symmetry

MDPI

Article

Results on Minkowski-Type Inequalities for Weighted Fractional Integral Operators

Hari Mohan Srivastava [1,2,3,*], Soubhagya Kumar Sahoo [4], Pshtiwan Othman Mohammed [5], Artion Kashuri [6] and Nejmeddine Chorfi [7]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
2 Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Seoul 02447, Republic of Korea
3 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
4 Department of Mathematics, C.V. Raman Global University, Bhubaneswar 752054, India
5 Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Iraq; pshtiwansangawi@gmail.com
6 Department of Mathematics, Faculty of Technical and Natural Sciences, University "Ismail Qemali", 9400 Vlora, Albania
7 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: harimsri@math.uvic.ca

Abstract: This article considers a general family of weighted fractional integral operators and utilizes this general operator to establish numerous reverse Minkowski inequalities. When it comes to understanding and investigating convexity and inequality, symmetry is crucial. It provides insightful explanations, clearer explanations, and useful methods to help with the learning of key mathematical ideas. The kernel of the general family of weighted fractional integral operators is related to a wide variety of extensions and generalizations of the Mittag-Leffler function and the Hurwitz-Lerch zeta function. It delves into the applications of fractional-order integral and derivative operators in mathematical and engineering sciences. Furthermore, this article derives specific cases for selected functions and presents various applications to illustrate the obtained results. Additionally, novel applications involving the Digamma function are introduced.

Keywords: weighted fractional integral operators; reverse Minkowski integral inequality; digamma (or ψ-) function; Mittag-Leffler functions; Hurwitz-Lerch zeta function

check for updates

Citation: Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Kashuri, A.; Chorfi, N. Results on Minkowski-Type Inequalities for Weighted Fractional Integral Operators. *Symmetry* **2023**, *15*, 1522. https://doi.org/10.3390/sym15081522

Academic Editor: Dongfang Li

Received: 23 June 2023
Revised: 20 July 2023
Accepted: 25 July 2023
Published: 2 August 2023

1. Introduction

Fractional calculus is a field within mathematical analysis that expands the notions of differentiation and integration beyond integer orders. Unlike in traditional calculus, where the order of differentiation or integration is limited to positive integers, fractional calculus allows for orders that include real numbers, including fractions. This field has broad-ranging practical applications in several disciplines, such as physics [1], engineering [2], finance [3], and signal processing [4]. One of the key features of fractional calculus is its ability to describe systems with long-range memory and non-locality. Fractional derivatives and integrals have been used to model phenomena such as diffusion [5], wave propagation [6], and viscoelasticity [7], among others. The theory of fractional calculus is still an active area of research, and new applications and generalizations are being discovered regularly (see [8–11]).

Fractional order derivatives and integrals are the subject of fractional calculus. Due to the usefulness of calculus, many researchers are interested in exploring its origins and the fundamental principles of fractional calculus. For further reading on this subject, refer to [12–16]. Fractional integral inequalities have a significant role in establishing uniqueness of both conventional and fractional differential equations. These inequalities find applications in various areas. For example, Anastassiou [17] defined a Caputo-like discrete nabla fractional difference and produced discrete nabla fractional Taylor formulae, and then presented fractional Opial-, Ostrowski-, Poincaré and Sobolev-type inequalities, Zheng [18] established Gronwall-Bellman-type discrete fractional difference inequalities and fractional sum inequalities for solutions to discrete fractional difference equations, and Agarwal et al. [19] established new variants of Hermite-Hadamard-type inequalities using a convex function.

Definition 1 (see [15,20])**.** *The left and right Riemann-Liouville fractional integrals are defined as follows for* $\mathrm{f} \in \mathcal{L}_1$ *(the Lebesgue measurable functions on* $[\mathrm{a}, \mathrm{b}]$*):*

$$\left(\mathrm{I}^{\alpha}_{\mathrm{a}^+}\mathrm{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathrm{a}}^{x}(x-\mathrm{z})^{\alpha-1}\mathrm{f}(\mathrm{z})dz \qquad (x > \mathrm{a})$$

and

$$\left(\mathrm{I}^{\alpha}_{\mathrm{b}^-}\mathrm{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathrm{b}}(\mathrm{z}-x)^{\alpha-1}\mathrm{f}(\mathrm{z})dz \qquad (x < \mathrm{b}),$$

respectively, for the order $\alpha > 0$.

Fractional derivatives and integrals are effective in characterizing the memory and hereditary properties of numerous materials and processes. Successful applications of fractional differential equations and inequality models can be found, among others, in engineering, physics, biomathematics, viscoelasticity, aerodynamics, electrodynamics of complex media, electrical circuits, electroanalytical chemistry, control theory, etc.

Baleanu and Fernandez [21] categorized fractional-calculus operators into distinct classes based on their varying properties. Meanwhile, Hilfer et al. [22] proposed utilizing desiderata to define an operator as a fractional derivative or integral. However, they did not establish a single and definitive approach to defining fractional derivatives or integrals, nor did they specify their desiderata axioms. Instead, they wanted to encourage the field by offering recommendations based on a selected group of well-known and respected criteria. Many other recent developments on the reverse Minkowski-type inequalities as well as their connections with other families of integral inequalities can be found in (for example) [23–26].

The Chebyshev inequality is the first fundamental integral inequality which we will study in this article. It is expressed as follows (see [27,28]):

Let f_1 *and* f_2 *be a set of integrable and synchronous functions on the closed interval* $[\mathrm{a}, \mathrm{b}]$. *Then, the following integral inequality holds true*:

$$\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}_1(\mathrm{z})\mathrm{f}_2(\mathrm{z})\,\mathrm{dz} \geqq \left(\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}_1(\mathrm{z})\,\mathrm{dz}\right)\left(\frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}_2(\mathrm{z})\,\mathrm{dz}\right). \tag{1}$$

Definition 2. *Two functions are said to be synchronous if the following inequality is satisfied:*

$$[\mathrm{f}_1(x) - \mathrm{f}_1(y)] \cdot [\mathrm{f}_2(x) - \mathrm{f}_2(y)] \geqq 0,$$

for all $x, y \in [\mathrm{a}, \mathrm{b}]$.

One significant class of inequality that is connected to the synchronous functions is the so-called Chebyshev inequality [28]. Many different inequalities for expectation and variance for cumulative distribution functions were obtained using a version of Chebyshev's inequality, according to Liu [29]. It is significant to note that, according to Ozdemir and Pachpatte [30,31], Chebyshev's inequality (1) has been expanded to include functions whose derivatives are located in $\mathcal{L}_p$ spaces. The Chebyshev inequality (1) is also incorporated with fractional calculus by Set et al. [32] and differentiable functions by Pachpatte [31].

The reverse Minkowski inequality is a fundamental result in mathematics that concerns the volumes of convex sets in the Euclidean space. It states that the volume of a convex set in the n-dimensional Euclidean space raised to the power of $\frac{1}{n}$ is bounded below by a constant times the $\mathcal{L}_p$ norm of the support function of the set raised to the power of $\frac{1}{p}$, where p is the dual exponent of the $\mathcal{L}_p$ norm. This inequality has important applications in convex geometry [33], differential geometry [34], analysis [35], probability theory [36], and mathematical finance [37]. Many variations of the inequality have been established for different classes of convex sets and measures. The reverse Minkowski inequality is a powerful tool in mathematics that has enabled researchers to derive many important results in various areas of research.

The development of reverse Minkowski fractional integral inequalities can be seen in the article of Dahmani et al. [38]. Set et al. [39] used the Riemann-Liouville fractional integrals to derive reverse Minkowski inequalities. The reverse Minkowski inequality was considered by Chinchane et al. [40] using the Hadamard fractional integral operators. Sousa et al. [41] derived Minkowski-type inequalities by using some variations of the Erdélyi-Kober-type fractional integral operators, and Rahman et al. [42] used some generalized proportional fractional integral operators to derive Minkowski-type inequalities.

Recently, there has been a significant increase in the utilization of fractional calculus concerning integral inequalities. In order to advance further in this field, it becomes imperative to integrate the notions of weighted fractional calculus to introduce novel inequalities. The exploration of the reverse Minkowski inequality for weighted fractional integrals has been relatively limited in existing research. To address this knowledge gap, our objective is to propose a fresh weighted fractional integral operator and subsequently establish a set of Reverse Minkowski-type inequalities, along with their enhancements, by leveraging fractional calculus, fuzzy calculus, and quantum calculus. In this article, we initiate our investigation by incorporating the introduced weighted fractional calculus to present our main findings. Future articles will delve into these concepts within the aforementioned areas of focus.

The research paper aims to thoroughly study and understand the properties and uses of weighted fractional integral operators, the reverse Minkowski integral inequality, the digamma (or ψ-) function, Mittag-Leffler functions, and the Hurwitz-Lerch zeta function. It seeks to explore how these mathematical ideas are connected to each other, their unique characteristics, and how they can be applied in various areas like fractional calculus, inequalities, special functions, and related fields. The ultimate goal is to contribute new knowledge, theorems, or approaches that can enhance our understanding and application of these concepts in practical and theoretical contexts.

The new weighted fractional integral operator is introduced in Section 2 of this article. Section 3 is devoted towards developing reverse Minkowski inequalities for the weighted fractional integral operators. To emphasize the importance of our findings, we identify a number of significant special situations for proper function choices. In Section 4, we illustrate a number of implementations of our findings. Applications related to the Digamma (or the ψ-) function are presented in Section 5. Finally, in Section 6, a brief conclusion and some thoughts for future research are discussed.

2. Models Based upon Operators of Fractional Calculus

The definition of the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$, generated by the series below, is provided in [43–45]:

$$
{}_p\Psi_q\left[\begin{array}{c}(\mathrm{i}_1,\mathrm{U}_1),\cdots,(\mathrm{i}_p,\mathrm{U}_p);\\ \\ (\mathrm{j}_1,\mathrm{V}_1),\cdots,(\mathrm{j}_q,\mathrm{V}_q);\end{array} z\right] := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}\Gamma(\mathrm{i}_j+\mathrm{U}_j n)}{\prod\limits_{k=1}^{q}\Gamma(\mathrm{j}_k+\mathrm{V}_k n)} \frac{z^n}{n!}, \tag{2}
$$

where
$$\mathrm{i}_j, \mathrm{j}_k \in \mathbb{C} \qquad (j = 1, \cdots, p;\ k = 1, \cdots, q),$$
with
$$\mathrm{U}_1, \cdots, \mathrm{U}_p \in \mathbb{R}^+ \qquad \text{and} \qquad \mathrm{V}_1, \cdots, \mathrm{V}_q \in \mathbb{R}^+,$$
satisfying the following condition:
$$1 + \sum_{k=1}^{q} \mathrm{V}_k - \sum_{j=1}^{p} \mathrm{U}_j \geqq 0, \tag{3}$$
in which the equality holds true only for the appropriate constraint on the argument z given by
$$|z| < \nabla := \left(\prod_{j=1}^{p} \mathrm{U}_j^{-\mathrm{U}_j} \right) \cdot \left(\prod_{j=1}^{q} \mathrm{V}_j^{\mathrm{V}_j} \right).$$

We will now shift our focus to a general version of the Fox-Wright function ${}_p\Psi_q(z)$ in (2), known as $\mathrm{W}^{\sigma}_{\eta,\lambda}(z)$, which was introduced by Wright on Page 424 of [46]. The function is defined as follows:
$$\mathrm{W}^{\sigma}_{\eta,\lambda}(z) = \mathrm{W}^{\sigma(0),\sigma(1),\cdots}_{\eta,\lambda}(z) := \sum_{n=0}^{\infty} \frac{\sigma(n)}{\Gamma(\eta n + \lambda)}\, z^n, \tag{4}$$
where $\eta, \lambda > 0$ and $|z| < R$, with the bounded sequence $\{\sigma(n)\}_{n \in \mathbb{N}_0}$ in the real-number set $\mathbb{R}$. As previously noted in, for example, [47–49], this same function $\mathrm{W}^{\sigma}_{\eta,\lambda}$ was recently replicated, but without giving due credit to Wright [46]. In his survey-cum-expository review articles, Srivastava systematically examined various families of fractional integral and fractional derivative operators by using the Wright function $\mathrm{W}^{\sigma}_{\eta,\lambda}$ in (4), as well as its several companions and extensions. Specifically, the following unification of the definition in (4) and a wide variety of extensions and generalizations of the familiar Hurwitz-Lerch zeta function $\Phi(z, s, \kappa)$ was introduced and studied by Srivastava (see, for details, [48,49]; see also [45]):
$$\mathcal{E}_{\alpha,\beta}(\chi; z, s, \kappa) := \sum_{n=0}^{\infty} \frac{\chi(n)}{(n+\kappa)^s\, \Gamma(\alpha n + \beta)}\, z^n \qquad (\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0), \tag{5}$$
where the suitably-bounded sequence $\{\chi(n)\}_{n=0}^{\infty}$ and the parameters s and κ are appropriately constrained. In fact, the function $\mathcal{E}_{\alpha,\beta}(\chi; z, s, \kappa)$ was successfully used as the kernel of a general family of operators of fractional calculus. Here, in this article, we recall the following case with the Wright function in (4) as the kernel.

Definition 3 (see [47–49])**.** *The left- and right-sided fractional integral operators are defined for $\lambda, \eta > 0$, and $\omega \in \mathbb{R}$ for a given $\mathcal{L}_1$-function f on an interval $[\mathrm{a}, \mathrm{b}]$, as follows:*
$$\left(\mathsf{S}^{\sigma}_{\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}\right)(x) = \int_{\mathrm{a}}^{x} (x-\mathrm{u})^{\lambda-1} \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(x-\mathrm{u})^{\eta}]\mathrm{f}(\mathrm{u})\, \mathrm{du} \qquad (x > \mathrm{a}) \tag{6}$$
and
$$\left(\mathsf{S}^{\sigma}_{\eta,\lambda,\mathrm{b}^-;\omega}\mathrm{f}\right)(x) = \int_{x}^{\mathrm{b}} (\mathrm{u}-x)^{\lambda-1} \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{u}-x)^{\eta}]\mathrm{f}(\mathrm{u})\, \mathrm{du} \qquad (x < \mathrm{b}), \tag{7}$$
where $\mathrm{W}^{\sigma}_{\eta,\lambda}$ is the Wright function defined by (4).

In what follows, we consider the function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the conditions:
$$\int_0^1 \frac{\varphi(\mathrm{u})}{\mathrm{u}}\, \mathrm{du} < \infty, \tag{8}$$

$$\frac{1}{A_1} \leqq \frac{\varphi(s)}{\varphi(r)} \leqq A_1 \ \text{ for } \ \frac{1}{2} \leqq \frac{s}{r} \leqq 2, \tag{9}$$

$$\frac{\varphi(r)}{r^2} \leqq A_2 \frac{\varphi(s)}{s^2} \ \text{ for } \ s \leqq r \tag{10}$$

and

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leqq A_3 |r - s| \, \frac{\varphi(r)}{r^2} \ \text{ for } \ \frac{1}{2} \leqq \frac{s}{r} \leqq 2, \tag{11}$$

where A_1, A_2 and $A_3 > 0$ are independent of $s, r > 0$.

Sarikaya et al. (see [50]) considered the following definition by using the above function $\varphi : [0, \infty) \to [0, \infty)$ that satisfies the conditions (8) to (11).

Definition 4 (see [50])**.** *The definition of generalized fractional integrals for a function* $f : [a, b] \longrightarrow \mathbb{R}$ *is given by*

$$_{a^+}I_\varphi f(x) = \int_a^x \frac{\varphi(x - u)}{x - u} f(u) \, du \qquad (x > a \geqq 0) \tag{12}$$

and

$$_{b^-}I_\varphi f(x) = \int_x^b \frac{\varphi(u - x)}{u - x} f(u) \, du \qquad (x < b), \tag{13}$$

respectively.

The following operators of fractional calculus were defined by Srivastava et al. [51].

Definition 5 (see [51])**.** *For a given* $\mathcal{L}_1$*-function* f *on an interval* $[a, b]$*, the generalized fractional integral operators for* $\lambda, \eta > 0$*, and* $\omega \in \mathbb{R}$*, are presented as follows:*

$$\left(J^{\varphi}_{\sigma,\eta,\lambda,a^+;\omega} f \right)(x) = \int_a^x \frac{\varphi(x - u)}{x - u} W^{\sigma}_{\eta,\lambda}[\omega(x - u)^\eta] f(u) \, du \qquad (x > a \geqq 0) \tag{14}$$

and

$$\left(J^{\varphi}_{\sigma,\eta,\lambda,b^-;\omega} f \right)(x) = \int_x^b \frac{\varphi(u - x)}{u - x} W^{\sigma}_{\eta,\lambda}[\omega(u - x)^\eta] f(u) \, du \qquad (x < b). \tag{15}$$

Definition 6 (see [52])**.** *The operators of weighted generalized fractional integrals are defined for a given* $\mathcal{L}_1$*-function* f *and a positive function* Φ *that has an inverse on the interval* $[a, b]$*, with* $\lambda, \eta > 0$ *and* $\omega \in \mathbb{R}$ *as follows:*

$$\begin{aligned} &\left(J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} f \right)(x) \\ &\quad = \Phi^{-1}(x) \int_a^x \frac{\varphi(x - u)}{x - u} \Phi(u) W^{\sigma}_{\eta,\lambda}[\omega(x - u)^\eta] f(u) \, du \qquad (x > a \geqq 0) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\left(J^{\varphi,\Phi}_{\sigma,\eta,\lambda,b^-;\omega} f \right)(x) \\ &\quad = \Phi^{-1}(x) \int_x^b \frac{\varphi(u - x)}{u - x} \Phi(u) W^{\sigma}_{\eta,\lambda}[\omega(u - x)^\eta] f(u) \, du \qquad (x < b). \end{aligned} \tag{17}$$

Inspired by the above Definition 6, we will now introduce a new weighted generalized fractional integral operator in Definition 7 below.

Definition 7. *The weighted generalized fractional integral operators are defined below for a given positive continuous function* f *defined on an interval* $[a, b]$*, along with an increasing positive function* $Q : [a, b] \longrightarrow [0, \infty)$ *having a continuous derivative* Q'*, and a positive function* Φ *that has* *an inverse on* $[a,b]$*. We assume also that* $a \geqq 0$ *and define the operators for* $\lambda, \eta > 0$ *and* $\omega \in \mathbb{R}$ *as follows:*

$$\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}f\right)(x) = \Phi^{-1}(x)\int_a^x \frac{\varphi\big(Q(x)-Q(u)\big)}{Q(x)-Q(u)}\Phi(u) \cdot W^{\sigma}_{\eta,\lambda}[\omega(Q(x)-Q(u))^{\eta}]Q'(u)f(u)\,du \quad (x > a) \tag{18}$$

and

$$\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,b^-;\omega}f\right)(x) = \Phi^{-1}(x)\int_x^b \frac{\varphi\big(Q(u)-Q(x)\big)}{Q(u)-Q(x)}\Phi(u) \cdot W^{\sigma}_{\eta,\lambda}[\omega(Q(u)-Q(x))^{\eta}]Q'(u)f(u)\,du \quad (x < b). \tag{19}$$

Remark 1.

Taking $Q(u) = u$ *for all* $u \in [a,b]$ *in Definition 7, we obtain Definition 6.*

Taking $Q(u) = u$ *and* $\Phi(u) \equiv 1$ *for all* $u \in [a,b]$ *in Definition 7, we obtain Definition 5.*

Remark 2. *Some important special cases of Definition 7, in which the function* $\varphi : [0,\infty) \to [0,\infty)$ *satisfies the conditions (8) to (11) are recorded here as follows:*

(I) *Let* $\varphi(u) = u$*. Then*

$$\left({}^{Q}C^{\Phi}_{\sigma,\eta,\lambda,a^+;\omega}f\right)(x) = \Phi^{-1}(x)\int_a^x \Phi(u)W^{\sigma}_{\eta,\lambda}[\omega(Q(x)-Q(u))^{\eta}]Q'(u)f(u)\,du \qquad (x > a) \tag{20}$$

and

$$\left({}^{Q}C^{\Phi}_{\sigma,\eta,\lambda,b^-;\omega}f\right)(x) = \Phi^{-1}(x)\int_x^b \Phi(u)W^{\sigma}_{\eta,\lambda}[\omega(Q(u)-Q(x))^{\eta}]Q'(u)f(u)\,du \qquad (x < b), \tag{21}$$

for all $u \in [a,b]$*, where* $a \geqq 0$*.*

(II) *Let* $\varphi(u) = \frac{u^{\alpha}}{\Gamma(\alpha)}$ *and* $\alpha \in (0,1]$*. Then*

$$\left({}^{Q}C^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}f\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_a^x [Q(x)-Q(u)]^{\alpha-1}\Phi(u) \cdot W^{\sigma}_{\eta,\lambda}[\omega(Q(x)-Q(u))^{\eta}]Q'(u)f(u)\,du \tag{22}$$

and

$$\left({}^{Q}C^{\alpha,\Phi}_{\sigma,\eta,\lambda,b^-;\omega}f\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_x^b [Q(u)-Q(x)]^{\alpha-1}\Phi(u) \cdot W^{\sigma}_{\eta,\lambda}[\omega(Q(u)-Q(x))^{\eta}]Q'(u)f(u)\,du, \tag{23}$$

for all $u \in [a,b]$*, where* $a \geqq 0$*.*

(III) *For* $\varphi(u) = \frac{u^{\alpha}}{\Gamma(\alpha)}$*,* $Q(u) = u$ *and* $\alpha \in (0,1]$*, we have*

$$\left(C^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}f\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_a^x (x-u)^{\alpha-1}\Phi(u)W^{\sigma}_{\eta,\lambda}[\omega(x-u)^{\eta}]f(u)\,du \quad (x > a) \tag{24}$$

and

$$\left(\mathsf{C}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{b}^-;\omega}\mathrm{f}\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_x^{\mathrm{b}}(\mathrm{u}-x)^{\alpha-1}\Phi(\mathrm{u})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{u}-x)^{\eta}]\mathrm{f}(\mathrm{u})\,\mathrm{du} \quad (x<\mathrm{b}), \tag{25}$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where* $\mathrm{a}\geqq 0$.

(IV) *For* $\varphi(\mathrm{u})=\frac{\mathrm{u}^{\alpha}}{\Gamma(\alpha)}$, $\mathrm{Q}(\mathrm{u})=\ln\mathrm{u}$ *and* $\alpha\in(0,1]$, *we have*

$$\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_{\mathrm{a}}^{x}(\ln x-\ln\mathrm{u})^{\alpha-1}\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\ln x-\ln\mathrm{u})^{\eta}]\mathrm{f}(\mathrm{u})\,\frac{\mathrm{du}}{\mathrm{u}} \quad (x>\mathrm{a}) \tag{26}$$

and

$$\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{b}^-;\omega}\mathrm{f}\right)(x) = \frac{\Phi^{-1}(x)}{\Gamma(\alpha)}\int_x^{\mathrm{b}}(\ln\mathrm{u}-\ln x)^{\alpha-1}\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\ln\mathrm{u}-\ln x)^{\eta}]\mathrm{f}(\mathrm{u})\,\frac{\mathrm{du}}{\mathrm{u}} \quad (x<\mathrm{b}), \tag{27}$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where* $\mathrm{a}\geqq 0$.

(V) *Taking* $\varphi(\mathrm{u})=\mathrm{u}(\mathrm{b}-\mathrm{u})^{\alpha-1}$ *and* $\alpha\in(0,1]$, *we have*

$$\left({}^{\mathrm{Q}}\mathsf{T}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}\right)(x) = \Phi^{-1}(x)\int_{\mathrm{a}}^{x}[\mathrm{Q}(\mathrm{u})+\mathrm{Q}(\mathrm{b})-\mathrm{Q}(x)]^{\alpha-1}\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(x)-\mathrm{Q}(\mathrm{u}))^{\eta}]\mathrm{f}(\mathrm{u})\mathrm{Q}'(\mathrm{u})\mathrm{du} \tag{28}$$

and

$$\left({}^{\mathrm{Q}}\mathsf{T}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{b}^-;\omega}\mathrm{f}\right)(x) = \Phi^{-1}(x)\int_x^{\mathrm{b}}[\mathrm{Q}(x)+\mathrm{Q}(\mathrm{b})-\mathrm{Q}(\mathrm{u})]^{\alpha-1}\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(x))^{\eta}]\mathrm{f}(\mathrm{u})\mathrm{Q}'(\mathrm{u})\mathrm{du}, \tag{29}$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where* $\mathrm{a}\geqq 0$.

(VI) *Choosing* $\varphi(\mathrm{u})=\frac{\mathrm{u}}{\alpha}\exp(-\mathcal{M}\mathrm{u})$ *and* $\alpha\in(0,1]$, *we obtain*

$$\left({}^{\mathrm{Q}}\mathsf{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}\right)(x) = \frac{\Phi^{-1}(x)}{\alpha}\int_{\mathrm{a}}^{x}\exp(-\mathcal{M}(\mathrm{Q}(x)-\mathrm{Q}(\mathrm{u})))\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(x)-\mathrm{Q}(\mathrm{u}))^{\eta}]\mathrm{f}(\mathrm{u})\mathrm{Q}'(\mathrm{u})\mathrm{du} \tag{30}$$

and

$$\left({}^{\mathrm{Q}}\mathsf{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{b}^-;\omega}\mathrm{f}\right)(x) = \frac{\Phi^{-1}(x)}{\alpha}\int_x^{\mathrm{b}}\exp\Big(-\mathcal{M}\big(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(x)\big)\Big)\Phi(\mathrm{u}) \cdot \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(x))^{\eta}]\mathrm{f}(\mathrm{u})\mathrm{Q}'(\mathrm{u})\mathrm{du}, \tag{31}$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where*

$$\mathrm{a}\geqq 0 \qquad \textit{and} \qquad \mathcal{M}=\frac{1-\alpha}{\alpha}.$$

We now explore some more special cases of Definition 7 by applying different conditions of $\Phi(\mathrm{u})$ and $\varphi(\mathrm{u})$, where the function $\varphi:[0,\infty)\to[0,\infty)$ satisfies the conditions (8) to (11).

Remark 3.

(I) *Taking* $\Phi(\mathsf{u}) \equiv 1$ *and* $\varphi(\mathsf{u}) = \mathsf{u}$, *we have*

$$\left({}^{\mathsf{Q}}\mathsf{C}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x) = \int_{\mathsf{a}}^{x} \mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(x)-\mathsf{Q}(\mathsf{u}))^{\eta}]\mathsf{Q}'(\mathsf{u})\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x > \mathsf{a}) \tag{32}$$

and

$$\left(\mathsf{C}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \int_{x}^{\mathsf{b}} \mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(x))^{\eta}]\mathsf{Q}'(\mathsf{u})\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \quad (x < \mathsf{b}), \tag{33}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

(II) *For* $\Phi(\mathsf{u}) \equiv 1$ *and* $\mathsf{Q}(\mathsf{u}) = \mathsf{u}$, *we have*

$$\left(\mathsf{C}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x) = \int_{\mathsf{a}}^{x} \frac{\varphi(x-\mathsf{u})}{x-\mathsf{u}}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(x-\mathsf{u})^{\eta}]\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x > \mathsf{a}) \tag{34}$$

and

$$\left(\mathsf{C}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \int_{x}^{\mathsf{b}} \frac{\varphi(\mathsf{u}-x)}{\mathsf{u}-x}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{u}-x)^{\eta}]\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x < \mathsf{b}), \tag{35}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

(III) *For* $\Phi(\mathsf{u}) \equiv 1$ *and* $\varphi(\mathsf{u}) = \frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)}$ *and* $\alpha \in (0,1]$, *we have*

$$\begin{aligned}\left({}^{\mathsf{Q}}\mathsf{C}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathsf{a}}^{x}&[\mathsf{Q}(x)-\mathsf{Q}(\mathsf{u})]^{\alpha-1}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(x)-\mathsf{Q}(\mathsf{u}))^{\eta}]\\ &\cdot \mathsf{Q}'(\mathsf{u})\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x > \mathsf{a})\end{aligned} \tag{36}$$

and

$$\begin{aligned}\left({}^{\mathsf{Q}}\mathsf{C}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathsf{b}}&[\mathsf{Q}(\mathsf{u})-\mathsf{Q}(x)]^{\alpha-1}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(x))^{\eta}]\\ &\cdot \mathsf{Q}'(\mathsf{u})\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x < \mathsf{b}),\end{aligned} \tag{37}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

(IV) *For* $\Phi(\mathsf{u}) \equiv 1$, $\mathsf{Q}(\mathsf{u}) = \mathsf{u}$, $\varphi(\mathsf{u}) = \frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)}$ *and* $\alpha \in (0,1]$, *we have*

$$\left(\mathsf{C}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathsf{a}}^{x}(x-\mathsf{u})^{\alpha-1}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(x-\mathsf{u})^{\eta}]\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x > \mathsf{a}) \tag{38}$$

and

$$\left(\mathsf{C}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathsf{b}}(\mathsf{u}-x)^{\alpha-1}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{u}-x)^{\eta}]\mathsf{f}(\mathsf{u})\,\mathsf{d}\mathsf{u} \qquad (x < \mathsf{b}), \tag{39}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

(V) *For* $\Phi(\mathsf{u}) \equiv 1$, $\mathsf{Q}(\mathsf{u}) = \ln \mathsf{u}$, $\varphi(\mathsf{u}) = \frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)}$ *and* $\alpha \in (0,1]$, *we have*

$$\begin{aligned}&\left(\mathsf{K}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x)\\ &\quad= \frac{1}{\Gamma(\alpha)}\int_{\mathsf{a}}^{x}(\ln x-\ln \mathsf{u})^{\alpha-1}\mathsf{W}^{\sigma}_{\eta,\lambda}[\omega(\ln x-\ln \mathsf{u})^{\eta}]\mathsf{f}(\mathsf{u})\,\frac{\mathsf{d}\mathsf{u}}{\mathsf{u}} \qquad (x > \mathsf{a})\end{aligned} \tag{40}$$

and

$$\left(\mathsf{K}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\mathsf{b}} (\ln \mathsf{u} - \ln x)^{\alpha-1} W^{\sigma}_{\eta,\lambda}[\omega(\ln \mathsf{u} - \ln x)^{\eta}]\mathsf{f}(\mathsf{u})\,\frac{d\mathsf{u}}{\mathsf{u}} \quad (x < \mathsf{b}), \tag{41}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

(VI) *For* $\Phi(\mathsf{u}) \equiv 1$, $\mathsf{Q}(\mathsf{u}) = \mathsf{u}$, $\varphi(\mathsf{u}) = \frac{\mathsf{u}}{\alpha}\exp(-\mathcal{M}\mathsf{u})$ *and* $\alpha \in (0,1]$, *we have*

$$\left(\mathrm{E}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}\right)(x) = \frac{1}{\alpha}\int_{\mathsf{a}}^{x} \exp\big(-\mathcal{M}(x-\mathsf{u})\big) W^{\sigma}_{\eta,\lambda}[\omega(x-\mathsf{u})^{\eta}]\mathsf{f}(\mathsf{u})\,d\mathsf{u} \qquad (x > \mathsf{a}) \tag{42}$$

and

$$\left(\mathrm{E}^{\alpha}_{\sigma,\eta,\lambda,\mathsf{b}^-;\omega}\mathsf{f}\right)(x) = \frac{1}{\alpha}\int_{x}^{\mathsf{b}} \exp\big(-\mathcal{M}(\mathsf{u}-x)\big) W^{\sigma}_{\eta,\lambda}[\omega(\mathsf{u}-x)^{\eta}]\mathsf{f}(\mathsf{u})\,d\mathsf{u} \qquad (x < \mathsf{b}), \tag{43}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

3. Main Results

We make the following assumptions for the rest of this paper: $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a set of non-negative real numbers and $\varphi : [0,\infty) \to [0,\infty)$ satisfies Conditions (8) to (11). Additionally, p, q, v and V are positive real numbers with $\mathsf{v} < \mathrm{V}$, and Φ is a positive function with an inverse Φ^{-1}.

Theorem 1. *Assuming that* $\mathsf{Q} : [\mathsf{a},\mathsf{b}] \longrightarrow [0,\infty)$ *is an increasing positive function with a continuous derivative* Q' *on* $\mathsf{a} \geqq 0$, *and* $p \geqq 1$, $\lambda, \eta > 0$ *and* $\omega \in \mathbb{R}$, *consider the positive functions* f_1 *and* f_2 *defined on* $[\mathsf{a},\infty)$ *such that*

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u}) < \infty \qquad \text{and} \qquad \left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u}) < \infty$$

for all $\mathsf{u} > \mathsf{a}$. *If*

$$0 < \mathsf{v} \leqq \frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V},$$

where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, *then*

$$\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}} \leqq \frac{1+\mathrm{V}(\mathsf{v}+2)}{(\mathsf{v}+1)(\mathrm{V}+1)}\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathsf{f}_1+\mathsf{f}_2)^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}}.$$

Proof. Using the condition $\frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V}$, where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, we have

$$(\mathrm{V}+1)^{p}\mathsf{f}_1{}^{p}(\mathsf{z}) \leqq \mathrm{V}^{p}(\mathsf{f}_1+\mathsf{f}_2)^{p}(\mathsf{z}). \tag{44}$$

Upon multiplication of both sides of (44) by

$$\Phi^{-1}(\mathsf{u})\frac{\varphi\big(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})\big)}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})}\Phi(\mathsf{z})W^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}]\mathsf{Q}'(\mathsf{z}),$$

with $r \in (\mathsf{a}, \mathsf{u})$, we find that

$$(\mathrm{V}+1)^p \Phi^{-1}(\mathsf{u}) \frac{\varphi(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})} \Phi(\mathsf{z}) \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}] \mathsf{Q}'(\mathsf{z}) \mathsf{f_1}^p(\mathsf{z})$$
$$\leqq \mathrm{V}^p \Phi^{-1}(\mathsf{u}) \frac{\varphi(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})} \Phi(\mathsf{z}) \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}] \mathsf{Q}'(\mathsf{z}) (\mathsf{f_1}+\mathsf{f_2})^p(\mathsf{z}).$$

Consequently, upon integration over $\mathsf{z} \in (\mathsf{a}, \mathsf{u})$, we get

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} \mathsf{f_1}^p\right)(\mathsf{u}) \leqq \left(\frac{\mathrm{V}}{\mathrm{V}+1}\right)^p \left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} (\mathsf{f_1}+\mathsf{f_2})^p\right)(\mathsf{u}).$$

Hence, clearly, we obtain

$$\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} \mathsf{f_1}^p\right)(\mathsf{u})\right]^{\frac{1}{p}} \leqq \frac{\mathrm{V}}{\mathrm{V}+1} \left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} (\mathsf{f_1}+\mathsf{f_2})^p\right)(\mathsf{u})\right]^{\frac{1}{p}}. \tag{45}$$

In a similar way as above and using the condition $\mathsf{v} \leqq \frac{\mathsf{f_1}(\mathsf{z})}{\mathsf{f_2}(\mathsf{z})}$, where $\mathsf{z} \in (\mathsf{a}, \mathsf{u})$, we have

$$\left(1+\frac{1}{\mathsf{v}}\right) \mathsf{f_2}(\mathsf{z}) \leqq \frac{1}{\mathsf{v}} (\mathsf{f_1}+\mathsf{f_2})(\mathsf{z}).$$

Consequently, we get

$$\left(1+\frac{1}{\mathsf{v}}\right)^p \mathsf{f_2}^p(\mathsf{z}) \leqq \left(\frac{1}{\mathsf{v}}\right)^p (\mathsf{f_1}+\mathsf{f_2})^p(\mathsf{z}), \tag{46}$$

which, upon multiplication of both sides of

$$\Phi^{-1}(\mathsf{u}) \frac{\varphi(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})} \Phi(\mathsf{z}) \mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}] \mathsf{Q}'(\mathsf{z}),$$

with $r \in (\mathsf{a}, \mathsf{u})$, and integrating the obtained results over $\mathsf{z} \in (\mathsf{a}, \mathsf{u})$, yields

$$\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} \mathsf{f_2}^p\right)(\mathsf{u})\right]^{\frac{1}{p}} \leqq \frac{1}{\mathsf{v}+1} \left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} (\mathsf{f_1}+\mathsf{f_2})^p\right)(\mathsf{u})\right]^{\frac{1}{p}}. \tag{47}$$

Finally, by adding the inequalities in the Equations (45) and (47), we complete the proof of Theorem 1. □

Corollary 1. *Suppose that the assumptions of Theorem 1 are satisfied with*

$$\mathsf{Q}(\mathsf{u}) = \ln \mathsf{u}, \quad \varphi(\mathsf{u}) = \frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)} \quad \textit{and} \quad \alpha \in (0,1].$$

Then

$$\left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} \mathsf{f_1}^p\right)(\mathsf{u})\right]^{\frac{1}{p}} + \left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} \mathsf{f_2}^p\right)(\mathsf{u})\right]^{\frac{1}{p}}$$
$$\leqq \frac{1+\mathrm{V}(\mathsf{v}+2)}{(\mathsf{v}+1)(\mathrm{V}+1)} \left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega} (\mathsf{f_1}+\mathsf{f_2})^p\right)(\mathsf{u})\right]^{\frac{1}{p}},$$

for all $\mathsf{u} \in [\mathsf{a}, \mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

Corollary 2. *Suppose that the conditions of Theorem 1 are satisfied with*

$$\varphi(\mathsf{u}) = \frac{\mathsf{u}}{\alpha} \exp(-\mathcal{M}\mathsf{u}) \quad \textit{and} \quad \alpha \in (0,1].$$

Then

$$\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}+\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq\frac{1+\mathrm{V}(\mathrm{v}+2)}{(\mathrm{v}+1)(\mathrm{V}+1)}\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}},$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where*

$$\mathrm{a}\geqq 0\qquad\textit{and}\qquad\mathcal{M}=\frac{1-\alpha}{\alpha}.$$

We now state and prove Theorem 2 below.

Theorem 2. *Assuming that* $\mathrm{Q}:[\mathrm{a},\mathrm{b}]\longrightarrow[0,\infty)$ *is an increasing positive function with a continuous derivative* Q' *and that* $\mathrm{a}\geqq 0$, $p\geqq 1$, $\lambda,\eta>0$ *and* $\omega\in\mathbb{R}$, *let* $\mathrm{f_1}$ *and* $\mathrm{f_2}$ *be positive functions on* $[\mathrm{a},\infty)$ *such that*

$$\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})<\infty\qquad\textit{and}\qquad\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})<\infty$$

for all $\mathrm{u}>\mathrm{a}$. *If*

$$0<\mathrm{v}\leqq\frac{\mathrm{f_1}(\mathrm{z})}{\mathrm{f_2}(\mathrm{z})}\leqq\mathrm{V},$$

where $\mathrm{z}\in(\mathrm{a},\mathrm{u})$, *then*

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{2}{p}}+\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{2}{p}}$$
$$\geqq\left(\frac{(\mathrm{v}+1)(\mathrm{V}+1)}{\mathrm{V}}-2\right)\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}.$$

Proof. By multiplying the inequalities (45) and (47), we obtain

$$\frac{(\mathrm{v}+1)(\mathrm{V}+1)}{\mathrm{V}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq\left(\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\right)^2.\tag{48}$$

Next, by using the hypotheses of the Minkowski inequality to the right-hand side of (48), we have

$$\left(\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\right)^2$$
$$\leqq\left(\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}+\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\right)^2,$$

which implies that

$$\left(\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\right)^2$$
$$\leqq\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{2}{p}}+\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{2}{p}}$$
$$+2\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}.\tag{49}$$

In view of the above inequalities (48) and (49), the proof of Theorem 2 is complete. □

Corollary 3. *Under the hypotheses of Theorem 2 with*

$$\mathsf{Q}(\mathsf{u}) = \ln \mathsf{u}, \quad \varphi(\mathsf{u}) = \frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)} \qquad \textit{and} \qquad \alpha \in (0,1],$$

it is asserted that

$$\begin{aligned}&\left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u})\right]^{\frac{2}{p}} + \left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u})\right]^{\frac{2}{p}} \\ &\qquad \geqq \left(\frac{(\mathsf{v}+1)(\mathrm{V}+1)}{\mathrm{V}} - 2\right)\left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}}\left[\left(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}},\end{aligned}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a} \geqq 0$.

Corollary 4. *Suppose that the assumptions of Theorem 2 with*

$$\varphi(\mathsf{u}) = \frac{\mathsf{u}}{\alpha}\exp(-\mathcal{M}\mathsf{u}) \qquad \textit{and} \qquad \alpha \in (0,1],$$

are satisfied. Then

$$\begin{aligned}&\left[\left(\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u})\right]^{\frac{2}{p}} + \left[\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u})\right]^{\frac{2}{p}} \\ &\qquad \geqq \left(\frac{(\mathsf{v}+1)(\mathrm{V}+1)}{\mathrm{V}} - 2\right)\left[\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^{p}\right)(\mathsf{u})\right]^{\frac{1}{p}},\end{aligned}$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where*

$$\mathsf{a} \geqq 0 \qquad \textit{and} \qquad \mathcal{M} = \frac{1-\alpha}{\alpha}.$$

Theorem 3. *Assume that* $\mathsf{Q} : [\mathsf{a},\mathsf{b}] \longrightarrow [0,\infty)$ *is an increasing positive function with a continuous derivative* Q', *and let* $\mathsf{a} \geqq 0$,

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (p > 1),$$

$\lambda, \eta > 0$, *and* $\omega \in \mathbb{R}$. *Also let* f_1 *and* f_2 *be positive functions defined on* $[\mathsf{a},\infty)$ *such that*

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1\right)(\mathsf{u}) < \infty \qquad \textit{and} \qquad \left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2\right)(\mathsf{u}) < \infty,$$

for all $\mathsf{u} > \mathsf{a}$. *If*

$$0 < \mathsf{v} \leqq \frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V},$$

where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, *then*

$$\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1\right)(\mathsf{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2\right)(\mathsf{u})\right]^{\frac{1}{q}} \leqq \left(\frac{\mathrm{V}}{\mathsf{v}}\right)^{\frac{1}{pq}}\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{\frac{1}{p}}\mathsf{f}_2{}^{\frac{1}{q}}\right)(\mathsf{u}).$$

Proof. Since $\frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V}$, where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, we have

$$[\mathsf{f}_2(\mathsf{z})]^{\frac{1}{q}} \geqq \mathrm{V}^{-\frac{1}{q}}[\mathsf{f}_1(\mathsf{z})]^{\frac{1}{q}}.$$

It follows that

$$[\mathsf{f}_1(\mathsf{z})]^{\frac{1}{p}}[\mathsf{f}_2(\mathsf{z})]^{\frac{1}{q}} \geqq \mathrm{V}^{-\frac{1}{q}}[\mathsf{f}_1(\mathsf{z})]^{\frac{1}{p}}[\mathsf{f}_1(\mathsf{z})]^{\frac{1}{q}} = \mathrm{V}^{-\frac{1}{q}}\mathsf{f}_1(\mathsf{z}). \tag{50}$$

Upon multiplication of both sides of (50) by

$$\Phi^{-1}(\mathrm{u})\frac{\varphi(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))}{\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z})}\Phi(\mathrm{z})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))^{\eta}]\mathrm{Q}'(\mathrm{z}),$$

with $r \in (\mathrm{a},\mathrm{u})$, if we integrate the obtained results over $\mathrm{z} \in (\mathrm{a},\mathrm{u})$, we have

$$\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^{\frac{1}{p}}\mathrm{f}_2{}^{\frac{1}{q}}\right)(\mathrm{u}) \geqq \mathrm{V}^{-\frac{1}{q}}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1\right)(\mathrm{u}).$$

Consequently, we obtain

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^{\frac{1}{p}}\mathrm{f}_2{}^{\frac{1}{q}}\right)(\mathrm{u})\right]^{\frac{1}{p}} \geqq \mathrm{V}^{-\frac{1}{pq}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1\right)(\mathrm{u})\right]^{\frac{1}{p}}. \tag{51}$$

Thus, if $\mathrm{v}\mathrm{f}_2(\mathrm{z}) \leqq \mathrm{f}_1(\mathrm{z})$, where $\mathrm{z} \in (\mathrm{a},\mathrm{u})$, then

$$[\mathrm{f}_1(\mathrm{z})]^{\frac{1}{p}} \geqq \mathrm{v}^{\frac{1}{p}}[\mathrm{f}_2(\mathrm{z})]^{\frac{1}{p}},$$

which readily yields

$$[\mathrm{f}_1(\mathrm{z})]^{\frac{1}{p}}[\mathrm{f}_2(\mathrm{z})]^{\frac{1}{q}} \geqq \mathrm{v}^{\frac{1}{p}}[\mathrm{f}_2(\mathrm{z})]^{\frac{1}{p}}[\mathrm{f}_2(\mathrm{z})]^{\frac{1}{q}} = \mathrm{v}^{\frac{1}{p}}\mathrm{f}_2(\mathrm{z}). \tag{52}$$

We now apply both sides of (52) by

$$\Phi^{-1}(\mathrm{u})\frac{\varphi(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))}{\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z})}\Phi(\mathrm{z})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))^{\eta}]\mathrm{Q}'(\mathrm{z}),$$

with $r \in (\mathrm{a},\mathrm{u})$. If we then integrate the obtained results over r, we have

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^{\frac{1}{p}}\mathrm{f}_2{}^{\frac{1}{q}}\right)(\mathrm{u})\right]^{\frac{1}{q}} \geqq \mathrm{v}^{\frac{1}{pq}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2\right)(\mathrm{u})\right]^{\frac{1}{q}}. \tag{53}$$

Finally, multiplying the above inequalities (51) and (53), we get the desired result asserted by Theorem 3. □

Corollary 5. *Let the assumptions of Theorem 3 with*

$$\mathrm{Q}(\mathrm{u}) = \ln \mathrm{u}, \quad \varphi(\mathrm{u}) = \frac{\mathrm{u}^{\alpha}}{\Gamma(\alpha)} \qquad \textit{and} \qquad \alpha \in (0,1],$$

be satisfied. Then

$$\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2\right)(\mathrm{u})\right]^{\frac{1}{q}} \leqq \left(\frac{\mathrm{V}}{\mathrm{v}}\right)^{\frac{1}{pq}}\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^{\frac{1}{p}}\mathrm{f}_2{}^{\frac{1}{q}}\right)(\mathrm{u}),$$

for all $\mathrm{u} \in [\mathrm{a},\mathrm{b}]$, *where* $\mathrm{a} \geqq 0$.

Corollary 6. *If the conditions of Theorem 3 with*

$$\varphi(\mathrm{u}) = \frac{\mathrm{u}}{\alpha}\exp(-\mathcal{M}\mathrm{u}) \qquad \textit{and} \qquad \alpha \in (0,1],$$

are satisfied, then

$$\left[\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1\right)(\mathsf{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2\right)(\mathsf{u})\right]^{\frac{1}{q}} \leqq \left(\frac{\mathrm{V}}{\mathsf{v}}\right)^{\frac{1}{pq}}\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^{\frac{1}{p}}\mathsf{f}_2{}^{\frac{1}{q}}\right)(\mathsf{u}),$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where*

$$\mathsf{a} \geqq 0 \qquad \textit{and} \qquad \mathcal{M} = \frac{1-\alpha}{\alpha}.$$

Theorem 4. *Suppose that* $\mathsf{Q} : [\mathsf{a},\mathsf{b}] \longrightarrow [0,\infty)$ *is an increasing positive function with a continuous derivative* Q' *and* $\mathsf{a} \geqq 0$,

$$\frac{1}{p}+\frac{1}{q} = 1 \qquad (p>1),$$

$\lambda, \eta > 0$ *and* $\omega \in \mathbb{R}$. *Let* f_1 *and* f_2 *be positive functions defined on* $[\mathsf{a},\infty)$ *satisfying the following conditions*:

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^p\right)(\mathsf{u}) < \infty \qquad \textit{and} \qquad \left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_2{}^q\right)(\mathsf{u}) < \infty,$$

for all $\mathsf{u} > \mathsf{a}$. *If*

$$0 < \mathsf{v} \leqq \frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V},$$

where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, *then*

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1\mathsf{f}_2\right)(\mathsf{u}) \leqq \frac{2^{p-1}\mathrm{V}^p}{p(\mathrm{V}+1)^p}\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathsf{f}_1{}^p+\mathsf{f}_2{}^p)\right)(\mathsf{u}) + \frac{2^{q-1}}{q(\mathsf{v}+1)^q}\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathsf{f}_1{}^q+\mathsf{f}_2{}^q)\right)(\mathsf{u}).$$

Proof. Since $\frac{\mathsf{f}_1(\mathsf{z})}{\mathsf{f}_2(\mathsf{z})} \leqq \mathrm{V}$, where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, we have

$$(\mathrm{V}+1)^p[\mathsf{f}_1(\mathsf{z})]^p \leqq \mathrm{V}^p[\mathsf{f}_1(\mathsf{z})+\mathsf{f}_2(\mathsf{z})]^p. \tag{54}$$

Upon multiplication of both sides of (54) by

$$\Phi^{-1}(\mathsf{u})\frac{\varphi\big(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})\big)}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})}\Phi(\mathsf{z})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}]\mathsf{Q}'(\mathsf{z}),$$

with $r \in (\mathsf{a},\mathsf{u})$, if we integrate the obtained results over $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, we get

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathsf{f}_1{}^p\right)(\mathsf{u}) \leqq \frac{\mathrm{V}^p}{(\mathrm{V}+1)^p}\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathsf{f}_1+\mathsf{f}_2)^p\right)(\mathsf{u}). \tag{55}$$

Consequently, if $\mathsf{v}\mathsf{f}_2(\mathsf{z}) \leqq \mathsf{f}_1(\mathsf{z})$, where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, then

$$(\mathsf{v}+1)^q[\mathsf{f}_2(\mathsf{z})]^q \leqq [\mathsf{f}_1(\mathsf{z})+\mathsf{f}_2(\mathsf{z})]^q. \tag{56}$$

We first multiply both sides of (56) by

$$\Phi^{-1}(\mathsf{u})\frac{\varphi\big(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})\big)}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})}\Phi(\mathsf{z})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}]\mathsf{Q}'(\mathsf{z}),$$

with $r \in (a, u)$ and then integrate the obtained results over $z \in (a, u)$. We thus find that

$$\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} {f_2}^q\right)(u) \leqq \frac{1}{(v+1)^q}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}(f_1+f_2)^q\right)(u). \tag{57}$$

Now, in view of Young's inequality, we have

$$f_1(z)f_2(z) \leqq \frac{[f_1(z)]^p}{p} + \frac{[f_2(z)]^q}{q}. \tag{58}$$

Upon multiplication of both sides of (58) by

$$\Phi^{-1}(u)\frac{\varphi\big(Q(u)-Q(z)\big)}{Q(u)-Q(z)}\Phi(z)W^{\sigma}_{\eta,\lambda}[\omega(Q(u)-Q(z))^{\eta}]Q'(z),$$

with $r \in (a, u)$, and then integrating the obtained results over $z \in (a, u)$, we obtain

$$\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} f_1 f_2\right)(u) \leqq \frac{1}{p}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} {f_1}^p\right)(u) + \frac{1}{q}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} {f_2}^q\right)(u). \tag{59}$$

Substituting from the above inequalities (55) and (57) into (59), we have

$$\begin{aligned}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} f_1 f_2\right)(u) \leqq{}& \frac{V^p}{p(V+1)^p}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}(f_1+f_2)^p\right)(u) \\ &+ \frac{1}{q(v+1)^q}\left({}^{Q}J^{\varphi,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}(f_1+f_2)^q\right)(u).\end{aligned} \tag{60}$$

Finally, since

$$(\mu_1+\mu_2)^{\lambda} \leqq 2^{\lambda-1}\left(\mu_1^{\lambda}+\mu_2^{\lambda}\right) \qquad (\lambda > 1;\ \mu_1, \mu_2 > 0),$$

in (60), we get the desired result asserted by Theorem 4. □

Corollary 7. *Suppose that the conditions of Theorem 4 with*

$$Q(u) = \ln u, \quad \varphi(u) = \frac{u^{\alpha}}{\Gamma(\alpha)} \qquad \text{and} \qquad \alpha \in (0, 1],$$

are satisfied. Then

$$\begin{aligned}\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} f_1 f_2\right)(u) \leqq{}& \frac{2^{p-1}V^p}{p(V+1)^p}\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}({f_1}^p+{f_2}^p)\right)(u) \\ &+ \frac{2^{q-1}}{q(v+1)^q}\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega}({f_1}^q+{f_2}^q)\right)(u),\end{aligned}$$

for all $u \in [a, b]$, *where* $a \geqq 0$.

Corollary 8. *Assume that the conditions of Theorem 4 with*

$$\varphi(u) = \frac{u}{\alpha}\exp(-\mathcal{M}u) \qquad \text{and} \qquad \alpha \in (0, 1],$$

are satisfied. Then

$$\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathrm{f}_1\mathrm{f}_2\right)(\mathsf{u}) \leqq \frac{2^{p-1}\mathrm{V}^p}{p(\mathrm{V}+1)^p}\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathrm{f}_1{}^p+\mathrm{f}_2{}^p)\right)(\mathsf{u}) + \frac{2^{q-1}}{q(\mathsf{v}+1)^q}\left({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathrm{f}_1{}^q+\mathrm{f}_2{}^q)\right)(\mathsf{u}),$$

for all $\mathsf{u} \in [\mathsf{a},\mathsf{b}]$, *where*

$$\mathsf{a} \geqq 0 \qquad \textit{and} \qquad \mathcal{M} = \frac{1-\alpha}{\alpha}.$$

Theorem 5. *Let* $\mathsf{Q} : [\mathsf{a},\mathsf{b}] \longrightarrow [0,\infty)$ *be an increasing positive function having a continuous derivative* Q' *with* $\mathsf{a} \geqq 0$, $p \geqq 1$, $\lambda,\eta > 0$ *and* $\omega \in \mathbb{R}$. *Assume that* f_1 *and* f_2 *are positive functions on* $[\mathsf{a},\infty)$ *such that*

$$\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathsf{u}) < \infty \qquad \textit{and} \qquad \left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathsf{u}) < \infty,$$

for all $\mathsf{u} > \mathsf{a}$. *If*

$$0 < \mathrm{g} < \mathsf{v} \leqq \frac{\mathrm{f}_1(\mathsf{z})}{\mathrm{f}_2(\mathsf{z})} \leqq \mathrm{V},$$

where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, *then*

$$\begin{aligned}\left(\frac{\mathrm{V}+1}{\mathrm{V}-\mathrm{g}}\right)&\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^p\right)(\mathsf{u})\right]^{\frac{1}{p}}\\ &\leqq \left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathsf{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathsf{u})\right]^{\frac{1}{p}}\\ &\leqq \left(\frac{\mathsf{v}+1}{\mathsf{v}-\mathrm{g}}\right)\left[\left({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^+;\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^p\right)(\mathsf{u})\right]^{\frac{1}{p}}.\end{aligned}$$

Proof. Since

$$0 < \mathrm{g} < \mathsf{v} \leqq \frac{\mathrm{f}_1(\mathsf{z})}{\mathrm{f}_2(\mathsf{z})} \leqq \mathrm{V},$$

where $\mathsf{z} \in (\mathsf{a},\mathsf{u})$, we have

$$\mathrm{g}\mathsf{v} \leqq \mathrm{g}\mathrm{V}, \tag{61}$$

which implies that

$$\frac{\mathrm{V}+1}{\mathrm{V}-\mathrm{g}} \leqq \frac{\mathsf{v}+1}{\mathsf{v}-\mathrm{g}}.$$

Furthermore, we obtain

$$\mathsf{v}-\mathrm{g} \leqq \frac{\mathrm{f}_1(\mathsf{z})-\mathrm{g}\mathrm{f}_2(\mathsf{z})}{\mathrm{f}_2(\mathsf{z})} \leqq \mathrm{V}-\mathrm{g}.$$

Hence, clearly, we find that

$$\frac{[\mathrm{f}_1(\mathsf{z})-\mathrm{g}\mathrm{f}_2(\mathsf{z})]^p}{(\mathrm{V}-\mathrm{g})^p} \leqq [\mathrm{f}_2(\mathsf{z})]^p \leqq \frac{[\mathrm{f}_1(\mathsf{z})-\mathrm{g}\mathrm{f}_2(\mathsf{z})]^p}{(\mathsf{v}-\mathrm{g})^p}. \tag{62}$$

Moreover, we have

$$\frac{1}{\mathrm{V}} \leqq \frac{\mathrm{f}_2(\mathsf{z})}{\mathrm{f}_1(\mathsf{z})} \leqq \frac{1}{\mathsf{v}},$$

which implies that

$$\left(\frac{\mathrm{V}}{\mathrm{V}-\mathrm{g}}\right)^p[\mathrm{f}_1(\mathsf{z})-\mathrm{g}\mathrm{f}_2(\mathsf{z})]^p \leqq [\mathrm{f}_1(\mathsf{z})]^p \leqq \left(\frac{\mathsf{v}}{\mathsf{v}-\mathrm{g}}\right)^p[\mathrm{f}_1(\mathsf{z})-\mathrm{g}\mathrm{f}_2(\mathsf{z})]^p. \tag{63}$$

Upon multiplication of both sides of (62) by

$$\Phi^{-1}(\mathrm{u}) \, \frac{\varphi(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))}{\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z})} \, \Phi(\mathrm{z}) W^{\sigma}_{\eta,\lambda}\left[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))^{\eta}\right]\mathrm{Q}'(\mathrm{z}),$$

with $r \in (\mathrm{a}, \mathrm{u})$, if we then integrate the obtained results over $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, we have

$$\begin{aligned} \frac{1}{\mathrm{V}-\mathrm{g}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} &\leqq \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_2{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\leqq \frac{1}{\mathrm{v}-\mathrm{g}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}}. \end{aligned} \tag{64}$$

Again, if we first multiply both sides of (63) by

$$\Phi^{-1}(\mathrm{u}) \, \frac{\varphi(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))}{\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z})} \, \Phi(\mathrm{z}) W^{\sigma}_{\eta,\lambda}[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))^{\eta}]\mathrm{Q}'(\mathrm{z}),$$

with $r \in (\mathrm{a}, \mathrm{u})$, and then integrate the obtained results over $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, we get

$$\begin{aligned} \frac{\mathrm{V}}{\mathrm{V}-\mathrm{g}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} &\leqq \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_1{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\leqq \frac{\mathrm{v}}{\mathrm{v}-\mathrm{g}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}}. \end{aligned} \tag{65}$$

Finally, by adding the inequalities (64) and (65), the proof of Theorem 5 is completed. □

Corollary 9. *If the conditions of Theorem 5 with*

$$\mathrm{Q}(\mathrm{u}) = \ln \mathrm{u}, \quad \varphi(\mathrm{u}) = \frac{\mathrm{u}^{\alpha}}{\Gamma(\alpha)} \qquad \textit{and} \qquad \alpha \in (0,1],$$

are satisfied, then

$$\begin{aligned} &\left(\frac{\mathrm{V}+1}{\mathrm{V}-\mathrm{g}}\right)\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\quad\leqq \left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_1{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_2{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\quad\leqq \left(\frac{\mathrm{v}+1}{\mathrm{v}-\mathrm{g}}\right)\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}}, \end{aligned}$$

for all $\mathrm{u} \in [\mathrm{a}, \mathrm{b}]$, *where* $\mathrm{a} \geqq 0$.

Corollary 10. *Let the conditions of Theorem 5 with*

$$\varphi(\mathrm{u}) = \frac{\mathrm{u}}{\alpha}\exp(-\mathcal{M}\mathrm{u}) \qquad \textit{and} \qquad \alpha \in (0,1],$$

be satisfied. Then

$$\begin{aligned} &\left(\frac{\mathrm{V}+1}{\mathrm{V}-\mathrm{g}}\right)\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\quad\leqq \left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_1{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}\mathrm{f}_2{}^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\quad\leqq \left(\frac{\mathrm{v}+1}{\mathrm{v}-\mathrm{g}}\right)\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^{+};\omega}(\mathrm{f}_1-\mathrm{g}\mathrm{f}_2)^{p}\right)(\mathrm{u})\right]^{\frac{1}{p}}, \end{aligned}$$

for all $\mathrm{u} \in [\mathrm{a}, \mathrm{b}]$, *where*

$$\mathrm{a} \geqq 0 \quad \text{and} \quad \mathcal{M} = \frac{1-\alpha}{\alpha}.$$

Theorem 6. *Suppose that* $\mathrm{Q} : [\mathrm{a}, \mathrm{b}] \longrightarrow [0, \infty)$ *is an increasing positive function with a continuous derivative* Q', *where* $\mathrm{a} \geqq 0$, $p \geqq 1$, $\lambda, \eta > 0$, *and* $\omega \in \mathbb{R}$. *Let* f_1 *and* f_2 *be positive functions on* $[\mathrm{a}, \infty)$ *that satisfy the following conditions*:

$$\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathrm{u}) < \infty \quad \text{and} \quad \left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathrm{u}) < \infty,$$

for all $\mathrm{u} > \mathrm{a}$. *If*

$$0 \leqq \mu_1 \leqq \mathrm{f}_1(\mathrm{z}) \leqq \mu_2 \quad \text{and} \quad 0 \leqq \nu_1 \leqq \mathrm{f}_2(\mathrm{z}) \leqq \nu_2,$$

where $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, *then*

$$\begin{aligned} &\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \\ &\qquad \leqq \left(\frac{\mu_2(\mu_1+\nu_2)+\nu_2(\nu_1+\mu_2)}{(\mu_2+\nu_1)(\nu_2+\mu_1)}\right)\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f}_1+\mathrm{f}_2)^p\right)(\mathrm{u})\right]^{\frac{1}{p}}. \end{aligned}$$

Proof. Given the stated assumptions, it follows that

$$\frac{1}{\nu_2} \leqq \frac{1}{\mathrm{f}_2(\mathrm{z})} \leqq \frac{1}{\nu_1}. \tag{66}$$

The product of (66) with $0 \leqq \mu_1 \leqq \mathrm{f}_1(\mathrm{z}) \leqq \mu_2$ yields

$$\frac{\mu_1}{\nu_2} \leqq \frac{\mathrm{f}_1(\mathrm{z})}{\mathrm{f}_2(\mathrm{z})} \leqq \frac{\mu_2}{\nu_1}, \tag{67}$$

so that

$$[\mathrm{f}_2(\mathrm{z})]^p \leqq \left[\frac{\nu_2}{\mu_1+\nu_2}\right]^p [\mathrm{f}_1(\mathrm{z})+\mathrm{f}_2(\mathrm{z})]^p \tag{68}$$

and

$$[\mathrm{f}_1(\mathrm{z})]^p \leqq \left(\frac{\mu_2}{\nu_1+\mu_2}\right)^p [\mathrm{f}_1(\mathrm{z})+\mathrm{f}_2(\mathrm{z})]^p. \tag{69}$$

Multiplying the Equations (68) and (69) by

$$\Phi^{-1}(\mathrm{u})\,\frac{\varphi(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))}{\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z})}\,\Phi(\mathrm{z})\mathrm{W}^{\sigma}_{\eta,\lambda}\left[\omega(\mathrm{Q}(\mathrm{u})-\mathrm{Q}(\mathrm{z}))^{\eta}\right]\mathrm{Q}'(\mathrm{z}),$$

with $r \in (\mathrm{a}, \mathrm{u})$, and then integrating them over $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, we find that

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \leqq \frac{\nu_2}{\mu_1+\nu_2}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f}_1+\mathrm{f}_2)^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \tag{70}$$

and

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \leqq \frac{\mu_2}{\nu_1+\mu_2}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f}_1+\mathrm{f}_2)^p\right)(\mathrm{u})\right]^{\frac{1}{p}}. \tag{71}$$

Finally, by adding the inequalities (70) and (71), the proof of Theorem 6 is complete. □

Corollary 11. *Under the hypotheses of Theorem 6 with*

$$\mathrm{Q}(\mathrm{u}) = \ln \mathrm{u}, \quad \varphi(\mathrm{u}) = \frac{\mathrm{u}^{\alpha}}{\Gamma(\alpha)} \quad \text{and} \quad \alpha \in (0,1],$$

it is asserted that

$$\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}+\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq\left(\frac{\mu_2(\mu_1+\nu_2)+\nu_2(\nu_1+\mu_2)}{(\mu_2+\nu_1)(\nu_2+\mu_1)}\right)\left[\left(\mathrm{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}},$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where* $\mathrm{a}\geqq 0$.

Corollary 12. *Let the assumptions of Theorem 6 with*

$$\varphi(\mathrm{u})=\frac{\mathrm{u}}{\alpha}\exp(-\mathcal{M}\mathrm{u})\qquad\text{and}\qquad\alpha\in(0,1],$$

be satisfied. Then

$$\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}+\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq\left(\frac{\mu_2(\mu_1+\nu_2)+\nu_2(\nu_1+\mu_2)}{(\mu_2+\nu_1)(\nu_2+\mu_1)}\right)\left[\left({}^{\mathrm{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^p\right)(\mathrm{u})\right]^{\frac{1}{p}},$$

for all $\mathrm{u}\in[\mathrm{a},\mathrm{b}]$, *where*

$$\mathrm{a}\geqq 0\qquad\text{and}\qquad\mathcal{M}=\frac{1-\alpha}{\alpha}.$$

Theorem 7. *Let* $\mathrm{Q}:[\mathrm{a},\mathrm{b}]\longrightarrow[0,\infty)$ *be an increasing positive function having a continuous derivative* Q' *with* $\mathrm{a}\geqq 0$, $\lambda,\eta>0$ *and* $\omega\in\mathbb{R}$. *Assume that* $\mathrm{f_1}$ *and* $\mathrm{f_2}$ *are two positive functions on* $[\mathrm{a},\infty)$ *such that*

$$\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}\right)(\mathrm{u})<\infty\qquad\text{and}\qquad\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_2}\right)(\mathrm{u})<\infty,$$

for all $\mathrm{u}>\mathrm{a}$. *If*

$$0<\mathrm{v}\leqq\frac{\mathrm{f_1}(\mathrm{z})}{\mathrm{f_2}(\mathrm{z})}\leqq\mathrm{V},$$

where $\mathrm{z}\in(\mathrm{a},\mathrm{u})$, *then*

$$\frac{1}{\mathrm{V}}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}\mathrm{f_2}\right)(\mathrm{u})$$
$$\leqq\frac{1}{(\mathrm{v}+1)(\mathrm{V}+1)}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{f_1}+\mathrm{f_2})^2\right)(\mathrm{u})$$
$$\leqq\frac{1}{\mathrm{v}}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f_1}\mathrm{f_2}\right)(\mathrm{u}).$$

Proof. Since

$$0<\mathrm{v}\leqq\frac{\mathrm{f_1}(\mathrm{z})}{\mathrm{f_2}(\mathrm{z})}\leqq\mathrm{V},$$

where $\mathrm{z}\in(\mathrm{a},\mathrm{u})$, we have

$$(\mathrm{v}+1)\mathrm{f_2}(\mathrm{z})\leqq\mathrm{f_1}(\mathrm{z})+\mathrm{f_2}(\mathrm{z})\leqq(\mathrm{V}+1)\mathrm{f_2}(\mathrm{z})\tag{72}$$

and

$$\left(\frac{\mathrm{V}+1}{\mathrm{V}}\right)\mathrm{f_1}(\mathrm{z})\leqq\mathrm{f_1}(\mathrm{z})+\mathrm{f_2}(\mathrm{z})\leqq\left(\frac{\mathrm{v}+1}{\mathrm{v}}\right)\mathrm{f_1}(\mathrm{z}).\tag{73}$$

Upon multiplying the inequalities (72) and (73), we get

$$\frac{\mathrm{f_1}(\mathrm{z})\mathrm{f_2}(\mathrm{z})}{\mathrm{V}}\leqq\frac{(\mathrm{f_1}(\mathrm{z})+\mathrm{f_2}(\mathrm{z}))^2}{(\mathrm{v}+1)(\mathrm{V}+1)}\leqq\frac{\mathrm{f_1}(\mathrm{z})\mathrm{f_2}(\mathrm{z})}{\mathrm{v}}.\tag{74}$$

Again, if we multiply the inequality (74) by

$$\Phi^{-1}(\mathsf{u})\frac{\varphi\big(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})\big)}{\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z})}\Phi(\mathsf{z})\mathrm{W}^{\sigma}_{\eta,\lambda}[\omega(\mathsf{Q}(\mathsf{u})-\mathsf{Q}(\mathsf{z}))^{\eta}]\mathsf{Q}'(\mathsf{z}),$$

with $r\in(\mathsf{a},\mathsf{u})$, and integrate the obtained results over $\mathsf{z}\in(\mathsf{a},\mathsf{u})$, the proof of Theorem 7 is concluded. □

Corollary 13. *Suppose that the assumptions of Theorem 7 with*

$$\mathsf{Q}(\mathsf{u})=\ln\mathsf{u},\quad \varphi(\mathsf{u})=\frac{\mathsf{u}^{\alpha}}{\Gamma(\alpha)}\qquad \textit{and}\quad \alpha\in(0,1],$$

are satisfied. Then

$$\begin{aligned}\frac{1}{\mathrm{V}}\Big(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}\mathrm{f}_1\mathrm{f}_2\Big)(\mathsf{u}) &\leqq \frac{1}{(\mathsf{v}+1)(\mathrm{V}+1)}\Big(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}(\mathrm{f}_1+\mathrm{f}_2)^2\Big)(\mathsf{u})\\ &\leqq \frac{1}{\mathsf{v}}\Big(\mathsf{K}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}\mathrm{f}_1\mathrm{f}_2\Big)(\mathsf{u}),\end{aligned}$$

for all $\mathsf{u}\in[\mathsf{a},\mathsf{b}]$, *where* $\mathsf{a}\geqq 0$.

Corollary 14. *Assume that the conditions of Theorem 7 with*

$$\varphi(\mathsf{u})=\frac{\mathsf{u}}{\alpha}\exp(-\mathcal{M}\mathsf{u})\qquad \textit{and}\qquad \alpha\in(0,1],$$

are satisfied. Then

$$\begin{aligned}\frac{1}{\mathrm{V}}\Big({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}\mathrm{f}_1\mathrm{f}_2\Big)(\mathsf{u}) &\leqq \frac{1}{(\mathsf{v}+1)(\mathrm{V}+1)}\Big({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}(\mathrm{f}_1+\mathrm{f}_2)^2\Big)(\mathsf{u})\\ &\leqq \frac{1}{\mathsf{v}}\Big({}^{\mathsf{Q}}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}\mathrm{f}_1\mathrm{f}_2\Big)(\mathsf{u}),\end{aligned}$$

for all $\mathsf{u}\in[\mathsf{a},\mathsf{b}]$, *where*

$$\mathsf{a}\geqq 0\qquad \textit{and}\qquad \mathcal{M}=\frac{1-\alpha}{\alpha}.$$

Theorem 8. *Let* $\mathsf{Q}:[\mathsf{a},\mathsf{b}]\longrightarrow[0,\infty)$ *be an increasing positive function having a continuous derivative* Q' *with* $\mathsf{a}\geqq 0$, $p\geqq 1$, $\lambda,\eta>0$ *and* $\omega\in\mathbb{R}$. *Assume that* f_1 *and* f_2 *are positive functions on* $[\mathsf{a},\infty)$ *such that*

$$\Big({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}{\mathrm{f}_1}^{p}\Big)(\mathsf{u})<\infty\qquad \textit{and}\qquad \Big({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}{\mathrm{f}_2}^{p}\Big)(\mathsf{u})<\infty,$$

for all $\mathsf{u}>\mathsf{a}$. *If*

$$0<\mathsf{v}\leqq\frac{\mathrm{f}_1(\mathsf{z})}{\mathrm{f}_2(\mathsf{z})}\leqq\mathrm{V},$$

where $\mathsf{z}\in(\mathsf{a},\mathsf{u})$, *then*

$$\begin{aligned}&\Big[\Big({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}{\mathrm{f}_1}^{p}\Big)(\mathsf{u})\Big]^{\frac{1}{p}}+\Big[\Big({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}{\mathrm{f}_2}^{p}\Big)(\mathsf{u})\Big]^{\frac{1}{p}}\\ &\quad\leqq 2\big[\big({}^{\mathsf{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathsf{a}^{+};\omega}\Upsilon^{p}(\mathrm{f}_1,\mathrm{f}_2)\big)(\mathsf{u})\big]^{\frac{1}{p}},\end{aligned}$$

where

$$\Upsilon\big(\mathrm{f}_1(\mathsf{z}),\mathrm{f}_2(\mathsf{z})\big):=\max\left\{\left(\frac{\mathrm{V}}{\mathsf{v}}+1\right)\mathrm{f}_1(\mathsf{z})-\mathrm{V}\mathrm{f}_2(\mathsf{z}),\frac{(\mathrm{V}+\mathsf{v})\mathrm{f}_2(\mathsf{z})-\mathrm{f}_1(\mathsf{z})}{\mathsf{v}}\right\}.$$

Proof. Since

$$0 < v \leqq \frac{f_1(z)}{f_2(z)} \leqq V,$$

where $z \in (a,u)$, we have

$$0 < v \leqq V + v - \frac{f_1(z)}{f_2(z)} \leqq V. \tag{75}$$

We also find from the inequality (75) that

$$f_2(z) \leqq \frac{(V+v)f_2(z) - f_1(z)}{v} \leqq Y\big(f_1(z), f_2(z)\big). \tag{76}$$

Similarly, we have

$$\frac{1}{V} \leqq \frac{1}{V} + \frac{1}{v} - \frac{f_2(z)}{f_1(z)} \leqq \frac{1}{v}. \tag{77}$$

which leads us to the following inequality:

$$\frac{1}{V} \leqq \frac{\left(\frac{1}{V} + \frac{1}{v}\right) f_1(z) - f_2(z)}{f_1(z)} \leqq \frac{1}{v}. \tag{78}$$

Consequently, from the inequality (78), we have

$$f_1(z) \leqq Y\big(f_1(z), f_2(z)\big),$$

that is,

$$[f_1(z)]^p \leqq Y^p\big(f_1(z), f_2(z)\big). \tag{79}$$

In a similar manner, from the inequality (76), we obtain

$$[f_2(z)]^p \leqq Y^p\big(f_1(z), f_2(z)\big). \tag{80}$$

Upon multiplication of (79) and (80) by

$$\Phi^{-1}\,(u) \frac{\varphi\big(Q(u) - Q(z)\big)}{Q(u) - Q(z)} \Phi(z) W^{\sigma}_{\eta,\lambda}\left[\omega\big(Q(u) - Q(z)\big)^{\eta}\right] Q'(z),$$

with $r \in (a,u)$, if we integrate the obtained results over $z \in (a,u)$ and add them together, we have the result asserted by Theorem 8. □

Corollary 15. *Let the hypotheses of Theorem 8 with*

$$Q(u) = \ln u, \quad \varphi(u) = \frac{u^{\alpha}}{\Gamma(\alpha)} \qquad \textit{and} \qquad \alpha \in (0,1],$$

be satisfied. Then

$$\left[\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} {f_1}^p\right)(u)\right]^{\frac{1}{p}} + \left[\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} {f_2}^p\right)(u)\right]^{\frac{1}{p}}$$
$$\leqq 2\left[\left(K^{\alpha,\Phi}_{\sigma,\eta,\lambda,a^+;\omega} Y^p(f_1,f_2)\right)(u)\right]^{\frac{1}{p}},$$

for all $u \in [a,b]$, *where* $a \geqq 0$.

Corollary 16. *If the conditions of Theorem 8 with*

$$\varphi(u) = \frac{u}{\alpha}\exp(-\mathcal{M}u) \qquad \textit{and} \qquad \alpha \in (0,1],$$

are satisfied, then

$$\left[\left({}^{Q}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_1{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{Q}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{f}_2{}^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq 2\left[\left({}^{Q}\mathrm{E}^{\alpha,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\Upsilon^p(\mathrm{f}_1,\mathrm{f}_2)\right)(\mathrm{u})\right]^{\frac{1}{p}}$$

for all $\mathrm{u} \in [\mathrm{a},\mathrm{b}]$, *where*

$$\mathrm{a} \geqq 0 \qquad \textit{and} \qquad \mathcal{M} = \frac{1-\alpha}{\alpha}.$$

Remark 4. *For suitable choices of function, i.e.,* $\varphi(\mathrm{u}) = \frac{\mathrm{u}^\alpha}{\Gamma(\alpha)}$ *and* $\varphi(\mathrm{u}) = \mathrm{u}(\mathrm{b}-\mathrm{u})^{\alpha-1}$, *interested researchers can obtain similar results as are obtained for all the above theorems.*

4. A Set of Examples

In this section, we choose to present each of the following illustrative examples of the findings in this paper.

Example 1. *For all* $\mathrm{u} > \mathrm{a} \geqq 1$ *and* $\mathrm{z} \in (\mathrm{a},\mathrm{u})$, *let us assume that* $p \geqq 1$, $\lambda, \eta, a > 0$ *and* $\omega \in \mathbb{R}$. *Then*

$$\left[\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(r+a)^p\right)(\mathrm{u})\right]^{\frac{1}{p}}$$
$$\leqq \frac{3a+4}{2(a+2)}\left[\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(2\mathrm{z}+a)^p\right)(\mathrm{u})\right]^{\frac{1}{p}}.$$

Proof. By selecting

$$\mathrm{f}_1(\mathrm{z}) = \mathrm{z} + a \qquad \text{and} \qquad \mathrm{f}_2(\mathrm{z}) = \mathrm{z},$$

we obtain $\mathrm{v} = 1$ and $\mathrm{V} = a+1$, respectively. The proof follows by implementing Theorem 1. □

Example 2. *For any* $\mathrm{u} > \mathrm{a} \geqq 1$ *and* $\mathrm{z} \in (\mathrm{a},\mathrm{u})$, *let* $\frac{1}{p} + \frac{1}{q} = 1 \quad (p > 1)$, *and suppose that* $\lambda, \eta, a > 0$ *and* $\omega \in \mathbb{R}$. *Then*

$$\left[\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}+a)\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}\right)(\mathrm{u})\right]^{\frac{1}{q}} \leqq (a+1)^{\frac{1}{pq}}\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}+a)^{\frac{1}{p}}\mathrm{z}^{\frac{1}{q}}\right)(\mathrm{u}).$$

Proof. By putting

$$\mathrm{f}_1(\mathrm{z}) = \mathrm{z} + a \qquad \text{and} \qquad \mathrm{f}_2(\mathrm{z}) = \mathrm{z},$$

we obtain $\mathrm{v} = 1$ and $\mathrm{V} = a+1$, respectively. The proof follows now by applying Theorem 3. □

Example 3. *For any* $\lambda, \eta, a > 0$ *and* $\omega \in \mathbb{R}$, *let* $\frac{1}{p} + \frac{1}{q} = 1 \quad (p > 1)$. *Then, for all* $\mathrm{u} > \mathrm{a} \geqq 1$ *and* $\mathrm{z} \in (\mathrm{a},\mathrm{u})$, *the following inequality holds true*:

$$\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}(\mathrm{z}+a)\right)(\mathrm{u}) \leqq \frac{2^{p-1}(a+1)^p}{p(a+2)^p}\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}^p + (\mathrm{z}+a)^p)\right)(\mathrm{u})$$
$$+ \frac{1}{2q}\left({}^{Q}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}^q + (\mathrm{z}+a)^q)\right)(\mathrm{u}).$$

Proof. By putting $\mathrm{f}_1(\mathrm{z}) = \mathrm{z} + a$ and $\mathrm{f}_2(\mathrm{z}) = \mathrm{z}$, we find that $\mathrm{v} = 1$ and $\mathrm{V} = a+1$, respectively. The proof follows by implementing Theorem 4. □

Example 4. *For any* $p \geqq 1$, $\lambda, \eta, a > 0$ *and* $\omega \in \mathbb{R}$, *and for all* $\mathrm{u} > \mathrm{a} \geqq 1$, $\mathrm{g} \in (0,1)$ *and* $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, *it is asserted that*

$$\begin{aligned}
&\left(\frac{a+2}{a+1-\mathrm{g}}\right)\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}(1-\mathrm{g})+a)^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \\
&\quad \leqq \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}^p\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}+a)^p\right)(\mathrm{u})\right]^{\frac{1}{p}} \\
&\quad \leqq \left(\frac{2}{1-\mathrm{g}}\right)\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\mathrm{z}(1-\mathrm{g})+a)^p\right)(\mathrm{u})\right]^{\frac{1}{p}}.
\end{aligned}$$

Proof. By setting $\mathrm{f}_1(\mathrm{z}) = \mathrm{z} + a$ and $\mathrm{f}_2(\mathrm{z}) = \mathrm{z}$, we see that $\mathrm{v} = 1$ and $\mathrm{V} = a + 1$, respectively. The proof follows by implementing Theorem 5. □

Example 5. *For any* $\mathrm{u} > \mathrm{a} \geqq 0$ *and* $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, *where* $\lambda, \eta > 0$ *and* $\omega \in \mathbb{R}$ *with* $p \geqq 1$, *the following inequality holds true*:

$$\begin{aligned}
&\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\sin^{2p} r\right)(\mathrm{u})\right]^{\frac{1}{p}} + \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\cos^{2p} r\right)(\mathrm{u})\right]^{\frac{1}{p}} \\
&\quad \leqq 2\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}1\right)(\mathrm{u})\right]^{\frac{1}{p}}.
\end{aligned}$$

Proof. By choosing $\mathrm{f}_1(\mathrm{z}) = \sin^2 \mathrm{z}$ and $\mathrm{f}_2(\mathrm{z}) = \cos^2 \mathrm{z}$, we obtain $\mu_1 = \nu_1 = 0$ and $\mu_2 = \nu_2 = 1$, respectively. The proof follows by applying Theorem 6. □

5. Applications Involving the Digamma Functions

In this section, before giving some applications regarding Digamma function, let us recall the following definition and theorem.

Definition 8 ([53])**.** *The Digamma (or the* ψ*-) function* $\psi(\mathrm{z})$ *defined for all real or complex* $\mathrm{z}\ \ (\mathrm{z} \neq 0, -1, -2, \cdots)$ *by*

$$\psi(\mathrm{z}) := \frac{\mathrm{d}}{\mathrm{dz}}\{\log \Gamma(\mathrm{z})\} = \frac{\Gamma'(\mathrm{z})}{\Gamma(\mathrm{z})} \qquad \text{or} \qquad \log \Gamma(\mathrm{z}) = \int_1^{\mathrm{z}} \psi(t)\,\mathrm{d}t$$

is known to be strictly increasing and strictly concave on the interval $(0, \infty)$*. Here,* $\Gamma(z)$ *is the widely-recognized Gamma function defined for all real or complex* $z\ \ (z \neq 0, -1, -2, \cdots)$.

We recall the following well-known asymptotic expansion as Laurent series (see, for example, [53]; see also page 433 in [54]):

$$\psi'(\mathrm{z}) = \frac{1}{\mathrm{z}} + \frac{1}{2\mathrm{z}^2} + \frac{1}{6\mathrm{z}^3} + \frac{\lambda}{30\mathrm{z}^5} \qquad (\lambda \in [0,1]),$$

which shows clearly that $\psi'(\mathrm{z}) > 0$ for every $\mathrm{z} > 0$.

Theorem 9 ([55])**.** *For every* $\mathrm{z} > 0$, *the following inequality holds true*:

$$\frac{1}{\mathrm{z}^2} < \psi'(\mathrm{z})\psi'(\mathrm{z}+1) < \frac{2}{\mathrm{z}^2}.$$

Using Theorem 9 and the fact that $\psi'(\mathrm{z}) > 0$ for every $\mathrm{z} > 0$, we can deduce the following results.

Proof. By selecting

$$\mathrm{f}_1(\mathrm{z}) = \psi'(\mathrm{z})\psi'(\mathrm{z}+1) \qquad \text{and} \qquad \mathrm{f}_2(\mathrm{z}) = \frac{1}{\mathrm{z}^2},$$

we have $\mathrm{v} = 1$ and $\mathrm{V} = 2$, respectively. Thus, by utilizing Theorem 9 in Theorem 1, the intended outcome is achieved. □

Proposition 1. *Let $p \geqq 1$, with λ, $\eta > 0$ and $\omega \in \mathbb{R}$. Then, for any $\mathrm{u} > \mathrm{a} \geqq 0$ and $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, the following inequality is valid:*

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\psi'(\mathrm{z})\psi'(\mathrm{z}+1))^p\right)(\mathrm{u})\right]^{\frac{2}{p}} + \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}^{-2p}\right)(\mathrm{u})\right]^{\frac{2}{p}}$$
$$\geqq \left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\psi'(\mathrm{z})\psi'(\mathrm{z}+1))^p\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}^{-2p}\right)(\mathrm{u})\right]^{\frac{1}{p}}.$$

Proof. By setting $\mathrm{f}_1(\mathrm{z}) = \psi'(\mathrm{z})\psi'(\mathrm{z}+1)$ and $\mathrm{f}_2(\mathrm{z}) = \frac{1}{\mathrm{z}^2}$, we get $\mathrm{v} = 1$ and $\mathrm{V} = 2$, respectively. Therefore, by applying Theorem 9 to Theorem 2, the intended outcome is achieved. □

Proposition 2. *Suppose that $\frac{1}{p} + \frac{1}{q} = 1 \quad (p > 1)$ with λ, $\eta > 0$ and $\omega \in \mathbb{R}$. Then, for any $\mathrm{u} > \mathrm{a} \geqq 0$ and $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, the following inequality is valid:*

$$\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\psi'(\mathrm{z})\psi'(\mathrm{z}+1)\right)(\mathrm{u})\right]^{\frac{1}{p}}\left[\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\mathrm{z}^{-2}\right)(\mathrm{u})\right]^{\frac{1}{q}}$$
$$\leqq \sqrt[pq]{2}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}(\psi'(\mathrm{z})\psi'(\mathrm{z}+1))^{\frac{1}{p}}\mathrm{z}^{-\frac{2}{q}}\right)(\mathrm{u}).$$

Proof. By choosing $\mathrm{f}_1(\mathrm{z}) = \psi'(\mathrm{z})\psi'(\mathrm{z}+1)$ and $\mathrm{f}_2(\mathrm{z}) = \frac{1}{\mathrm{z}^2}$, we obtain $\mathrm{v} = 1$ and $\mathrm{V} = 2$, respectively. So, by utilizing Theorem 9 in Theorem 3, the intended outcome is achieved. □

Proposition 3. *Suppose that $\frac{1}{p} + \frac{1}{q} = 1 \quad (p > 1)$ with λ, $\eta > 0$ and $\omega \in \mathbb{R}$. Then, for any $\mathrm{u} > \mathrm{a} \geqq 0$ and $\mathrm{z} \in (\mathrm{a}, \mathrm{u})$, the following inequality is valid:*

$$\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\frac{\psi'(\mathrm{z})\psi'(\mathrm{z}+1)}{\mathrm{z}^2}\right)(\mathrm{u}) \leqq \frac{1}{2p}\left(\frac{4}{3}\right)^p\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\left(\mathrm{z}^{-2p} + [\psi'(\mathrm{z})\psi'(\mathrm{z}+1))^p]\right)(\mathrm{u})$$
$$+ \frac{1}{2q}\left({}^{\mathrm{Q}}\mathrm{J}^{\varphi,\Phi}_{\sigma,\eta,\lambda,\mathrm{a}^+;\omega}\left(\mathrm{z}^{-2q} + [\psi'(\mathrm{z})\psi'(\mathrm{z}+1)]^q\right)\right)(\mathrm{u}).$$

Proof. By putting $\mathrm{f}_1(\mathrm{z}) = \psi'(\mathrm{z})\psi'(\mathrm{z}+1)$ and $\mathrm{f}_2(\mathrm{z}) = \frac{1}{\mathrm{z}^2}$, we obtain $\mathrm{v} = 1$ and $\mathrm{V} = 2$, respectively. Therefore, by utilizing Theorem 9 in Theorem 4, the intended outcome is achieved. □

6. Conclusions

The study highlights the relevance of the weighted fractional integral operators and their application in establishing reverse Minkowski inequalities. Additionally, the manuscript also mentions various novel applications and examples involving Digamma functions for suitable functions. Future research aims to integrate these operators with existing mathematical concepts, such as Chebyshev, Markov, and Minkowski inequalities, utilizing advanced tools like quantum calculus and interval-valued analysis. The objective is to enhance the effectiveness and applicability of the newly developed operators by incorporating special functions.

We conclude this presentation by drawing the attention of the interested reader toward some recent developments (see, for example, [56–58]) on the reverse Minkowski-type and other types of integral inequalities.

Author Contributions: Conceptualization, S.K.S. and P.O.M.; Data curation, H.M.S.; Formal analysis, A.K.; Funding acquisition, S.K.S.; Investigation, H.M.S., S.K.S., P.O.M., A.K. and N.C.; Methodology, S.K.S. and A.K.; Project administration, H.M.S. and P.O.M.; Resources, N.C.; Software, S.K.S.; Supervision, H.M.S.; Validation, A.K.; Visualization, N.C.; Writing—original draft, S.K.S. and P.O.M.;

Writing—review & editing, A.K. and N.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Researchers Supporting Project number (RSP2023R153), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
2. Tenreiro Machado, J.A.; Silva, M.F.; Barbosa, R.S.; Jesus, I.S.; Reis, C.M.; Marcos, M.G.; Galhano, A.F. Some applications of fractional calculus in engineering. *Math. Prob. Eng.* **2010**, *2010*, 639801.
3. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Phys. A Stat. Mech. Appl.* **2000**, *284*, 376–384.
4. Assaleh, K.; Ahmad, W.M. Modeling of speech signals using fractional calculus. In Proceedings of the 2007 9th International Symposium on Signal Processing and Its Applications, Sharjah, United Arab Emirates, 12–15 February 2007; IEEE: New York, NY, USA, 2007; pp. 1–4.
5. Atanackovic, T.M.; Pilipovic, S.; Stankovic, B.; Zorica, D. *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*; John Wiley & Sons: Hoboken, NJ, USA, 2014.
6. Atanackovic, T.M.; Pilipovic, S.; Stankovic, B.; Zorica, D. *Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles*; John Wiley & Sons: Hoboken, NJ, USA, 2014.
7. Koeller, R. Applications of fractional calculus to the theory of viscoelasticity. *J. Appl. Mech.* **1984**, *51*, 299–307.
8. Cai, Z.W.; Huang, J.H.; Huang, L.H. Periodic orbit analysis for the delayed Filippov system. *Proc. Am. Math. Soc.* **2018**, *146*, 4667–4682.
9. Chen, T.; Huang, L.H.; Yu, P.; Huang, W.T. Bifurcation of limit cycles at infinity in piecewise polynomial systems. *Nonlinear Anal. Real World Appl.* **2018**, *41*, 82–106.
10. Rahman, R.U.; Al-Maaitah, A.F.; Qousini, M.; Az-Zo'bi, E.A.; Eldin, S.M.; Abuzar, M. New soliton solutions and modulation instability analysis of fractional Huxley equation. *Results Phys.* **2023**, *44*, 106163.
11. Ur Rahman, R.; Faridi, W.A.; El-Rahman, M.A.; Taishiyeva, A.; Myrzakulov, R.; Az-Zo'bi, E.A. The sensitive visualization and generalized fractional solitons' construction for regularized long-wave governing model. *Fractal Fract.* **2023**, *7*, 136.
12. Adjabi, Y.; Jarad, F.; Baleanu, D.; Abdeljawad, T. On Cauchy problems with Caputo-Hadamard fractional derivatives. *Math. Meth. Appl. Sci.* **2016**, *40*, 661–681.
13. Zhou, X.S.; Huang, C.X.; Hu, H.J.; Liu, L. Inequality estimates for the boundedness of multilinear singular and fractional integral operators. *J. Inequal. Appl.* **2013**, *2013*, 303.
14. Liu, F.W.; Feng, L.B.; Anh, V.; Li, J. Unstructured-mesh Galerkin finite element method for the two-dimensional multi-term time-space fractional Bloch–Torrey equations on irregular convex domains. *Comput. Math. Appl.* **2019**, *78*, 1637–1650.
15. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
16. Kashuri, A.; Sahoo, S.K.; Mohammed, P.O.; Al-Sarairah, E.; Hamed, Y.S. Some New Hermite-Hadamard Type Inequalities Pertaining to Fractional Integrals with an Exponential Kernel for Subadditive Functions. *Symmetry* **2023**, *15*, 748. [CrossRef]
17. Anastassiou, G.A. Nabla discrete fractional calculus and nabla inequalities. *Math. Comput. Model.* **2010**, *51*, 562–571. [CrossRef]
18. Zheng, B. Some new discrete fractional inequalities and their applications in fractional difference equations. *J. Math. Inequal.* **2015**, *9*, 823–839.
19. Ahmad, B.; Alsaedi, A.; Kirane, M.;Torebek, B.T. Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. *J. Comput. Appl. Math.* **2019**, *353*, 120–129. [CrossRef]
20. Podlubny, I. Geometric and physical interpretations of fractional integration and differentiation. *Fract. Calc. Appl. Anal.* **2001**, *5*, 230–237.
21. Baleanu, D.; Fernandez, A. On fractional operators and their classifications. *Mathematics* **2019**, *7*, 830.
22. Hilfer, R.; Luchko, Y.F. Desiderata for fractional derivatives and integrals. *Mathematics* **2019**, *7*, 149.
23. Böröcsky, K.J.; Hug, D. A reverse Minkowski-type inequality. *Proc. Am. Math. Soc.* **2020**, *148*, 4907–4922. [CrossRef]
24. Benaissa, B. On the reverse Minkowski's integral inequality. *Kragujevac J. Math.* **2022**, *46*, 407–416.
25. Benaissa, B. A further generalization of the reverse Minkowski-type inequality via Hölder and Jensen inequalities. *J. Sib. Fed. Univ. Math. Phys.* **2022**, *15*, 319–328.
26. Krasopoulos, P.T.; Bougoffa, L. Reverse Hölder and Minkowski-type inequalities for n functions. *Austral. J. Math. Anal. Appl.* **2022**, *19*, 9.
27. Alsmeyer, G. Chebyshev's Inequality. In *International Encyclopedia of Statistical Science*; Lovric, M., Ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2011.

28. Chebyshev, P.L. Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. *Proc. Math. Soc. Charkov* **1882**, *2*, 93–98.
29. Liu, Z. A variant of Chebyshev inequality with applications. *J. Math. Inequal.* **2013**, *7*, 551–561.
30. Özdemir, M.E.; Set, E.; Akdemir, A.O.; Sarikaya, M.Z. Some new Chebyshev type inequalities for functions whose derivatives belongs to $\mathcal{L}_p$ spaces. *Afr. Mat.* **2015**, *26*, 1609–1619.
31. Pachpatte, B.G. A note on Chebyshev-Grüss type inequalities for differential functions. *Tamsui Oxf. J. Math. Sci.* **2006**, *22*, 29–36.
32. Set, E.; Kashuri, A.; Mumcu, İ. Chebyshev type inequalities by using generalized proportional Hadamard fractional integrals via Polya–Szegö inequality with applications. *Chaos Solit. Fractals* **2021**, *146*, 110860.
33. Gardner, R.J. The Brunn-Minkowski inequality. *Bull. Am. Math. Soc.* **2002**, *39*, 355–405. [CrossRef]
34. Zhao, C.J. Reverse Lp-dual Minkowski's inequality. *Differ. Geom. Appl.* **2015**, *40*, 243–251. [CrossRef]
35. Bennett, G.; Carbery, A.; Christ, M.; Tao, T. The Brascamp-Lieb inequalities: Finiteness, structure and extremals. *Geom. Funct. Anal.* **2008**, *17*, 1343–1415.
36. Bobkov, S.G.; Ledoux, M. On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures. *J. Funct. Anal.* **2000**, *171*, 532–583.
37. Jiao, Y.; Peng, L.; Xu, M. Reverse Minkowski inequality for risk measures. *J. Optim. Theory Appl.* **2017**, *174*, 232–244.
38. Dahmani, Z. On Minkowski and Hermite–Hadamard integral inequalities via fractional integral. *Ann. Funct. Anal.* **2010**, *1*, 51–58. [CrossRef]
39. Set, E.; Özdemir, M.; Dragomir, S.S. On the Hermite–Hadamard inequality and other integral inequalities involving two functions. *J. Inequal. Appl.* **2010**, *2010*, 148102. [CrossRef]
40. Chinchane, V.L.; Pachpatte, D.B. New fractional inequalities via Hadamard fractional integral. *Int. J. Funct. Anal. Oper. Theory Appl.* **2013**, *5*, 165–176.
41. Da C. Sousa, J.V.; Capelas de Oliveira, E. The Minkowski's inequality by means of a generalized fractional integral. *AIMS Ser. Appl. Math.* **2018**, *3*, 131–147.
42. Rahman, G.; Khan, A.; Abdeljawad, T.; Nisar, K.S. The Minkowski inequalities via generalized proportional fractional integral operators. *Adv. Differ. Equ.* **2019**, *2019*, 287. [CrossRef]
43. Fox, C. The asymptotic expansion of generalized hypergeometric functions. *Proc. Lond. Math. Soc.* **1928**, *27*, 389–400.
44. Wright, E.M. The asymptotic expansion of the generalized hypergeometric function. *J. Lond. Math. Soc.* **1935**, *10*, 286–293. [CrossRef]
45. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294.
46. Wright, E.M. The asymptotic expansion of integral functions defined by Taylor series. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1940**, *238*, 423–451.
47. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Baleanu, D. Fractional integral inequalities for exponentially nonconvex functions and their applications. *Fractal Fract.* **2021**, *5*, 80. [CrossRef]
48. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Engrg. Comput.* **2021**, *5*, 135–166. [CrossRef]
49. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
50. Sarikaya, M.Z.; Ertuğral, F. On the generalized Hermite–Hadamard inequalities. *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **2020**, *47*, 193–213.
51. Srivastava, H.M.; Kashuri, A.; Mohammed, P.O.; Nonlaopon, K. Certain inequalities pertaining to some new generalized fractional integral operators. *Fractal Fract.* **2021**, *5*, 160. [CrossRef]
52. Liko, R.; Mohammed, P.O.; Kashuri, A.; Hamed, Y.S. Reverse Minkowski inequalities pertaining to new weighted generalized fractional integral operators. *Fractal Fract.* **2022**, *6*, 131.
53. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Applied Mathematics Series 55; National Bureau of Standards: Washington, DC, USA, 1964; Dover Publications: New York, NY, USA, 1965.
54. English, B.J.; Rousseau, G. Bounds for certain harmonic sums. *J. Math. Anal. Appl.* **1997**, *206*, 428–441. [CrossRef]
55. Farhangdoost, M.R.; Dolatabadi, M.K. New inequalities for gamma and digamma functions. *J. Appl. Math.* **2014**, *2014*, 264652. [CrossRef]
56. Pan, J.; Yang, S.; Wang, W. Reverse form of the Minkowski inequalities with applications. *J. Math. Inequal.* **2022**, *16*, 1051–1059. [CrossRef]
57. Yewale, B.R.; Pachpatte, D.B. On some reverses of Minkowski's, Hölder's and Hardy's type inequalities using ψ-fractional integral operators. *South East Asian J. Math. Math. Sci.* **2022**, *18*, 97–112.
58. Zhao, C.-J.; Cheung, W.S. On reverse Hölder and Minkowski inequalities. *Math. Slovaca* **2020**, *70*, 821–828.