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Concircular Vector Fields on Radical Anti-Invariant Lightlike Hypersurfaces of Almost Product-like Statistical Manifolds

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Abstract: The motivation of the present study is to describe the main relations of the radical anti-invariant lightlike hypersurfaces of almost product-like statistical manifolds. We provide concircular vector fields on radical anti-invariant lightlike hypersurfaces and obtain some results involving these vector fields.

Keywords: Hermite-like manifold; product-like manifold; lightlike hypersurface

1. Introduction

In lightlike geometry, a fundamental problem is the introduction of new kinds of lightlike submanifolds admitting various differentiable structures such as almost complex, contact or product structures, etc. The theory of lightlike submanifolds was presented by K. L. Duggal and his colleagues [1–3]. Since the metric of a lightlike submanifold is degenerate, examining these submanifolds includes complicated and rich geometric properties. In this manner, various lightlike submanifolds of almost product-like semi-Riemannian manifolds were investigated by several authors [4,5]. The geometry of radical anti-invariant lightlike submanifolds of semi-Riemannian product manifolds was discussed by E. Kılıç and B. Şahin in [6].

On the other hand, the theory of statistical manifolds has important application areas, and various properties of statistical structures in geometric and physical terms have been studied intensively in recent years. Statistical manifolds were first introduced by S. Amari [7] in 1985, as follows:

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold furnished with the metric tensor $\tilde{g}$ and let $\tilde{\nabla}$ be a torsion-free affine connection on $(\tilde{M}, \tilde{g})$. If $\tilde{\nabla}\tilde{g}$ is symmetric, then $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called a statistical manifold. In [8], K. Takano introduced the notion of Hermite-like manifolds. This notion is a more general form than Hermitian geometry. Inspired by the definition of Hermite-like manifolds, almost product-like manifolds were investigated in [9] and screen semi-invariant lightlike hypersurfaces of almost product-like manifolds were studied in [10].

Based on the above facts, it is possible to study the radical anti-invariant lightlike hypersurfaces of almost product-like statistical manifolds. Furthermore, in this paper, we find some results on these hypersurfaces with the aid of concircular vector fields.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a statistical manifold and $\tilde{g}$ be a semi-Riemannian metric $\tilde{g}$ with constant index q . In this case, there exists an affine connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$ such that

$$X_1\tilde{g}(X_2, X_3) = \tilde{g}(\tilde{\nabla}_{X_1}X_2, X_3) + \tilde{g}(X_2, \tilde{\nabla}_{X_1}^*X_3) \quad (1)$$



Citation: Erkan, E. Concircular Vector Fields on Radical Anti-Invariant Lightlike Hypersurfaces of Almost Product-like Statistical Manifolds. *Symmetry* **2023**, *15*, 1531. <https://doi.org/10.3390/sym15081531>

Academic Editors: Yanlin Li and Tiehong Zhao

Received: 19 July 2023

Revised: 31 July 2023

Accepted: 1 August 2023

Published: 3 August 2023



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holds for any $X_1, X_2, X_3 \in \Gamma(TM)$; then, $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called a statistical manifold. Here, $\tilde{\nabla}^*$ is another affine connection, which is called the dual connection of $\tilde{\nabla}$, and we have

$$\tilde{\nabla}_{X_1}^0 X_2 = \frac{1}{2}(\tilde{\nabla}_{X_1} X_2 + \tilde{\nabla}_{X_1}^* X_2), \quad (2)$$

where $\tilde{\nabla}^0$ indicates the Levi-Civita connection of $(\tilde{M}, \tilde{g})$ [7].

Let (M, g) be a hypersurface of $(\tilde{M}, \tilde{g})$. If the induced metric g is degenerate on M , then (M, g) is called a lightlike hypersurface. Due to the degeneracy of g , there exists a one-dimensional distribution $Rad(TM)$, which is known as the radical distribution. The radical space is defined by

$$Rad(TM) = span\{\xi : g(\xi, X) = 0 \text{ for any } X \in \Gamma(TM)\} \quad (3)$$

and the complementary distribution of $Rad(TM)$ such that we write

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad (4)$$

where $\oplus_{orth}$ is the orthogonal direct sum. The distribution $S(TM)$ is called the screen distribution of (M, g) and a lightlike hypersurface is indicated by $(M, g, S(TM))$ [1,3].

For any lightlike hypersurface, there is a unique 1-dimensional vector bundle $ltr(TM) = span\{N\}$. Therefore, we can write

$$T\tilde{M} = S(TM) \oplus_{orth} (Rad(TM) \oplus ltr(TM)), \quad (5)$$

where $\oplus$ indicates the direct sum, which is not orthogonal.

The Gauss and Weingarten formulae are specified by

$$\tilde{\nabla}_{X_1}^0 X_2 = \nabla_{X_1}^0 X_2 + B^0(X_1, X_2)N \quad (6)$$

and

$$\tilde{\nabla}_{X_1}^0 N = -A_N^0 X_1 + \tau^0(X_1)N, \quad (7)$$

where $\nabla_{X_1}^0 X_2, A_N^0(X_1) \in \Gamma(TM)$ [11].

The Gauss and Weingarten type formulas with respect to statistical connections could be derived as follows:

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + B(X_1, X_2)N, \quad (8)$$

$$\tilde{\nabla}_{X_1} N = -A_N^* X_1 + \tau^*(X_1)N, \quad (9)$$

$$\tilde{\nabla}_{X_1}^* X_2 = \nabla_{X_1}^* X_2 + B^*(X_1, X_2)N, \quad (10)$$

$$\tilde{\nabla}_{X_1}^* N = -A_N X_1 + \tau(X_1)N, \quad (11)$$

where $\nabla_{X_1} X_2, \nabla_{X_1}^* X_2, A_N^* X_1, A_N X_1 \in \Gamma(TM)$ and τ, τ^* are 1-forms. The above formulas could be written on $S(TM)$ as follows:

$$\nabla_{X_1} PX_2 = \bar{\nabla}_{X_1} PX_2 + C(X_1, PX_2)\xi, \quad (12)$$

$$\nabla_{X_1} \xi = -\bar{A}_\xi X_1 - \tau(X_1)\xi, \quad (13)$$

$$\nabla_{X_1}^* PX_2 = \bar{\nabla}_{X_1}^* PX_2 + C^*(X_1, PX_2)\xi, \quad (14)$$

$$\nabla_{X_1}^* \xi = -\bar{A}_\xi^* X_1 - \tau^*(X_1)\xi, \quad (15)$$

where $P : \Gamma(TM) \rightarrow \Gamma(S(TM))$ is the natural projection and $\bar{\nabla}_{X_1} PX_2, \bar{\nabla}_{X_1}^* PX_2, \bar{A}_\xi X_1, \bar{A}_\xi^* X_1 \in \Gamma(S(TM))$.

For any lightlike hypersurface $(M, g, S(TM))$, the following equalities occur:

$$B(X_1, \xi) + B^*(X_1, \xi) = 0, \quad (16)$$

$$C(X_1, PX_2) = g(A_N X_1, PX_2), \quad (17)$$

$$C^*(X_1, PX_2) = g(A_N^* X_1, PX_2), \quad (18)$$

$$B(X_1, X_2) = g(\bar{A}_\xi^* X_1, X_2) - B^*(X_1, \xi) \tilde{g}(X_2, N), \quad (19)$$

$$B^*(X_1, X_2) = g(\bar{A}_\xi X_1, X_2) - B(X_1, \xi) \tilde{g}(X_2, N). \quad (20)$$

A lightlike hypersurface is called totally geodesic with respect to $\tilde{\nabla}$ if $B = 0$, totally geodesic with respect to $\tilde{\nabla}^*$ if $B^* = 0$, totally tangential umbilical with respect to $\tilde{\nabla}$ if $B(X, Y) = \kappa g(X, Y)$ is satisfied, totally tangential umbilical with respect to $\tilde{\nabla}^*$ if $B^*(X, Y) = \kappa^* g(X, Y)$ is satisfied, totally normally umbilical with respect to $\tilde{\nabla}$ if $A_N X = \kappa X$ or $C^*(X, Y) = \kappa^* g(X, Y)$ is satisfied, and totally normally umbilical with respect to $\tilde{\nabla}^*$ if $A_N X = \kappa X$ or $C(X, Y) = \kappa g(X, Y)$ is satisfied. Here, κ and κ^* are smooth functions on M [12,13].

3. Lightlike Hypersurfaces of Almost Product-like Statistical Manifolds

Let $\tilde{M}$ be a differentiable manifold and F, F^* be two tensor fields satisfying $F^2 = (F^*)^2 = I$, where I identifies the identity map. If there is a semi-Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(FX_1, X_2) = \tilde{g}(X_1, F^* X_2), \quad (21)$$

then $(\tilde{M}, \tilde{g}, F)$ is said to be an almost product-like statistical manifold. For any almost product-like manifold $(\tilde{M}, \tilde{g}, F)$, if $F = F^*$ then (M, g, F) becomes an almost product-like semi-Riemannian manifold. Putting $F^* Y$ instead of Y in (21), we find

$$\tilde{g}(FX_1, F^* X_2) = \tilde{g}(X_1, X_2) \quad (22)$$

is satisfied on $(\tilde{M}, \tilde{g}, F, F^*)$.

Example 1. Let us consider $\mathbb{R}^4$ with a local coordinate system $\{x_1, x_2, x_3, x_4\}$. Define

$$F = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad F^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-x_1} \\ 0 & 0 & e^{-x_1} & 0 \end{pmatrix}$$

and

$$\tilde{g} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have $F^2 = (F^*)^2 = I_4$ and $(\tilde{M}, \tilde{g}, F)$ is an almost product-like Lorentzian manifold.

Definition 1. Let $(\tilde{M}, \tilde{g}, F)$ be an almost product-like semi-Riemannian manifold. If there exists a torsion-free connection $\tilde{\nabla}$ such that $\tilde{\nabla} \tilde{g}$ is symmetric, then $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ is called an almost product-like statistical manifold.

Example 2. Consider $(\mathbb{R}^4, \tilde{g}, F)$ in Example 1. Define a connection $\tilde{\nabla}$ on $(\mathbb{R}^4, \tilde{g}, F)$ satisfying (1), with

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^4 \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

and

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}}^* \frac{\partial}{\partial x_j} = \sum_{k=1}^4 \Gamma_{ij}^{*k} \frac{\partial}{\partial x_k}$$

where Γ_{ij}^k and Γ_{ij}^{*k} are smooth functions for $i, j, k \in \{1, \dots, 4\}$. Using the fact that

$$\frac{\partial}{\partial x_i} \tilde{g} \left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_s} \right) = \tilde{g} \left(\sum_{k=1}^4 \Gamma_{ij}^{*k} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_s} \right) + \tilde{g} \left(\frac{\partial}{\partial x_j}, \sum_{k=1}^4 \Gamma_{is}^{*k} \frac{\partial}{\partial x_k} \right),$$

we obtain

$$\begin{aligned} \Gamma_{11}^1 + \Gamma_{11}^{*1} &= 0, & \Gamma_{11}^2 + \Gamma_{12}^{*1} &= 0, & \Gamma_{11}^3 + \Gamma_{13}^{*1} &= 0, & \Gamma_{11}^4 + \Gamma_{14}^{*1} &= 0, \\ \Gamma_{21}^1 + \Gamma_{21}^{*1} &= 0, & \Gamma_{21}^2 + \Gamma_{22}^{*1} &= 0, & \Gamma_{21}^3 + \Gamma_{23}^{*1} &= 0, & \Gamma_{21}^4 + \Gamma_{24}^{*1} &= 0, \\ \Gamma_{31}^1 + \Gamma_{31}^{*1} &= 0, & \Gamma_{31}^2 + \Gamma_{32}^{*1} &= 0, & \Gamma_{31}^3 + \Gamma_{33}^{*1} &= 0, & \Gamma_{31}^4 + \Gamma_{34}^{*1} &= 0, \\ \Gamma_{41}^1 + \Gamma_{41}^{*1} &= 0, & \Gamma_{41}^2 + \Gamma_{42}^{*1} &= 0, & \Gamma_{41}^3 + \Gamma_{43}^{*1} &= 0, & \Gamma_{41}^4 + \Gamma_{44}^{*1} &= 0, \\ \Gamma_{12}^1 + \Gamma_{11}^{*2} &= 0, & \Gamma_{12}^2 + \Gamma_{12}^{*2} &= 0, & \Gamma_{12}^3 + \Gamma_{13}^{*2} &= 0, & \Gamma_{12}^4 + \Gamma_{14}^{*2} &= 0, \\ \Gamma_{22}^1 + \Gamma_{21}^{*2} &= 0, & \Gamma_{22}^2 + \Gamma_{22}^{*2} &= 0, & \Gamma_{22}^3 + \Gamma_{23}^{*2} &= 0, & \Gamma_{22}^4 + \Gamma_{24}^{*2} &= 0, \\ \Gamma_{32}^1 + \Gamma_{31}^{*2} &= 0, & \Gamma_{32}^2 + \Gamma_{32}^{*2} &= 0, & \Gamma_{32}^3 + \Gamma_{33}^{*2} &= 0, & \Gamma_{32}^4 + \Gamma_{34}^{*2} &= 0, \\ \Gamma_{42}^1 + \Gamma_{41}^{*2} &= 0, & \Gamma_{42}^2 + \Gamma_{42}^{*2} &= 0, & \Gamma_{42}^3 + \Gamma_{43}^{*2} &= 0, & \Gamma_{42}^4 + \Gamma_{44}^{*2} &= 0, \\ \Gamma_{13}^1 + \Gamma_{11}^{*3} &= 0, & \Gamma_{13}^2 + \Gamma_{12}^{*3} &= 0, & \Gamma_{13}^3 + \Gamma_{13}^{*3} &= e^{x_1}, & \Gamma_{13}^4 + \Gamma_{14}^{*3} &= 0, \\ \Gamma_{23}^1 + \Gamma_{21}^{*3} &= 0, & \Gamma_{23}^2 + \Gamma_{22}^{*3} &= 0, & \Gamma_{23}^3 + \Gamma_{23}^{*3} &= 0, & \Gamma_{23}^4 + \Gamma_{24}^{*3} &= 0, \\ \Gamma_{33}^1 + \Gamma_{31}^{*3} &= 0, & \Gamma_{33}^2 + \Gamma_{32}^{*3} &= 0, & \Gamma_{33}^3 + \Gamma_{33}^{*3} &= 0, & \Gamma_{33}^4 + \Gamma_{34}^{*3} &= 0, \\ \Gamma_{43}^1 + \Gamma_{41}^{*3} &= 0, & \Gamma_{43}^2 + \Gamma_{42}^{*3} &= 0, & \Gamma_{43}^3 + \Gamma_{43}^{*3} &= 0, & \Gamma_{43}^4 + \Gamma_{44}^{*3} &= 0, \\ \Gamma_{14}^1 + \Gamma_{11}^{*4} &= 0, & \Gamma_{14}^2 + \Gamma_{12}^{*4} &= 0, & \Gamma_{14}^3 + \Gamma_{13}^{*4} &= 0, & \Gamma_{14}^4 + \Gamma_{14}^{*4} &= 0, \\ \Gamma_{24}^1 + \Gamma_{21}^{*4} &= 0, & \Gamma_{24}^2 + \Gamma_{22}^{*4} &= 0, & \Gamma_{24}^3 + \Gamma_{23}^{*4} &= 0, & \Gamma_{24}^4 + \Gamma_{24}^{*4} &= 0, \\ \Gamma_{34}^1 + \Gamma_{31}^{*4} &= 0, & \Gamma_{34}^2 + \Gamma_{32}^{*4} &= 0, & \Gamma_{34}^3 + \Gamma_{33}^{*4} &= 0, & \Gamma_{34}^4 + \Gamma_{34}^{*4} &= 0, \\ \Gamma_{44}^1 + \Gamma_{41}^{*4} &= 0, & \Gamma_{44}^2 + \Gamma_{42}^{*4} &= 0, & \Gamma_{44}^3 + \Gamma_{43}^{*4} &= 0, & \Gamma_{44}^4 + \Gamma_{44}^{*4} &= 0. \end{aligned}$$

In this case, $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$ is an almost product-like statistical manifold.

For any almost product-like statistical manifold, there is a dual connection $\tilde{\nabla}^*$ of $\tilde{\nabla}$ such that Equations (1) and (2) are satisfied. An almost product-like statistical manifold is called a locally product-like statistical manifold if

$$\tilde{\nabla} F = 0 \quad (23)$$

is satisfied. We note that for any locally product-like statistical manifold,

$$\tilde{\nabla}^* F^* = 0 \quad (24)$$

is also satisfied. For more details, we refer to [9].

Example 3. Let us consider $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$ given in Example 2. If $\tilde{\nabla} F = 0$, then we obtain the following identities:

$$\begin{aligned}
\tilde{\nabla}_{\partial_1}\partial_1 &= \Gamma_{11}^1\partial_1 + \Gamma_{11}^3\partial_3 - \Gamma_{11}^3\partial_4, \\
\tilde{\nabla}_{\partial_1}\partial_2 &= \Gamma_{12}^1\partial_2 + \Gamma_{12}^3\partial_3 + \Gamma_{12}^3\partial_4, \\
\tilde{\nabla}_{\partial_1}\partial_3 &= \Gamma_{13}^1\partial_1 + \Gamma_{13}^2\partial_2 + \Gamma_{13}^3\partial_3 + \Gamma_{13}^4\partial_4, \\
\tilde{\nabla}_{\partial_1}\partial_4 &= -\Gamma_{13}^1\partial_1 + \Gamma_{13}^2\partial_2 + \Gamma_{13}^4\partial_3 + \Gamma_{13}^3\partial_4, \\
\tilde{\nabla}_{\partial_2}\partial_1 &= \Gamma_{21}^1\partial_1 + \Gamma_{21}^3\partial_3 - \Gamma_{21}^3\partial_4, \\
\tilde{\nabla}_{\partial_2}\partial_2 &= \Gamma_{22}^2\partial_2 + \Gamma_{22}^3\partial_3 + \Gamma_{22}^3\partial_4, \\
\tilde{\nabla}_{\partial_2}\partial_3 &= \Gamma_{23}^1\partial_1 + \Gamma_{23}^2\partial_2 + \Gamma_{23}^3\partial_3 + \Gamma_{23}^4\partial_4, \\
\tilde{\nabla}_{\partial_2}\partial_4 &= -\Gamma_{23}^1\partial_1 + \Gamma_{23}^2\partial_2 + \Gamma_{23}^4\partial_3 + \Gamma_{23}^3\partial_4, \\
\tilde{\nabla}_{\partial_3}\partial_1 &= \Gamma_{31}^1\partial_1 + \Gamma_{31}^3\partial_3 - \Gamma_{31}^3\partial_4, \\
\tilde{\nabla}_{\partial_3}\partial_2 &= \Gamma_{32}^2\partial_2 + \Gamma_{32}^3\partial_3 + \Gamma_{32}^4\partial_4, \\
\tilde{\nabla}_{\partial_3}\partial_3 &= \Gamma_{33}^1\partial_1 + \Gamma_{33}^2\partial_2 + \Gamma_{33}^3\partial_3 + \Gamma_{33}^4\partial_4, \\
\tilde{\nabla}_{\partial_3}\partial_4 &= -\Gamma_{33}^1\partial_1 + \Gamma_{33}^2\partial_2 + \Gamma_{33}^4\partial_3 + \Gamma_{33}^3\partial_4, \\
\tilde{\nabla}_{\partial_4}\partial_1 &= \Gamma_{41}^1\partial_1 + \Gamma_{41}^3\partial_3 - \Gamma_{41}^3\partial_4, \\
\tilde{\nabla}_{\partial_4}\partial_2 &= \Gamma_{42}^2\partial_2 + \Gamma_{42}^3\partial_3 + \Gamma_{42}^3\partial_4, \\
\tilde{\nabla}_{\partial_4}\partial_3 &= \Gamma_{43}^1\partial_1 + \Gamma_{43}^2\partial_2 + \Gamma_{43}^3\partial_3 + \Gamma_{43}^4\partial_4, \\
\tilde{\nabla}_{\partial_4}\partial_4 &= -\Gamma_{43}^1\partial_1 + \Gamma_{43}^2\partial_2 + \Gamma_{43}^4\partial_3 + \Gamma_{43}^3\partial_4.
\end{aligned}$$

Using the Christoffel symbols, components of $\tilde{\nabla}^*$ could be obtained. Therefore, $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$ is a locally product-like statistical manifold.

Now, we shall state the following definition:

Definition 2. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$. If $F(Rad(TM))$ belongs to $ltr(TM)$, then $(M, g, S(TM))$ is called a radical anti-invariant lightlike hypersurface.

For any radical anti-invariant lightlike hypersurface, we write

$$F\xi = \lambda N, \quad (25)$$

where λ is a non-zero differentiable function. Using (25) in (21), we have

$$F^*\xi = \lambda N, \quad (26)$$

which shows that $F^*(Rad(TM))$ belongs to $ltr(TM)$. Additionally, based on (21), (25) and (26), we can conclude that

$$FN = F^*N = \frac{1}{\lambda}\xi, \quad (27)$$

which shows that $F(ltr(TM))$ and $F^*(ltr(TM))$ belong to $Rad(TM)$.

Let X be a vector field on $\Gamma(TM)$. Then, we can write

$$X = PX + \eta(X)\xi, \quad (28)$$

where $PX \in \Gamma(S(TM))$ and $\eta(X) = \tilde{g}(X, N)$. Therefore, we put

$$\begin{aligned}
FX &= \psi PX + \eta(X)F\xi \\
&= \psi PX + \mu_1\xi + \eta(X)\lambda N,
\end{aligned} \quad (29)$$

where μ_1 is a differential function. Using (21), (27) and (29), we find

$$\mu_1 = \tilde{g}(FX, N) = \tilde{g}(X, F^*N) = \lambda g(X, \xi) = 0. \quad (30)$$

Thus, we can write (29) as

$$FX = \varphi X + \eta(X)\lambda N \quad (31)$$

such that $\varphi = \psi P$ and $\phi X \in \Gamma(S(TM))$. In a similar way to (31), we can write

$$F^*X = \varphi^*X + \eta(X)\lambda N \quad (32)$$

such that $\varphi^*X \in \Gamma(S(TM))$. Considering (21), (22), (31) and (32), we obtain the following lemma:

Lemma 1. *For any radical anti-invariant lightlike hypersurface, the following equations hold for any $X_1, X_2 \in \Gamma(TM)$:*

$$g(\varphi X_1, \varphi^* X_2) = g(X_1, X_2), \quad (33)$$

$$g(\varphi X_1, X_2) = g(X_1, \varphi^* X_2). \quad (34)$$

Example 4. Let M be a hypersurface of $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$, which is given in Example 3. If M is defined by

$$M = \{(x_1, x_2, x_3, x_4) : x_1 = x_2\},$$

then the induced metric of M can be expressed as

$$g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, we have

$$Rad(TM) = span\{\xi = (1, 1, 0, 0)\},$$

$$ltr(TM) = span\{N = \frac{1}{2}(-1, 1, 0, 0)\},$$

$$S(TM) = span\{\partial_3, \partial_4\}.$$

So, $(M, g, S(TM))$ represents a lightlike hypersurface of $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$.

Since $F\xi = N$ and $F^*\xi = N$, we can determine that $(M, g, S(TM))$ is a radical anti-invariant lightlike hypersurface of $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$.

Lemma 2. *For any radical anti-invariant lightlike hypersurface, the following equation occurs:*

$$\varphi^2 X = (\varphi^*)^2 X = -X + \eta(X)\xi. \quad (35)$$

Proof. Based on (27) and (31), we obtain

$$\begin{aligned} F^2 X &= F\varphi X + \eta(X)\lambda FN \\ &= \varphi^2 X + \eta(X)\xi. \end{aligned} \quad (36)$$

Considering $F^2 X = X$ in (36), we can deduce that

$$\varphi^2 X = -X + \eta(X)\xi. \quad (37)$$

Based on (27) and (32), we also find

$$(\varphi^*)^2 X = -X + \eta(X)\xi. \quad (38)$$

It is easy to follow the proof of (35) by referring to (37) and (38). $\square$

Proposition 1. Let $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ be a locally product-like statistical manifold and $(M, g, S(TM))$ be a radical anti-invariant lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$. Then, the following relations are satisfied for any $X \in \Gamma(TM)$:

$$\tau(X) + \tau^*(X) = \lambda \tilde{\nabla}_X \left(\frac{1}{\lambda} \right) = \lambda \tilde{\nabla}_X^* \left(\frac{1}{\lambda} \right), \quad (39)$$

$$\varphi A_N^* X = \frac{1}{\lambda} \bar{A}_\xi X, \quad (40)$$

$$B(X, \xi) = \lambda^2 \eta(A_N^* X), \quad (41)$$

$$\varphi^* A_N X = \frac{1}{\lambda} \bar{A}_\xi^* X, \quad (42)$$

$$B^*(X, \xi) = \lambda^2 \eta(A_N X). \quad (43)$$

Proof. Based on (27), we write

$$\tilde{\nabla}_X F N = \tilde{\nabla}_X \left(\frac{1}{\lambda} \right) \xi + \frac{1}{\lambda} \tilde{\nabla}_X \xi. \quad (44)$$

From the information provided in (8), (13) and (44), we can derive

$$\tilde{\nabla}_X F N = \tilde{\nabla}_X \left(\frac{1}{\lambda} \right) \xi - \frac{1}{\lambda} \bar{A}_\xi X - \frac{1}{\lambda} \tau(X) \xi + \frac{1}{\lambda} B(X, \xi) N. \quad (45)$$

On the other hand, from (27) and (31), we can determine

$$F \tilde{\nabla}_X N = -\varphi A_N^* X - \eta(A_N^* X) \lambda N + \tau^*(X) \frac{1}{\lambda} \xi. \quad (46)$$

Since $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ is a locally product-like statistical manifold, we have (40), (41) and

$$\tau(X) + \tau^*(X) = \lambda \tilde{\nabla}_X \left(\frac{1}{\lambda} \right). \quad (47)$$

Based on (9), (15) and (27), we have

$$\tilde{\nabla}_X^* F^* N = \nabla_X^* \left(\frac{1}{\lambda} \right) \xi - \frac{1}{\lambda} \bar{A}_\xi^* X - \frac{1}{\lambda} \tau^*(X) \xi + \frac{1}{\lambda} B^*(X, \xi) N \quad (48)$$

and

$$F^* \tilde{\nabla}_X^* N = -\varphi^* A_N X - \eta(A_N X) \lambda N + \frac{1}{\lambda} \tau(X) \xi. \quad (49)$$

By utilizing the information that $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ is a locally product-like statistical manifold and applying the equations (48) and (49), we can derive (42), (43) and

$$\tau(X) + \tau^*(X) = \lambda \tilde{\nabla}_X^* \left(\frac{1}{\lambda} \right). \quad (50)$$

It is easy to follow the proof of (39) by referring to (47) and (50). This completes the proof. $\square$

Proposition 2. For any radical anti-invariant lightlike hypersurface $(M, g, S(TM))$ of a locally product-like statistical manifold, we have

$$\tilde{\nabla}_X \lambda + \lambda \tau^*(X) = -\eta(\bar{A}_\xi X) \lambda - \tau(X) \lambda, \quad (51)$$

$$\lambda A_N^* X = \varphi \bar{A}_\xi X - B(X, \xi) \frac{1}{\lambda} \xi, \quad (52)$$

$$\tilde{\nabla}_X^* \lambda + \lambda \tau(X) = -\eta(\bar{A}_\xi^* X) \lambda - \tau^*(X) \lambda, \quad (53)$$

$$\lambda A_N X = \varphi^* \bar{A}_\xi^* X - B^*(X, \xi) \frac{1}{\lambda} \xi. \quad (54)$$

Proof. From the equations (9) and (25), we find

$$\tilde{\nabla}_X F \xi = (\tilde{\nabla}_X \lambda) N - \lambda A_N^* X + \lambda \tau^*(X) N. \quad (55)$$

On the other hand, we can conclude from (9), (13), and (31) that

$$F \tilde{\nabla}_X \xi = -\varphi \bar{A}_\xi X - \eta(\bar{A}_\xi X) \lambda N - \tau(X) \lambda N + B(X, \xi) \frac{1}{\lambda} \xi. \quad (56)$$

By examining the tangential and transversal components of the Equations (55) and (56), we can derive (51) and (52).

Proving (53) and (54) is easy when applying the same techniques utilized in proving (51) and (52). $\square$

Theorem 1. Let $(M, g, S(TM))$ be a radical anti-invariant lightlike hypersurface of a locally product-like statistical manifold. Then, A_N and A_N^* are $S(TM)$ -valued.

Proof. When we substitute the value of (43) into (54), the result is

$$\lambda A_N X = \varphi^* \bar{A}_\xi^* X - \lambda \eta(A_N X) \xi. \quad (57)$$

Based on (57), we find

$$-\eta(A_N X) = \eta(A_N X),$$

which implies $\lambda = 0$ or $\eta(A_N X) = 0$. Since $(M, g, S(TM))$ is a radical anti-invariant lightlike hypersurface, we have $\lambda \neq 0$. As a result, we obtain $\eta(A_N X) = 0$. This fact demonstrates that A_N is $S(TM)$ -valued. Based on (41) and (52), we find

$$-\eta(A_N^* X) = \eta(A_N^* X),$$

which implies that A_N^* is also $S(TM)$ -valued. $\square$

As a result of Theorem 1, we find the following corollary:

Corollary 1. For any radical anti-invariant lightlike hypersurface of a locally product-like statistical manifold, B and B^* vanish on $\text{Rad}(TM)$.

Now, we shall give some relations involving Riemannian curvature tensor fields.

Let $\tilde{R}$ and $\tilde{R}^*$ denote the Riemannian curvature tensor fields with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Then, we arrive at

$$\tilde{g}(\tilde{R}(X_1, X_2)X_3, X_4) = -\tilde{g}(\tilde{R}^*(X_1, X_2)X_4, X_3) \quad (58)$$

for any $X_1, X_2, X_3, X_4 \in \Gamma(TM)$.

We recall the following lemma [12]:

Lemma 3. Let $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ be a statistical manifold and $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then,

$$\begin{aligned}\tilde{R}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 - B(X_2, X_3)A_N^*X_1 + B(X_1, X_3)A_N^*X_2 \\ &\quad + [B(X_2, X_3)\tau^*(X_1) - B(X_1, X_3)\tau^*(X_2)]N \\ &\quad + [(\nabla_{X_1}B)(X_2, X_3) - (\nabla_{X_2}B)(X_1, X_3)]N\end{aligned}\quad (59)$$

and

$$\begin{aligned}\tilde{R}^*(X_1, X_2)X_3 &= R^*(X_1, X_2)X_3 - B^*(X_2, X_3)A_NX_1 + B^*(X_1, X_3)A_NX_2 \\ &\quad + [B^*(X_2, X_3)\tau(X_1) - B^*(X_1, X_3)\tau(X_2)]N \\ &\quad + [(\nabla_{X_1}^*B^*)(X_2, X_3) - (\nabla_{X_2}^*B^*)(X_1, X_3)]N,\end{aligned}\quad (60)$$

where R and R^* are the induced Riemannian curvature tensor fields on $(M, g, S(TM))$ with respect to ∇ and ∇^* .

Now, we consider a null plane section $\pi = \text{span}\{X, \xi\}$. Then, the null sectional curvatures of $(M, g, S(TM))$ and $(\tilde{M}, \tilde{g})$ are given by

$$K_0^{\text{null}}(\pi) = \frac{g(R^0(X, \xi)\xi, X)}{g(X, X)} \quad \text{and} \quad \tilde{K}_0^{\text{null}}(\pi) = \frac{\tilde{g}(\tilde{R}^0(X, \xi)\xi, X)}{\tilde{g}(X, X)},$$

respectively. Here, $\tilde{R}^0$ indicates the Riemannian curvature tensor field of $(\tilde{M}, \tilde{g})$ with respect to the Levi-Civita connection $\tilde{\nabla}^0$ and R^0 indicates the Riemannian curvature tensor field of $(M, g, S(TM))$ with respect to the induced connection ∇^0 .

Considering the definition of null sectional curvature, we can express the following definition:

Definition 3. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, the null sectional curvatures of a null plane section π with respect to $\tilde{\nabla}$ are given by

$$\begin{aligned}\tilde{K}^{\text{null}}(\pi) &= \frac{\tilde{g}(\tilde{R}(X, \xi)\xi, X)}{\tilde{g}(X, X)}, \\ K^{\text{null}}(\pi) &= \frac{g(R(X, \xi)\xi, X)}{g(X, X)}, \\ \tilde{K}^{*\text{null}}(\pi) &= \frac{\tilde{g}(\tilde{R}^*(X, \xi)\xi, X)}{\tilde{g}(X, X)}\end{aligned}$$

and

$$K^{*\text{null}}(\pi) = \frac{g(R(X, \xi)\xi, X)}{g(X, X)}.$$

Theorem 2. Let $(M, g, S(TM))$ be a radical anti-invariant lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then,

$$\tilde{K}^{\text{null}}(\pi) = K^{\text{null}}(\pi)$$

and

$$\tilde{K}^{*\text{null}}(\pi) = K^{*\text{null}}(\pi)$$

are satisfied for any null plane section π .

Based on (58) and Theorem 2, the following corollary can be derived:

Corollary 2. For any radical anti-invariant lightlike hypersurface of a locally product-like statistical manifold, we have

$$K^{null}(\pi) + K^{*null}(\pi) = 0.$$

Now, we shall recall the following proposition: cf. Proposition 7 in [13].

Proposition 3. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then,

$$K^{*null}(\pi) = B(X, \xi)C(\xi, X)$$

holds for any null plane section $\pi = \text{span}\{X, \xi\}$, where $X \in \Gamma(S(TM))$.

Based on Corollary 2 and Proposition 3, the following corollary can be derived:

Corollary 3. Let $(M, g, S(TM))$ be a radical anti-invariant lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$. Then,

$$K^{*null}(\pi) = K^{null}(\pi) = 0$$

is satisfied for any null plane section π .

4. Conircular Vector Fields

First, we provide the definition of concircular vector fields:

Definition 4. Let $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ be a statistical manifold. A vector field v is called concircular with respect to $\tilde{\nabla}$ if there exists a smooth function θ such that

$$\tilde{\nabla}_X v = \theta X \quad (61)$$

is satisfied for any $X \in \Gamma(T\tilde{M})$. In addition, it is called concircular with respect to $\tilde{\nabla}^*$ if there exists a smooth function θ^* such that

$$\tilde{\nabla}_X^* v = \theta^* X \quad (62)$$

is satisfied for any $X \in \Gamma(T\tilde{M})$.

It is clear that if v is concircular with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, then v is also concircular with respect to $\tilde{\nabla}^0$. For some applications of concircular vector fields, we recommend checking out [14–22].

Example 5. Taking into account $(\mathbb{R}^4, \tilde{g}, \tilde{\nabla}, F)$ as shown in Example 3, if we set $\Gamma_{31}^3 = \Gamma_{32}^2 = \Gamma_{32}^4 = \Gamma_{33}^1 = \Gamma_{33}^2 = \Gamma_{33}^4 = 0$, then we find that ∂_3 is a concircular vector field.

Additional examples can be extracted.

Proposition 4. Let v be a concircular vector field with respect to $\tilde{\nabla}$ on a lightlike hypersurface $(M, g, S(TM))$ of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, the following situations occur:

1. v is also concircular with respect to ∇ .
2. $B(X, v) = 0$ for any $X \in \Gamma(TM)$.

Proof. Based on (8) and (61), we conclude

$$\tilde{\nabla}_X v = \nabla_X v + B(X, v)N = \theta X$$

indicating that v is concircular with respect to ∇ and $B(X, v) = 0$. $\square$

Proposition 5. Let v be a concircular vector field with respect to $\tilde{\nabla}^*$ on a lightlike hypersurface $(M, g, S(TM))$ of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Then, the following situations occur:

1. v is also concircular with respect to ∇^* .
2. $B^*(X, v) = 0$ for any $X \in \Gamma(TM)$.

Assume that v is concircular lying on $\Gamma(TM)$. In this case, we can express

$$v = v^T + fN, \quad (63)$$

where $v^T \in \Gamma(TM)$ and $f = g(v, \xi)$.

Proposition 6. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If v is concircular with respect to $\tilde{\nabla}$, then the following relations are satisfied for any $X \in \Gamma(TM)$:

$$\theta X = \nabla_X v^T - fA_N^* X, \quad (64)$$

$$B(X, v^T) = -f\tau^*(X) - \tilde{\nabla}_X f. \quad (65)$$

Proof. According to (61) and (63), we can express

$$\tilde{\nabla}_X v = \tilde{\nabla}_X v^T + \tilde{\nabla}_X fN = \theta X, \quad (66)$$

which is equivalent to

$$\tilde{\nabla}_X v^T + (\tilde{\nabla}_X f)N + f\tilde{\nabla}_X N = \theta X. \quad (67)$$

By using (8) and (9) in (67), it follows that

$$\theta X = \nabla_X v^T + B(X, v^T)N + (\tilde{\nabla}_X f)N - fA_N^* X + f\tau^*(X)N. \quad (68)$$

By examining the tangential and transversal components of (68), we can derive (64) and (65). $\square$

Proposition 7. Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. If v is concircular with respect to $\tilde{\nabla}^*$, then the following relations are satisfied for any $X \in \Gamma(TM)$:

$$\theta^* X = \nabla_X^* v^T - fA_N X, \quad (69)$$

$$B^*(X, v^T) = -f\tau(X) - \tilde{\nabla}_X^* f. \quad (70)$$

Theorem 3. Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost product-like statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$.

1. If v is a concircular vector field with respect to $\tilde{\nabla}$ on $\Gamma(S(TM))$, then

$$\overline{\nabla}_\xi v = 0, \quad C(v, \xi) = 0. \quad (71)$$

2. If v is a concircular vector field with respect to $\tilde{\nabla}^*$ on $\Gamma(S(TM))$, then

$$\overline{\nabla}_\xi^* v = 0, \quad C^*(v, \xi) = 0. \quad (72)$$

Proof. If we put ξ instead of X in (61), then we have

$$\tilde{\nabla}_\xi v = \theta \xi. \quad (73)$$

By utilizing the value of (8) in (73), we conclude that

$$\nabla_\xi v + B(\xi, v)N = \theta \xi. \quad (74)$$

From Proposition 4, we write $B(\xi, v) = 0$. Therefore, we determine that

$$\nabla_\xi v = \theta \xi. \quad (75)$$

Considering the fact that ν lies on $\Gamma(S(TM))$ and (12), we obtain

$$\bar{\nabla}_{\xi}\nu + C(\xi, \nu)\xi = \theta\xi. \quad (76)$$

From the last equation, we find (71).

Using a similar method as in proof (71), one can arrive at (72). $\square$

Statement 1 of Theorem 3 leads us to the following corollary:

Corollary 4. *Let $(M, g, S(TM))$ be a $S(TM)$ totally geodesic or totally normally umbilical lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ with respect to $\tilde{\nabla}^*$. Then, there does not exist any concircular vector field with respect to $\tilde{\nabla}$, which lies on $\Gamma(S(TM))$.*

Statement 2 of Theorem 3 leads us to the following corollary:

Corollary 5. *Let $(M, g, S(TM))$ be a $S(TM)$ totally geodesic or totally normally umbilical lightlike hypersurface of $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ with respect to $\tilde{\nabla}$. Then, there does not exist any concircular vector field with respect to $\tilde{\nabla}$, which lies on $\Gamma(S(TM))$.*

Proposition 8. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost product-like statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$. If $\nu = FN$ is a concircular vector field on $(M, g, S(TM))$, then the following equalities are satisfied for all $X \in \Gamma(TM)$:*

$$B(X, \xi) = 0 \quad \text{and} \quad (\tilde{\nabla}_X \frac{1}{\lambda})\xi + \frac{1}{\lambda}\nabla_X \xi = \theta X. \quad (77)$$

In particular, if $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ is a locally product-like statistical manifold, then

$$\eta(A_N^* X) = 0, \quad (78)$$

$$-\varphi A_N^* X + \tau^*(X)\frac{1}{\lambda}\xi = \theta X. \quad (79)$$

Proof. If we write $FN = \nu$ in (61), then we obtain

$$(\tilde{\nabla}_X \frac{1}{\lambda})\xi + \frac{1}{\lambda}\tilde{\nabla}_X \xi = \theta X. \quad (80)$$

It is easy to prove (77) based on (13) and (80). In particular, if $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$ is a locally product-like statistical manifold, we can determine that

$$\tilde{\nabla}_X FN = F\tilde{\nabla}_X N = \theta X. \quad (81)$$

It follows from (31) and (81) that

$$-\varphi A_N^* X + \eta(A_N^* X)\lambda N + \tau^*(X)\frac{1}{\lambda}\xi = \theta X \quad (82)$$

is satisfied. Based on (82), proving (78) and (79) becomes straightforward. $\square$

Through a similar approach to the proof of Proposition 8, one can obtain the following proposition:

Proposition 9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost product-like statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla}, F)$. If $\nu = F^*N$ is a concircular vector field with respect to $\tilde{\nabla}^*$ on $(M, g, S(TM))$, then the following equalities are satisfied for all $X \in \Gamma(TM)$:*

$$B^*(X, \xi) = 0 \quad \text{and} \quad (\tilde{\nabla}_X^* \frac{1}{\lambda})\xi + \frac{1}{\lambda}\nabla_X^* \xi = \theta^* X. \quad (83)$$

In particular, if $(M, g, S(TM))$ is a locally product-like statistical manifold, then

$$\eta(A_N X) = 0, \quad (84)$$

$$-\varphi^* A_N X + \tau(X) \frac{1}{\lambda} \zeta = \theta^* X. \quad (85)$$

As a result of Proposition 8 and Proposition 9, we can derive the following corollaries:

Corollary 6. Let $(M, g, S(TM))$ be a lightlike hypersurface of a locally product-like statistical manifold. If $v = FN$ is a concircular vector field with respect to $\tilde{\nabla}$ or $\tilde{\nabla}^*$, then the null sectional curvatures K and K^* vanish identically.

Corollary 7. Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost product-like statistical manifold. If $v = FN$ is a concircular vector field with respect to $\tilde{\nabla}$ (or $\tilde{\nabla}^*$), then $A_N^* X$ (or $A_N X$) belongs to $\Gamma(S(TM))$.

Corollary 8. Let $(M, g, S(TM))$ be a totally normally umbilical with respect to $\tilde{\nabla}$ (or $\tilde{\nabla}^*$). Then, $v = FN$ is not a concircular vector field with respect to $\tilde{\nabla}^*$ (or $\tilde{\nabla}$).

5. Conclusions and Future Works

Various relations on a Riemannian manifold can be obtained by examining the geometric properties of some special types of vector fields. The most used of these vector fields are geodesic vector fields, Killing vector fields, concurrent vector fields, concircular vector fields and torse-forming vector fields, etc. In this study, we obtained some results on radical anti-invariant lightlike hypersurfaces of almost product-like statistical manifolds, with the help of concircular vector fields and concurrent vector fields.

In future works, the lightlike hypersurfaces of almost product-like statistical manifolds with geodesic vector fields, Killing vector fields and torse-forming vector fields may be studied. Considering [23–29], some applications of these hypersurfaces in singularity theory and submanifold theory could be obtained.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: I am grateful to the editor and the anonymous referees for their constructive comments and valuable suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Duggal, K.L.; Bejancu, A. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*; Springer: Dordrecht, The Netherlands, 1996.
2. Duggal, K.L.; Jin, D.H. *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*; World Scientific Publishing Company: Singapore, 2007.
3. Duggal, K.L.; Şahin, B. *Differential Geometry of Lightlike Submanifolds*; Springer Science, Business Media: Berlin/Heidelberg, Germany, 2011.
4. Atçeken, M.; Kılıç, E. Semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. *Kodai Math. J.* **2007**, *30*, 361–378. [CrossRef]
5. Kılıç, E.; Bahadır, O. Lightlike hypersurfaces of a semi-Riemannian product manifold and quarter-symmetric nonmetric connections. *Int. J. Math. Math. Sci.* **2012**, *2012*, 178390. [CrossRef]
6. Kılıç, E.; Şahin, B. Radical anti-invariant lightlike submanifolds of semi-Riemannian product manifolds. *Turk. J. Math.* **2008**, *32*, 429–449.
7. Amari, S. *Differential-Geometrical Methods in Statistics, Lecture Notes in Statistics* 28, 1st ed.; Springer: New York, NY, USA, 1985.
8. Takano, T. Statistical manifolds with almost complex structures. *Tensor New Ser.* **2010**, *72*, 225–231.

9. Erkan, E.; Takano, K.; Gülbahar, M. Locally product-like statistical manifolds and their hypersurfaces. *Int. Electron. J. Geom.* **2023**, *submitted*.
10. Aksu, Ö.; Gülbahar, M.; Erkan, E. Lightlike hypersurfaces of almost product-like semi-Riemannian manifolds. *Symmetry* **2022**, *15*, 77. [\[CrossRef\]](#)
11. Duggal, K.L.; Jin, D.H. Totally umbilical lightlike submanifolds. *Kodai Math. J.* **2003**, *26*, 49–68. [\[CrossRef\]](#)
12. Bahadır, O.; Tripathi, M.M. Geometry of lightlike hypersurfaces of a statistical manifold. *arXiv* **2019**, arXiv:1901.09251.
13. Bahadır, O.; Siddiqui, A.N.; Gülbahar, M.; Alkhaldi, A.H. Main curvatures identities on lightlike hypersurfaces of statistical manifolds and their characterizations. *Mathematics* **2022**, *10*, 2290. [\[CrossRef\]](#)
14. Chen, B.Y. Some results on concircular vector fields and their applications to Ricci solitons. *Bull. Korean Math. Soc.* **2015**, *52*, 1535–1547. [\[CrossRef\]](#)
15. Chen, B.Y.; Deshmukh, S. Some results about concircular vector fields on Riemannian manifolds. *Filomat* **2020**, *34*, 835–842. [\[CrossRef\]](#)
16. Deshmukh, S.; Ilarslan, K.; Alsodais, H.; De, U.C. Spheres and Euclidean spaces via concircular vector fields. *Mediterr. J. Math.* **2021**, *18*, 209. [\[CrossRef\]](#)
17. Li, Y.; Abolarinwa, A.; Alkhaldi, A.H.; Ali, A. Some Inequalities of Hardy Type Related to Witten-Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580. [\[CrossRef\]](#)
18. Li, Y.; Haseeb, A.; Ali, M. LP-Kenmotsu manifolds admitting η -Ricci solitons and spacetime. *J. Math.* **2022**, *2022*, 6605127. [\[CrossRef\]](#)
19. Li, Y.; Ganguly, D. Kenmotsu metric as conformal η -Ricci soliton. *Mediterr. J. Math.* **2023**, *20*, 193. [\[CrossRef\]](#)
20. Li, Y.; Kumara, H.A.; Siddesha, M.S.; Naik, D.M. Characterization of Ricci almost soliton on Lorentzian manifolds. *Symmetry* **2023**, *15*, 1175. [\[CrossRef\]](#)
21. Li, Y.; Ganguly, D.; Dey, S.; Bhattacharyya, A. Conformal η -Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math.* **2022**, *7*, 5408–5430. [\[CrossRef\]](#)
22. Yano, K. On the torse-forming direction in Riemannian spaces. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 340–345. [\[CrossRef\]](#)
23. Li, Y.; Caliskan, A. Quaternionic Shape Operator and Rotation Matrix on Ruled Surfaces. *Axioms* **2023**, *12*, 486. [\[CrossRef\]](#)
24. Li, Y.; Gezer, A.; Karakaş, E. Some notes on the tangent bundle with a Ricci quarter-symmetric metric connection. *AIMS Math.* **2023**, *8*, 17335–17353. [\[CrossRef\]](#)
25. Li, Y.; Bhattacharyya, S.; Azami, S.; Saha, A.; Hui, S.K. Harnack Estimation for Nonlinear, Weighted, Heat-Type Equation along Geometric Flow and Applications. *Mathematics* **2023**, *11*, 2516. [\[CrossRef\]](#)
26. Li, Y.; Eren, K.; Ersoy, S. On simultaneous characterizations of partner-ruled surfaces in Minkowski 3-space. *AIMS Math.* **2023**, *8*, 22256–22273. [\[CrossRef\]](#)
27. Li, Y.; Alkhaldi, A.; Ali, A.; Abdel-Baky, R.A.; Saad, M.K. Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-space. *AIMS Math.* **2023**, *8*, 13875–13888. [\[CrossRef\]](#)
28. Li, Y.; Laurian-Ioan, P.; Alqahtani, L.S.; Alkhaldi, A.H.; Ali, A. Zermelo's navigation problem for some special surfaces of rotation. *AIMS Math.* **2023**, *8*, 16278–16290. [\[CrossRef\]](#)
29. Li, Y.; Srivastava, S.K.; Mofarreh, F.; Kumar, A.; Ali, A. Ricci soliton of CR-warped product manifolds and their classifications. *Symmetry* **2023**, *15*, 976. [\[CrossRef\]](#)

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