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# Infinitely Many Positive Solutions to Nonlinear First-Order Iterative Systems of Singular BVPs on Time Scales

**Famei Zheng, Xiaojing Wang, Xiwang Cheng and Bo Du ***

School of Mathematics and Statistics, Huaiyin Normal University, Huaian 223300, China
* Correspondence: dubo7307@163.com

**Abstract:** Iterative differential equations provide a new idea to study functional differential equations. The study of iterative equations can provide new methods for the study of differential equations with state-dependent delays. In this paper, we are concerned with proving the existence of infinitely many positive solutions to nonlinear first-order iterative systems of singular BVPs on time scales by using Krasnoselskii's cone fixed point theorem in a Banach space. It is worth pointing out that in this paper, we can use the symmetry of the iterative process and Green's function to transform the considered differential equation into an equivalent integral equation, which plays a key role in the proof of the theorem in this paper.

**Keywords:** positive solution; time scales; existence; iterative system

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## 1. Introduction

In this paper, we consider the following nonlinear first-order iterative system of singular BVPs on time scales:

$$\begin{cases} x_i^{\Delta}(t) + p(t)x_i(\sigma(t)) = \lambda(t)f_i(x_{i+1}(t)), \ \ t \in (0,a]_{\mathbb{T}}, 1 \le i \le n, \\ x_{n+1}(t) = x_1(t), \\ x_i(0) = x_i(\sigma(a)), \end{cases} \tag{1}$$

where $i \in \mathbb{N}$, $\lambda(t)$ has a singularity in $(0, \frac{\sigma(a)}{2}]_{\mathbb{T}}$, $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and $p : [0,a]_{\mathbb{T}} \to \mathbb{R}^+$ is right-dense continuous.

Iterative differential equations belong to functional differential equations. In particular, iterative equations can be used to describe functional differential equations with state-dependent delays, see [1–3]. Iterative differential equations provide a new idea to study functional differential equations. Zhao and Liu [4] considered a class of iterative differential equation

$$c_0x''(t) + c_1x'(t) + c_2x(t) = x(p(t) + bx(t)) + h(t) \tag{2}$$

and obtained the existence, uniqueness, and stability of periodic solutions for Equation (2). Eder [5] studied the iterative functional differential equation

$$x'(t) = x^{[2]}(t)$$

and obtained asymptotic properties of solutions. Fečkan [6] further studied the following nonlinear iterative equation

$$x'(t) = f(x^{[2]}(t))$$

with initial condition $x(0) = 0$. In [7], Wang considered the existence of a solution for the following iterative differential equation:

$$x'(t) = f(x(x(t))),$$

where $f$ is a smooth function that maps a closed interval $I$ into itself where this interval contains a fixed point of the unknown function $x$. Cheng, Si, and Wang [8] studied an iterative functional differential equation

$$x'(t) = f(x(x \cdots (x(t))))$$

and obtained a local existence theorem by means of the Schauder fixed point theorem.

We focus on the study of iterative equations on time scales. The dynamic equations on time scales can unify continuous and discrete systems, so studying them can obtain results with a wider applicability, see [9–11]. Khuddush et al. [12] studied the following second-order iterative system of boundary value problems with singularities on time scales:

$$\begin{cases} x_l^{\Delta\nabla}(t) + \lambda(t) g_l(x_{l+1}(t)) = 0, \ t \in (0, \sigma(a)]_{\mathbb{T}}, 1 \le l \le n, \\ x_{n+1}(t) = x_1(t), \\ x_l^{\Delta}(0) = 0, \ \ x_l(\sigma(a)) = \sum_{k=1}^{n-2} c_k x_l(\xi_k) \end{cases} \tag{3}$$

and some sufficient conditions for the existence of infinitely many positive solutions were obtained by applying Krasnoselskii's cone fixed point theorem in a Banach space. When $p(t) = 0$ in system (1), then Equation (3) is similar to system (1). Therefore, the results of this article generalize the results of article [12]. In [13], the authors studied positive solutions for the iterative system of dynamic equations

$$\begin{cases} u_i^{\Delta^{(n)}}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0, \ t \in [a, b]_{\mathbb{T}}, 1 \le i \le n, \\ u_{n+1}(t) = u_1(t), \\ u_i^{\Delta^{(m)}}(a) = 0, \ \ u_i(\sigma^n(b)) = 0, \ 0 \le m \le n-2. \end{cases}$$

For an iterative system of conformable fractional order dynamic boundary value problems on time scales, see [14]; for nondecreasing and convex $C^2$ solutions of an iterative functional differential equation, see [15]. In this paper, we will study system (1), which is a singular dynamic system, and obtain the existence of infinitely many positive solutions by applying Krasnoselskii's cone fixed point theorem in a Banach space. To the best of our knowledge, there are few results for the existence of a positive solution to system (1).

The main contributions are summarized in the following two aspects:

(1) We study a class more extensive iterative system with singularity, which generalizes the results of article [12].
(2) We innovatively use Krasnoselskii's cone fixed point theorem on time scales to study the existence of solutions for iterative system (1).

The following sections are organized as follows. Section 2 gives some preliminaries. The existence of positive solutions of system (1) is obtained in Section 3. In Section 4, an example is given to show the feasibility of our results. Section 4 concludes this article with a summary of our results.

## 2. Preliminaries

The time scales theory, which has received much more attention, was firstly introduced by Hilger [16]. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We give the following notations, and their means can be found in [17]: the forward jump $\sigma(t)$ , backward jump operator $\rho(t)$, regressive rd-continuous function $\mathcal{R}$, and the delta derivative $x^{\Delta}(t)$ of $x(t)$. We also give the following notations: $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}, \ a \le t \le b\}$, the intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ and $[a, b]_{\mathbb{T}}$ are defined similarly. Let $p, q \in \mathcal{R}$. The exponential function is defined by

$$e_p(t, s) = \exp\left( \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right),$$

where $\xi_h(z)$ is the so-called cylinder transformation. If $p, q \in \mathcal{R}$, a circle plus addition is defined by $(p \oplus q) := p + q + \mu pq$. For $p \in \mathcal{R}$, we defined a circle minus $p$ by $(\ominus p) := \frac{-p}{1+\mu p}$.

**Lemma 1.** [17].

*[i]* $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
*[ii]* $e_p(\rho(t),s) = (1-\mu(t)p(t))e_p(t,s)$;
*[iii]* $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
*[iv]* $e_p(t,s)e_p(s,r) = e_p(t,r)$;
*[v]* $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$.

**Lemma 2** (Krasnoselskii's [18]). *Let $P$ be a cone in a Banach space $E$ and $\Omega_1, \Omega_2$ are open sets with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $\mathcal{L} : E \cap (\overline{\Omega}_2 \setminus \Omega_1) \to E$ be a completely continuous operator such that*
*(i) $||\mathcal{L}u|| \leq ||u||$, $u \in E \cap \partial\Omega_1$, and $||\mathcal{L}u|| \geq ||u||$, $u \in E \cap \partial\Omega_2$ or*
*(ii) $||\mathcal{L}u|| \geq ||u||$, $u \in E \cap \partial\Omega_1$, and $||\mathcal{L}u|| \leq ||u||$, $u \in E \cap \partial\Omega_2$.*
*Then $\mathcal{L}$ has a fixed point in $E \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Let $\mathbb{C} = \{u | u : [0,\sigma(a)]_{\mathbb{T}} \to \mathbb{R} \text{ is continuous}\}$ with the norm $||u|| = \max_{t\in[0,\sigma(a)]_{\mathbb{T}}} |u(t)|$.

**Lemma 3.** *For any $y \in \mathbb{C}$, the boundary value problem*

$$\begin{cases} x_1^{\Delta}(t) + p(t)x_1(\sigma(t)) = y(t), \ \ t \in [0,a]_{\mathbb{T}}, \\ x_1(0) = x_1(\sigma(a)), \end{cases} \tag{4}$$

*has a unique solution*

$$x_1(t) = \int_0^{\sigma(a)} G(s,t)y(s)ds,$$

*where*

$$G(s,t) = \begin{cases} (A+1)e_p(s,t), & \text{if } 0 \leq s \leq t \leq \sigma(a) \\ Ae_p(s,t), & \text{if } 0 \leq t < s \leq \sigma(a), \end{cases}$$

$A = \frac{1}{e_p(\sigma(a),0)-1}$.

**Proof.** Similar to the proof of [19], BVP (4) has a unique solution

$$x_1(t) = \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0)y(s)ds + A\int_0^{\sigma(a)} e_p(s,0)y(s)ds\right].$$

Thus,

$$\begin{aligned} x_1(t) &= (A+1)\int_0^t e_p(s,t)y(s)ds + A\int_t^{\sigma(a)} e_p(s,t)y(s)ds \\ &= \int_0^{\sigma(a)} G(s,t)y(s)ds. \end{aligned}$$

□

**Lemma 4.** *Let $\eta \in (0, \frac{\sigma(a)}{2})_{\mathbb{T}}$. The kernel $G(s,t)$ has the following properties:*
*(i)* $0 \leq G(s,t) \leq G_L$ *for all* $t,s \in [0,\sigma(a)]$, *where* $G_L = \min\{A+1, Ae_p(\sigma(a),0)\}$;
*(ii)* $G(s,t) \geq G_l$ *for all* $s \in [\eta, \sigma(a)-\eta]$, $t \in [0,\sigma(a)]$, *where* $G_l = \max\{(A+1)e_p(\eta - \sigma(a),0), A\}$.

**Proof.** We first show that case (i) holds. If $0 \leq s \leq t \leq \sigma(a)$, we have

$$G(s,t) \leq A+1. \tag{5}$$

If $0 \leq t < s \leq \sigma(a)$, we have

$$G(s,t) \leq Ae_p(\sigma(a),0). \tag{6}$$

From (5) and (6), we obtain that

$$G(s,t) \leq \min\{A+1, Ae_p(\sigma(a),0)\} = G_L.$$

Next, we show that case (ii) holds. Let $s \in [\eta, \sigma(a)-\eta]$. If $s \leq t$, we have

$$G(s,t) \geq (A+1)e_p(\eta-\sigma(a),0). \tag{7}$$

If $s > t$, we have

$$G(s,t) \geq A. \tag{8}$$

From (7) and (8), we obtain that

$$G(s,t) \leq \max\{(A+1)e_p(\eta-\sigma(a),0), A\} = G_l.$$

□

Obviously, $(x_1(t), x_2(t), \cdots, x_n(t))$ is a solution of the iterative BVP (1) if and only if

$$x_i(t) = \int_0^{\sigma(a)} G(s,t)\lambda(s)f_i(x_{i+1}(s))\Delta s$$

and

$$x_{n+1}(t) = x_1(t),\ t \in (0,a]_{\mathbb{T}},\ 1 \leq i \leq n.$$

Then, we have

$$x_1(t) = \int_0^{\sigma(a)} G(s_1,t)\lambda(s_1)f_1\Bigg[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2)f_2\Bigg[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \times f_{n-1}\Bigg[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n)f_n(x_1(s_n))\Delta s_n\Bigg]\cdots\Delta s_3\Bigg]\Delta s_2\Bigg]\Delta s_1.$$

For $\eta \in (0, \frac{\sigma(a)}{2})_{\mathbb{T}}$, we define a cone $P_\eta \subset \mathbb{C}$ by

$$P_\eta = \Bigg\{x \in \mathbb{C} : x(t) \text{ is nonnegative and } \min_{t\in[\eta,\sigma(a)-\eta]_{\mathbb{T}}} x(t) \geq \frac{G_l}{G_L}||x||\Bigg\},$$

where $G_l$ and $G_L$ are defined by Lemma 4. For any $x_1 \in P_\eta$, we define an operator $\mathcal{L} : P_\eta \to \mathbb{C}$ by

$$(\mathcal{L}x_1)(t) = \int_0^{\sigma(a)} G(s_1,t)\lambda(s_1)f_1\Bigg[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2)f_2\Bigg[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \times f_{n-1}\Bigg[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n)f_n(x_1(s_n))\Delta s_n\Bigg]\cdots\Delta s_3\Bigg]\Delta s_2\Bigg]\Delta s_1.$$

**Lemma 5.** *For each $\eta \in (0, \frac{\sigma(a)}{2})_{\mathbb{T}}$, $\mathcal{L} : P_\eta \to P_\eta$ is completely continuous.*

**Proof.** By Lemma 4, we obtain that $G(s,t) \geq 0$ and $(\mathcal{L}x_1)(t)$ for all $s,t \in [0,\sigma(a)]_{\mathbb{T}}$. Again by Lemma 4, for $x_1 \in P_\eta$, we have

$$\begin{aligned}||\mathcal{L}x_1|| &= \max_{t\in[0,\sigma(a)]_{\mathbb{T}}} \int_0^{\sigma(a)} G(s_1,t)\lambda(s_1) f_1\Bigg[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2) f_2\Bigg[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \\ &\times f_{n-1}\Bigg[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n) f_n(x_1(s_n))\Delta s_n\Bigg]\cdots \Delta s_3\Bigg]\Delta s_2\Bigg]\Delta\, s_1 \\ &\leq G_L \int_0^{\sigma(a)} \lambda(s_1) f_1\Bigg[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2) f_2\Bigg[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \\ &\times f_{n-1}\Bigg[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n) f_n(x_1(s_n))\Delta s_n\Bigg]\cdots \Delta s_3\Bigg]\Delta s_2\Bigg]\Delta\, s_1\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[\eta,\sigma(a)-\eta]_{\mathbb{T}}}\{(\mathcal{L}x_1)(t)\} &\geq G_l \int_0^{\sigma(a)} \lambda(s_1) f_1\Bigg[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2) f_2\Bigg[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \\ &\times f_{n-1}\Bigg[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n) f_n(x_1(s_n))\Delta s_n\Bigg]\cdots \Delta s_3\Bigg]\Delta s_2\Bigg]\Delta\, s_1 \\ &\geq \frac{G_l}{G_L}||\mathcal{L}x_1||.\end{aligned}$$

Hence, $\mathcal{L}x_1 \in P_\eta$ and $\mathcal{L}(P_\eta) \subset P_\eta$. It follows by the Arzela–Ascoli theorem that $\mathcal{L}$ is completely continuous. □

## 3. Main Results

**Theorem 1.** *Suppose that the following assumption holds:*

*($H_1$) there exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $0 < t_{j+1} < t_j < \frac{\sigma(a)}{2}$ and*

$$\lim_{j\to\infty} t_j = t^* < \frac{\sigma(a)}{2},\ \lim_{t\to t_j}\lambda(t) = +\infty.$$

*Let $\{\eta_j\}_{j=1}^{\infty}$ with $0 < t_{j+1} < \eta_j < t_j$. Let $\{\Gamma_j\}_{j=1}^{\infty}$ and $\{\Lambda_j\}_{j=1}^{\infty}$ be such that*

$$\Gamma_{j+1} < \frac{G_l}{G_L}\Lambda_j < \Lambda_j < \theta\Lambda_j < \Gamma_j,\ j \in \mathbb{N},$$

*where $\theta \geq \frac{1}{G_l\delta(\sigma(a)-2\eta_1)}$. Assume that $f_i$ statisfies*

*($H_2$) $f_i(u(t)) \leq \Xi\Gamma_j$ for $t \in (0,\sigma(a)]_{\mathbb{T}}$, $0 \leq u \leq \Gamma_j$, where $\Xi \leq \frac{1}{G_L\int_0^{\sigma(a)}\lambda(s)\Delta s}$.*

*($H_3$) $f_i(u(t)) \geq \theta\Lambda_j$ for $t \in [\eta_j,\sigma(a)-\eta_j]_{\mathbb{T}}$, $\frac{G_l}{G_L}\Lambda_j \leq u \leq \Lambda_j$.*

*Then the iterative BVP (1) has infinitely many solutions $\{(x_1^{[j]}, x_2^{[j]},\cdots, x_n^{[j]})\}_{j=1}^{\infty}$ such that $x_i^{[j]} \geq 0$ on $(0,\sigma(a)]_{\mathbb{T}}$ for $i = 1,2,\cdots,n$.*

**Proof.** It follows that by ($H_1$), there exists $\delta > 0$ such that $\lambda(t) > \delta$. Let

$$\Omega_{1,j} = \{x \in \mathbb{C} : ||x|| < \Gamma_j\},\ \Omega_{2,j} = \{x \in \mathbb{C} : ||x|| < \Lambda_j\}.$$

By assumption ($H_1$), we have

$$0 < t^* < t_{j+1} < \eta_j < t_j < \frac{\sigma(a)}{2} \text{ for } j \in \mathbb{N}.$$

For $j \in \mathbb{N}$, we define the cone $P_{\eta_j}$ by

$$P_{\eta_j} = \left\{ x \in \mathbb{C} : x(t) \text{ is nonnegative and } \min_{t \in [\eta_j, \sigma(a)-\eta_j]_{\mathbb{T}}} x(t) \geq \frac{G_l}{G_L} ||x|| \right\}.$$

Let $x_1 \in P_{\eta_j} \cap \partial\Omega_{1,j}$. Then, $x_1(s) \leq \Gamma_j$ for all $s \in (0, \sigma(a)]_{\mathbb{T}}$. For $s_{n-1} \in (0, \sigma(a)]_{\mathbb{T}}$, by $(H_2)$, we have

$$\begin{aligned} &\int_0^{\sigma(a)} G(s_n, s_{n-1}) \lambda(s_n) f_n(x_1(s_n)) \Delta\, s_n \\ &\leq G_L \Xi \Gamma_j \int_0^{\sigma(a)} \lambda(s_n) \Delta\, s_n \\ &\leq \Gamma_j. \end{aligned}$$

Similar to the above proof, for $s_{n-2} \in (0, \sigma(a)]_{\mathbb{T}}$, by $(H_2)$, we also have

$$\begin{aligned} &\int_0^{\sigma(a)} G(s_{n-1}, s_{n-2}) \lambda(s_{n-1}) f_{n-1} \left[ \int_0^{\sigma(a)} G(s_n, s_{n-1}) \lambda(s_n) f_n(x_1(s_n)) \Delta s_n \right] \Delta s_{n-1} \\ &\leq \int_0^{\sigma(a)} G(s_{n-1}, s_{n-2}) \lambda(s_{n-1}) f_{n-1}(\Gamma_j) \Delta s_{n-1} \\ &\leq G_L \Xi \Gamma_j \int_0^{\sigma(a)} \lambda(s_{n-1}) \Delta s_{n-1} \\ &\leq \Gamma_j. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\int_0^{\sigma(a)} G(s_1, t) \lambda(s_1) f_1 \left[ \int_0^{\sigma(a)} G(s_2, s_1) \lambda(s_2) f_2 \left[ \int_0^{\sigma(a)} G(s_3, s_2) \cdots \right. \right. \\ &\times f_{n-1} \left[ \int_0^{\sigma(a)} G(s_n, s_{n-1}) \lambda(s_n) f_n(x_1(s_n)) \Delta s_n \right] \cdots \Delta s_3 \Big] \Delta s_2 \Big] \Delta\, s_1 \\ &\leq \Gamma_j \end{aligned}$$

and $(\mathcal{L}x_1)(t) \leq \Gamma_j$. By $\Gamma_j = ||x_1||$ for $x_1 \in P_{\eta_j} \cap \partial\Omega_{1,j}$, we have

$$||\mathcal{L}x_1|| \leq ||x_1||. \tag{9}$$

Next, for $t \in [\eta_j, \sigma(a) - \eta_j]_{\mathbb{T}}$, we have

$$\Lambda_j = ||x_1|| \geq x_1(t) \geq \min_{t \in [\eta_j, \sigma(a)-\eta_j]_{\mathbb{T}}} x_1(t) \geq \frac{G_l}{G_L} ||x_1||.$$

For $s_{n-1} \in [\eta_j, \sigma(a) - \eta_j]_{\mathbb{T}}$, by $(H_3)$, we have

$$\begin{aligned} &\int_0^{\sigma(a)} G(s_n, s_{n-1}) \lambda(s_n) f_n(x_1(s_n)) \Delta\, s_n \\ &\geq \int_{\eta_j}^{\sigma(a)-\eta_j} G(s_n, s_{n-1}) \lambda(s_n) f_n(x_1(s_n)) \Delta\, s_n \\ &\geq G_l \theta \Lambda_j \delta(\sigma(a) - 2\eta_j) \\ &\geq G_l \theta \Lambda_j \delta(\sigma(a) - 2\eta_1) \\ &\geq \Lambda_j. \end{aligned}$$

Continuing with the bootstrapping argument, we obtain

$$\int_0^{\sigma(a)} G(s_1,t)\lambda(s_1)f_1\Big[\int_0^{\sigma(a)} G(s_2,s_1)\lambda(s_2)f_2\Big[\int_0^{\sigma(a)} G(s_3,s_2)\cdots \times f_{n-1}\Big[\int_0^{\sigma(a)} G(s_n,s_{n-1})\lambda(s_n)f_n(x_1(s_n))\Delta s_n\Big]\cdots\Delta s_3\Big]\Delta s_2\Big]\Delta\, s_1 \geq \Lambda_j$$

and $(\mathcal{L}x_1)(t) \geq \Lambda_j$. By $\Lambda_j = ||x_1||$ for $x_1 \in P_{\eta_j} \cap \partial\Omega_{2,j}$, we have

$$||\mathcal{L}x_1|| \geq |||x_1||. \tag{10}$$

It is easy to see that $0 \in \Omega_{2,j} \subset \overline{\Omega}_{2,j} \subset \Omega_{1,j}$. In view of (9) and (10), the operator $\mathcal{L}$ has a fixed point $x_1^{[j]} \in P_{\eta_j} \cap (\overline{\Omega}_{1,j}\backslash)\Omega_{2,j}$ such that $x_1^{[j]} \geq 0$ on $(0,a]_{\mathbb{T}}$. Let $x_{n+1} = x_1$. We obtain infinitely many positive solutions $\{(x_1^{[j]}, x_2^{[j]}, \cdots, x_n^{[j]})\}_{j=1}^{\infty}$ of BVP (1) given iteratively by

$$x_i(t) = \int_0^{\sigma(a)} G(s,t)\lambda(s)f_i(x_{i+1}(s))\Delta s,\ i = 1,2,\cdots,n.$$

□

**Remark 1.** *In system (1), the initial conditions are different from the ones in system (2). Obviously, the initial conditions in system (2) are more complicated. In future work, we will study the following iterative system:*

$$\begin{cases} x_l^{\Delta\nabla}(t) + p(t)x_l(\sigma(t)) + \lambda(t)g_l(x_{l+1}(t)) = 0,\ t \in (0,\sigma(a)]_{\mathbb{T}}, 1 \leq l \leq n, \\ x_{n+1}(t) = x_1(t), \\ x_l^{\Delta}(0) = 0,\ \ x_l(\sigma(a)) = \sum_{k=1}^{n-2} c_k x_l(\xi_k), \end{cases}$$

*where* $l \in \mathbb{N}$, $c_k \in \mathbb{R}^+$ with $\sum_{k=1}^{n-2} c_k < 1$, $0 < \xi_k < \frac{\sigma(a)}{2}$, $\lambda(t) = \Pi_{i=1}^{m}\lambda_i(t)$ *and each* $\lambda_i(t) \in L_{\nabla}^{p_i}\Big((0,\sigma(a)]_{\mathbb{T}}\Big)$ $(p_1 \geq 1)$ *has a singularity in* $(0, \frac{\sigma(a)}{2}]_{\mathbb{T}}$, $g_l : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and* $p : [0,a]_{\mathbb{T}} \to \mathbb{R}^+$ *is right-dense continuous.*

## 4. Example

Let $\mathbb{T} = \mathbb{R}$, consider the following BVP of model (1):

$$\begin{cases} x_i^{\Delta}(t) + p(t)x_i(t) = \lambda(t)f_i(x_{i+1}(t)),\ \ t \in (0,1], 1 \leq i \leq 4, \\ x_5(t) = x_1(t), \\ x_i(0) = x_i(1), \end{cases} \tag{11}$$

Choose

$$p(t) = 2,\ \ \lambda(t) = \frac{1}{|t-0.25|^{\frac{1}{2}}}$$

then $\delta = \left(\frac{4}{3}\right)^{\frac{1}{2}}$. For $j \in \mathbb{N}, i = 1,2,3,4$, let

$$f_i(x) = \begin{cases} 0.02 \times 10^{-4j},\ \ x \in [10^{-4j}, \infty), \\ \frac{50\times10^{-(4j+3)} - 0.02\times10^{-4j}}{10^{-(4j+3)} - 10^{-4j}}(x - 10^{-4j}) + 0.02 \times 10^{-6j},\ \ x \in [10^{-(4j+3)}, 10^{-4j}], \\ 50 \times 10^{-(4j+3)},\ \ x \in [0.01 \times 10^{-(4j+3)}, 10^{-(4j+3)}], \\ \frac{50\times10^{-(4j+3)} - 0.02\times10^{-6j}}{0.02\times10^{-(4j+3)} - 10^{-4j}}(x - 10^{-(4j+4)}) + 0.02 \times 10^{-6j},\ \ x \in [10^{-(4j+4)}, 0.02 \times 10^{-(4j+3)}], \\ 0,\ \ x = 0. \end{cases}$$

Let

$$t_j = \frac{33}{70} - \sum_{k=1}^{j} \frac{1}{4(k+1)^4},\ \eta_j = 0.5(t_j + t_{j+1}),\ \ j \in \mathbb{N},$$

then

$$\eta_1 = \frac{493}{1120} - \frac{1}{628} < \frac{493}{1120}$$

and

$$t_{j+1} < \eta_j < t_j.$$

Obviously,

$$t_1 = \frac{493}{1120} < 0.5,\ t_j - t_{j+1} = \frac{1}{4(j+2)^4}.$$

Using $\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}$, we have

$$t^* = \lim_{j\to\infty} t_j = \frac{33}{70} - \sum_{k=1}^{\infty} \frac{1}{4(k+1)^4} = \frac{33}{70} + \frac{1}{64} - \frac{\pi^4}{360} \approx 0.45.$$

After a simple calculation, we obtain

$$A = 0.18,\ G_L = \min\{A+1, Ae_p(\sigma(a),0)\} \approx 0.28$$

$$G_l = \max\{(A+1)e_p(\eta - \sigma(a),0), A\} \approx 0.18,\ \theta \geq \frac{1}{G_l \delta(\sigma(a) - 2\eta_1)} \approx 40,$$

$$\Xi \leq \frac{1}{G_L \int_0^{\sigma(a)} \lambda(s)\Delta s} \approx 1.307.$$

Choose $\Xi = 1.2$. Furthermore, if we take

$$\Gamma_j = 10^{-4j} \text{ and } \Lambda_j = 10^{-(4j+3)},$$

then

$$\Gamma_{j+1} = 10^{-(4j+4)} < \frac{G_l}{G_L}\Lambda_j = 0.64 \times 10^{-(4j+3)}$$
$$< \Lambda_j = 10^{-(4j+3)} < \theta\Lambda_j = 40 \times 10^{-(4j+3)} < \Gamma_j = 10^{-4j}.$$

Obviously, $f_i,\ (i = 1,2,3,4)$ satisfies the following growth conditions:

(i) $f_i(x) \leq \Xi\Gamma_j = 1.2 \times 10^{-4j}$ for $x \in [0, 10^{-4j}]$,

(ii) $f_i(x) \geq \theta\Lambda_j = 40 \times 10^{-(4j+3)}$ for $x \in [0.64 \times 10^{-(4j+3)}, 10^{-(4j+3)}]$.

Then, all the conditions of Theorem 1 are satisfied. Therefore, by Theorem 1, the BVP (4.1) has infinitely many solutions $\{(x_1^{[j]}, x_2^{[j]}, \cdots, x_n^{[j]})\}_{j=1}^{\infty}$ such that $x_i^{[j]} \geq 0$ on $(0, \sigma(a)]_{\mathbb{T}}$ for $i = 1,2,3,4$.

## 5. Discussion and Conclusions

In the past few years, iterative differential equations have received extensive attention. The concept of iterative equations was firstly introduced by Babbage [20] in 1815. Babbage found a function equalling its $n-$th iterate. The real development of iterative equations began in the twentieth century, for more details, see [21] and related references. Iterative equations are a particular type of delay differential equations that depend on both the time $t$ and the state variable $x$, which are defined implicitly by the iterates. The theory of time scale calculus unifies the calculus of the theory of continuous equations with that of discrete equations, which gives a new way to study hybrid discrete–continuous dynamical systems and has more applications in any field that requires simultaneous modeling of discrete and continuous data. This paper considers the existence of infinitely many positive solutions of nonlinear first-order iterative systems of singular BVPs on time scales by using

Krasnoselskii's cone fixed point theorem in a Banach space. It should be pointed out that the properties of Green's function are the basis of this study. We believe that the properties of Green's function obtained in this paper can be used to study other types of equations. In future work, we will study the iterative system in Remark 1. Furthermore, we will further investigate iterative systems with impulses and fractional order iterative systems.

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