

Article

Studying the Dynamics of the Rumor Spread Model with Fractional Piecewise Derivative

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Abstract: Sensitively altered news, commonly referred to as rumors, can lead an individual, organization, or nation astray, potentially resulting in harm, even to the extent of causing violence among large groups of people. In this digital age, news can be easily twisted and rapidly spread through the internet and social media. It becomes challenging for consumers to discern whether the information they encounter online has been manipulated. Unfortunately, the rise of internet forgeries has facilitated the dissemination of false or distorted information by unscrupulous individuals, particularly on sensitive matters, to serve their own interests. Once a rumor is generated and made public on the internet, it quickly spreads through sharing and discussions by anonymous individuals, sometimes intentionally, without thorough fact-checking. In this manuscript, we investigate the dynamical model of rumor propagation in a social network using the classical Caputo piecewise derivative. We examine the existence and uniqueness of a solution for the aforementioned problem and analyze the equilibrium, stability, boundedness, and positivity of the model. To obtain the numerical simulation of the piecewise derivative, we employ various fractional orders, and the approximate solution of the considered model is found using the fractional piecewise numerical iterative approach of the Newton polynomial. This approach allows us to gain valuable insights into the dynamics of rumor propagation and its effects within a social network.

Keywords: rumor spread model; piecewise derivative; classical derivative; Caputo operator



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1. Introduction

Rumors are a complex phenomenon that has eluded mankind throughout history, involving a vast range of components and behaviors, including ecological, sociological, economic, and psychological factors. Over the years, many rumors have formed in communities and circulated widely between them. Leaders of various civilizations have engaged with and studied rumors throughout history [1]. People have created and declared rumor models for government, economic, and social purposes [2] and experimented for economic gain either to win battles by introducing fear and submission in the competitor or by enduring faith in their supreme leaders. Rumors have gone through a collection of changes in their composition, in keeping with the variations well lived by communities and the assessment of daily life, due to advancing technologies along with innovation in communication technologies. This method has been applied to an extraordinary spike in reputation, as well as to stimulate dissemination. On the other hand, this rise has had wonderful effects and far-reaching impacts. The progression of rumor conditions, as well as the power of its effect and impact within communities, provides a new feature [3]. This new feature is utilized by electronic media and intelligence in conflicts between countries and propaganda by exposing news with false information to control the rights of voters by deception and increase or decrease misunderstandings between political parties [4].

Mathematical modeling is the main branch of applied mathematics to investigate different types of difficulties to help researchers express events in the form of systems of equations. The

first research on rumor transmission models was conducted in 1960. The authors in [5] studied the spreading of a rumor model in the form of mathematical epidemiology. The authors in [6] constructed a new mathematical rumor model that is more popular in this new area. The authors divided both models into three different categories: the first, which is awareness of the rumor and dissemination; the second, which is unawareness of the rumor; and the third, which is awareness of the rumor but no interest in dissemination. Several researchers have studied different mathematical models to investigate the transmission dynamics of rumors [7] and methods to control spreading [5,8–11]. Ndii et al. [12] presented and studied different models of rumors regarding spreading. The authors in [13] documented a study using the concept of mathematical modeling to show rumor transmission in societies. Novel components have evolved following the growth of civilizations and the creation of modern technological techniques, which have clarified the phenomenon of rumors. Many researchers have started working on rumor models to improve them more clearly. The authors in [5] constructed a new model by adding more classes to the *SIR* and *SEIZR* models, which was very useful for adding new information. The considered classes are for the uninformed, spreaders, and stiflers, which divides the individual into three, reflecting the initial performers in the transmissions of rumor spreading [14].

Using a system of four ODEs, Arindam and Md. Haider [15] proposed the following novel mathematical model of rumor propagation via online media. This topic was investigated as the dynamics of misinformation diffusion across the internet, with these ODEs serving as the model's four compartments. The authors used this system to depict a shift in population that is prone to spreading rumors and determine who prevents them from spreading and who is neutral and may disseminate rumors without confirming them.

$$\begin{aligned}\frac{dS}{dt} &= IS\alpha - SZ\beta, \\ \frac{d\mathcal{E}}{dt} &= -\mathcal{E}\sigma - \theta\mathcal{E}Z - \mathcal{E}\mathcal{I}\rho + IS\alpha(q_1 - 1) + SZ\beta(1 - \Omega), \\ \frac{d\mathcal{I}}{dt} &= \mathcal{E}\sigma + \mathcal{E}\mathcal{I}\rho + IS\alpha q_2, \\ \frac{dZ}{dt} &= \theta\mathcal{E}Z + SZ\beta\Omega.\end{aligned}\tag{1}$$

Susceptible (S) refers to someone who has not heard the news yet but might be notified at any time. Exposed ($\mathcal{E}$) is a person who has been exposed to news from somebody but has taken time to post it owing to an exposure delay. Infected ($\mathcal{I}$) refers to a person who has disseminated news on purpose; they can generate or share the news after or before learning about it. Protestors (Z) are users who have heard about the news but wish to verify and validate it. The description of parameters used in the considered system (1) is presented in Table 1.

Several important properties, including hereditary fractional mathematical models, are much better than integer-order mathematical models to express the behavior of pandemic- and epidemic-type diseases. Another important property is known as the extra degree of freedom. The authors in [16] investigated the propagation of a mathematical rumor model in a social network under the ABC operator. The authors in [17] considered a rumor transmission model in a mobile social network in the framework of the fractional operator. They also developed the system from an underlying physical stochastic process. The authors' considered model was examined under the fractional operator for which the power law was taken. Using a multifaceted approach, the dynamics of mathematical models were investigated by employing various types of operators. Diverse mathematical tools and techniques were integrated to gain a comprehensive understanding of the behaviors exhibited by these models [18–24]. Through the application of different operators, including fractional, differential, and integral operators, it was explored how each approach influenced model behavior. This systematic investigation allowed for a more nuanced analysis

of complex phenomena, offering insights into how various mathematical operations impact the dynamics of the models.

Table 1. Details of the used parameters of the considered system (1).

Variables	Description
α	Contact rate from S to I
b	Contact from S to Z
ρ	Contact from E to I
σ	Incubation rate
θ	Contact rate from E to Z
βL	Transmission rate from S to Z
$\alpha\rho$	Transmission rate from S to I
$\beta(1-L)$	Transmission rate from S to E through contact with Z
$\alpha(1-p_1)$	Infection rate from S to E through contact with I
Ω	$S \rightarrow Z$ Probability rate through contact with protesters
$1-\Omega$	$S \rightarrow E$ Probability rate through contact with protesters
q_1	$S \rightarrow I$ Probability rate through contact with infected
$1-q_1$	$S \rightarrow E$ Probability rate through contact with infected

Additionally, researchers have extensively explored a wide range of operators involving fractal and fractional orders, encompassing both singular and nonsingular kernels. Moreover, the concept of fractal–fractional derivatives has garnered considerable attention and investigation from the scientific community. These endeavors have led to valuable insights and applications across diverse fields of study. The ongoing research in this area reflects the dynamic nature of scientific exploration, continually unveiling new mathematical tools to better comprehend complex phenomena and real-world problems. As the study of fractal and fractional calculus progresses, it is expected that further advancements will emerge, contributing to our understanding of intricate systems and their behavior [23,25–30]. On the contrary, the power law and the Mittag–Leffler law have shown limitations in expressing the crossover behavior observed at different time scales. Although these mathematical models have proven valuable in capturing certain dynamic behaviors, they may fall short when attempting to describe the intricate and complex phenomena that occur during crossovers at various time intervals. Atangana and Seda presented a new technique of piecewise differential and integrals to solve this difficulty for the researchers [31]. They introduced the classical and global piecewise derivatives and explained them with some examples. Numerous researchers have recently embarked on exploring the crossover behavior approach and have extensively studied various models utilizing this concept. By focusing on understanding crossover phenomena, these researchers aim to gain deeper insights into the dynamic behaviors of complex systems operating across different time scales. The study of crossover behavior has spurred the development of innovative mathematical and computational models, enabling a more accurate representation of the intricate dynamics observed in real-world scenarios [32–35].

2. Basic Results

This section presents the basic results from the literature on fractional calculus.

Definition 1. The Caputo derivative of a function $\mathbf{M}(t)$ with ω order is defined as

$${}_0^C D_t^\omega \mathbf{M}(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\eta)^{-\omega} \mathbf{M}'(\eta) d\eta,$$

while the Riemann–Liouville fractional integral is expressed as

$${}_0^{\text{RL}} I_t^\omega \mathbf{M}(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\eta)^{\omega-1} d\eta, \quad \omega > 0.$$

Definition 2 ([31]). Let $\mathbf{M}(t)$ be a piecewise differentiable function. Then, the piecewise classical and fractional derivative is defined as

$${}_0^{PF}D_t^\omega \mathbf{M}(t) = \begin{cases} \mathbf{M}'(t), & 0 < t \leq t_1, \\ {}_0^CD_t^\omega \mathbf{M}(t) & t_1 < t \leq t_2, \end{cases}$$

and its equivalent integration is given as

Definition 3.

$${}_0^{PF}I_t \mathbf{M}(t) = \begin{cases} \int_0^t \mathbf{M}(\tau) d\tau, & 0 < t \leq t_1, \\ \frac{1}{\Gamma(\omega)} \int_{t_1}^t (t - \eta)^{\omega-1} \mathbf{M}(\eta) d\eta & t_1 < t \leq t_2. \end{cases}$$

Here, ${}_0^{PF}D_t^\omega$ and ${}_0^{PF}I_t$ represents the piecewise fractional derivative and the integral operator, respectively.

Lemma 1 ([31]). The equivalent form of a PW equation

$${}_0^{PCC}D_t^\omega \mathbf{M}(t) = \mathbf{K}(t, \mathbf{M}(t)), \quad 0 < r \leq 1$$

is given as

$$\mathbf{M}(t) = \begin{cases} \mathbf{M}_0 + \int_0^t \mathbf{M}(\tau) d\tau, & 0 < t \leq t_1 \\ \mathbf{M}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^t (t - \eta)^{\omega-1} \mathbf{M}(\eta) d\eta & t_1 < t \leq t_2. \end{cases}$$

3. Piecewise Derivative for the Considered System

In piecewise derivative form, the considered system (1) is given below:

$$\begin{aligned} {}_0^{PCC}D_t^\omega S(t) &= \mathcal{I}S\alpha - S\mathcal{Z}\beta, \\ {}_0^{PCC}D_t^\omega \mathcal{E}(t) &= -\mathcal{E}\sigma - \theta\mathcal{E}\mathcal{Z} - \mathcal{E}\mathcal{I}\rho + \mathcal{I}S\alpha(q_1 - 1) + S\mathcal{Z}\beta(1 - \Omega), \\ {}_0^{PCC}D_t^\omega \mathcal{I}(t) &= \mathcal{E}\sigma + \mathcal{E}\mathcal{I}\rho + \mathcal{I}S\alpha q_2, \\ {}_0^{PCC}D_t^\omega \mathcal{Z}(t) &= \theta\mathcal{E}\mathcal{Z} + S\mathcal{Z}\beta\Omega. \end{aligned} \quad (2)$$

For more simplification, System (2) can be written as

$$\begin{aligned} {}_0^{PCC}D_t^\omega (S(t)) &= \begin{cases} {}_0D_t(S(t)) = \frac{d}{dt} \mathbf{K}_1(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & 0 < t \leq t_1, \\ {}_0^CD_t^\omega (S(t)) = {}^C \mathbf{K}_1(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & t_1 < t \leq t_2, \end{cases} \\ {}_0^{PCC}D_t^\omega (\mathcal{E}(t)) &= \begin{cases} {}_0D_t(\mathcal{E}(t)) = \frac{d}{dt} \mathbf{K}_2(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & 0 < t \leq t_1, \\ {}_0^CD_t^\omega (\mathcal{E}(t)) = {}^C \mathbf{K}_2(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & t_1 < t \leq t_2, \end{cases} \\ {}_0^{PCC}D_t^\omega (\mathcal{I}(t)) &= \begin{cases} {}_0D_t(\mathcal{I}(t)) = \frac{d}{dt} \mathbf{K}_3(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & 0 < t \leq t_1, \\ {}_0^CD_t^\omega (\mathcal{I}(t)) = {}^C \mathbf{K}_3(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & t_1 < t \leq t_2, \end{cases} \\ {}_0^{PCC}D_t^\omega (\mathcal{Z}(t)) &= \begin{cases} {}_0D_t(\mathcal{Z}(t)) = \frac{d}{dt} \mathbf{K}_4(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & 0 < t \leq t_1, \\ {}_0^CD_t^\omega (\mathcal{Z}(t)) = {}^C \mathbf{K}_4(S, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t), & t_1 < t \leq t_2. \end{cases} \end{aligned} \quad (3)$$

The notations ${}_0D_t$ and ${}_0^CD_t^\omega$ are used for both the classical and Caputo derivatives.

4. Existence and Uniqueness

The current section is devoted to the results of the existence and uniqueness of a solution for the aforementioned piecewise system. For this, the considered model (2) along with Lemma 1 is represented below.

$${}^{\rho}_{0}D_t^{\omega} \mathbf{M}(t) = \mathbf{K}(t, \mathbf{M}), \quad 0 < \rho \leq 1$$

is

$$\mathbf{M}(t) = \begin{cases} \mathbf{M}_0 + \int_0^t \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta, & 0 < t \leq t_1 \\ \mathbf{M}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d(\vartheta), & t_1 < t \leq t_2, \end{cases} \quad (4)$$

where

$$\mathbf{M}(t) = \begin{cases} \mathcal{S}(t) \\ \mathcal{E}(t) \\ \mathcal{I}(t) \\ \mathcal{Z}(t) \end{cases} \quad \mathbf{M}_0 = \begin{cases} \mathcal{S}(0) \\ \mathcal{E}(0) \\ \mathcal{I}(0) \\ \mathcal{Z}(0) \end{cases} \quad \mathbf{M}_{t_1} = \begin{cases} \mathcal{S}_{t_1} \\ \mathcal{E}_{t_1} \\ \mathcal{I}_{t_1} \\ \mathcal{Z}_{t_1} \end{cases} \quad \mathbf{K}(t, \mathbf{M}(t)) = \begin{cases} \mathbf{K}_1 = \begin{cases} \frac{d}{dt} \mathbf{K}_1(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \\ {}^C \mathbf{K}_1(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \end{cases} \\ \mathbf{K}_2 = \begin{cases} \frac{d}{dt} \mathbf{K}_2(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \\ {}^C \mathbf{K}_2(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \end{cases} \\ \mathbf{K}_3 = \begin{cases} \frac{d}{dt} \mathbf{K}_3(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \\ {}^C \mathbf{K}_3(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \end{cases} \\ \mathbf{K}_4 = \begin{cases} \frac{d}{dt} \mathbf{K}_4(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \\ {}^C \mathbf{K}_4(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, t) \end{cases} \end{cases}$$

Let the Banach space $E_1 = C[0, T]$ with norm $\|\mathbf{M}\| = \max_{t \in [0, T]} |\mathbf{M}(t)|$, and $0 < t_1 < t \leq t_2 < \infty$. To obtain the solution, we consider the growth condition as

(C1) There exists $L_{\mathbf{M}} > 0$. For all $\mathbf{K}, \bar{\mathbf{M}} \in E$, we have

$$|\mathbf{K}(t, \mathbf{M}) - \mathbf{K}(t, \bar{\mathbf{M}})| \leq L_{\mathbf{K}} |\mathbf{M} - \bar{\mathbf{M}}|,$$

(C2) $\exists C_{\mathbf{K}} > 0$ & $M_{\mathbf{K}} > 0$,

$$|\mathbf{K}(t, \mathbf{M}(t))| \leq C_{\mathbf{K}} |\mathbf{M}| + M_{\mathbf{K}}.$$

Theorem 1. A piecewise function $\mathbf{K}$ continuous on subinterval $0 < t \leq t_1$ and $t_1 < t \leq t_2$ on $[0, T]$ with Assumption (C2). Then, the above system (3) has at least one solution on every subinterval.

Proof. We will start the proof from the famous theorem of the fixed point Schauder theorem. For this, we define a closed subset on both subintervals of 0 and T as B of E as

$$B = \{\mathbf{M} \in E : \|\mathbf{M}\| \leq R_{1,2}, R > 0\}.$$

Next, by taking an operator $F : B \rightarrow B$ and using (4) as

$$F(\mathbf{M}) = \begin{cases} \mathbf{M}_0 + \int_0^t \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta, & 0 < t \leq t_1 \\ \mathbf{M}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d(\vartheta), & t_1 < t \leq t_2. \end{cases} \quad (5)$$

For $\mathbf{M} \in B$, we have

$$\begin{aligned}
 |F(\mathbf{M})(t)| &\leq \begin{cases} |\mathbf{M}_0| + \int_0^{t_1} |\mathbf{K}(\vartheta, \mathbf{M}(\vartheta))| d\vartheta, \\ |\mathbf{M}_{t_1}| + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} |\mathbf{K}(\vartheta, \mathbf{M}(\vartheta))| d(\vartheta), \end{cases} \\
 &\leq \begin{cases} |\mathbf{M}_0| + \int_0^{t_1} [C_K |\mathbf{M}| + M_K] d\vartheta, \\ |\mathbf{M}_{t_1}| + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} [C_K |\mathbf{M}| + M_K] d(\vartheta), \end{cases} \\
 &\leq \begin{cases} |\mathbf{M}_0| + t_1 [C_K |\mathbf{M}| + M_K] = R_1, \quad 0 < t \leq t_1, \\ |\mathbf{M}_{t_1}| + \frac{(t_2 - t_1)^\omega}{\Gamma(\omega + 1)} [C_K |\mathbf{M}| + M_K] = R_2, \quad t_1 < t \leq t_2, \end{cases} \\
 &\leq \begin{cases} R_1, \quad 0 < t \leq t_1, \\ R_2, \quad t_1 < t \leq t_2. \end{cases}
 \end{aligned}$$

From the last relation, we see that $\mathbf{M} \in B$, so $F(B)$ is a subset of B . Thus, the operator F is closed and complete. Further, to show the complete continuity, we take $t_j < t_i \in [0, t_1]$ for the classical derivative we consider the first interval as

$$\begin{aligned}
 |F(\mathbf{M})(t_i) - F(\mathbf{M})(t_j)| &= \left| \int_0^{t_i} \mathbf{K}(\vartheta, \mathbf{M}) d\vartheta - \int_0^{t_j} \mathbf{K}(\vartheta, \mathbf{M}) d\vartheta \right| \\
 &\leq \int_0^{t_i} |\mathbf{K}(\vartheta, \mathbf{M})| d\vartheta - \int_0^{t_j} |\mathbf{K}(\vartheta, \mathbf{M})| d\vartheta \\
 &\leq \left[\int_0^{t_i} (C_K |\mathbf{M}| + M_K) - \int_0^{t_j} (C_K |\mathbf{M}| + M_K) \right] \\
 &\leq (C_K \mathbf{M} + M_K)[t_i - t_j].
 \end{aligned} \tag{6}$$

Further, from Equation (6) above, we obtain $t_j \rightarrow t_i$. Then,

$$|F(\mathbf{M})(t_i) - F(\mathbf{M})(t_j)| \rightarrow 0, \text{ as } t_j \rightarrow t_i.$$

Therefore, the operator F is equicontinuous in the interval $[0, t_1]$. Now, we consider the second interval $t_i, t_j \in [t_1, T]$ in the sense of ABC

$$\begin{aligned}
 |F(\mathbf{M})(t_i) - F(\mathbf{M})(t_j)| &= \left| \frac{1}{\Gamma(\omega)} \int_0^{t_i} (t_i - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta \right. \\
 &\quad \left. - \frac{1}{\Gamma(\omega)} \int_0^{t_j} (t_j - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta \right| \\
 &\leq \frac{1}{\Gamma(\omega)} \int_0^{t_j} [(t_j - \vartheta)^{\omega-1} - (t_i - \vartheta)^{\omega-1}] |\mathbf{K}(\vartheta, \mathbf{M}(\vartheta))| d\vartheta \\
 &\quad + \frac{1}{\Gamma(\omega)} \int_{t_j}^{t_i} (t_i - \vartheta)^{\omega-1} |\mathbf{K}(\vartheta, \mathbf{M}(\vartheta))| d\vartheta \\
 &\leq \frac{1}{\Gamma(\omega)} \left[\int_0^{t_j} [(t_j - \vartheta)^{\omega-1} - (t_i - \vartheta)^{\omega-1}] d\vartheta \right. \\
 &\quad \left. + \int_{t_j}^{t_i} (t_i - \vartheta)^{\omega-1} d\vartheta \right] (C_G |\mathbf{M}| + M_G) \\
 &\leq \frac{(C_G \mathbf{M} + M_G)}{\Gamma(\omega + 1)} [t_i^\omega - t_j^\omega + 2(t_i - t_j)^\omega].
 \end{aligned} \tag{7}$$

Thus, from Equation (7), we obtain $t_j \rightarrow t_i$ as

$$|F(\mathbf{M})(t_i) - F(\mathbf{M})(t_j)| \rightarrow 0, \text{ as } t_j \rightarrow t_i.$$

Thus, the operator F is equicontinuous in $[t_1, t_2]$ interval. Therefore, F is an equicontinuous mapping. With the help of the “Arzelà-Ascoli theorem”, F is continuous completely and so continuous uniform and bounded. Hence, using Schauder’s fixed-point theorem, the proposed problem (3) has at least one solution for the intervals. $\square$

Theorem 2. Under Assumption (C1), the proposed problem solution is unique if F is a contractive operator.

Proof. Consider a piecewise continuous map $F : B \rightarrow B$, let $\mathbf{M}$ and $\bar{\mathbf{M}} \in B$ on $[0, t_1]$ for the classical derivative

$$\begin{aligned} \|F(\mathbf{M}) - F(\bar{\mathbf{M}})\| &= \max_{t \in [0, t_1]} \left| \int_0^{t_1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta - \int_0^{t_1} \mathbf{K}(\vartheta, \bar{\mathbf{M}}(\vartheta)) d\vartheta \right| \\ &\leq t_1 L_K \|\mathbf{M} - \bar{\mathbf{M}}\|. \end{aligned} \quad (8)$$

From (8), we have

$$\|F(\mathbf{M}) - F(\bar{\mathbf{M}})\| \leq t_1 L_K \|\mathbf{M} - \bar{\mathbf{M}}\|. \quad (9)$$

Hence, we have the operator F , which is contraction. As a result of the contraction theorem of Banach, the proposed system solution is unique for the considered subinterval. Next, the second interval $t \in [t_1, t_2]$ for the Caputo derivative is

$$\begin{aligned} \|F(\mathbf{M}) - F(\bar{\mathbf{M}})\| &= \max_{t \in [t_1, t_2]} \left| \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \mathbf{M}(\vartheta)) d\vartheta - \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}(\vartheta, \bar{\mathbf{M}}(\vartheta)) d\vartheta \right| \\ &\leq \frac{(t_2 - t_1)^\omega}{\Gamma(\omega + 1)} L_K \|\mathbf{M} - \bar{\mathbf{M}}\|. \end{aligned} \quad (10)$$

From (10), we have

$$\|F(\mathbf{M}) - F(\bar{\mathbf{M}})\| \leq \frac{(t_2 - t_1)^\omega}{\Gamma(\omega + 1)} L_K \|\mathbf{M} - \bar{\mathbf{M}}\|. \quad (11)$$

Thus, the operator F is a contraction. From the solution of the Banach contraction theorem, the aforementioned system solution is unique also in the second subinterval. Therefore, from Equations (9) and (11), the piecewise derivable system solution is unique for both subintervals. $\square$

5. Equilibrium Points

To find the equilibrium points of the model, consider

$${}_0^{PCC} D_0^\omega \mathcal{S}(t) = {}_0^{PCC} D_0^\omega \mathcal{E}(t) = {}_0^{PCC} D_0^\omega \mathcal{I}(t) = {}_0^{PCC} D_0^\omega \mathcal{Z}(t) = 0.$$

We obtain

$$\mathcal{E}^*(\mathcal{S}^*, \mathcal{E}^*, \mathcal{I}^*, \mathcal{Z}^*) = \left\{ -\frac{\theta}{bl}, 1, -\frac{b\sigma l}{bl\rho - \alpha q_1 \theta'}, \frac{\alpha \sigma l}{bl\rho - \alpha q_1 \theta} \right\}.$$

$\mathcal{S}, \mathcal{E}, \mathcal{I}$, and $\mathcal{Z}$ are all non-negative for $\mathcal{S}(0) > 0, \mathcal{E}(0) > 0, \mathcal{I}(0) > 0, \mathcal{Z}(0) > 0$.

If $\theta \leq 1$, the equilibrium point is locally asymptotically stable and unstable if $\theta > 1$. Now,

$$\det |\mathcal{J} - \theta I| = \mathcal{B}_0 \theta^4 + \mathcal{B}_1 \theta^3 + \mathcal{B}_2 \theta^2 + \mathcal{B}_3 \theta = 0$$

$$\begin{aligned}
\mathcal{B}_0 &= \frac{b^2 l^2 \rho - b \alpha l \varrho_1 \theta}{bl(bl\rho - \alpha \varrho_1 \theta)} \\
\mathcal{B}_1 &= -\frac{b^2 l^2 \rho^2 - \sigma b \alpha l^2 \theta - 2b \alpha l \varrho_1 \rho \theta + \sigma b \alpha l \varrho_1 \theta + \alpha^2 \varrho_1^2 \theta^2}{bl(bl\rho - \alpha \varrho_1 \theta)} \\
\mathcal{B}_2 &= -\frac{\sigma b^2 \alpha l^2 \theta - \sigma \varrho_1 b \alpha^2 l \theta + \sigma \rho b \alpha l^2 \theta - \sigma b \alpha l \theta^2 - \sigma \varrho_1 \alpha^2 l \theta^2 + \sigma \varrho_1 \alpha^2 \theta^2}{bl(bl\rho - \alpha \varrho_1 \theta)} \\
\mathcal{B}_3 &= \frac{\sigma \rho b^2 \alpha l^2 \theta - \sigma \varrho_1 b \alpha^2 l \theta^2 - \sigma \rho b \alpha l \theta^2 + \sigma \varrho_1 \alpha^2 \theta^3}{bl(bl\rho - \alpha \varrho_1 \theta)}.
\end{aligned}$$

If $\mathcal{B}_1 \mathcal{B}_2 > \mathcal{B}_0 \mathcal{B}_3$, then all the roots have negative real parts.

6. Numerical Scheme

A numerical scheme is derived for the considered system (3) in this part of the study. The numerical scheme is constructed for both subintervals of $[0, T]$ in the framework of classical and Caputo derivatives. The considered technique for the aforementioned model is presented in the form of integer-order numerical technique [31]. The piecewise integral for model (3) in the framework of classical and Caputo derivatives is given as

$$\begin{aligned}
\mathcal{S}(t) &= \begin{cases} \mathcal{S}_0 + \int_0^{t_1} \mathbf{K}_1(\vartheta, \mathcal{S}) d\vartheta, & 0 < t \leq t_1, \\ \mathcal{S}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}_1(\vartheta, \mathcal{S}) d\vartheta, & t_1 < t \leq t_2, \end{cases} \\
\mathcal{E}(t) &= \begin{cases} \mathcal{E}_0 + \int_0^{t_1} \mathbf{K}_2(\vartheta, \mathcal{E}) d\vartheta, & 0 < t \leq t_1, \\ \mathcal{E}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}_2(\vartheta, \mathcal{E}) d\vartheta, & t_1 < t \leq t_2, \end{cases} \\
\mathcal{I}(t) &= \begin{cases} \mathcal{I}_0 + \int_0^{t_1} \mathbf{K}_3(\vartheta, \mathcal{I}) d\vartheta, & 0 < t \leq t_1, \\ \mathcal{I}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}_3(\vartheta, \mathcal{I}) d\vartheta, & t_1 < t \leq t_2, \end{cases} \\
\mathcal{Z}(t) &= \begin{cases} \mathcal{Z}_0 + \int_0^{t_1} \mathbf{K}_4(\vartheta, \mathcal{Z}) d\vartheta, & 0 < t \leq t_1, \\ \mathcal{Z}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_2} (t - \vartheta)^{\omega-1} \mathbf{K}_4(\vartheta, \mathcal{Z}) d\vartheta, & t_1 < t \leq t_2, \end{cases}. \quad (12)
\end{aligned}$$

In the next step, we derive the scheme of the first equation for Problem (12) and will perform the same technique for the renaming compartments.

At $t = t_{n+1}$

$$\mathcal{E}(t_{n+1}) = \begin{cases} \mathcal{E}_0 + \int_0^{t_1} \mathbf{K}_1(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, \vartheta) d\vartheta, & 0 < t \leq t_1, \\ \mathcal{E}(t_1) + \frac{1}{\Gamma(\omega)} \int_{t_1}^{t_{n+1}} (t - \vartheta)^{\omega-1} \mathbf{K}_1(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{Z}, \vartheta) d\vartheta, & t_1 < t \leq t_2, \end{cases}. \quad (13)$$

Equation (13) above, expressing the form of Newtons' interpolation method [31], is given below

$$S(t_{n+1}) = \left\{ \begin{array}{l} S_0 + \left\{ \sum_{a=2}^i \left[\frac{5}{12} K_1(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \Delta t \right. \right. \\ \left. \left. - \frac{4}{3} K_1(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \Delta t + K_1(S^a, \mathcal{E}^a, \mathcal{I}^a, Z^a, t_a) \right] \right\}, \\ \\ S(t_1) + \left\{ \begin{array}{l} \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+1)} \sum_{a=i+3}^n \left[K_1(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Pi \\ + \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+2)} \sum_{a=i+3}^n \left[K_1(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \right. \\ \left. - K_1(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Sigma \\ + \frac{\omega(\Delta t)^{\omega-1}}{2\Gamma(\omega+3)} \sum_{a=i+3}^n \left[K_1(S^a, \mathcal{E}^a, \mathcal{I}^a, Z^a, t_a) - 2K_1(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \right. \\ \left. + K_1(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Delta \end{array} \right\} \end{array} \right. \quad (14)$$

Similarly, the same procedure is used for the remaining compartments

$$\mathcal{E}(t_{n+1}) = \left\{ \begin{array}{l} \mathcal{E}(0) + \left\{ \sum_{a=2}^i \left[\frac{5}{12} K_2(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \Delta t \right. \right. \\ \left. \left. - \frac{4}{3} K_2(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \Delta t + K_2(S^a, \mathcal{E}^a, \mathcal{I}^a, Z^a, t_a) \right] \right\}, \\ \\ \mathcal{E}(t_1) + \left\{ \begin{array}{l} \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+1)} \sum_{a=i+3}^n \left[K_2(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Pi \\ + \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+2)} \sum_{a=i+3}^n \left[K_2(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \right. \\ \left. - K_2(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Sigma \\ + \frac{\omega(\Delta t)^{\omega-1}}{2\Gamma(\omega+3)} \sum_{a=i+3}^n \left[K_2(S^a, \mathcal{E}^a, \mathcal{I}^a, Z^a, t_a) - 2K_2(S^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, Z^{a-1}, t_{a-1}) \right. \\ \left. + K_2(S^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, Z^{a-2}, t_{a-2}) \right] \Delta \end{array} \right\} \end{array} \right. \quad (15)$$

$$\mathcal{I}(t_{n+1}) = \left\{ \begin{array}{l} \mathcal{I}_0 + \left\{ \begin{array}{l} \sum_{a=2}^i \left[\frac{5}{12} \mathbf{K}_3(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \Delta t \right. \\ \left. - \frac{4}{3} \mathbf{K}_3(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \Delta t + \mathbf{K}_3(\mathcal{S}^a, \mathcal{E}^a, \mathcal{I}^a, \mathcal{Z}^a, t_a) \right], \\ \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+1)} \sum_{a=i+3}^n \left[\mathbf{K}_3(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \Pi \\ + \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+2)} \sum_{a=i+3}^n \left[\mathbf{K}_3(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \right. \\ \left. - \mathbf{K}_3(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \sum \\ + \frac{\omega(\Delta t)^{\omega-1}}{2\Gamma(\omega+3)} \sum_{a=i+3}^n \left[\mathbf{K}_3(\mathcal{S}^a, \mathcal{E}^a, \mathcal{I}^a, \mathcal{Z}^a, t_a) - 2\mathbf{K}_3(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \right. \\ \left. + \mathbf{K}_3(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \Delta \end{array} \right\} \end{array} \right. \quad (16)$$

$$\mathcal{Z}(t_{n+1}) = \left\{ \begin{array}{l} \mathcal{Z}_0 + \left\{ \begin{array}{l} \sum_{a=2}^i \left[\frac{5}{12} \mathbf{K}_4(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \Delta t \right. \\ \left. - \frac{4}{3} \mathbf{K}_4(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \Delta t + \mathbf{K}_4(\mathcal{S}^a, \mathcal{E}^a, \mathcal{I}^a, \mathcal{Z}^a, t_a) \right], \\ \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+1)} \sum_{a=i+3}^n \left[\mathbf{K}_4(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \Pi \\ + \frac{(\Delta t)^{\omega-1}}{\Gamma(\omega+2)} \sum_{a=i+3}^n \left[\mathbf{K}_4(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \right. \\ \left. - \mathbf{K}_4(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \sum \\ + \frac{\omega(\Delta t)^{\omega-1}}{2\Gamma(\omega+3)} \sum_{a=i+3}^n \left[\mathbf{K}_4(\mathcal{S}^a, \mathcal{E}^a, \mathcal{I}^a, \mathcal{Z}^a, t_a) - 2\mathbf{K}_4(\mathcal{S}^{a-1}, \mathcal{E}^{a-1}, \mathcal{I}^{a-1}, \mathcal{Z}^{a-1}, t_{a-1}) \right. \\ \left. + \mathbf{K}_4(\mathcal{S}^{a-2}, \mathcal{E}^{a-2}, \mathcal{I}^{a-2}, \mathcal{Z}^{a-2}, t_{a-2}) \right] \Delta. \end{array} \right\} \end{array} \right. \quad (17)$$

where

$$\begin{aligned} \Pi &= (1+k+n)^\delta \left(2(-k+n)^2 + (3\delta+10)(-k+n) + 2\delta^2 + 9\delta + 12 \right) \\ &\quad - (-k+n) \left(2(-k+n)^2 + (5\delta+10)(-k+n) + 6\delta^2 + 18\delta + 12 \right), \end{aligned}$$

$$\sum = (1+k+n)^\delta \left(3 + 2\delta - k + n \right) - (-k+n) \left(-k + n + 3\delta + 3 \right),$$

$$\Delta = (1+k+n)^\delta - (-k+n)^\delta.$$

7. Results

This section is devoted to showing the numerical simulations and approximate solution for Problem (2), as aforementioned, which we considered in the framework of classical and Caputo piecewise derivatives. To perform the simulation, we took values for the parameter, which are described in Table 2. For the given compartments, we selected the initial values as $S(0) = 0.1$, $E(0) = 0.6$, $I(0) = 0.2$, and $Z(0) = 0.1$. In the obtained results, both subintervals are considered $(0, t_1] = (0, 14]$ and $[t_1, T] = [14, 200]$ for the below figures. The classical derivative was selected for the first interval, while for the second interval, the fractional Caputo derivative was used. Hence, the red-colored curves in the first four figures show the behavior of the classical dynamics of the said system, which shows the first subinterval, and the other colored lines present the dynamical behavior of the considered system having different fractional orders. Figures 1–4 are plotted for the second interval by using various fractional orders, such as (*green*, 0.98), (*blue*, 0.96), and (*purple*, 0.94). The susceptible class and its population dynamics are presented in Figure 1. Figures 2 and 3 depict the behavior of the exposed and infected populace, respectively. The dynamical behavior of the rumor protesters class is shown in Figure 4. The decrease in the fractional order shows decreases in the $\mathcal{S}$ class and stability at time $t = 30$, as can be seen in Figure 1. Moreover, we also noticed that the exposed class decreases and becomes stable when compared to the high values of β . In the same fashion, we recognize from Figure 3 that the infected class decreases with time and becomes stable at $t = 35$, and in Figure 4, the protesters decrease as the model moves from integer to fractional order.

Table 2. Values of parameters occurring in model (2).

Parameters	Values	Parameters	Values
α	0.1	β	0.4
ρ	0.1	σ	0.1
θ	0.5	Ω	0.4
ϱ_1	0.03		

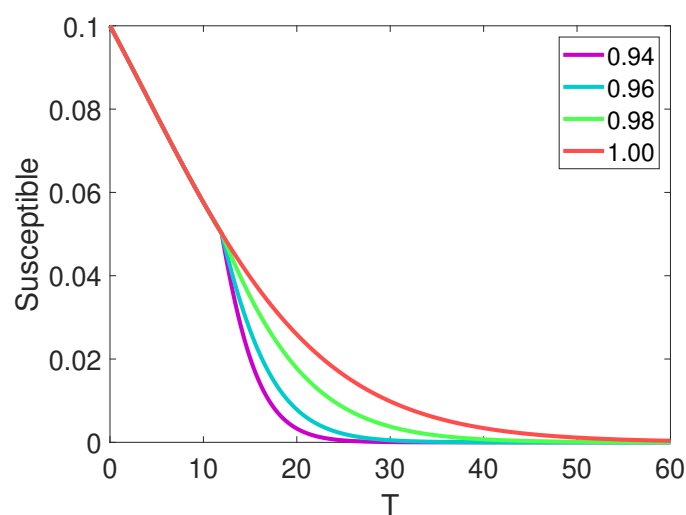


Figure 1. Dynamical behavior of class $\mathcal{S}$ for the considered system (2) with $t_1 = 14$.

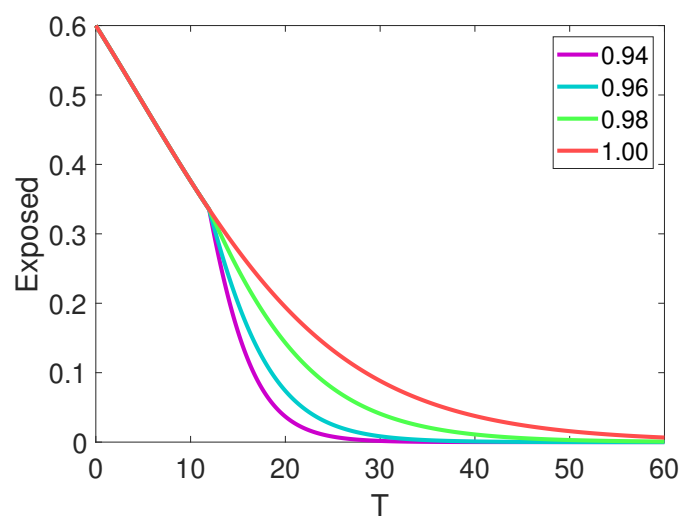


Figure 2. Dynamical behavior of class $\mathcal{E}$ for the considered system (2) with $t_1 = 14$.

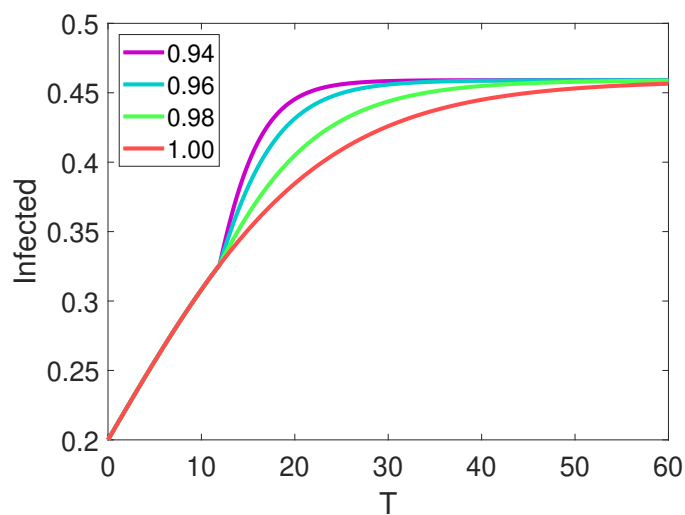


Figure 3. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$.

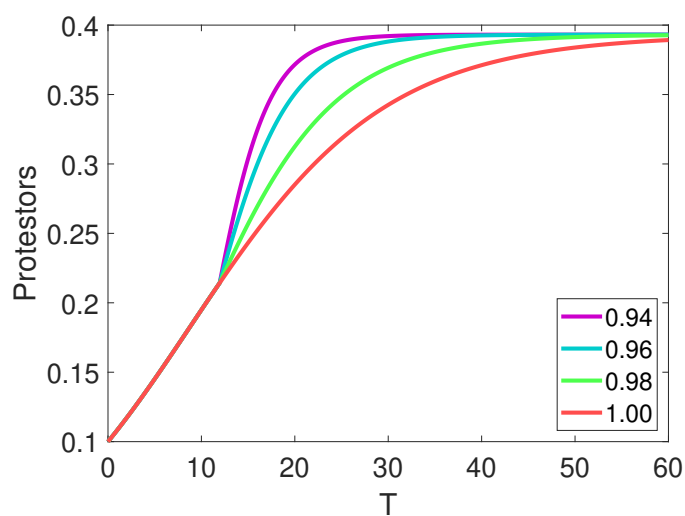


Figure 4. Dynamical behavior of class $\mathcal{Z}$ for the considered system (2) with $t_1 = 14$.

In Figures 5–7, the dynamics of the infected individuals are presented with various values of parameters. In Figure 5, θ is considered as 0.5, 0.4, 0.3, 0.1, and ρ is used as 0.2, 0.3, 0.1. Further, in Figure 6, α is considered as 0.6, 0.5, 0.4, 0.3, and b is used as 0.4, 0.5, 0.6, 0.7. Similarly, in Figure 7, p_1 is considered as 0.2, 0.4, 0.6, 0.8. Next, in Figures 8–10, the effects of various parameters on the dynamics of the protesters are presented, where in Figure 8, θ is taken as 0.5, 0.4, 0.3, 0.1, and ρ is used as 0.2, 0.3, 0.1. Furthermore, in Figure 9, α is considered as 0.6, 0.5, 0.4, 0.3, and β is used as 0.4, 0.5, 0.6, 0.7. Finally, in Figure 10, q_1 is considered as 0.2, 0.4, 0.6, 0.8. From the effects of parameters on the dynamics of the infected individuals and protesters, it is observed that an increase in α decreases the number of the infected individuals and increases the number of protesters.

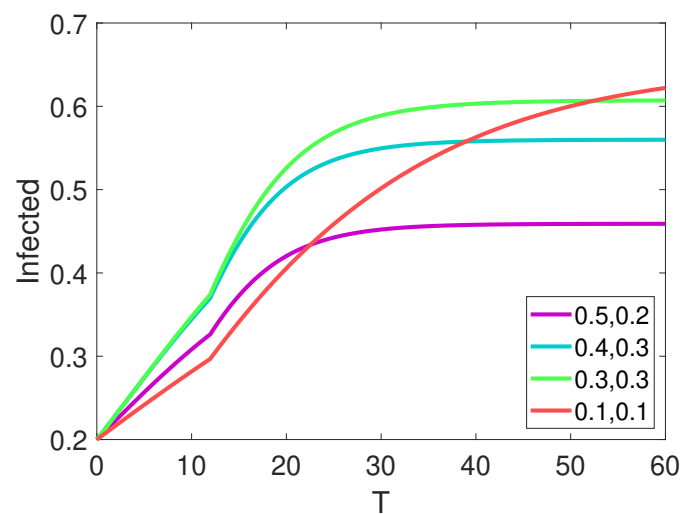


Figure 5. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

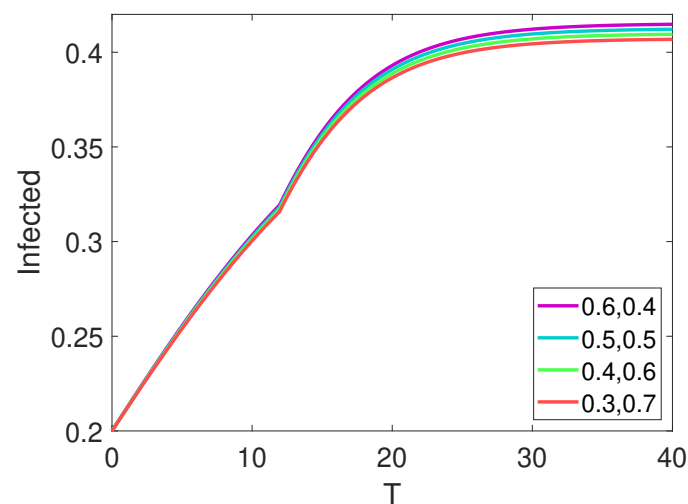


Figure 6. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

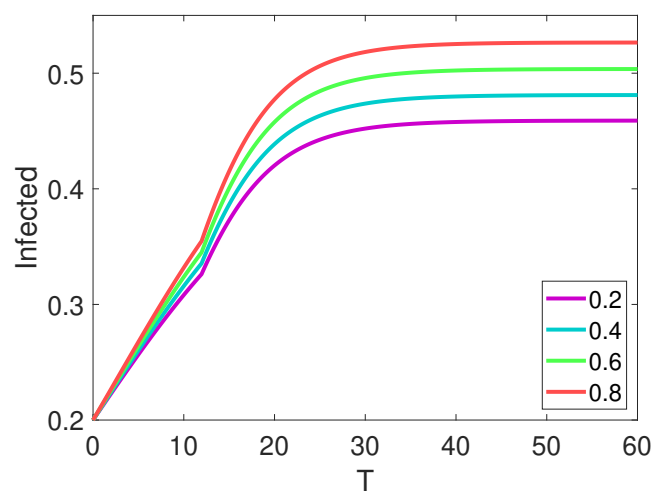


Figure 7. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

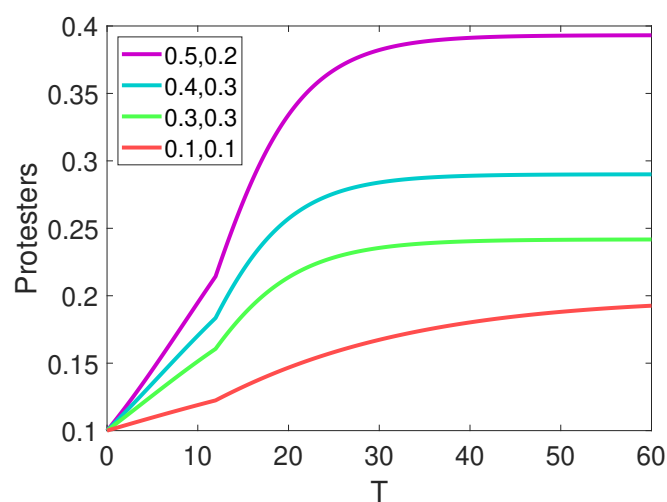


Figure 8. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

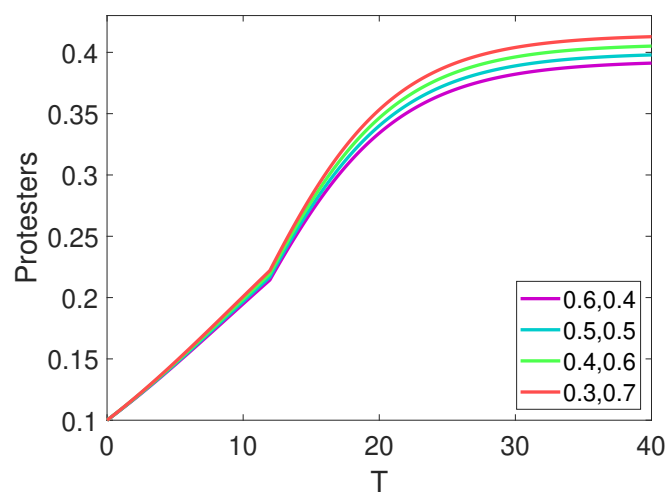


Figure 9. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

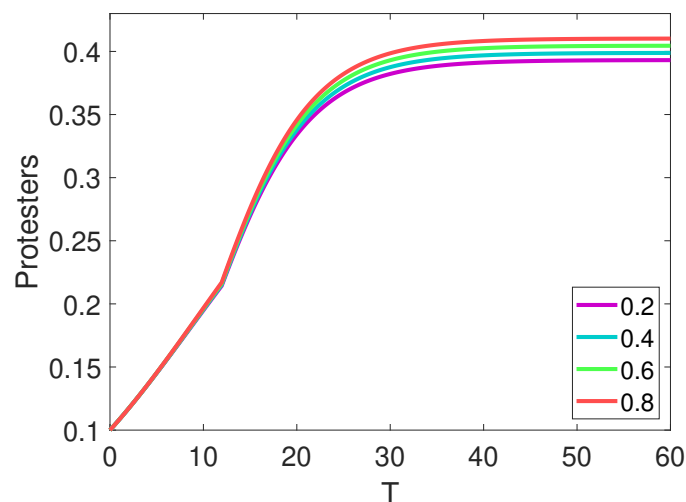


Figure 10. Dynamical behavior of class $\mathcal{I}$ for the considered system (2) with $t_1 = 14$ and different fractional orders.

8. Concluding Remarks

The current study analyzed the dynamics of rumor propagation in a social network using the proposed system, incorporating both classical and Caputo piecewise operators. The model was carefully examined to ascertain the existence and uniqueness of the solution through the application of a fixed-point scheme. To find an approximate solution, the system was analyzed using the Newton polynomial approach. The numerical simulation employs different fractional orders to test the considered system. The results indicate that as the fractional order α increases, the number of infected individuals decreases, while the number of protesters increases. Graphical representations demonstrate that the piecewise operators outperform the classical ones, showcasing better results for the system. By using the new concept of the piecewise global operator, this study advances our understanding of the dynamics of crossover behavior, offering improved insights into the spread of rumors in social networks. The findings contribute to the ongoing research in this field and may have practical implications for managing information dissemination in online communities.

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