i^{-1}(y_{i+2}, y_i) \mapsto x_{i+1} a_i(y_i, y_{i+1}) c_i^{-1}(y_{i+2}, y_i), \\ x_{i+2} \mapsto x_{i+2} \mapsto x_{i+1} c_{i+1}^{-1}(y_{i+2}, y_{i+1}) \mapsto x_i a_i^{-1}(y_i, y_{i+1}) c_{i+1}^{-1}(y_{i+2}, y_{i+1}), \end{cases}$$

and

$$s_{i+1} r_i r_{i+1} : \begin{cases} x_i \mapsto x_{i+1} c_i(y_i, y_{i+1}) \mapsto x_{i+2} c_{i+1}(y_{i+1}, y_{i+2}) c_i(y_i, y_{i+2}), \\ x_{i+1} \mapsto x_{i+2} a_{i+1}(y_{i+1}, y_{i+2}) \mapsto x_{i+2} a_{i+1}(y_i, y_{i+2}) \\ \qquad \mapsto x_{i+1} c_{i+1}^{-1}(y_{i+2}, y_{i+1}) a_{i+1}(y_i, y_{i+1}), \\ x_{i+2} \mapsto x_{i+1} a_{i+1}^{-1}(y_{i+1}, y_{i+2}) \mapsto x_i c_i^{-1}(y_{i+1}, y_i) a_{i+1}^{-1}(y_i, y_{i+2}) \\ \qquad \mapsto x_i c_i^{-1}(y_{i+2}, y_i) a_{i+1}^{-1}(y_i, y_{i+1}). \end{cases}$$

Thus, to fulfill the relation (8), it is necessary and sufficient that

$$a_{i+1}(y_i, y_{i+1})c_i(y_{i+2}, y_i) = c_{i+1}(y_{i+2}, y_{i+1})a_i(y_i, y_{i+1}), \tag{16}$$

holds for all $1 \leq i \leq n-2$.

A similar consideration of the relations (6) leads to the equalities

$$c_{i+1}(y_i, y_{i+2})c_i(y_{i+1}, y_{i+2}) = c_{i+1}(y_{i+1}, y_{i+2})c_i(y_i, y_{i+2}), \tag{17}$$

for all $1 \leq i \leq n-2$. This is only possible if $c_i(y_i, y_{i+1}) = y_{i+1}^{m_i}$ for some $m_i \in \mathbb{Z}$ with $1 \leq i \leq n-1$. Using (16) and (17) we obtain

$$a_{i+1}(y_i, y_{i+1}) = y_{i+1}^{m_{i+1}} a_i(y_i, y_{i+1}) y_i^{-m_i}, \qquad 1 \leq i \leq n-2.$$

The fulfillment of the relations (2) is checked directly.

(2) Let us now show that the forbidden relations do not hold under the map $\Theta_n$. Indeed, we have:

$$y_i \xmapsto{r_i} y_{i+1} \xmapsto{s_{i+1}} y_{i+1} \xmapsto{s_i} y_{i+1},$$
$$y_i \xmapsto{s_{i+1}} y_i \xmapsto{s_i} y_i \xmapsto{r_{i+1}} y_i,$$

therefore, $r_i s_{i+1} s_i \neq s_{i+1} s_i r_{i+1}$. Similarly

$$y_i \xmapsto{s_i} y_i \xmapsto{s_{i+1}} y_i \xmapsto{r_i} y_{i+1},$$
$$y_i \xmapsto{r_{i+1}} y_i \xmapsto{s_i} y_i \xmapsto{s_{i+1}} y_i,$$

therefore, $s_i s_{i+1} r_i \neq r_{i+1} s_i s_{i+1}$. $\square$

Thus the representation $\Theta_n$ given by the formula (14) depends on the word $a_1(A, B) = w(A, B) \in \mathbb{F}_2 = \langle A, B \rangle$, into which we substitute $y_i$ and $y_{i+1}$ instead of $A$ and $B$ respectively, and a set of integers $m = (m_1, \ldots, m_{n-1})$. To emphasize this dependence of the representation on $w$ and $m$, we denote it $\Theta_n^{w,m}$:

$$\Theta_n^{w,m}(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1} \prod\limits_{k=i}^{2} y_{i+1}^{m_k} \, w(y_i, y_{i+1}) \prod\limits_{k=1}^{i-1} y_i^{-m_k}, \\ x_{i+1} \mapsto x_i \prod\limits_{k=i-1}^{1} y_i^{m_k} \, (w(y_i, y_{i+1}))^{-1} \prod\limits_{k=2}^{i} y_{i+1}^{-m_k}, \end{cases} \tag{18}$$

and

$$\Theta_n^{w,m}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1} y_{i+1}^{m_i}, \\ x_{i+1} \mapsto x_i y_i^{-m_i}, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases} \tag{19}$$

where in the products $\prod\limits_{k=i}^{2}$ and $\prod\limits_{k=i-1}^{1}$ it is assumed that $i \geq 2$ and the indices are decreasing, and in the products $\prod\limits_{k=1}^{i-1}$ and $\prod\limits_{k=2}^{i}$ it is assumed $i \geq 2$ and the indices are increasing.

The word $w$ is called the defining word for the homomorphism $\Theta_n^{w,m}$. In the particular case when $m_i = 0$ for all $i = 1, \ldots, n-1$, we will write $\Theta_n^w : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ assuming that

$$\Theta_n^w(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1} w(y_i, y_{i+1}), \\ x_{i+1} \mapsto x_i (w(y_i, y_{i+1}))^{-1}, \end{cases} \qquad \Theta_n^w(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i. \end{cases} \tag{20}$$

Note that the homomorphism (12) constructed in [46] can be represented as $\eta_n = \Theta_n^w$ for $w(A, B) = B$.

Let $\beta \in FVB_n$ and $x \in \mathbb{F}_{2n}$. Further, to simplify the notation, by $\beta(x)$ we mean $\Theta_n^{w,m}(\beta)(x)$, where the word $w$ and the set $m$ are assumed to be clear from the context.

## 3. The Kernel of Homomorphism $\Theta_n^{w,m}$ and $FVK_n$ Group

In this section we show that the kernel of the homomorphism $\Theta_n^{w,m}$ lies in the intersection of the group of flat virtual pure braids and the group of flat virtual kure braids group $FVK_n$ defined below.

Consider the subgroup $S_n = \langle \sigma_1 \ldots \sigma_{n-1} \rangle$ of $FVB_n$, which is isomorphic to the permutation group of an $n$-element set. The map $\pi_n : FVB_n \to S_n$ defined on the generators $\sigma_i, \rho_i$ according to the rule:

$$\begin{aligned} \pi_n(\sigma_i) &= \sigma_i, \qquad & 1 \le i \le n-1, \\ \pi_n(\rho_i) &= \sigma_i, \qquad & 1 \le i \le n-1, \end{aligned}$$

is obviously a homomorphism.

**Definition 2.** *Denote $FVP_n = \mathrm{Ker}(\pi_n)$ and call it flat virtual pure braid group on $n$ strands.*

Similarly, the subgroup $S'_n = \langle \rho_1 \ldots \rho_{n-1} \rangle$ of $FVB_n$ is isomorphic to the permutation group of an $n$-element set, and the map $\nu_n : FVB_n \to S'_n$ defined on generators $\sigma_i$, $\rho_i$ as follows:

$$\begin{aligned} \nu_n(\sigma_i) &= 1, \qquad & 1 \le i \le n-1, \\ \nu_n(\rho_i) &= \rho_i, \qquad & 1 \le i \le n-1, \end{aligned}$$

is also a homomorphism.

**Definition 3.** *Denote $FVK_n = \mathrm{Ker}(\nu_n)$ and call it flat virtual kure braid group on $n$ strands.*

Here, we use the term flat virtual kure braid since the term kure virtual braid group was used in [47] for kernel of the map $\pi_K : VB_n \to S_n$ which is defined analogously to $\nu_n : FVB_n \to S'$. The group $FVK_n = \mathrm{Ker}(\nu_n)$ also was denoted by $FH_n$ in [46] since it is the flat analog of the Rabenda's group $H_n$ from [48] (Prop. 17).

**Lemma 1** ([46]). *(Prop. 4) The group $FVK_n$ admits a presentation with generators $x_{i,j}$, $1 \le i \ne j \le n$ and defining relations*

$$x_{i,j}^2 = 1, \qquad x_{i,j}\, x_{k,l} = x_{k,l}\, x_{i,j}, \qquad x_{i,k}\, x_{k,j}\, x_{i,k} = x_{k,j}\, x_{i,k}\, x_{k,j}, \tag{21}$$

*where different letters stand for different indices.*

**Corollary 1.** *The group $FVK_n$ is a Coxeter group with generators $x_{i,j}$, $1 \le i \ne j \le n$ and defining relations*

$$x_{i,j}^2 = 1, \qquad (x_{i,j}\, x_{k,l})^2 = 1, \qquad (x_{i,k}\, x_{k,j})^3 = 1, \tag{22}$$

*where different letters stand for different indices.*

The following property is a generalization of the property established in [46] (Prop. 9) for the word $w(A,B) = B$.

**Lemma 2.** *Let $n \geq 2$. For any word $w \in \mathbb{F}_2$ and any set of integers $m = (m_1, \ldots, m_{n-1})$ $\mathrm{Ker}(\Theta_n^{w,m}) \leq FVP_n \cap FVK_n$.*

**Proof.** Let $g \in \mathrm{Ker}(\Theta_n^{w,m})$. Then $y_i = g(y_i) = \nu_n(g)(y_i)$, since all $\sigma_i$ act identically on $y_i$. But then $\nu_n(g)$ is the identity permutation of the set $\{y_1, \ldots, y_n\}$, which by definition means $g \in FVK_n$.

Next, we show that $g \in FVP_n$. Denote by $G$ the normal closure of the subgroup $\langle y_1, \ldots, y_n \rangle$ in $\mathbb{F}_{2n}$. It is clear that $G$ is a $\Theta_n^{w,m}(FVB_n)$–invariant subgroup. Then $\Theta_n^{w,m}$ induces a homomorphism $\Psi_n^{w,m} \colon FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n}/G) = \mathrm{Aut}(\mathbb{F}_n)$, where $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$. From Formulas (14) we can write out the action of $\Psi_n^{w,m}$ on the generators of the group $FVB_n$:

$$\Psi_n^{w,m}(\sigma_i) \colon \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \Psi_n^{w,m}(\rho_i) \colon \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad 1 \leq i \leq n-1.$$

Now it is easy to see that the image of $FVB_n$ under the map $\Psi_n^{w,m}$ is a permutation of the set $\{x_1, \ldots, x_n\}$. It remains to note that if $g \in \mathrm{Ker}(\Theta_n^{w,m})$, then $g \in \mathrm{Ker}(\Psi_n^{w,m}) = FVP_n$. $\square$

Since $S_n' \leq FVB_n$, then the decomposition of $FVB_n = FVK_n \rtimes S_n'$ follows directly from the definition of $FVK_n$. Considering the restriction of the homomorphism $\pi_n$ to $FVK_n$, we obtain the homomorphism $\xi_n \colon FVK_n \to S_n$. Note that its kernel is $X_n = \mathrm{Ker}(\xi_n) = FVP_n \cap FVK_n$. Further, since $S_n' \leq FVK_n$, we obtain the decomposition $FVK_n = X_n \rtimes S_n$. Thus, we have the following decomposition of the group of flat virtual braids:

$$FVB_n = (X_n \rtimes S_n) \rtimes S_n'.$$

As it invented in [48], we denote

$$\begin{aligned} &\lambda_{i,i+1} = \rho_i \sigma_i, && 1 \leq i \leq n-1, \\ &\lambda_{i,j} = \rho_{j-1}\rho_{j-2} \ldots \rho_{i+1}\lambda_{i,i+1}\rho_{i+1} \ldots \rho_{j-2}\rho_{j-1}, && j - i \geq 2. \end{aligned}$$

Element $\lambda_{i,j}$ is presented geometrically in Figure 7,

**Lemma 3** ([48])**.** *The group $FVP_n$ is generated by the elements $\lambda_{i,j}$, $1 \leq i < j \leq n$ and the defining relations are:*

$$\lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j}, \tag{23}$$

$$\lambda_{k,i}\lambda_{k,j}\lambda_{i,j} = \lambda_{i,j}\lambda_{k,j}\lambda_{k,i}, \tag{24}$$

*where $i, j, k, l$ correspond to different indices.*

Let us consider the case $n = 3$ in more details. As shown in [48],

$$FVP_3 = \langle a, b, c \mid [a, c] = 1 \rangle, \tag{25}$$

where $a = \lambda_{2,3}\lambda_{1,3}$, $b = \lambda_{2,3}$ and $c = \lambda_{2,3}\lambda_{1,2}^{-1}$. These elements presented geometrically in Figure 8.

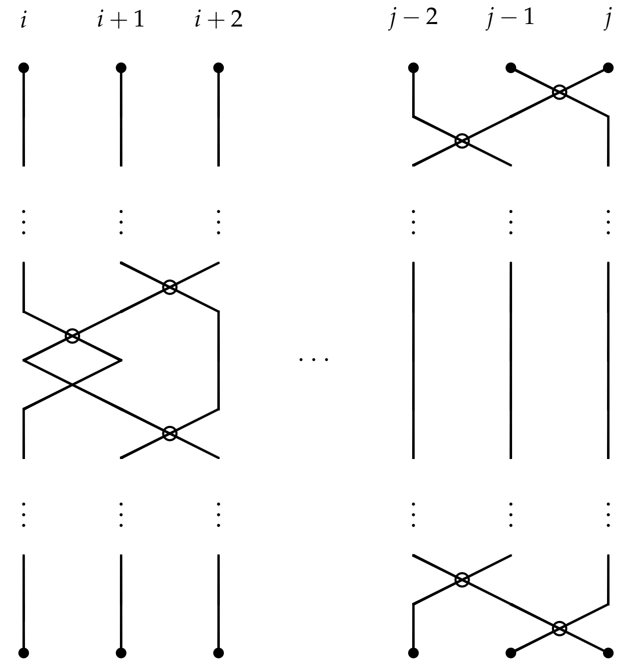


**Figure 7.** Element $\lambda_{i,j} \in FVB_n$.

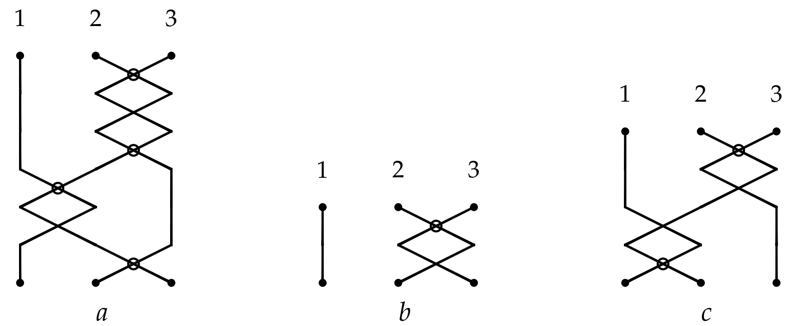


**Figure 8.** Elements $a = \lambda_{2,3}\lambda_{1,3}$, $b = \lambda_{2,3}$, $c = \lambda_{2,3}\lambda_{1,2}^{-1} \in FVB_3$.

**Theorem 2.** *We have the following decomposition*

$$X_3 = \mathbb{Z}^2 * \mathbb{F}_3 * \Gamma,$$

*where* $\mathbb{F}_3$ *is a free group of rank* 3 *and* $\Gamma = \langle x, y, u, v, p, q \mid xy = uv, vu = pq, qp = yx \rangle$.

**Proof.** Consider the restriction of the homomorphism $\nu_3 \colon FVB_3 \to S_3'$ to $FVP_3$. Let us denote it by $\varphi \colon FVP_3 \to S_n'$. Then $X_3 = \mathrm{Ker}(\varphi)$.

To find the generators and relations of the $X_3$ group, we use the Reidemeisetr–Schreier rewriting process, see for example [49]. Let us write out the system of Schreier representatives for $\mathrm{Ker}(\varphi)$ using the generators indicated in the presentation (25): $T = \{1, a, ab, c, cb, b\}$. For an element $g$, we denote its representative in $T$ by $\overline{g}$. Then the kernel $\mathrm{Ker}(\varphi)$ is generated by the following elements:

$$
\begin{aligned}
a \cdot a \cdot (\overline{a^2})^{-1} &= a^2c^{-1} = t, & a \cdot c \cdot (\overline{ac})^{-1} &= ac = m, \\
ab \cdot a \cdot (\overline{aba})^{-1} &= abab^{-1} = v, & ab \cdot b \cdot (\overline{ab^2})^{-1} &= ab^2a^{-1} = w, \\
ab \cdot c \cdot (\overline{abc})^{-1} &= abcb^{-1}c^{-1} = p, & c \cdot a \cdot (\overline{ca})^{-1} &= ca = r, \\
c \cdot c \cdot (\overline{c^2})^{-1} &= c^2a^{-1} = g, & cb \cdot a \cdot (\overline{cba})^{-1} &= cbab^{-1}a^{-1} = q, \\
cb \cdot b \cdot (\overline{cb^2})^{-1} &= cb^2c^{-1} = h, & cb \cdot c \cdot (\overline{cbc})^{-1} &= cbcb^{-1} = y, \\
b \cdot a \cdot (\overline{ba})^{-1} &= bab^{-1}c^{-1} = x, & b \cdot b \cdot (\overline{b^2})^{-1} &= b^2 = f, \\
b \cdot c \cdot (\overline{bc})^{-1} &= bcb^{-1}a^{-1} = u.
\end{aligned}
$$

Further, the relations $taca^{-1}c^{-1}t^{-1}$ for $t \in T$ must be rewritten in new generators. For example, for $t = ab$ we obtain:

$$
\begin{aligned}
ab(aca^{-1}c^{-1})b^{-1}a^{-1} \; &= \; vbca^{-1}c^{-1}b^{-1}a^{-1} \; = \; vuaba^{-1}c^{-1}b^{-1}a^{-1} \\
&= \; vuq^{-1}cbc^{-1}b^{-1}a^{-1} \; = \; vuq^{-1}p^{-1}.
\end{aligned}
$$

The rest of the relations are found similarly. As a result, we obtain:

$$
\begin{aligned}
&m = r, && m = tg, && r = gt, \\
&xy = uv, && vu = pq, && qp = yx.
\end{aligned}
$$

It is now clear that the elements $g, t$ generate $\mathbb{Z}^2$, the elements $w, h, f$ generate $\mathbb{F}_3$, and the group generated by the elements $x$, $y$, $u$, $v$, $p$, and $q$ we denote by $\Gamma$. $\square$

**Lemma 4** ([46])**.** *Let $G_n$ be a subgroup of $FVP_n$ generated by the elements:*

$$t_{i,j} = \lambda_{i,j}^2, \qquad 1 \leq i < j \leq n, \tag{26}$$

$$d_{i,j,k} = \lambda_{j,k}^{-1}\lambda_{i,j}^{-1}\lambda_{j,k}\lambda_{i,k}, \qquad 1 \leq i < j < k \leq n, \tag{27}$$

$$e_{i,j,k} = \lambda_{j,k}^{-1}\lambda_{i,j}^{-1}\lambda_{i,k}\lambda_{i,j}, \qquad 1 \leq i < j < k \leq n. \tag{28}$$

*Then the normal closure of $G_n$ in $FVP_n$ coincides with $X_n$.*

Let us describe the action of $\Theta_n^w$ on the generators indicated in Lemma 4.

**Lemma 5.** *The homomorphism $\Theta_n^w : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$, defined by the word $w \in \mathbb{F}_2$, maps the generators of the group $G_n$ to the following automorphisms:*

$$\Theta_n^w(t_{i,j}) : \begin{cases} x_i \mapsto x_i\, w_{i,j}^{-1}\, w_{j,i}^{-1}, \\ x_j \mapsto x_j\, w_{i,j}\, w_{j,i}, \end{cases} \qquad 1 \leq i < j \leq n, \tag{29}$$

$$\Theta_n^w(d_{i,j,k}) : \begin{cases} x_j \mapsto x_j\, w_{j,i}^{-1}\, w_{i,k}^{-1}\, w_{j,k}, \\ x_k \mapsto x_k\, w_{i,k}\, w_{j,i}\, w_{j,k}^{-1}, \end{cases} \qquad 1 \leq i < j < k \leq n, \tag{30}$$

$$\Theta_n^w(e_{i,j,k}) : \begin{cases} x_i \mapsto x_i\, w_{i,j}^{-1}\, w_{j,k}^{-1}\, w_{i,k}, \\ x_j \mapsto x_j\, w_{i,j}\, w_{i,k}^{-1}\, w_{j,k}, \end{cases} \qquad 1 \leq i < j < k \leq n. \tag{31}$$

*where $w_{i,j} = w(y_i, y_j)$ for all $i, j$.*

**Proof.** Let, as before, $s_i = \Theta_n^w(\sigma_i)$ and $r_i = \Theta_n^w(\rho_i)$. First of all, let us establish some auxiliary formulas. Let $1 \leq i < j-1 \leq n-1$, then

$$b_{i,j} := \Theta_n^w(\rho_{j-1}\dots\rho_{i+1}) : \begin{cases} x_{i+1} \xmapsto{r_{i+1}} x_{i+2}, \\ \vdots \\ x_{j-1} \xmapsto{r_{j-1}} x_j, \\ x_j \xmapsto{r_{j-1}} x_{j-1} \xmapsto{r_{j-2}} x_{j-2} \dots \xmapsto{r_{i+1}} x_{i+1}, \\ y_{i+1} \xmapsto{r_{i+1}} y_{i+2}, \\ \vdots \\ y_{j-1} \xmapsto{r_{j-1}} y_j, \\ y_j \xmapsto{r_{j-1}} x_{j-1} \xmapsto{r_{j-2}} x_{j-2} \dots \xmapsto{r_{i+1}} y_{i+1}. \end{cases}$$

Further,

$$a_{i,i+1} := \Theta_n^w(\lambda_{i,i+1}) = \Theta_n^w(\rho_i\sigma_i) : \begin{cases} x_i \xmapsto{r_i} x_{i+1} \xmapsto{s_i} x_i\, w_{i,i+1}^{-1}, \\ x_{i+1} \xmapsto{r_i} x_i \xmapsto{s_i} x_{i+1}\, w_{i,i+1}, \\ y_i \xmapsto{r_i} y_{i+1} \xmapsto{s_i} y_{i+1}, \\ y_{i+1} \xmapsto{r_i} y_i \xmapsto{s_i} y_i. \end{cases}$$

Let us show that for $1 \leq i < j \leq n$ the formulas

$$\Theta_n^w(\lambda_{i,j}) : \begin{cases} x_i \mapsto x_i\, w_{i,j}^{-1}, \\ x_j \mapsto x_j\, w_{i,j}, \\ y_i \mapsto y_j, \\ y_j \mapsto y_i, \end{cases} \qquad \Theta_n^w(\lambda_{i,j}^{-1}) : \begin{cases} x_i \mapsto x_i\, w_{j,i}, \\ x_j \mapsto x_j\, w_{j,i}^{-1}, \\ y_i \mapsto y_j, \\ y_j \mapsto y_i \end{cases}$$

hold. Indeed, we have

$$\Theta_n^w(\lambda_{i,j}) : \begin{cases} x_i \xmapsto{b_{i,j}} x_i \xmapsto{a_{i,i+1}} x_i\, w_{i,i+1}^{-1} \xmapsto{b_{i,j}^{-1}} x_i\, w_{i,j}^{-1}, \\ x_{i+1} \xmapsto{b_{i,j}} x_{i+2} \xmapsto{a_{i,i+1}} x_{i+2} \xmapsto{b_{i,j}^{-1}} x_{i+1}, \\ \vdots \\ x_{j-1} \xmapsto{b_{i,j}} x_j \xmapsto{a_{i,i+1}} x_j \xmapsto{b_{i,j}^{-1}} x_{j-1}, \\ x_j \xmapsto{b_{i,j}} x_{i+1} \xmapsto{a_{i,i+1}} x_{i+1}\, w_{i,i+1} \xmapsto{b_{i,j}^{-1}} x_j\, w_{i,j}, \\ y_i \xmapsto{b_{i,j}} y_i \xmapsto{a_{i,i+1}} y_{i+1} \xmapsto{b_{i,j}^{-1}} y_j, \\ y_{i+1} \xmapsto{b_{i,j}} y_{i+2} \xmapsto{a_{i,i+1}} y_{i+2} \xmapsto{b_{i,j}^{-1}} y_{i+1}, \\ \vdots \\ y_{j-1} \xmapsto{b_{i,j}} y_j \xmapsto{a_{i,i+1}} y_j \xmapsto{b_{i,j}^{-1}} y_{j-1}, \\ y_j \xmapsto{b_{i,j}} y_{i+1} \xmapsto{a_{i,i+1}} y_i \xmapsto{b_{i,j}^{-1}} y_i. \end{cases}$$

We are now ready to prove Formulas (29)–(31). For example, let us establish (29):

$$\Theta_n^w(t_{i,j}) = \Theta_n^w(\lambda_{i,j}^2) : \begin{cases} x_i \xmapsto{\lambda_{i,j}} x_i\, w_{i,j}^{-1} \xmapsto{\lambda_{i,j}} x_i\, w_{i,j}^{-1}\, w_{j,i}^{-1}, \\ x_j \xmapsto{\lambda_{i,j}} x_i\, w_{i,j} \xmapsto{\lambda_{i,j}} x_i\, w_{i,j}\, w_{j,i}, \\ y_i \xmapsto{\lambda_{i,j}} y_j \xmapsto{\lambda_{i,j}} y_i, \\ y_j \xmapsto{\lambda_{i,j}} y_i \xmapsto{\lambda_{i,j}} y_j. \end{cases}$$

The Formulas (30) and (31) are proved in the same way. □

The following statement answers the question about the faithfulness of the representations $\Theta_n^{w,m}$ in the case $n = 2$.

**Theorem 3.** *The nontrivial representation* $\Theta_2^{w,m} : FVB_2 \to \mathrm{Aut}(\mathbb{F}_4)$ *is not exact if the defining word* $w$ *is*

$$w(A, B) = A^{k_1} B^{k_2} \dots A^{k_m} B^{-k_m} \dots A^{-k_2} B^{-k_1} A^{m_1},$$

*where all* $k_i$ *are nonzero integers except possibly only for* $k_1$ *and* $k_m$*. In this case,* $\mathrm{Ker}(\Theta_2^{w,m}) = X_2 \simeq \mathbb{Z}$*. The representation of* $\Theta_2^{w,m}$ *is exact for other* $w$*.*

**Proof.** Lemma 2 implies that $\mathrm{Ker}(\Theta_2^{w,m}) \leq X_2$. It is easy to show that $X_2$ is generated by the element $t_{1,2}$.

In the case of $n = 2$, the set $m$ consists of a single integer $m = \{m_1\}$. For any $k \in \mathbb{Z}$

$$\Theta_n^{w,m}(t_{1,2}^k) \colon \begin{cases} x_1 \mapsto x_1\, (w_{1,2}^{-1}\, y_2^{m_1}\, w_{2,1}^{-1}\, y_1^{m_1})^k, \\ x_2 \mapsto x_2\, (w_{1,2}\, y_1^{-m_1}\, w_{2,1}\, y_2^{-m_1})^k. \end{cases}$$

Thus $\Theta_n^w(t_{1,2}^k) = \mathrm{id}$ iff either $k = 0$ or

$$w_{1,2}^{-1}\, y_2^{m_1}\, w_{2,1}^{-1}\, y_1^{m_1} = 1,$$

i.e., $f(y_1, y_2) = f^{-1}(y_2, y_1)$ for the word $f(y_1, y_2) = w(y_1, y_2) y_1^{-m_1}$.

Let $f(A, B) = A^{k_1} B^{k_2} \dots B^{k_s} A^{k_{s+1}}$. Then

$$A^{k_1} B^{k_2} \dots B^{k_s} A^{k_{s+1}} = B^{-k_{s+1}} A^{-k_s} \dots A^{-k_2} B^{-k_1},$$

therefore $f(A, B) = A^{k_1} B^{k_2} \dots A^{k_m} B^{-k_m} \dots A^{-k_2} B^{-k_1}$, where all $k_i$ — nonzero integers except maybe $k_1$ and $k_m$. But then

$$w(A, B) = A^{k_1} B^{k_2} \dots A^{k_m} B^{-k_m} \dots A^{-k_2} B^{-k_1} A^{m_1}.$$

This completes the proof. □

## 4. Nontriviality of the Kernel $\mathrm{Ker}(\Theta_n^{w,m})$ for $n \geq 3$

Consider the following subgroups of the $FVB_n$ group:

$$Q_n^i = \langle t_{i,i+1},\ e_{i,i+1,i+2} \rangle, \qquad 1 \leq i \leq n-2, \tag{32}$$

$$M_n^{i+1} = \langle t_{i+1,i+2},\ d_{i,i+1,i+2} \rangle, \qquad 1 \leq i \leq n-2, \tag{33}$$

$$P_n^{i+2} = \langle t_{i,i+2} \rangle, \qquad 1 \leq i \leq n-2. \tag{34}$$

**Lemma 6.** *Let* $n \geq 3$ *and* $w(A, B) \in \mathbb{F}_2(A, B)$*. Then, for all* $i$*,* $1 \leq i \leq n-2$*, we obtain the inclusion*

$$\left[Q_n^i, \left[M_n^{i+1}, P_n^{i+2}\right]\right] \leq \mathrm{Ker}(\Theta_n^w). \tag{35}$$

**Proof.** Note that $Q_n^i$ acts non-trivially only on generators $x_i$ and $x_{i+1}$, $M_n^{i+1}$ acts non-trivially only on $x_{i+1}$ and $x_{i+2}$, while $P_n^{i+2}$ acts non-trivially only on $x_i$ and $x_{i+2}$. Consider the element

$$h = q(mpm^{-1}p^{-1})q^{-1}(mpm^{-1}p^{-1})^{-1},$$

where $q \in Q_n^i$, $m \in M_n^{i+1}$ and $p \in P_n^{i+2}$. This element acts non-trivially only on the generators $x_i$, $x_{i+1}$ and $x_{i+2}$. Write out its action:

$$h : x_i \xmapsto{q} qx_i \xmapsto{m} qx_i \xmapsto{p} pqx_i \xmapsto{m^{-1}} pqx_i \xmapsto{p^{-1}} qx_i \xmapsto{q^{-1}} x_i \xmapsto{p} px_i \xmapsto{m} px_i \xmapsto{p^{-1}} x_i \xmapsto{m^{-1}} x_i;$$

$$h : x_{i+1} \xmapsto{q} qx_{i+1} \xmapsto{m} mqx_{i+1} \xmapsto{p} mqx_{i+1} \xmapsto{m^{-1}} qx_{i+1} \xmapsto{p^{-1}} qx_{i+1} \xmapsto{q^{-1}} x_{i+1} \xmapsto{p} x_{i+1}$$
$$\xmapsto{m} mx_{i+1} \xmapsto{p^{-1}} mx_{i+1} \xmapsto{m^{-1}} x_{i+1};$$

$$h : x_{i+2} \xmapsto{q} x_{i+2} \xmapsto{m} mx_{i+2} \xmapsto{p} pmx_{i+2} \xmapsto{m^{-1}} m^{-1}pmx_{i+2} \xmapsto{p^{-1}} p^{-1}m^{-1}pmx_{i+2} \xmapsto{q^{-1}}$$
$$p^{-1}m^{-1}pmx_{i+2} \xmapsto{pmp^{-1}m^{-1}} x_{i+2}$$

Thus, $h \in \mathrm{Ker}(\Theta_n^w)$ and the inclusion (35) is proved. □

**Theorem 4.** *Let $n \geq 3$. For any defining word $w(A,B) \in \mathbb{F}_2(A,B)$ the kernel $\mathrm{Ker}(\Theta_n^w)$ contains a subgroup isomorphic to a free group of rank 2.*

**Proof.** Denote a free group of rank 2 by $\mathbb{F}_2$. By Lemma 6, it suffices to show that $\mathbb{F}_2 \leq \left[Q_3^1, \left[M_3^2, P_3^3\right]\right] \leq FVP_3$.

Consider the elements

$$h_0 = \left[t_{1,2}, \left[t_{2,3}, t_{1,3}\right]\right] \qquad \text{and} \qquad h_1 = \left[t_{1,2}, \left[d_{1,2,3}, t_{1,3}\right]\right].$$

We write them down in terms of the generators of $FVP_3$ group, see (25):

$$h_0 = \left[(c^{-1}b)^2, [b^2, (b^{-1}a)^2]\right] \qquad \text{and} \qquad h_1 = \left[(c^{-1}b)^2, [b^{-2}ca, (b^{-1}a)^2]\right]. \tag{36}$$

Let us prove that $h_0$ and $h_1$ generate $\mathbb{F}_2$. To achieve this, it suffices to show that there are no relations between them. Let $\psi : FVP_3 \to \langle a,b \rangle$ be the homomorphism given by the mapping $\psi(a) = a$, $\psi(b) = b$ and $\psi(c) = 1$. Denote $\bar{h}_0 = \psi(h_0)$ and $\bar{h}_1 = \psi(h_1)$. Then

$$\bar{h}_0 = b^3ab^{-1}ab^{-2}a^{-1}ba^{-1}b^{-2}ab^{-1}ab^2a^{-1}ba^{-1}b^{-1},$$
$$\bar{h}_1 = ab^{-1}aba^{-1}ba^{-1}b^{-2}ab^{-1}ab^{-1}a^{-1}ba^{-1}b^2.$$

The elements $\bar{h}_0$ and $\bar{h}_1$ lie in the free group $\langle a,b \rangle$. Hence the group $\langle \bar{h}_0, \bar{h}_1 \rangle$ is either isomorphic to $\mathbb{Z}$ or isomorphic to $\mathbb{F}_2$. The first case means that $\bar{h}_0$ and $\bar{h}_1$ must be powers of the same element, i.e., $\bar{h}_0 = g(a,b)^k$ and $\bar{h}_1 = g(a,b)^s$ for some word $g(a,b) \in \langle a,b \rangle$ and nonzero $k,s \in \mathbb{Z}$. Let $g(a,b) = f \cdot w \cdot f^{-1}$, where $w(a,b)$ is the cyclic reduced word in $\langle a,b \rangle$. Then $g^s = fw^sf^{-1} = \bar{h}_1$ and since $\bar{h}_1$ is itself cyclically reduced, we obtain $f = 1$. But then $g^k = fw^kf^{-1} = w^k = \bar{h}_0$ must be cyclically reduced, which is not the case. Thus $\langle \bar{h}_0, \bar{h}_1 \rangle \cong \mathbb{F}_2$ and hence $\langle h_0, h_1 \rangle \cong \mathbb{F}_2$. □

**Corollary 2.** *Let $n \geq 3$, then $\mathrm{Ker}(\Theta_n^{w,m})$ contains a subgroup isomorphic to a free group of rank 2 for any integer tuple $m = (m_1, \ldots, m_{n-1})$ and arbitrary defining word $w(A,B) \in \mathbb{F}_2(A,B)$.*

**Proof.** There are three points which are sufficient to prove the corollary:

- $t_{1,2}$ acts non-trivially only on generators $x_1$ and $x_2$,
- $t_{2,3}$ and $d_{1,2,3}$ act non-trivially only on elements $x_2$ and $x_3$,
- $t_{1,3}$ acts non-trivially only on $x_1$ and $x_3$.

Recall we have $t_{i,i+1} = \lambda^2_{i,i+1} = (\rho_i\sigma_i)^2$ by Formula (26). Therefore, $t_{1,2}$ and $t_{2,3}$ satisfy property above.

Let us check that $t_{1,3} = (\rho_2\rho_1\sigma_1\rho_2)^2$ leaves $x_2$, $y_1$, $y_2$ and $y_3$ in place. The element $t_{1,3}$ acts trivially on the generators $y_i$ for $1 \le i \le 3$, because using Formulas (18) and (19) we obtain $t_{1,3} \cdot y_i = (\rho_2\rho_1\rho_2)^2 \cdot y_i = 1 \cdot y_i = y_i$. The trivial action on $x_2$ follows from the fact that $\rho_2\rho_1\sigma_1\rho_2$ leaves this element in place. Indeed, according to (18) and (19), we obtain

$$x_2 \xmapsto{\rho_2} x_3y_3^{m_2} \xmapsto{\rho_1} x_3y_3^{m_2} \xmapsto{\sigma_1} x_3y_3^{m_2} \xmapsto{\rho_2} x_2y_2^{-m_2}y_2^{m_2} = x_2.$$

Similarly, we check the action for

$$\begin{gathered} x_1 \xmapsto{\sigma_2\rho_2} x_1 \xmapsto{\sigma_1} x_2w(y_1,y_2) \xmapsto{\rho_1} x_1y_1^{-m_1}w(y_2,y_1) \xmapsto{\rho_2} x_1y_1^{-m_1}w(y_3,y_1) \xmapsto{\sigma_2} \\ \xmapsto{\sigma_2} x_1y_1^{-m_1}w(y_3,y_1) \xmapsto{\rho_2} x_1y_1^{-m_1}w(y_2,y_1) \xmapsto{\rho_1} \\ \xmapsto{\rho_1} x_2y_2^{m_1}y_2^{-m_1}w(y_1,y_2) \xmapsto{\sigma_1} x_1w^{-1}(y_1,y_2)w(y_1,y_2) \xmapsto{\rho_2} x_1, \\ d_{1,2,3} \cdot y_i = \rho_1\rho_2\rho_1\rho_1\rho_2\rho_1 \cdot y_i = y_i, \quad 1 \le i \le 3. \end{gathered}$$

This completes the proof. □

## 5. Examples of Non-Homogeneous Representations

Following [12,13] we recall the concept of a local representation of the braid group by automorphisms of the free group. Let $\mathbb{F}_n$ be the free group of rank $n$ generated by $x_1, \ldots, x_n$. For $i = 1, \ldots, n-1$, an automorphism $T_i : \mathbb{F}_n \to \mathbb{F}_n$ is said to be $i$-local if $T_i(x_j) = x_j$ for $j \neq i, i+1$ and $T_i(\langle x_i, x_{i+1}\rangle) = \langle x_i, x_{i+1}\rangle$, where $\langle x_i, x_{i+1}\rangle$ denotes the subgroup of $\mathbb{F}_n$ generated by $x_i$ and $x_{i+1}$. In other words, an automorphism $T_i \in \mathrm{Aut}(\mathbb{F}_n)$ is $i$-local if and only if there exists an automorphism $t_i \in \mathrm{Aut}(\mathbb{F}_2)$ of the free group $\mathbb{F}_2 = \langle A, B\rangle$ of rank 2 such that

$$T_i = \mathrm{id}_{\mathbb{F}_{i-1}} * t_i * \mathrm{id}_{\mathbb{F}_{n-i-1}} : \mathbb{F}_n = \mathbb{F}_{i-1} * \mathbb{F}_2 * \mathbb{F}_{n-i-1} \to \mathbb{F}_{i-1} * \mathbb{F}_2 * \mathbb{F}_{n-i-1} = \mathbb{F}_n.$$

In this case we say that $t_i$ is the core of $T_i$.

A representation $\Theta : B_n \to \mathrm{Aut}(\mathbb{F}_n)$ is said to be local if the automorphism $\Theta(\sigma_i)$ is $i$-local for all $i = 1, \ldots, n-1$. If a local representation $\Theta$ is such that $t_1 = \ldots = t_{n-1}$, then it is said to be a homogeneous representation or Wada representation.

We introduce the concept of a local representation of the group of flat virtual braids by automorphisms of the free group as follows. Let $\mathbb{F}_{2n}$ be the free group of rank $2n$ generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. For $i = 1, \ldots, n-1$, an automorphism $T_i : \mathbb{F}_{2n} \to \mathbb{F}_{2n}$ is said to be $i$-local if:

- $T_i(x_j) = x_j$ and $T_i(y_j) = y_j$ for $j \neq i, i+1$;
- $T_i(\langle y_i, y_{i+1}\rangle) = \langle y_i, y_{i+1}\rangle$;
- $T_i(x_i) \in \langle x_{i+1}, y_i, y_{i+1}\rangle$ and $T_i(x_{i+1}) \in \langle x_i, y_i, y_{i+1}\rangle$.

A representation $\Theta : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ is said to be local if the automorphisms $\Theta(\sigma_i)$ and $\Theta(\rho_i)$ are $i$-local for all $i = 1, \ldots, n-1$. In this case, each automorphism $\Theta(\sigma_i)$ and $\Theta(\rho_i)$ of $\mathbb{F}_{2n}$ corresponds to automorphism $t_i$ and $\tau_i$ of $\mathbb{F}_4 = \langle A, B, C, D\rangle$. In this case we say that $t_i$ and $\tau_i$ are $i$-cores of $\Theta$. A local representation $\Theta : FVB_n \to \mathrm{Aut}(\mathbb{F}_{2n})$ is said to be homogeneous if $t_1 = \ldots = t_{n-1}$ and $\tau_1 = \ldots = \tau_{n-1}$.

Below we will focus on the non-homogeneous representations of the group $FVB_3$ by automorphisms of the free group $\mathbb{F}_6 = \langle x_1, x_2, x_3,\ y_1, y_2, y_3\rangle$ of rank 6. Recall that in this case, a representation $\Theta : FVB_3 \to \mathrm{Aut}(\mathbb{F}_6)$ is locally non-homogeneous if the automorphisms corresponding to the generators $\sigma_1$, $\sigma_2$, and $\rho_1$, $\rho_2$ are induced by distinct automorphisms of the free group $\mathbb{F}_4$.

Above, we considered representations of the group of flat virtual braids $FVB_n$ in which the image of the generator $\sigma_i$ acted trivially on $y$, and $y_{i+1}$ for all $1 \le i \le n-1$, see formulae (18) and (19). Now we relax this condition. Taking into account Ito's classification result in [13] and the involutiveness of generators of $FVB_n$, we obtain two possible scenarios for the action of the image of the generators $\sigma_i$ or $\rho_i$ on $y_i$ and $y_{i+1}$:

$$\begin{cases} y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases} \quad \text{or} \quad \begin{cases} y_i \mapsto y_{i+1}^{-1}, \\ y_{i+1} \mapsto y_i^{-1}. \end{cases}$$

The following result can be considered as an analog of Theorem 1 in the case of local non-homogeneous representations of the group $FVB_3$.

**Theorem 5.** *Let $e_i, \varepsilon_i, \alpha_i, a_i \in \{\pm 1\}$ for $1 \leq i \leq 2$ and coefficients $\beta_{i,j}, \gamma_{i,j}, b_{i,j}, g_{i,j} \in \mathbb{Z}$ for $1 \leq i, j \leq 2$. Consider the map $\Phi_3 : FVB_3 \to \mathrm{Aut}(\mathbb{F}_6)$, defined on the generators as follows:*

$$\Phi_3(\sigma_1)\colon \begin{cases} x_1 \mapsto y_2^{\beta_{11}} x_2^{\alpha_1} y_2^{\beta_{12}}, \\ x_2 \mapsto y_1^{\beta_{21}} x_1^{\alpha_1} y_1^{\beta_{22}}, \\ y_1 \mapsto y_2^{\varepsilon_1}, \\ y_2 \mapsto y_1^{\varepsilon_1}, \end{cases} \qquad \Phi_3(\rho_1)\colon \begin{cases} x_1 \mapsto y_2^{\gamma_{11}} x_2^{\alpha_2} y_2^{\gamma_{12}}, \\ x_2 \mapsto y_1^{\gamma_{21}} x_1^{\alpha_2} y_1^{\gamma_{22}}, \\ y_1 \mapsto y_2^{\varepsilon_2}, \\ y_2 \mapsto y_1^{\varepsilon_2}, \end{cases}$$

$$\Phi_3(\sigma_2)\colon \begin{cases} x_2 \mapsto y_3^{b_{11}} x_3^{a_1} y_3^{b_{12}}, \\ x_3 \mapsto y_2^{b_{21}} x_2^{a_1} y_2^{b_{22}}, \\ y_2 \mapsto y_3^{e_1}, \\ y_3 \mapsto y_2^{e_1}, \end{cases} \qquad \Phi_3(\rho_2)\colon \begin{cases} x_2 \mapsto y_3^{g_{11}} x_3^{a_2} y_3^{g_{12}}, \\ x_3 \mapsto y_2^{g_{21}} x_2^{a_2} y_2^{g_{22}}, \\ y_2 \mapsto y_3^{e_2}, \\ y_3 \mapsto y_2^{e_2}, \end{cases}$$

*Then $\Phi_3$ is a representation if and only if $e_1 e_2 = \varepsilon_1 \varepsilon_2$ and one of the following conditions is satisfied:*

*(1)* $\alpha_1 = \alpha_2 = a_1 = a_2 = 1$ *or* $\alpha_1 = \alpha_2 = -a_1 = -a_2 = 1$, *where*

$$b_{11} - g_{11} = e_2\beta_{11} - e_1\gamma_{11},$$
$$b_{12} - g_{12} = e_2\beta_{12} - e_1\gamma_{12},$$

*(2)* $\alpha_1 = -\alpha_2 = a_1 = -a_2 = 1$ *or* $\alpha_1 = -\alpha_2 = -a_1 = a_2 = 1$, *where*

$$b_{11} + g_{12} = -e_2\beta_{12} + e_1\gamma_{12},$$
$$b_{12} + g_{11} = -e_2\beta_{11} + e_1\gamma_{11},$$

*(3)* $-\alpha_1 = \alpha_2 = a_1 = -a_2 = 1$ *or* $-\alpha_1 = \alpha_2 = -a_1 = a_2 = 1$, *where*

$$b_{11} + g_{12} = e_2\beta_{11} - e_1\gamma_{11},$$
$$b_{12} + g_{11} = e_2\beta_{12} - e_1\gamma_{12},$$

*(4)* $-\alpha_1 = -\alpha_2 = a_1 = a_2 = 1$ *or* $-\alpha_1 = -\alpha_2 = -a_1 = -a_2 = 1$, *where*

$$b_{11} - g_{11} = -e_2\beta_{12} + e_1\gamma_{12},$$
$$b_{12} - g_{12} = -e_2\beta_{12} + e_1\gamma_{11}.$$

*In all cases, the condition of involutiveness uniquely determines the coefficients*

$$\beta_{21}, \beta_{22}, \gamma_{21}, \gamma_{22}$$

*in terms of the remaining coefficients.*

**Proof.** The condition $\varepsilon_1\varepsilon_2 = e_1 e_2$ follows directly from relation $\rho_1\rho_2\sigma_1 = \sigma_2\rho_1\rho_2$.

The homomorphism $\Phi_3$ induces an action $\widetilde{\Phi_3}$ on the quotient group of $\mathbb{F}_{2n}$ by relations $y_1 = y_2 = y_3 = 1$:

$$\widetilde{\Phi}_3(\sigma_1)\colon \begin{cases} x_1 \mapsto x_2^{\alpha_1}, \\ x_2 \mapsto x_1^{\alpha_1}, \end{cases} \qquad \widetilde{\Phi}_3(\rho_1)\colon \begin{cases} x_1 \mapsto x_2^{\alpha_2}, \\ x_2 \mapsto x_1^{\alpha_2}, \end{cases}$$

$$\widetilde{\Phi}_3(\sigma_2)\colon \begin{cases} x_2 \mapsto x_3^{a_1}, \\ x_3 \mapsto x_2^{a_1}, \end{cases} \qquad \widetilde{\Phi}_3(\rho_2)\colon \begin{cases} x_2 \mapsto x_3^{a_2}, \\ x_3 \mapsto x_2^{a_2}. \end{cases}$$

Then, the condition $\alpha_1 a_2 = a_1 \alpha_2$ follows similarly from relation $\rho_1\rho_2\sigma_1 = \sigma_2\rho_1\rho_2$.

Further, we consider a case $e_1 = e_2 = \varepsilon_1 = \varepsilon_2 = 1$ and $a_1 = a_2 = \alpha_1 = \alpha_2 = 1$. In this case, we have the following formulae for $\Phi_3$:

$$\Phi_3(\sigma_1)\colon \begin{cases} x_1 \mapsto y_2^{\beta_{11}} x_2 y_2^{\beta_{12}}, \\ x_2 \mapsto y_1^{-\beta_{11}} x_1 y_1^{-\beta_{12}}, \\ y_1 \mapsto y_2, \\ y_2 \mapsto y_1, \end{cases} \qquad \Phi_3(\rho_1)\colon \begin{cases} x_1 \mapsto y_2^{\gamma_{11}} x_2 y_2^{\gamma_{12}}, \\ x_2 \mapsto y_1^{-\gamma_{11}} x_1 y_1^{-\gamma_{12}}, \\ y_1 \mapsto y_2, \\ y_2 \mapsto y_1, \end{cases}$$

$$\Phi_3(\sigma_2)\colon \begin{cases} x_2 \mapsto y_3^{b_{11}} x_3 y_3^{b_{12}}, \\ x_3 \mapsto y_2^{-b_{11}} x_2 y_2^{-b_{12}}, \\ y_2 \mapsto y_3, \\ y_3 \mapsto y_2, \end{cases} \qquad \Phi_3(\rho_2)\colon \begin{cases} x_2 \mapsto y_3^{g_{11}} x_3 y_3^{g_{12}}, \\ x_3 \mapsto y_2^{-g_{11}} x_2 y_2^{-g_{12}}, \\ y_2 \mapsto y_3, \\ y_3 \mapsto y_2, \end{cases}$$

here we use involutiveness of generators of $FVB_3$.

Finally, we can write conditions for relation $\rho_1\rho_2\sigma_1 = \sigma_2\rho_1\rho_2$:

$$\Phi_3(\rho_1\rho_2\sigma_1)\colon \begin{cases} x_1 \mapsto y_3^{\gamma_{11}+g_{11}} x_3 y_3^{\gamma_{12}+g_{12}}, \\ x_2 \mapsto y_2^{-\gamma_{11}+\beta_{11}} x_2 y_2^{-\gamma_{12}+\beta_{12}}, \\ x_3 \mapsto y_1^{-g_{11}-\beta_{11}} x_1 y_1^{-g_{12}-\beta_{12}}, \\ y_1 \mapsto y_3, \\ y_2 \mapsto y_2, \\ y_3 \mapsto y_1, \end{cases}$$

$$\Phi_3(\sigma_2\rho_1\rho_2)\colon \begin{cases} x_1 \mapsto y_3^{\gamma_{11}+g_{11}} x_3 y_3^{\gamma_{12}+g_{12}}, \\ x_2 \mapsto y_2^{b_{11}-g_{11}} x_2 y_2^{b_{12}-g_{12}}, \\ x_3 \mapsto y_1^{-b_{11}-\gamma_{11}} x_1 y_1^{-b_{12}-\gamma_{12}}, \\ y_1 \mapsto y_3, \\ y_2 \mapsto y_2, \\ y_3 \mapsto y_1, \end{cases}$$

therefore, we obtain

$$b_{11} - g_{11} = \beta_{11} - \gamma_{11},$$
$$b_{12} - g_{12} = \beta_{12} - \gamma_{12}.$$

Other cases hold by analogous considerations. □

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