symmetry

Article

Generalized Common Best Proximity Point Results in Fuzzy Metric Spaces with Application

Umar Ishtiaq [1], Fahad Jahangeer [2], Doha A. Kattan [3] and Ioannis K. Argyros [4,*]

[1] Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan; umar.ishtiaq@umt.edu.pk
[2] Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan; fahad.msma719@iiu.edu.pk
[3] Department of Mathematics, Faculty of Sciences and Arts, King Abdulaziz University, Rabigh 21589, Saudi Arabia; dakattan@kau.edu.sa
[4] Department of Computing and Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
* Correspondence: iargyros@cameron.edu

Abstract: The symmetry of fuzzy metric spaces has benefits for flexibility, ambiguity tolerance, resilience, compatibility, and applicability. They provide a more comprehensive description of similarity and offer a solid framework for working with ambiguous and imprecise data. We give fuzzy versions of some celebrated iterative mappings. Further, we provide different concrete conditions on the real valued functions $\mathcal{J}, \mathcal{S} : (0,1] \to \mathbb{R}$ for the existence of the best proximity point of generalized fuzzy $(\mathcal{J}, \mathcal{S})$-iterative mappings in the setting of fuzzy metric space. Furthermore, we utilize fuzzy versions of $(\mathcal{J}, \mathcal{S})$-proximal contraction, $(\mathcal{J}, \mathcal{S})$-interpolative Reich–Rus–Ciric-type proximal contractions, $(\mathcal{J}, \mathcal{S})$-Kannan type proximal contraction and $(\mathcal{J}, \mathcal{S})$-interpolative Hardy Roger's type proximal contraction to examine the common best proximity points in fuzzy metric space. Also, we establish several non-trivial examples and an application to support our results.

Keywords: fixed point; best proximity point; fuzzy metric spaces; integral equations

check for updates

Citation: Ishtiaq, U.; Jahangeer, F.; Kattan, D.A.; Argyros, I.K. Generalized Common Best Proximity Point Results in Fuzzy Metric Spaces with Application. *Symmetry* **2023**, *15*, 1501. https://doi.org/10.3390/sym15081501

Academic Editor: Manuel Gadella

Received: 23 June 2023
Revised: 11 July 2023
Accepted: 26 July 2023
Published: 28 July 2023

1. Introduction

Fixed point theory is one of the most appealing areas of study. The techniques for determining a solution to a nonlinear equation of the pattern $\Upsilon\grave{u} = \grave{u}$, where Υ is self mapping, are discussed in fixed point theory. However, in various cases, the singular solution does not exist. Best approximation theorems and best proximity point theorems are helpful in solving the aforementioned problem. The best proximity point theorems have been generalized in a number of ways by numerous authors, and they provide an approximate optimal solution. If the mapping is self-mapping, then the best proximity point theorems become a fixed point.

In 1968, Kannan [1], introduced a new kind of contraction for discontinuous mappings and proved several fixed point results. He provided a new way for researchers to solve fixed point problems. Karapinar [2] introduced iterative Kannan–Mier-type contractions. Karapinar et al. [3] provided new results on Perov interpolative contractions of Suzuki type mappings. Karapinar and Agarwal [4] established interpolative Rus–Reich–Ciric-type contractions via simulation functions. Karapinar et al. [5] offered a new result for Hardy–Rogers-type interpolative contractions.

Altun et al. [6] gave some best proximity point results for p-proximal contractions. Further, Altun and Aysenur [7] proved some best proximity point results for interpolative proximal contractions. Shazad et al. [8] provided some common best proximity point results. Basha [9] developed common best proximity point results for global minimal solutions. Moreover, Basha [10] examined common best proximity point for multi-objective functions. Deep and Betra [11] introduced some common best proximity point results under proximal F-contraction. Mondal and Dey [12] proved some common best proximity point results in complete metric spaces. Shayanpour and Nematizadeh [13] presented some common best proximity point results in the setting of complete fuzzy metric space (in short, CFMS). Hierro [14] presented Proinov-type fixed point results in fuzzy metric spaces (FMS). Then, Zhou et al. [15] modified the results of [14] and introduced new Proinov-type fixed point results in FMS. Uddin et al. [16] proved several new results for a new extension to the intuitionistic FM-like spaces. Saleem et al. [17] provided a unique soltion for integral equations via intuitionistic extended fuzzy b-metric-like spaces. Saleem et al. [18] presented a result for graphical FMS applied to fractional differential equations. Hussain et al. [19] proved a result for fixed point in FMS. Nazam et al. [20] established several results for generalized interpolative contractions. Naseem et al. [21] worked on the analytical approximation of fractional delay differential equations.

In this paper, we introduce fuzzy versions of $(\mathcal{J},\mathcal{S})$-proximal contractions, $(\mathcal{J},\mathcal{S})$-interpolative Reich–Rus–Ciric-type proximal contractions, $(\mathcal{J},\mathcal{S})$-interpolative Kannan-type proximal contractions, and $(\mathcal{J},\mathcal{S})$-interpolative Hardy Roger's type proximal contractions to examine the common best proximity point in the setting of FMS. We provide several non-trivial examples and an application to integral equations to support our results.

2. Preliminaries

In this section, we provide definitions from the existing literature that will help readers to understand the main section.

Definition 1 ([9])**.** *Let $(\mathcal{B},\vartheta)$ be a metric space. The mappings $\Gamma:\mathcal{M}\to\mathcal{N}$ and $\Upsilon:\mathcal{M}\to\mathcal{N}$ are said to commute proximally if they satisfy the below condition*

$$[\vartheta(\breve{a},\Gamma\grave{u})=\vartheta(\breve{e},\Upsilon\grave{u})=\vartheta(\mathcal{M},\mathcal{N})]\Rightarrow\Gamma\breve{e}=\Upsilon\breve{a},$$

for all $\grave{u},\breve{a},\breve{e}$ in $\mathcal{M}$.

Definition 2 ([9])**.** *Let $(\mathcal{B},\vartheta)$ be a metric space. A mapping $\Upsilon:\mathcal{M}\to\mathcal{N}$ dominates proximally to a mapping $\Gamma:\mathcal{M}\to\mathcal{N}$ if there exists a non-negative number $\alpha<1$ such that*

$$\begin{aligned}\vartheta(\breve{a}_1,\Gamma\grave{u}_1)=\vartheta(\mathcal{M},\mathcal{N})=\vartheta(\breve{e}_1,\Upsilon\grave{u}_1)\\ \vartheta(\breve{a}_2,\Gamma\grave{u}_2)=\vartheta(\mathcal{M},\mathcal{N})=\vartheta(\breve{e}_2,\Upsilon\grave{u}_2)\end{aligned}$$

$$\vartheta(\breve{a}_1,\breve{a}_2)\le\alpha\vartheta(\breve{e}_1,\breve{e}_2).$$

for all $\breve{a}_1,\breve{a}_2,\breve{e}_1,\breve{e}_2,\grave{u}_1,\grave{u}_2\in\mathcal{M}$.

Definition 3 ([15])**.** *A binary operation $*:H\times H\to H$ (where $H=[0,1]$) is said to be a continuous t-norm (ctn) if it satisfies the below axioms:*

(1) $\breve{a}_1*\breve{a}_2=\breve{a}_1*\breve{a}_2$ *and* $\breve{a}_1*(\breve{a}_2*\breve{a}_3)=(\breve{a}_1*\breve{a}_2)*\breve{a}_3$ *for all* $\breve{a}_1,\breve{a}_2,\breve{a}_3\in H$;
(2) $*$ *is continuous;*
(3) $\breve{a}_1*1=\sigma$ *for all* $\breve{a}_1\in H$;
(4) $\breve{a}_1*\breve{a}_2\le\breve{a}_3*\breve{a}_4$ *when* $\breve{a}_1\le\breve{a}_3$ *and* $\breve{a}_2\le\breve{a}_4$, *with* $\breve{a}_1,\breve{a}_2,\breve{a}_3,\breve{a}_4\in H$.

Definition 4 ([15])**.** *A triplet $(\mathcal{B},\vartheta,*)$ is termed as FMS if $*$ is a ctn, $\mathcal{B}$ is arbitrary set, and ϑ is a fuzzy set on $\mathcal{B}\times\mathcal{B}\times(0,\infty)$ fulfilling the below conditions for all $\breve{a}_1,\breve{a}_2,\breve{a}_3\in\mathcal{B}$ and $\kappa,\varpi>0$:*

(i) $\vartheta(\breve{a}_1,\breve{a}_2,\kappa)>0$;
(ii) $\vartheta(\breve{a}_1,\breve{a}_2,\kappa)=0$ *if and only if* $\breve{a}_1=\breve{a}_2$;
(iii) $\vartheta(\breve{a}_1,\breve{a}_2,\kappa)=\vartheta(\breve{a}_2,\breve{a}_1,\kappa)$;
(iv) $\vartheta(\breve{a}_1,\breve{a}_3,\kappa+\varpi)\ge\vartheta(\breve{a}_1,\breve{a}_2,\kappa)*\vartheta(\breve{a}_2,\breve{a}_3,\varpi)$;
(v) $\vartheta(\breve{a}_1,\breve{a}_2,.):(0,\infty)\to[0,1]$.

Example 1. *Suppose $\mathcal{B} = \mathbb{R}^+$ and $\vartheta(\breve{a}_1, \breve{a}_2, \kappa) = \frac{\kappa}{\kappa + L^*(\breve{a}_1, \breve{a}_2)}$, consider a ctn as $m * n = mn$. Then, $\mathcal{B}$ is a FMS.*

Definition 5 ([13]). *A sequence $\{\breve{a}_n\}$ in a FMS $(\mathcal{B}, \vartheta, *)$ is said to be convergent to a point $a \in \mathcal{B}$ if for each $\varepsilon > 0$ and $\zeta \in (0,1)$, there exists $a_0(\varepsilon, \zeta) \in \mathbb{N}$ such that $\vartheta(\breve{a}, \breve{a}_n, \kappa) > 1 - \zeta$ for all $n \geq a_0(\varepsilon, \zeta)$ or $lim_{n\to\infty}\vartheta(\breve{a}, \breve{a}_n, \kappa) = 1$, for all $\kappa > 0$; in this case, we say that limit of sequence $\{\breve{a}_n\}$ exists.*

Definition 6 ([13]). *A sequence $\{\breve{a}_n\}$ in a FMS $(\mathcal{B}, \vartheta, *)$ is said to be convergent to a point $a \in \mathcal{B}$ if for each $\varepsilon > 0$ and $\zeta \in (0,1)$, there exists $a_0(\varepsilon, \zeta) \in \mathbb{N}$ such that $\vartheta\big(\breve{a}_n, \breve{a}_{n+p}, \kappa\big) > 1 - \zeta$ for all $n \geq a_0(\varepsilon, \zeta)$ and every $p \in \mathbb{N}$ or $lim_{n\to\infty}\vartheta\big(\breve{a}_n, \breve{a}_{n+p}, \kappa\big) = 1$, for all $\kappa > 0$ and $p \in \mathbb{N}$.*

Also, an FMS $(\mathcal{B}, \vartheta, *)$ is said to be complete if and only if every Cauchy sequence in $\mathcal{B}$ is convergent.

Definition 7 ([13]). *Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\mathcal{M}$,$\mathcal{N} \subseteq \mathcal{B}$. Then*

$$\vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \sup_{\breve{a}_1 \in \mathcal{M}, \breve{a}_1 \in \mathcal{N}} \vartheta(\breve{a}_1, \breve{a}_2, \kappa), \kappa > 0,$$

which is said to be a fuzzy distance between $\mathcal{M}$ and $\mathcal{N}$.

Definition 8 ([13]). *Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\mathcal{M}$,$\mathcal{N} \subseteq \mathcal{B}$. We define the following sets.*

$$\mathcal{M}_0 = \{\breve{a}_1 \in \mathcal{M}{:}\exists\ \breve{a}_2 \in \mathcal{N}\ s.t\ \forall\kappa > 0, \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa)\},$$
$$\mathcal{N}_0 = \{\breve{a}_2 \in \mathcal{N}{:}\exists\ \breve{a}_1 \in \mathcal{M}\ s.t\ \forall\kappa > 0, \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa)\}.$$

Definition 9 ([13]). *Let $(\mathcal{B}, \vartheta, *)$ be an FMS, $\mathcal{M}$,$\mathcal{N} \subseteq \mathcal{B}$ and $\Upsilon, \Gamma : \mathcal{M} \to \mathcal{N}$ be two mappings. We say that an element $\breve{a} \in \mathcal{M}$ is a common best proximity point of the mappings Υ and Γ, if*

$$\vartheta(\breve{a}, \Upsilon\breve{a}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}, \Gamma\breve{a}, \kappa).$$

Definition 10 ([13]). *Let $(\mathcal{B}, \vartheta, *)$ be a FMS, $\mathcal{M}$,$\mathcal{N} \subseteq \mathcal{B}$ and $\Upsilon, \Gamma : \mathcal{M} \to \mathcal{N}$ be two mappings. We say that Υ, Γ are commute proximally if*

$$\vartheta(\breve{a}_1, \Upsilon\breve{a}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_2, \Gamma\breve{a}, \kappa), \forall\kappa > 0,$$

then $\Upsilon\breve{a}_2 = \Gamma\breve{a}_1$, where $\breve{a}, \breve{a}_1, \breve{a}_2 \in \mathcal{M}$.

Definition 11 ([13]). *Let $(\mathcal{B}, \vartheta, *)$ be a FMS, $\mathcal{M}$,$\mathcal{N} \subseteq \mathcal{B}$ and $\Upsilon, \Gamma : \mathcal{M} \to \mathcal{N}$ be two mappings. We say that the mapping Υ is to dominate Γ proximally if*

$$\vartheta(\breve{a}_1, \Upsilon h_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(b_1, \Gamma h_2, \kappa)$$
$$\vartheta(\breve{a}_2, \Upsilon h_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(b_2, \Gamma h_2, \kappa)$$

for all $\kappa > 0$ then there exists $\alpha \in (0,1)$ such that for all $\kappa > 0$,

$$\vartheta(\breve{a}_1, \breve{a}_2, \alpha\kappa) \geq \vartheta(b_1, b_2, \kappa)$$

where $\breve{a}_1, \breve{a}_2, b_1, b_2$ and $h_1, h_2 \in \mathcal{M}$.

Definition 12 ([15]). *We denote by the Ĺ the family of the pairs $(\mathcal{J}, \mathcal{S})$ of a functions $\mathcal{J}, \mathcal{S}{:}(0,1] \to \mathbb{R}$ satisfying the given properties below:*

(s_1) $\mathcal{J}$ *is nondecreasing,*
(s_2) $\mathcal{S}(\breve{a}) > \mathcal{J}(\breve{a})$ *for any* $\breve{a} \in (0,1)$,
(s_3) $\lim_{\breve{a}\to T^-} \inf \mathcal{S}(\breve{a}) > \lim_{s\to T^-} \mathcal{J}(\breve{a})$ *for any* $T^- \in (0,1)$,

(s_4) *if* $\breve{a} \in (0,1)$ *is such that* $\mathcal{S}(\breve{a}) \geq \mathcal{J}(1)$ *then* $\breve{a} = 1$.

3. Main Results

In this section, we provide several common best proximity point results by utilizing generalized fuzzy interpolative contractions, and we prove non-trivial examples.

3.1. Fuzzy $(\mathcal{J}, \mathcal{S})$-Proximal Contraction

Let $\mathcal{M}$ and $\mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$. The mappings $\Upsilon : \mathcal{M} \to \mathcal{N}$ and $\Gamma : \mathcal{M} \to \mathcal{N}$ are called fuzzy $(\mathcal{J}, \mathcal{S})$-proximal if

$$\begin{array}{r} \vartheta(\breve{a}_1, \Gamma\grave{u}_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_1, \Upsilon\grave{u}_1, \kappa) \\ \vartheta(\breve{a}_2, \Gamma\grave{u}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_2, \Upsilon\grave{u}_2, \kappa) \\ \mathcal{J}(\vartheta(\breve{a}_1, \breve{a}_2, \kappa)) \geq \mathcal{S}(\vartheta(\breve{e}_1, \breve{e}_2, \kappa)) \end{array} \tag{1}$$

for all $\breve{a}_1, \breve{a}_2, \breve{e}_1, \breve{e}_2, \grave{u}_1, \grave{u}_2 \in \mathcal{M}$ and $\kappa > 0$.

Example 2. *Let* $(\mathcal{B}, \vartheta, *)$ *be a FMS with* $\vartheta(\grave{u}, \check{n}, \kappa) = e^{-\frac{|\grave{u}-\check{n}|}{\kappa}}$. *Let* $\mathcal{M} = \{0, 2, 4, 6, 8, 10\}$ *and* $\mathcal{N} = \{1, 3, 5, 7, 9, 11\}$. *Define mappings* $\Gamma : \mathcal{M} \to \mathcal{N}$ *and* $\Upsilon : \mathcal{M} \to \mathcal{N}$ *as*

$$\Upsilon(0) = 3, \Upsilon(2) = 5, \Upsilon(4) = 7, \Upsilon(6) = 3, \Upsilon(8) = 9, \Upsilon(10) = 11,$$

and

$$\Gamma(0) = 3, \Gamma(2) = 1, \Gamma(4) = 9, \Gamma(6) = 7, \Gamma(8) = 5, \Gamma(10) = 11.$$

Then, $\vartheta(\mathcal{M}, \mathcal{N}, \kappa) = e^{-\frac{1}{\kappa}}$, $\mathcal{M}_0 = \mathcal{M}$ *and* $\mathcal{N}_0 = \mathcal{N}$. *Then clearly* $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$. *Define the functions* $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ *by*

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

We show that Γ *and* Υ *are fuzzy* $(\mathcal{J}, \mathcal{S})$*-proximal in FMS. Consider* $\breve{a}_1 = 0$, $\breve{a}_2 = 8$, $\breve{e}_1 = 4$, $\breve{e}_2 = 6$ *and* $\grave{u}_1 = 2$, $\grave{u}_2 = 4$, $\kappa = 1$

$$\begin{array}{c} \vartheta(\breve{a}_1, \Gamma\grave{u}_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_1, \Upsilon\grave{u}_1, \kappa) \\ \vartheta(\breve{a}_2, \Gamma\grave{u}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_2, \Upsilon\grave{u}_2, \kappa), \end{array}$$

then

$$\begin{array}{c} \vartheta(0, \Gamma 2, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(4, \Upsilon 2, 1) \\ \vartheta(8, \Gamma 4, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(6, \Upsilon 4, 1). \end{array}$$

This implies that

$$\begin{array}{c} \mathcal{J}(\vartheta(\breve{a}_1, \breve{a}_2, \kappa)) \geq \mathcal{S}(\vartheta(\breve{e}_1, \breve{e}_2, \kappa)) \\ \mathcal{J}(\vartheta(0, 8, 1)) \geq \mathcal{S}(\vartheta(4, 6, 1)) \\ \mathcal{J}\left(e^{-\frac{|0-8|}{1}}\right) \geq \mathcal{S}\left(e^{-\frac{|4-6|}{1}}\right) \\ -0.1233 \geq -0.2500, \end{array}$$

and similar in other cases. This shows that mappings Γ *and* Υ *are fuzzy* $(\mathcal{J}, \mathcal{S})$*-proximal. However, the following shows that* Γ *and* Υ *are not proximal in FMS. We know that*

$$\begin{array}{c} \vartheta(0, \Gamma 2, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(4, \Upsilon 2, 1) \\ \vartheta(8, \Gamma 4, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(6, \Upsilon 4, 1). \end{array}$$

If there exists a non-negative number $\alpha = 0.5 \in (0, 1)$, *then*

$$\begin{array}{c} \vartheta(\breve{a}_1, \breve{a}_2, \alpha\kappa) \geq \vartheta(\breve{e}_1, \breve{e}_2, \kappa) \\ \vartheta(0, 8, (0.5)1) \geq \vartheta(4, 6, 1) \\ 0.00000 \geq 0.1353. \end{array}$$

This is a contradiction. Hence, mappings Γ *and* Υ *are not fuzzy proximal.*

Example 3. *Let* $(\mathcal{B}, \vartheta, *)$ *be a FMS define by* $\vartheta\ (\grave{u}, n, \kappa) = e^{-\frac{e^{|\grave{u}_1 - \grave{u}_2| + |n_1 - n_2|}}{\kappa}}$ *with ctn as* $s * t = st$. *Let* $\mathcal{M} = \{(0, n); n \in \mathbb{R}\}$ *and* $\mathcal{N} = \{(1, n); n \in \mathbb{R}\}$. *Define mappings* $\Gamma : \mathcal{M} \to \mathcal{N}$ *and* $\Upsilon : \mathcal{M} \to \mathcal{N}$ *as*

$$\Gamma(0, n) = \left(1, \frac{n}{2}\right)$$

and,

$$\Upsilon(0, n) = \left(1, \frac{n}{3}\right).$$

Then, $\vartheta(\mathcal{S}, \mathcal{N}, \kappa) = \vartheta(\grave{u}, n, \kappa) = e^{-\frac{1}{\kappa}}$, $\mathcal{M}_0 = \mathcal{S}$ *and* $\mathcal{N}_0 = \mathcal{N}$. *Then, clearly* $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$. *Define the functions* $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ *by*

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{2^{\ln 2t}} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{ and } \mathcal{S}(l) = \left\{ \begin{array}{c} \frac{1}{2^{\ln t}} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

The mappings Γ *and* Υ *are fuzzy* $(\mathcal{J}, \mathcal{S})$*-proximal. Here, we show that* Γ *and* Υ *are not fuzzy proximal. We have*

$$\begin{array}{c} \vartheta((0,0), \Gamma(0,0), 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta((0,0), \Upsilon(0,6), 1) \\ \vartheta((0,3), \Gamma(0,6), 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta((0,2), \Upsilon(0,6), 1). \end{array}$$

Then, there exists a non-negative number $\lambda = 0.2$ *such that*

$$\begin{array}{c} \vartheta(\breve{a}_1, \breve{a}_2, \lambda\kappa) \geq \vartheta(\breve{e}_1, \breve{e}_2, \kappa) \\ \vartheta((0,0), (0,3), 1(0.2)) \geq (\theta((0,0), (0,2), 1)) \\ 0.0000 \geq 0.1353, \end{array}$$

which is a contradiction. Hence, Γ *and* Υ *are not fuzzy proximal.*

To obtain the proofs of the key results, the following lemmas will be used.

Lemma 1 ([14])**.** Let $(\mathcal{B}, \vartheta, *)$ be a FMS and $\{\breve{a}_n\} \subset \mathcal{B}$ be a sequence verifying $\lim_{n\to\infty} \vartheta\ (\breve{a}_n, \breve{a}_{n+1}, \kappa) = 1$. If the sequence $\{q_n\}$ is not a Cauchy sequence, then there are subsequences $\{\breve{a}_{n_k}\}$, $\{\breve{a}_{q_k}\}$ and $\xi > 0$ such that

$$\lim_{k\to\infty} \vartheta(\breve{a}_{n_k+1}, \breve{a}_{q_k+1}, \kappa) = \xi. \tag{2}$$

$$\lim_{k\to\infty} \vartheta(\breve{a}_{n_k}, \breve{a}_{n_{q_k}}, \kappa) = \vartheta(\breve{a}_{n_k+1}, \breve{a}_{q_k}, \kappa) = \vartheta(\breve{a}_{n_k}, \breve{a}_{q_k+1}, \kappa) = \xi. \tag{3}$$

Lemma 2 ([14])**.** *Let* $\mathcal{J} : (0, 1] \to \mathbb{R}$. *Then the following conditions are equivalent:*

(i) $\inf_{t>\varepsilon} \mathcal{J}(t) > -\infty$ *for every* $\varepsilon \in (0, 1)$,
(ii) $\lim_{t\to\varepsilon-} \inf \mathcal{J}(t) > -\infty$ *for any* $\varepsilon \in (0, 1)$,
(iii) $\lim_{n\to\infty} \mathcal{J}(t_n) = -\infty$ *implies that* $\lim_{n\to\infty} t_n = 1$.

Lemma 3. *Assume* $\{\breve{a}_n\}$ *is a sequence such that* $\lim_{n\to\infty} \vartheta\ (\breve{a}_n, \breve{a}_{n+1}, \kappa) = 1$ *and the mappings* $\Upsilon : \mathcal{M} \to \mathcal{N}$ *and* $\Gamma : \mathcal{M} \to \mathcal{N}$ *satisfying (1). If the functions* $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ *with*

(1) $\limsup_{t\to\epsilon+} \mathcal{S}(t) < \mathcal{J}(\in +)$ *for any* $\epsilon > 0$.

Then $\{\breve{a}_n\}$ *is a Cauchy sequence.*

Proof. Let us suppose that the sequence $\{\breve{a}_n\}$ is not a Cauchy sequence; then, by Lemma 1, there exist two subsequences $\{\breve{a}_{n_k}\}$, $\{\breve{a}_{q_k}\}$ of $\{\breve{a}_n\}$ and $\epsilon > 0$ such that the Equations (2) and (3) hold. From Equation (2), we get that $\vartheta\ (\breve{a}_{n_k+1}, \breve{a}_{q_k+1}) > \epsilon$. Since, for $\breve{a}_{n_k}, \breve{a}_{n_k+1}, \breve{a}_{q_k}, \breve{a}_{q_k+1}, \grave{u}_{n_k}, \grave{u}_{q_k}, \grave{u}_{n_k+1}, \grave{u}_{q_k+1} \in \mathcal{M}$, we have

$$\vartheta(\breve{a}_{n_k+1}, \Gamma(\grave{u}_{n_k+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta\big(\breve{a}_{q_k+1}, \Gamma\big(\grave{u}_{q_k+1}\big), \kappa\big)$$
$$\vartheta(\breve{a}_{n_k}, \Upsilon(\grave{u}_{n_k+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta\big(\breve{a}_{q_k}, \Upsilon\big(\grave{u}_{q_k+1}\big), \kappa\big).$$

Thus, from Equation (1), we have

$$\mathcal{J}(\vartheta(\breve{a}_{n_k+1}, \breve{a}_{q_k+1}), \kappa) \geq \mathcal{S}(\vartheta(\breve{a}_{n_k}, \breve{a}_{q_k}, \kappa))$$

for all $k \geq 1$.Let $q_k = \vartheta\ (\breve{a}_{n_k+1}, \breve{a}_{q_k+1}, \kappa)$ and $q_{k-1} = \vartheta\ (\breve{a}_{n_k}, \breve{a}_{q_k}, \kappa)$, we have

$$\mathcal{J}(q_k) \geq \mathcal{S}(q_{k-1}), \text{ for any } k \geq 1. \tag{4}$$

From Equations (2) and (3), we have $\lim_{k\to\infty} q_k = \epsilon$ and $\lim_{k\to\infty} n_k = \epsilon$. From Equation (4), we get that

$$\mathcal{J}(\epsilon+) = \lim_{k\to\infty} \mathcal{J}(q_k) \geq \lim_{k\to\infty}\inf \mathcal{S}(q_{k-1}) \geq \lim_{c\to k}\inf \mathcal{S}(c). \tag{5}$$

This is a contradiction to condition (i). That is, $\{\breve{a}_n\}$ is a Cauchy sequence in $\mathcal{M}$. □

Theorem 1. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0$ and $\mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ satisfying the following conditions*

(i) Υ *dominates* Γ *and are fuzzy* $(\mathcal{J}, \mathcal{S})$*-proximal,*
(ii) Γ *and* Υ *are compact proximal,*
(iii) $\mathcal{J}$ *is non-decreasing function and* $\lim\inf_{t\to\epsilon+} \mathcal{S}(t) > \mathcal{J}(\epsilon+)$ *for any* $\epsilon > 0$*,*
(iv) Γ *and* Υ *are continuous,*
(v) $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Then, Υ *and* Γ *have a unique element* $\grave{u} \in \mathcal{M}$ *such that*

$$\vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa),$$
$$\vartheta(\grave{u}, \Gamma\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Proof. Let $\grave{u}_0 \in \mathcal{M}_0$. Since $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$ guarantees the existence of an element $\grave{u}_1 \in \mathcal{M}_0$ such that $\Gamma\grave{u}_0 = \Upsilon\grave{u}_1$. Also, we have $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$, $\exists$ an element $\grave{u}_2 \in \mathcal{M}_0$ such that $\Gamma\grave{u}_1 = \Upsilon\grave{u}_2$. This process of existence of points in $\mathcal{M}_0$ is implied to have a sequence $\{\grave{u}_n\} \subseteq \mathcal{M}_0$ such that

$$\Gamma\grave{u}_{n-1} = \Upsilon\grave{u}_n$$

for all positive integral values of n, since $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Since $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, $\exists$ an element $\breve{a}_n$ in $\mathcal{M}_0$ such that

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of $\grave{u}_n$ and $\breve{a}_n$ that

$$\vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa),$$
$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa).$$

If,

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \tag{6}$$

See that, if $\exists$ some $n \in \mathbb{N}$ such that $\breve{a}_n = \breve{a}_{n-1}$, then from Equation (6), the point $\breve{a}_n$ is a common best proximity point of the mappings Γ and Υ. On the other hand, if $\breve{a}_{n-1} \neq \breve{a}_n$ for all $n \in \mathbb{N}$, then from Equation (6), we have

$$\begin{aligned} \vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa) \\ \vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \end{aligned}$$

Thus, from Equation (1), we have

$$\mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) \geq \mathcal{S}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa)), \tag{7}$$

for all $\breve{a}_{n-1}, \breve{a}_n, \breve{a}_{n+1}, \grave{u}_{n+1}, \grave{u}_n \in \mathcal{M}$. Let $\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa) = q_n$, we have

$$\mathcal{J}(q_n) \geq \mathcal{S}(q_{n-1}) > \mathcal{J}(q_{n-1}).$$

Since $\mathcal{J}$ is non-decreasing, from Equation (7), we get $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is positive and strictly non-decreasing. Hence, it converges to some element $q \geq 0$. We show that $q = 0$. Suppose on the contrary that $q > 0$ and from Equation (7), we get the equation below:

$$\mathcal{J}(\varepsilon+) = \lim_{n\to\infty} \mathcal{J}(q_n) \geq \lim_{n\to\infty} \mathcal{S}(q_{n-1}) \geq \lim_{n\to q+} \inf \mathcal{S}(t).$$

This contradicts to assumption (iii), hence, $q = 1$ and $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}_{n+1}, \kappa) = 1$. By assumption (iii) and Lemma 3, we deduce that $\{\breve{a}_n\}$ is a Cauchy sequence. Since $(\mathcal{B}, \vartheta, *)$ is a CFMS, $\mathcal{M} \subseteq \mathcal{B}$. Since $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, there exists an element $\breve{a}^*$ in $\mathcal{M}$ such that $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}^*) = 0$. Moreover,

$$\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa).$$

Also,

$$\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Upsilon(\grave{u}_n), \kappa).$$

Therefore, $\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ and also $\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ as $n \to \infty$. As Γ and Υ commute proximally, $\Upsilon\breve{a}^*$ and $\Gamma\breve{a}^*$ are identical. Since $\mathcal{N}$ is AC with respect to $\mathcal{M}$, $\exists$ a subsequence $\{\Upsilon(\grave{u}_{n_k})\}$ of $\{\Upsilon(\grave{u}_n)\}$ and $\{\Gamma(\grave{u}_{nk})\}$ of $\{\Gamma(\grave{u}_n)\}$ such that $\Upsilon(\grave{u}_{n_k}) \to \check{e}^* \in \mathcal{N}$ and $\Gamma(\grave{u}_{n_k}) \to \check{e}^* \in \mathcal{N}$ as $k \to \infty$. Moreover, by letting $k \to \infty$ in the below equation,

$$\begin{aligned} \vartheta(\check{e}^*, \Gamma(\grave{u}_{n_k}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\check{e}^*, \Upsilon(\grave{u}_{n_k}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \end{aligned} \tag{8}$$

we have

$$\vartheta(\check{e}^*, \breve{a}^*, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Since, $\breve{a}^* \in \mathcal{M}_0$, so $\Gamma(\breve{a}^*) \in \Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\exists\, \xi \in \mathcal{M}_0$. Similarly $\breve{a}^* \in \mathcal{M}_0$, so $\Upsilon(\breve{a}^*) \in \Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\exists\, \xi \in \mathcal{M}_0$ such that

$$\begin{aligned} \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa), \\ \vartheta(\xi, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\xi, \Upsilon(\breve{a}^*), \kappa). \end{aligned} \tag{9}$$

Now, by Equations (8), (9) and (1), we have

$$\mathcal{J}(\vartheta(\breve{a}^*, \xi, \kappa)) \geq \mathcal{S}(\vartheta(\breve{a}^*, \xi, \kappa)) < \mathcal{J}(\vartheta(\breve{a}^*, \xi, \kappa)).$$

Since $\mathcal{J}$ is non-decreasing function, we have

$$\vartheta(\breve{a}^*, \xi, \alpha\kappa) \geq \vartheta(\breve{a}^*, \xi, \kappa) > \vartheta(\breve{a}^*, \xi, \kappa).$$

This implies $\breve{a}^*$ and ξ are identical. Finally, by Equation (6), we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of mappings Υ and Γ. □

Theorem 2. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0, \mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ satisfying the following conditions:*

(i) Υ *dominates* Γ *and are fuzzy* $(\mathcal{J}, \mathcal{S})$*-proximal.*
(ii) Γ *and* Υ *are compact proximal.*
(iii) $\mathcal{J}$ *is non-decreasing and* $\{\mathcal{J}(t_n)\}$ *and* $\{\mathcal{S}(t_n)\}$ *are convergent sequences such that* $\lim_{n\to\infty} \mathcal{J}(t_n) = \lim_{n\to\infty} \mathcal{S}(t_n)$, *then* $\lim_{n\to\infty} t_n = 1$.
(iv) Γ *and* Υ *are continuous.*
(v) $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Then Υ *and* Γ *have a unique element* $\grave{u} \in \mathcal{M}$ *such that*

$$\begin{aligned} \vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\grave{u}, \Gamma\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem 1, we get

$$\mathcal{J}(q_n) \geq \mathcal{S}(q_{n-1}) < \mathcal{J}(q_{n-1}). \tag{10}$$

By Equation (10), we infer that $\{\mathcal{J}(q_n)\}$ is a strictly nondecreasing sequence (in short, sds). We have two cases here; either the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, or not. If $\{\mathcal{J}(q_n)\}$ is not bounded above, then

$$\inf_{w_n > \varepsilon} \mathcal{J}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma 1 that $q_n \to 1$as $n \to \infty$. Secondly, if the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, then it is a convergent sequence. By Equation (10), the sequence $\{\mathcal{S}(q_n)\}$ also converges. Furthermore, both have the same limit. By condition (iii), we get $\lim_{n\to\infty} q_n = 1$, or $\lim_{n\to\infty} \vartheta(\breve{a}_n, \grave{u}_{n+1}, \kappa) = 1$, for any sequence $\{\breve{a}_n\}$ in $\mathcal{M}$. Now, following the proof of Theorem 1, we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of the mapping Υ and Γ. □

3.2. Fuzzy $(\mathcal{J}, \mathcal{S})$-Interpolative Reich–Rus–Ciric-Type Proximal Contractions

Let $\mathcal{M}$ and $\mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$. The mappings $\Upsilon : \mathcal{M} \to \mathcal{N}$ and $\Gamma : \mathcal{M} \to \mathcal{N}$ are called fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Reich–Rus–Ciric-type proximally contractiona if

$$\begin{aligned} \vartheta(\breve{a}_1, \Gamma\grave{u}_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) &= \vartheta(\check{e}_1, \Upsilon\grave{u}_1, \kappa) \\ \vartheta(\breve{a}_2, \Gamma\grave{u}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) &= \vartheta(\check{e}_2, \Upsilon\grave{u}_2, \kappa) \\ \mathcal{J}(\vartheta(\breve{a}_1, \breve{a}_2, \kappa)) \geq \mathcal{S}\Big((\vartheta(\check{e}_1, \check{e}_2, \kappa))^{\alpha} &(\vartheta(\check{e}_1, \breve{a}_1, \kappa))^{\beta} (\vartheta(\check{e}_2, \breve{a}_2, \kappa))^{1-\alpha-\beta} \Big) \end{aligned} \tag{11}$$

for all $\breve{a}_1, \breve{a}_2, \check{e}_1, \check{e}_2, \grave{u}_1, \grave{u}_2 \in \mathcal{M}$.

Example 4. *Let $(\mathcal{B}, \vartheta, *)$ be a CFMS defined as $\vartheta\,(\grave{u}, \check{n}, \kappa) = e^{-\frac{|\grave{u}-\check{n}|}{\kappa}}$. Let $\mathcal{M} = \{0, 2, 4, 6, 8, 10\}$ and $\mathcal{N} = \{1, 3, 5, 7, 9, 11\}$. Define mappings $\Gamma : \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ as*

$$\Upsilon(0) = 3, \Upsilon(2) = 5, \Upsilon(4) = 7, \Upsilon(6) = 3, \Upsilon(8) = 9, \Upsilon(10) = 11,$$

and

$$\Gamma(0) = 3, \Gamma(2) = 1, \Gamma(4) = 9, \Gamma(6) = 7, \Gamma(8) = 5, \Gamma(10) = 11.$$

Then, $\vartheta(\mathcal{M}, \mathcal{N}, \kappa) = e^{-\frac{1}{\kappa}}$, $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{N}_0 = \mathcal{N}$. Then clearly $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$. Define the functions $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

Under the conditions of Example 2, Γ and Υ are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Reich–Rus–Ciric-type proximal in FMS. However, the following shows that Γ and Υ are not fuzzy interpolative Reich–Rus–Ciric type proximal. We know that

$$\begin{aligned} \vartheta(0, \Gamma 2, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(4, \Upsilon 2, 1) \\ \vartheta(8, \Gamma 4, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(6, \Upsilon 4, 1). \end{aligned}$$

Then there exists a non negative number $\lambda \in (0, \frac{1}{2}]$ such that

$$\begin{aligned} \vartheta(\breve{a}_1, \breve{a}_2, \lambda\kappa) &\geq (\vartheta(\breve{e}_1, \breve{e}_2, \kappa))^{\alpha} (\vartheta(\breve{e}_1, \breve{a}_1, \kappa))^{\beta} (\vartheta(\breve{e}_2, \breve{a}_2))^{1-\alpha-\beta} \\ \vartheta(0, 8, (0.2)1) &\geq (\vartheta(4, 6, 1))^{\frac{1}{2}} (\vartheta(4, 0, 1))^{\frac{1}{3}} (\vartheta(6, 8, 1))^{1-\frac{1}{2}-\frac{1}{3}} \\ 0.0000 &\geq 0.0701 \\ 0 &\geq 0.0701, \end{aligned}$$

which is a contradiction. Hence, Γ and Υ are not fuzzy interpolative Reich–Rus–Ciric-type proximal.

Theorem 3. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0, \mathcal{N}_0 \neq \emptyset$. Let Γ: $\mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$, satisfying the following conditions*

(i) *Υ dominates Γ and are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Riech–Rus–Ciric-type proximal;*
(ii) *Γ and Υ are compact proximal;*
(iii) *$\mathcal{J}$ is a nondecreasing function, and $\liminf_{t\to\epsilon+} \mathcal{S}(t) > \mathcal{J}(\epsilon+)$ for any $\epsilon > 0$;*
(iv) *Γ and Υ are continuous;*
(v) *$\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.*

Then, Υ and Γ have a unique element $\grave{u} \in \mathcal{M}$ such that

$$\begin{aligned} \vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\grave{u}, \Gamma\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned}$$

Proof. Let $\grave{u}_0 \in \mathcal{M}_0$. Since $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$ guarantees the existence of an element $\grave{u}_1 \in \mathcal{M}_0$ such that, $\Gamma\grave{u}_0 = \Upsilon\grave{u}_1$. Also, we have $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$, $\exists$ an element $\grave{u}_2 \in \mathcal{M}_0$ such that $\Gamma\grave{u}_1 = \Upsilon\grave{u}_2$. This process of existence of points in $\mathcal{M}_0$ is implied to have a sequence $\{\grave{u}_n\} \subseteq \mathcal{M}_0$ such that

$$\Gamma\grave{u}_{n-1} = \Upsilon\grave{u}_n,$$

for all positive intergral values of n, because $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Since $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, $\exists$ an element $\breve{a}_n$ in $\mathcal{M}_0$ such that

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of $\grave{u}_n$ and $\breve{a}_n$ that

$$\begin{aligned} \vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa), \\ \vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa), \end{aligned}$$

if

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \tag{12}$$

See that if $\exists$ some $n \in \mathbb{N}$ such that $\breve{a}_n = \breve{a}_{n-1}$, then by Equation (12), the point $\breve{a}_n$ is a common best proximity point of the mappings Γ and Υ. On the other hand, if $\breve{a}_{n-1} \neq \breve{a}_n$ for all $n \in \mathbb{N}$, then by Equation (12), we get

$$\begin{aligned} \vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa), \\ \vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \end{aligned}$$

Thus, by Equation (11), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) \geq \mathcal{S}\Big((\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{\alpha}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\beta}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha-\beta}\Big). \\ \mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) \geq \mathcal{S}\Big((\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\beta}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha}\Big) \end{aligned} \tag{13}$$

for all $\breve{a}_{n-1}, \breve{a}_n, \breve{a}_{n+1}, \grave{u}_n, \grave{u}_{n+1} \in \mathcal{M}$. Since, $\mathcal{S}(t) > \mathcal{J}(t)$ for all $t > 0$, by Equation (13), we have

$$\mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n)) > \mathcal{J}\Big((\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\beta}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\beta}\Big).$$

Thus, $\mathcal{J}$ is non-decreasing function, and we get

$$\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa) > (\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\beta}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\beta}.$$

This implies that

$$(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{1-\beta} > (\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\beta}.$$

Let $(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) = q_n$, we have

$$\mathcal{J}(q_n) \geq \mathcal{S}\Big((q_n)^{\beta}(q_{n-1})^{1-\beta}\Big) > \mathcal{J}\Big((q_n)^{\beta}(q_{n-1})^{1-\beta}\Big).$$

This implies $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is PSD. Thus, it converges to some element $q \geq 1$. We show that $q = 1$. On the contrary, let $q > 1$ so that by Equation (13), we get the following:

$$\mathcal{J}(\varepsilon+) = \lim_{n\to\infty} \mathcal{J}(q_n) \geq \lim_{n\to\infty} \mathcal{S}\Big((q_n)^{\beta}(q_{n-1})^{1-\beta}\Big) \geq \lim_{t\to q+} \inf \mathcal{S}(t).$$

This contradicts condition (iii); hence, $q = 1$ and $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}_{n+1}, \kappa) = 1$. By the condition (iii) and Lemma 3, we deduce that $\{\breve{a}_n\}$ is a Cauchy sequence. Since $(\mathcal{B}, \vartheta, *)$ is a CFMS and $\mathcal{M} \subseteq \mathcal{B}$. Since, $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, there exists an element $\breve{a}^*$ in $\mathcal{M}$ such that $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}^*, \kappa) = 1$. Moreover,

$$\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa).$$

Also,

$$\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Upsilon(\grave{u}_n), \kappa).$$

Furthermore, $\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ and also $\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ as $n \to \infty$. As Γ and Υ CP, $\Upsilon\breve{a}^*$ and $\Gamma\breve{a}^*$ are identical. Since $\mathcal{N}$ is AC with respect to $\mathcal{M}$, there exists a subsequence $\{\Upsilon(\grave{u}_{n_k})\}$ of $\{\Upsilon(\grave{u}_n)\}$ and $\{\Gamma(\grave{u}_{nk})\}$ of $\{\Gamma(\grave{u}_n)\}$ Such that $\Upsilon(\grave{u}_{n_k}) \to \check{e}^* \in \mathcal{N}$ and $\Gamma(\grave{u}_{n_k}) \to \check{e}^* \in \mathcal{N}$ as $k \to \infty$. Moreover, by letting $k \to \infty$ in the below equation,

$$\begin{aligned} \vartheta(\check{e}^*, \Gamma(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) \\ \vartheta(\check{e}^*, \Upsilon(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned} \tag{14}$$

We have,

$$\vartheta(\check{e}^*, \breve{a}^*, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Since, $\breve{a}^* \in \mathcal{M}_0$, so $\Gamma(\breve{a}^*) \in \Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\exists\, \xi \in \mathcal{M}_0$. Similarly $\breve{a}^* \in \mathcal{M}_0$, so $\Upsilon(\breve{a}^*) \in \Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and there exists $\xi \in \mathcal{M}_0$ such that

$$\begin{aligned} \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa) = \vartheta(\xi, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\xi, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned} \tag{15}$$

Now, bearing in mind Equations (14) and (15), from (11), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\breve{a}^*, \xi, \kappa)) &\geq \mathcal{S}\Big((\vartheta(\breve{a}^*, \xi, \kappa))^{\alpha}(\vartheta(\breve{a}^*, \xi, \kappa))^{\beta}(\vartheta(\breve{a}^*, \xi, \kappa))^{1-\alpha-\beta}\Big) \\ &\geq \mathcal{S}(\vartheta(\breve{a}^*, \xi, \kappa)) > \vartheta(\breve{a}^*, \xi, \kappa). \end{aligned}$$

Since $\mathcal{J}$ is non-decreasing function, we have

$$\vartheta(\breve{a}^*, \xi, \alpha\kappa) \geq \vartheta(\breve{a}^*, \xi, \kappa) > \vartheta(\breve{a}^*, \xi, \kappa).$$

This implies $\breve{a}^*$ and ξ are identical. Finally, by Equation (12), we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of mappings Υ and Γ. □

Theorem 4. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0$, $\mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$, satisfying the following conditions*

(i) *Υ dominates Γ and are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Riech–Rus–Ciric-type proximal;*
(ii) *Γ and Υ are compact proximal;*
(iii) *$\mathcal{J}$ is non-decreasing and $\{\mathcal{J}(t_n)\}$ and $\{\mathcal{S}(t_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{J}(t_n) = \lim_{n\to\infty} \mathcal{S}(t_n)$, then $\lim_{n\to\infty} t_n = 1$;*
(iv) *Γ and Υ are continuous;*
(v) *$\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.*

Then, Υ and Γ have a unique element $\grave{u} \in \mathcal{M}$ such that

$$\begin{aligned} \vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) \\ \vartheta(\grave{u}, \Gamma\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned}$$

Proof. Proceeding from the proof of Theorem 3, we have

$$\mathcal{J}(q_n) \geq \mathcal{S}\Big((q_{n-1})^{1-\beta}(q_n)^{\beta}\Big) > \mathcal{J}\Big((q_{n-1})^{1-\beta}(q_n)^{\beta}\Big). \tag{16}$$

By Equation (16), we infer that $\{\mathcal{J}(q_n)\}$ is sds. We have two cases here; either the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, or it is not. If $\{\mathcal{J}(q_n)\}$ is not bounded below, then

$$\inf_{w_n > \varepsilon} \mathcal{J}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma 1 that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, then it is a convergent sequence. By Equation (16), the sequence

$\{\mathcal{S}(q_n)\}$ also converges. Furthermore, both have the same limit. By condition (iii), we get $\lim_{n\to\infty} q_n = 1$ or $\lim_{n\to\infty} \vartheta(\breve{a}_n, \grave{u}_{n+1}, \kappa) = 1$, for any sequence $\{\breve{a}_n\}$ in $\mathcal{M}$. Now, following the proof of Theorem 3, we obtain

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of the mapping Υ and Γ. □

3.3. Fuzzy $(\mathcal{J}, \mathcal{S})$-Kannan Type Proximal Contraction

Let $\mathcal{M}\ \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$. The mappings $\Upsilon : \mathcal{M} \to \mathcal{N}$ and $\Gamma : \mathcal{M} \to \mathcal{N}$ are called fuzzy $(\mathcal{J}, \mathcal{S})$-Kannan-type proximal contractions if

$$\begin{array}{c} \vartheta(\breve{a}_1, \Gamma\grave{u}_1, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\check{e}_1, \Upsilon\grave{u}_1, \kappa) \\ \vartheta(\breve{a}_2, \Gamma\grave{u}_2, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\check{e}_2, \Upsilon\grave{u}_2, \kappa) \\ \mathcal{J}(\vartheta(\breve{a}_1, \breve{a}_2)) \geq \mathcal{S}\Big((\vartheta(\check{e}_1, \breve{a}_1))^{\alpha}(\vartheta(\check{e}_2, \breve{a}_2))^{1-\alpha}\Big) \end{array} \tag{17}$$

for all $\breve{a}_1, \breve{a}_2, \check{e}_1, \check{e}_2, \grave{u}_1, \grave{u}_2 \in \mathcal{M}$.

Example 5. *Let $(\mathcal{B}, \vartheta, *)$ be a CFMS with $\vartheta\ (\grave{u}, \check{n}, \kappa) = e^{-\frac{|\grave{u}-\check{n}|}{\kappa}}$. Let $\mathcal{M} = \{0, 2, 4, 6, 8, 10\}$ and $\mathcal{N} = \{1, 3, 5, 7, 9, 11\}$. Define mappings $\Gamma : \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ as*

$$\Upsilon(0) = 3, \Upsilon(2) = 5, \Upsilon(4) = 7, \Upsilon(6) = 3, \Upsilon(8) = 9, \Upsilon(10) = 11,$$

and

$$\Gamma(0) = 3, \Gamma(2) = 1, \Gamma(4) = 9, \Gamma(6) = 7, \Gamma(8) = 5, \Gamma(10) = 11.$$

Then, $\vartheta(\mathcal{M}, \mathcal{N}, \kappa) = e^{-\frac{1}{\kappa}}$, $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{N}_0 = \mathcal{N}$. Then clearly $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$. Define the functions $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } t = 1 \end{array} \right\} \text{and } \mathcal{S}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } t = 1 \end{array} \right\}.$$

Under the conditions of Example 2, Γ and Υ are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Kannan-type proximal in FMS. However, the following shows that Γ and Υ are not fuzzy interpolative Kannan-type proximal. We know that

$$\begin{array}{c} \vartheta(0, \Gamma 2, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(4, \Upsilon 2, 1), \\ \vartheta(8, \Gamma 4, 1) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(6, \Upsilon 4, 1). \end{array}$$

Then there exists a non negative number $\lambda = 0.2$ such that

$$\begin{array}{r} \vartheta(\breve{a}_1, \breve{a}_2, \lambda\kappa) \geq \Big((\vartheta(\check{e}_1, \breve{a}_1, \kappa))^{\alpha}(\vartheta(\check{e}_2, \breve{a}_2, \kappa))^{1-\alpha}\Big) \\ \vartheta(0, 8, (0.2)1) \geq \Big((\vartheta(4, 2))^{\frac{1}{2}}(\vartheta(6, 8, \kappa))^{\frac{1}{2}}\Big) \\ 0.0000 \geq 0.0498 \\ 0 \geq 0.0498, \end{array}$$

which is a contradiction. Hence, Γ and Υ are not fuzzy interpolative Kannan-type proximal.

Theorem 5. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0$, $\mathcal{N}_0 \neq \emptyset$. Let Γ: $\mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ satisfying the following conditions:*

(i) *Υ dominates Γ and are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Kannan type proximal;*
(ii) *Γ and Υ are compact proximal;*
(iii) *$\mathcal{J}$ is non-decreasing function and $\liminf_{t\to\epsilon+} \mathcal{S}(t) > \mathcal{J}(\epsilon+)$ for any $\epsilon > 0$;*

(iv) Γ *and* Υ *are continuous;*
(v) $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Then, Υ *and* Γ *have a unique element* $\grave{u} \in \mathcal{M}$ *such that*

$$\vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa),$$
$$\vartheta(\grave{u}, \Gamma\grave{u}, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Proof. Let $\grave{u}_0 \in \mathcal{M}_0$. Since $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$ guarantees the existence of an element $\grave{u}_1 \in \mathcal{M}_0$ s.t. $\Gamma\grave{u}_0 = \Upsilon\grave{u}_1$. Also, we have $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$, $\exists$ an element $\grave{u}_2 \in \mathcal{M}_0$ such that $\Gamma\grave{u}_1 = \Upsilon\grave{u}_2$. This process of existence of points in $\mathcal{M}_0$ implies to have a sequence $\{\grave{u}_n\} \subseteq \mathcal{M}_0$ such that

$$\Gamma\grave{u}_{n-1} = \Upsilon\grave{u}_n,$$

for all positive intergral values of n, because $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Since $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, $\exists$ an element $\breve{a}_n$ in $\mathcal{M}_0$ such that

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of $\grave{u}_n$ and $\breve{a}_n$ that

$$\vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa),$$
$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa),$$

if

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \tag{18}$$

Notice that, if there exists some $n \in \mathbb{N}$ such that $\breve{a}_n = \breve{a}_{n-1}$, then from Equation (18), the point $\breve{a}_n$ is a common best proximity point of the mappings Γ and Υ. On the other hand, if $\breve{a}_{n-1} \neq \breve{a}_n$ for all $n \in \mathbb{N}$, then from Equation (18), we get

$$\vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_n), \kappa)$$
$$\vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_{n-1}), \kappa).$$

Thus, from Equation (17), we have

$$\mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) \geq \mathcal{S}\Big((\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\alpha}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha}\Big). \tag{19}$$

for all $\breve{a}_{n-1}, \breve{a}_n, \breve{a}_{n+1}, \grave{u}_n, \grave{u}_{n+1} \in \mathcal{M}$. Since $\mathcal{S}(t) > \mathcal{J}(t)$ for all $t > 0$, from Equation (19), we have

$$\mathcal{J}(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa)) > \mathcal{J}\Big((\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\alpha}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha}\Big).$$

Thus, $\mathcal{J}$ is non-decreasing function, and we get

$$\vartheta(\breve{a}_{n+1}, \breve{a}_n, \lambda\kappa) > (\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa))^{\alpha}(\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha}.$$

This implies that

$$(\vartheta(\breve{a}_{n+1}, \breve{a}_n, \lambda\kappa))^{1-\alpha} > (\vartheta(\breve{a}_n, \breve{a}_{n-1}, \kappa))^{1-\alpha}.$$

Let $\vartheta(\breve{a}_{n+1}, \breve{a}_n, \kappa) = q_n$, we have

$$\mathcal{J}(q_n) \geq \mathcal{S}\Big((q_n)^{\alpha}(q_{n-1})^{1-\alpha}\Big) > \mathcal{J}((q_n)^{\alpha}(q_{n-1})^{1-\alpha}).$$

This implies $q_n > q_{n-1}$ for all $n \in \mathbb{N}$. This shows that the sequence $\{q_n\}$ is positive and strictly non-decreasing. Hence, it converges to some element $q \geq 1$. We show that $q = 1$. On the contrary, let $q > 1$; from Equation (19), we get the following:

$$\mathcal{J}(\varepsilon+) = \lim_{n\to\infty} \mathcal{J}(q_n) \geq \lim_{n\to\infty} \mathcal{S}\Big((q_n)^{\alpha}(q_{n-1})^{1-\alpha}\Big) \geq \lim_{t\to q+} \inf \mathcal{S}(t).$$

This contradicts assumption (iii). Hence, $q = 1$ and $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}_{n+1}, \kappa) = 1$. By condition (iii) and Lemma 3, we deduce that $\{\breve{a}_n\}$ is a Cauchy sequence. Since $(\mathcal{B}, \vartheta, *)$ is a CFMS and $\mathcal{M} \subseteq \mathcal{B}$. Since, $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, there exists an element $\breve{a}^*$ in $\mathcal{M}$ such that $\lim_{n\to\infty} \vartheta(\breve{a}_n, \breve{a}^*) = 0$. Moreover,

$$\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa).$$

Also,

$$\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \geq \vartheta(\breve{a}^*, \breve{a}_n, \kappa).\vartheta(\breve{a}_n, \Upsilon(\grave{u}_n), \kappa).$$

Therefore, $\vartheta(\breve{a}^*, \Upsilon(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ and $\vartheta(\breve{a}^*, \Gamma(\grave{u}_n), \kappa) \to \vartheta(\breve{a}^*, \mathcal{N}, \kappa)$ as $n \to \infty$. As Γ and Υ are compact proximal, $\Upsilon\breve{a}^*$ and $\Gamma\breve{a}^*$ are identical. Since $\mathcal{N}$ is AC with respect to $\mathcal{M}$, there exists a subsequence $\{\Upsilon(\grave{u}_{n_k})\}$ of $\{\Upsilon(\grave{u}_n)\}$ and $\{\Gamma(\grave{u}_{nk})\}$ of $\{\Gamma(\grave{u}_n)\}$ such that $\Upsilon(\grave{u}_{n_k}) \to \breve{e}^* \in \mathcal{N}$ and $\Gamma(\grave{u}_{n_k}) \to \breve{e}^* \in \mathcal{N}$ as $k \to \infty$. Moreover, by letting $k \to \infty$ in the below equation,

$$\begin{aligned} \vartheta(\breve{e}^*, \Gamma(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) \\ \vartheta(\breve{e}^*, \Upsilon(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \end{aligned} \tag{20}$$

we have

$$\vartheta(\breve{e}^*, \breve{a}^*, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Since, $\breve{a}^* \in \mathcal{M}_0$, so $\Gamma(\breve{a}^*) \in \Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and there exists $\xi \in \mathcal{M}_0$. Similarly $\breve{a}^* \in \mathcal{M}_0$, so $\Upsilon(\breve{a}^*) \in \Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and there exists $\xi \in \mathcal{M}_0$ such that

$$\begin{aligned} \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa) &= \vartheta(\xi, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) \\ \vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) &= \vartheta(\xi, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned} \tag{21}$$

Now, bearing in mind Equations (20) and (21), from (17)), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\breve{a}^*, \xi), \kappa) &\geq \mathcal{S}\Big((\vartheta(\breve{a}^*, \xi, \kappa))^{\alpha}(\vartheta(\breve{a}^*, \xi, \kappa))^{1-\alpha}\Big) \\ &\geq \mathcal{S}(\vartheta(\breve{a}^*, \xi, \kappa)) > \vartheta(\breve{a}^*, \xi, \kappa). \end{aligned}$$

Since $\mathcal{J}$ is non-decreasing function, we have

$$\vartheta(\breve{a}^*, \xi, \alpha\kappa) \geq \vartheta(\breve{a}^*, \xi, \kappa) > \vartheta(\breve{a}^*, \xi, \kappa).$$

This implies $\breve{a}^*$ and ξ are identical. Finally, from Equation (18), we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of mappings Υ and Γ. □

Theorem 6. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Suppose that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0, \mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$, satisfying the following conditions:*

(i) *Υ dominates Γ and are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Kannan-type proximal;*
(ii) *Γ and Υ are compact proximal;*
(iii) *$\mathcal{J}$ is non-decreasing and $\{\mathcal{J}(t_n)\}$ and $\{\mathcal{S}(t_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{J}(t_n) = \lim_{n\to\infty} \mathcal{S}(t_n)$, then $\lim_{n\to\infty} t_n = 1$;*
(iv) *Γ and Υ are continuous;*
(v) *$\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.*

Then, Υ and Γ have a unique element $\grave{u} \in \mathcal{M}$ such that

$$\begin{aligned} \vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\grave{u}, \Gamma\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem 5, we get

$$\mathcal{J}(q_n) \leq \mathcal{S}\Big(\big(q_{n-1}\big)^{1-\beta}\big(q_n\big)^{\beta}\Big) < \mathcal{J}\Big(\big(q_{n-1}\big)^{1-\beta}\big(q_n\big)^{\beta}\Big). \tag{22}$$

From Equation (16), we infer that $\{\mathcal{J}(q_n)\}$ is sds. We have two cases here; either the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, or it is not. If $\{\mathcal{J}(q_n)\}$ is not bounded above, then

$$\inf_{w_n > \varepsilon} \mathcal{J}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma 1 that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, then it is a convergent sequence. From Equation (16), the sequence $\{\mathcal{S}(q_n)\}$ also converges. Furthermore, both have the same limit. From condition (iii), we get $\lim_{n\to\infty} q_n = 1$, or $\lim_{n\to\infty} \vartheta(\breve{a}_n, \grave{u}_{n+1}, \kappa) = 1$, for any sequence $\{\breve{a}_n\}$ in $\mathcal{M}$. Now, following the proof of Theorem 5, we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of the mapping Υ and Γ. □

3.4. Fuzzy $(\mathcal{J}, \mathcal{S})$-Interpolative Hardy–Rogers-Type Proximal Contraction

Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$. The mappings $\Upsilon : \mathcal{M} \to \mathcal{N}$ and $\Gamma : \mathcal{M} \to \mathcal{N}$ are called fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Hardy Roger's type proximal contraction if

$$\begin{aligned} \vartheta(\breve{a}_1, \Gamma\grave{u}_1, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_1, \Upsilon\grave{u}_1, \kappa) \\ \vartheta(\breve{a}_2, \Gamma\grave{u}_2, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{e}_2, \Upsilon\grave{u}_2, \kappa) \\ \mathcal{J}(\vartheta(\breve{a}_1, \breve{a}_2, \kappa)) &\leq \mathcal{S}\Big((\vartheta(\breve{e}_1, \breve{e}_2, \kappa))^{\alpha}(\vartheta(\breve{e}_1, \breve{a}_1, \kappa))^{\beta}(\vartheta(\breve{e}_2, \breve{a}_2, \kappa))^{\gamma}\big((\vartheta(\breve{e}_1, \breve{a}_2, \kappa))^{\delta}\vartheta(\breve{e}_2, \breve{a}_1, \kappa)\big)\big)^{1-\alpha-\beta-\gamma}\Big) \end{aligned} \tag{23}$$

for all $\breve{a}_1, \breve{a}_2, \breve{e}_1, \breve{e}_2, \grave{u}_1, \grave{u}_2 \in \mathcal{M}$.

Example 6. *Let $(\mathcal{B}, \vartheta, *)$ be a CFMS with $\vartheta\ (\grave{u}, \breve{n}, \kappa) = e^{-\frac{|\grave{u}-\breve{n}|}{\kappa}}$. Let $\mathcal{M} = \{0, 2, 4, 6, 8, 10\}$ and $\mathcal{N} = \{1, 3, 5, 7, 9, 11\}$. Define mappings $\Gamma : \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ as*

$$\Upsilon(0) = 3, \Upsilon(2) = 5, \Upsilon(4) = 7, \Upsilon(6) = 3, \Upsilon(8) = 9, \Upsilon(10) = 11,$$

and

$$\Gamma(0) = 3, \Gamma(2) = 1, \Gamma(4) = 9, \Gamma(6) = 7, \Gamma(8) = 5, \Gamma(10) = 11.$$

Then, $\vartheta(\mathcal{M}, \mathcal{N}, \kappa) = e^{-\frac{1}{\kappa}}$, $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{N}_0 = \mathcal{N}$. Then clearly, $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$. Define the functions $\mathcal{J}, \mathcal{S} : (0, 1] \to \mathbb{R}$ by

$$\mathcal{J}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t} \text{ if } 0 < t < 1 \\ 1 \text{ if } \ t = 1 \end{array} \right\} \text{and } \mathcal{S}(t) = \left\{ \begin{array}{c} \frac{1}{\ln t^2} \text{ if } 0 < t < 1 \\ 2 \text{ if } \ t = 1 \end{array} \right\}.$$

Under the conditions of Example 2, the mappings Γ and Υ are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Hardy–Rogers-type proximal in FMS. However, the following shows that Γ and Υ are not fuzzy interpolative Hardy–Rogers-type proximal. We know that

$$\begin{aligned} \vartheta(8, \Gamma 4, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(6, \Upsilon 4, \kappa) \\ \vartheta(0, \Gamma 2, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(4, \Upsilon 2, \kappa). \end{aligned}$$

If there exists a non-negative number $\lambda = 0.2$. Then,

$$
\begin{aligned}
&(\vartheta(\breve{a}_1, \breve{a}_2, \lambda\kappa)) \\
&\geq (\vartheta(\breve{e}_1, \breve{e}_2, \kappa))^{\alpha} (\vartheta(\breve{e}_1, \breve{a}_1, \kappa))^{\beta} (\vartheta(\breve{e}_2, \breve{a}_2, \kappa))^{\gamma} \big((\vartheta(\breve{e}_1, \breve{a}_2, \kappa))^{\delta} \vartheta(\breve{e}_2, \breve{a}_1, \kappa) \big) \big)^{1-\alpha-\beta-\gamma} \\
&\vartheta(8, 2, 1(0.2)) \\
&\geq (\vartheta(6, 4, 1))^{0.01} (\vartheta(4, 8, 1))^{0.02} (\vartheta(4, 0, 1))^{0.03} (\vartheta(6, 0, 1))^{0.04} (\vartheta(4, 8, 1))^{0.9} \\
&0.0000 \geq 0.0179,
\end{aligned}
$$

which is a contradiction. Hence, mappings Γ and Υ are not fuzzy interpolative Hardy–Rogers-type proximal.

Theorem 7. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ in a CFMS such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0$, $\mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$, satisfying the following conditions*

- *(i)* Υ *dominates* Γ *and are fuzzy* $(\mathcal{J}, \mathcal{S})$*-interpolative Hardy–Rogers-type proximal;*
- *(ii)* Γ *and* Υ *are compact proximal;*
- *(iii)* $\mathcal{J}$ *is nondecreasing function and* $\limsup_{t\to\epsilon+} \mathcal{S}(t) < \mathcal{J}(\epsilon+)$ *for any* $\epsilon > 0$*;*
- *(iv)* Γ *and* Υ *are continuous;*
- *(v)* $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ *and* $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Then, Υ *and* Γ *have a unique element* $\grave{u} \in \mathcal{M}$ *such that*

$$
\begin{aligned}
\vartheta(\grave{u}, \Upsilon\grave{u}) &= \vartheta(\mathcal{M}, \mathcal{N}), \\
\vartheta(\grave{u}, \Gamma\grave{u}) &= \vartheta(\mathcal{M}, \mathcal{N}).
\end{aligned}
$$

Proof. Let $\grave{u}_0 \in \mathcal{M}_0$. Since $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$ guarantees the existence of an element $\grave{u}_1 \in \mathcal{M}_0$ such that, $\Gamma\grave{u}_0 = \Upsilon\grave{u}_1$. Also, we have $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$, $\exists$ an element $\grave{u}_2 \in \mathcal{M}_0$ such that $\Gamma\grave{u}_1 = \Upsilon\grave{u}_2$. This process of existence of points in $\mathcal{M}_0$ is implied to have a sequence $\{\grave{u}_n\} \subseteq \mathcal{M}_0$ such that

$$\Gamma\grave{u}_{n-1} = \Upsilon\grave{u}_n,$$

for all positive integral values of n, because $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Since $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$, there exists an element $\breve{a}_n$ in $\mathcal{M}_0$ such that

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \text{ for all } n \in \mathbb{N}.$$

Further, it follows from the choice of $\grave{u}_n$ and $\breve{a}_n$ that

$$
\begin{aligned}
\vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa), \\
\vartheta(\breve{a}_n, \Upsilon(\grave{u}_{n+1}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa),
\end{aligned}
$$

if

$$\vartheta(\breve{a}_n, \Gamma\grave{u}_n, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_n), \kappa). \tag{24}$$

Notice that, if there exists some $n \in \mathbb{N}$ such that $\breve{a}_n = \breve{a}_{n-1}$, then by Equation (24), the point $\breve{a}_n$ is a common best proximity point of the mappings Γ and Υ. On the other hand, if $\breve{a}_{n-1} \neq \breve{a}_n$ for all $n \in \mathbb{N}$, then from Equation (24), we obtain

$$
\begin{aligned}
\vartheta(\breve{a}_{n+1}, \Gamma(\grave{u}_{n+1}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_n, \Upsilon(\grave{u}_n), \kappa), \\
\vartheta(\breve{a}_n, \Gamma(\grave{u}_n), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}_{n-1}, \Upsilon(\grave{u}_{n-1}), \kappa).
\end{aligned}
$$

Thus, from Equation (23), we have

$$\begin{aligned}\mathcal{J}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa)) &\geq \mathcal{S}\left(\begin{array}{c}(\vartheta(\breve{a}_n,\breve{a}_{n-1},\kappa))^{\alpha}(\vartheta(\breve{a}_n,\breve{a}_{n+1},\kappa))^{\beta}\\ (\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{\gamma}\left((\vartheta(\breve{a}_n,\breve{a}_n,\kappa))^{\delta}(\vartheta(\breve{a}_{n-1},\breve{a}_{n+1},\kappa))\right)^{1-\alpha-\beta-\gamma-\delta}\end{array}\right)\\ &\geq \mathcal{S}\left(\begin{array}{c}(\vartheta(\breve{a}_n,\breve{a}_{n-1},\kappa))^{\alpha}(\vartheta(\breve{a}_n,\breve{a}_{n+1},\kappa))^{\beta}\\ (\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{\gamma}((\vartheta(\breve{a}_{n-1},\breve{a}_{n+1},\kappa))^{1-\alpha-\beta-\gamma-\delta}\end{array}\right) \qquad (25)\end{aligned}$$

for all $\breve{a}_{n-1},\breve{a}_n,\breve{a}_{n+1},\grave{u}_n,\grave{u}_{n+1}\in\mathcal{M}$. Since $\mathcal{S}(t)>\mathcal{J}(t)$ for all $t>0$, from Equation (25), we have

$$\begin{aligned}&\mathcal{J}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))\\ &>\mathcal{J}\Big((\vartheta(\breve{a}_n,\breve{a}_{n-1},\kappa))^{\alpha}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\beta}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\gamma}((\vartheta(\breve{a}_{n-1},\breve{a}_{n+1},\kappa))^{1-\alpha-\beta-\gamma-\delta}\Big).\end{aligned}$$

Thus, $\mathcal{J}$ is a non-decreasing function, and we get

$$\begin{aligned}&\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))\\ &>(\vartheta(\breve{a}_n,\breve{a}_{n-1},\kappa))^{\alpha}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\beta}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\gamma}((\vartheta(\breve{a}_{n-1},\breve{a}_{n+1},\kappa))^{1-\alpha-\beta-\gamma-\delta}\\ &\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa)\\ &>(\vartheta(\breve{a}_n,\breve{a}_{n-1},\kappa))^{\alpha}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\beta}(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))^{\gamma}((\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa).\vartheta(\breve{a}_n,\breve{a}_{n+1},\kappa))^{1-\alpha-\beta-\gamma-\delta}\\ &\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa)\\ &>(\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{1-\beta-\gamma-\delta}(\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{1-\alpha-\delta}.\end{aligned}$$

This implies that

$$\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa)>(\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{1-\beta-\gamma-\delta}(\vartheta(\breve{a}_{n-1},\breve{a}_n,\kappa))^{1-\alpha-\delta}.$$

Let $(\vartheta(\breve{a}_{n+1},\breve{a}_n,\kappa))=q_n$, we have

$$\begin{aligned}\mathcal{J}(q_n)&\geq\mathcal{S}\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\\ &>\mathcal{J}\Big(\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\Big).\end{aligned}$$

Assume that $q_n>q_{n-1}$ for some $n\geq1$. Since $\mathcal{J}$ is non-decreasing, then by (25), we get $(q_n)>\Big(\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\Big)$. This is not possible. Hence, we obtain $q_n>q_{n-1}$ for all $n\geq1$. Thus, it converges to some element $q\geq1$. We show that $q=1$. On the contrary, let $q>1$; from Equation (25), we get the equation below:

$$\mathcal{J}(\varepsilon+)=\lim_{n\to\infty}\mathcal{J}(q_n)\geq\lim_{n\to\infty}\mathcal{S}\Big(\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\Big)\geq\lim_{t\to q+}\inf\mathcal{S}(t).$$

This contradicts condition (iii), hence $q=1$ and $\lim_{n\to\infty}\vartheta(\breve{a}_n,\breve{a}_{n+1},\kappa)=1$. By condition (iii) and Lemma 3, we deduce that $\{\breve{a}_n\}$ is a Cauchy sequence. Since $(\mathcal{B},\vartheta,*)$ is a CFMS and $\mathcal{M}\subseteq\mathcal{B}$. Since $\Gamma(\mathcal{M}_0)\subseteq\mathcal{N}_0$, $\exists$ an element $\breve{a}^*$ in $\mathcal{M}$ such that $\lim_{n\to\infty}\vartheta(\breve{a}_n,\breve{a}^*,\kappa)=1$. Moreover,

$$\vartheta(\breve{a}^*,\Gamma(\grave{u}_n),\kappa)\geq\vartheta(\breve{a}^*,\breve{a}_n,\kappa).\vartheta(\breve{a}_n,\Gamma(\grave{u}_n),\kappa).$$

Also,

$$\vartheta(\breve{a}^*,\Upsilon(\grave{u}_n),\kappa)\geq\vartheta(\breve{a}^*,\breve{a}_n).\vartheta(\breve{a}_n,\Upsilon(\grave{u}_n)).$$

Therefore, $\vartheta(\breve{a}^*,\Upsilon(\grave{u}_n),\kappa)\to\vartheta(\breve{a}^*,\mathcal{N},\kappa)$ and also $\vartheta(\breve{a}^*,\Gamma(\grave{u}_n),\kappa)\to\vartheta(\breve{a}^*,\mathcal{N},\kappa)$ as $n\to\infty$. As Γ and Υ are compact proximal, $\Upsilon\breve{a}^*$ and $\Gamma\breve{a}^*$ are identical. Since $\mathcal{N}$ is AC with respect to $\mathcal{M}$, $\exists$ a subsequence $\{\Upsilon(\grave{u}_{n_k})\}$ of $\{\Upsilon(\grave{u}_n)\}$ and $\{\Gamma(\grave{u}_{nk})\}$ of $\{\Gamma(\grave{u}_n)\}$ such that $\Upsilon(\grave{u}_{n_k}) \to \breve{e}^* \in \mathcal{N}$ and $\Gamma(\grave{u}_{n_k}) \to \breve{e}^* \in \mathcal{N}$ as $k \to \infty$. Moreover, by letting $k \to \infty$ in the below equation,

$$\begin{aligned} \vartheta(\breve{e}^*, \Gamma(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\breve{e}^*, \Upsilon(\grave{u}_{n_k}), \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \end{aligned} \tag{26}$$

we have

$$\vartheta(\breve{e}^*, \breve{a}^*, \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa).$$

Since, $\breve{a}^* \in \mathcal{M}_0$, so $\Gamma(\breve{a}^*) \in \Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\exists\, \xi \in \mathcal{M}_0$. Similarly, $\breve{a}^* \in \mathcal{M}_0$, so $\Upsilon(\breve{a}^*) \in \Upsilon(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\exists\, \xi \in \mathcal{M}_0$ such that

$$\begin{aligned} \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) \\ \vartheta(\xi, \Gamma(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\xi, \Upsilon(\breve{a}^*), \kappa). \end{aligned} \tag{27}$$

Now, bearing in mind Equations (26) and (27), from (23), we have

$$\begin{aligned} \mathcal{J}(\vartheta(\breve{a}^*, \xi, \kappa)) &\geq \mathcal{S}\Big((\vartheta(\breve{a}^*, \xi, \kappa))^{\alpha} (\vartheta(\breve{a}^*, \xi, \kappa))^{\beta} \Big) \\ &\geq \mathcal{S}(\vartheta(\breve{a}^*, \xi, \kappa)) \\ &> \vartheta(\breve{a}^*, \xi, \kappa). \end{aligned}$$

Since $\mathcal{J}$ is a non-decreasing function, we have

$$\vartheta(\breve{a}^*, \xi, \alpha\kappa) \geq \vartheta(\breve{a}^*, \xi, \kappa) > \vartheta(\breve{a}^*, \xi, \kappa).$$

This implies $\breve{a}^*$ and ξ are identical. Finally, from Equation (24), we have

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of mappings Γ and Υ. □

Theorem 8. *Let $\mathcal{M}, \mathcal{N} \subseteq (\mathcal{B}, \vartheta, *)$ such that $\mathcal{N}$ is AC with respect to $\mathcal{M}$. Also, assume that $\lim_{k\to\infty} \vartheta(\breve{a}_1, \breve{a}_2, \kappa) = 1$ and $\mathcal{M}_0$, $\mathcal{N}_0 \neq \emptyset$. Let $\Gamma\colon \mathcal{M} \to \mathcal{N}$ and $\Upsilon : \mathcal{M} \to \mathcal{N}$ satisfy the following conditions:*

(i) Υ dominates Γ and are fuzzy $(\mathcal{J}, \mathcal{S})$-interpolative Hardy–Rogers-type proximal;
(ii) Γ and Υ are compact proximal;
(iii) $\mathcal{J}$ is non-decreasing and $\{\mathcal{J}(t_n)\}$ and $\{\mathcal{S}(t_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathcal{J}(t_n) = \lim_{n\to\infty} \mathcal{S}(t_n)$, then $\lim_{n\to\infty} t_n = 1$;
(iv) Γ and Υ are continuous;
(v) $\Gamma(\mathcal{M}_0) \subseteq \mathcal{N}_0$ and $\Gamma(\mathcal{M}_0) \subseteq \Upsilon(\mathcal{M}_0)$.

Then, Υ and Γ have a unique element $\grave{u} \in \mathcal{M}$ such that

$$\begin{aligned} \vartheta(\grave{u}, \Upsilon\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa), \\ \vartheta(\grave{u}, \Gamma\grave{u}, \kappa) &= \vartheta(\mathcal{M}, \mathcal{N}, \kappa). \end{aligned}$$

Proof. Proceeding as in the proof of Theorem 7, we have

$$\begin{aligned} \mathcal{J}(q_n) &\geq \mathcal{S}\Big(\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\Big) \\ &> \mathcal{J}\Big(\Big((q_{n-1})^{1-\beta-\gamma-\delta}(q_n)^{1-\alpha-\delta}\Big)\Big). \end{aligned} \tag{28}$$

By Equation (28), we infer that $\{\mathcal{J}(q_n)\}$ is sds. We have two cases here; either the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, or it is not. If $\{\mathcal{J}(q_n)\}$ is not bounded above, then

$$\inf_{w_n > \varepsilon} \mathcal{J}(q_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows from Lemma 1 that $q_n \to 1$ as $n \to \infty$. Secondly, if the sequence $\{\mathcal{J}(q_n)\}$ is bounded above, then it is a convergent sequence. By Equation (28), the sequence $\{\mathcal{S}(q_n)\}$ also converges. Furthermore, both have the same limit. By condition (iii), we get $\lim_{n\to\infty} q_n = 1$, or $\lim_{n\to\infty} \vartheta(\breve{a}_n, \grave{u}_{n+1}, \kappa) = 1$, for any sequence $\{\breve{a}_n\}$ in $\mathcal{M}$. Now, following the proof of Theorem 7, we obtain

$$\vartheta(\breve{a}^*, \Upsilon(\breve{a}^*), \kappa) = \vartheta(\mathcal{M}, \mathcal{N}, \kappa) = \vartheta(\breve{a}^*, \Gamma(\breve{a}^*), \kappa).$$

This shows that the point $\breve{a}^*$ is a common best proximity point of the pair of the mapping Υ and Γ. $\square$

4. Application

In this part, we utilize Theorem 1 to find the existence and uniqueness of a solution to UIE:

$$\ell(\hbar) = f(\hbar) + \int\limits_{IR} k_1(\hbar, s, \ell(s))ds. \tag{29}$$

Depending on the integration region (IR), this integral equation involves both the Volterra integral equation (VIE) and the Fredholm integral equation (FIE). If $IR = (a, x)$, where a is fixed, then UIE is VIE. For this, we consider a common best proximity point approach. The common best proximity point technique is a straightforward and attractive way to demonstrate that each additional mathematical model has a singular solution.

Suppose IR is a set of finite measures, and

$$\pounds^2_{IR} = \left\{ \ell \int\limits_{IR} \mid \ell(s) \mid^2 ds < \infty \right\}.$$

Define the norm $\| \,.\, \|: \pounds^2_{IR} \to [0, \infty)$ by

$$\| \ell \| = \sqrt{\int\limits_{IR} \mid \ell(s) \mid^2 ds}, \text{ for all } \ell \in \pounds^2_{IR} \tag{30}$$

The following formula of an equivalent norm is given:

$$\| \ell \| = \sqrt{\sup\left\{ e^{-\nu \int\limits_{IR} \alpha(s)ds} \int\limits_{IR} \mid \ell(s) \mid^2 ds \right\}}, \text{ for all } \ell \in \pounds^2_{IR}, \nu > 1. \tag{31}$$

Then, $(\pounds^2_{IR}, \| \,.\, \|_{2,\nu})$ is a Banach space. Let $\mathcal{B} = \{\ell \in \pounds^2_{IR} : \ell(s) > 0 \text{ for almost every } s\}$. The FM ϑ_ν associated with the norm $\| \,.\, \|_{2,\nu}$ is given by $\vartheta_\nu(\ell,) = \vartheta_\nu(\ell, f, \kappa) = e^{-\frac{\|\ell - f\|}{\kappa}}$ for all $\ell, f \in \mathcal{B}$. Then $(\mathcal{B}, \vartheta, *)$ is an CFMS. Let

(A_1) The Kernal $k_1 : IR \times IR \times \mathbb{R} \to \mathbb{R}$ satisfies Carthodory conditions, and

$$\mid k_1(\hbar, s, \ell(s)) \mid \leq w(\hbar, s) + e(\hbar, s); w, e \in \pounds^2(IR \times IR), e(\hbar, s) > 0.$$

(A_2) The function $f : IR \to [1, \infty)$ is continuous and bounded on IR.

(A_3) There exists a positive constant C such that

$$\sup_{\hbar \in IR} \int_{IR} \mid k_1(\hbar, s)ds \leq C.$$

(A_4) Let $\ell_0 \in \mathcal{M}$. Since $\Gamma(\mathcal{M}) \subseteq \Upsilon(\mathcal{M})$ guarantees the existence of an element $\ell_1 \in \mathcal{M}$ such that, $\Gamma\ell_0 = \Upsilon\ell_1$. Also, we have $\Gamma(\mathcal{M}) \subseteq \Upsilon(\mathcal{M})$.

(A$_5$) There exists a nonnegative and measurable function $q : IR \times IR \to \mathbb{R}$ such that

$$\alpha(\hbar) = \int_{IR} q^2(\hbar, s)ds \leq \frac{1}{\nu M^2}$$

and integrable over IR with

$$e^{-\frac{|k_1(\hbar,s,\ell(s))-k_1(\hbar,s,f(s))|}{\kappa}} \geq e^{-\frac{q(\hbar,s)}{\kappa}} e^{-\frac{|\ell(s)-f(s)|}{\kappa}}$$

for all $\hbar, s \in IR$ and $\ell, f \in \mathcal{M}$.

Theorem 9. *Suppose the mapping f and k_1 mentioned above verify the conditions (A$_1$)–(A$_2$), then, the UIE Equation (29) has a unique solution.*

Proof. Define the pair of mappings $\Gamma, \Upsilon : \mathcal{M} \to \mathcal{N}$, in accordance with the abovementioned notations, by

$$(\vartheta\ell)(\hbar) = f(\hbar) + \int_{IR} k_1(\hbar, s, \ell(s))ds. \tag{32}$$

Let $\ell, j \in \mathcal{M}$, since, for almost every $\hbar \in IR$

$$(\vartheta\ell)(\hbar) = f(\hbar) + \int_{IR} k_1(\hbar, s, \ell(s))ds \geq 1.$$

Conditions (A$_1$)–(A$_2$) imply that ϑ is continuous and compact mapping from $\mathcal{M}$ to $\mathcal{N}$. By (A$_4$) we will check the contractive condition of Equation (7) of Theorem 1 in the next lines. By (A$_5$) and the Holder inequality, we have

$$\begin{aligned} e^{-\frac{|(\vartheta\ell)(\hbar)-(\vartheta f)(\hbar)|^2}{\kappa}} &= e^{-\frac{|\int_{IR} k_1(\hbar,s,\ell(s))ds - \int_{IR} k_1(\hbar,s,f(s))ds|^2}{\kappa}} \\ &\geq e^{\frac{-\left(\int_{IR}|k_1(\hbar,s,\ell(s))-k_1(\hbar,s,f(s))|ds\right)^2}{\kappa}} \\ &\geq e^{-\frac{\left(\int_{IR} q(\hbar,s)|l(s)-f(s)|ds\right)^2}{\kappa}} \\ &\geq e^{-\frac{\int_{IR} q^2(\hbar,s)ds}{\kappa}} . e^{\frac{-\int_{IR}|\ell(s)-f(s)|^2 ds}{\kappa}} \\ &\geq e^{-\frac{\alpha(\hbar)}{\kappa}} . e^{-\frac{\int_{IR}|\ell(s)-f(s)|^2 ds}{\kappa}}. \end{aligned}$$

This implies, by integrating with respect to $\hbar$,

$$\begin{aligned} e^{-\frac{|(\vartheta\ell)(\hbar)-(\vartheta f)(\hbar)|^2 d\hbar}{\kappa}} &\geq e^{-\frac{\int_{IR}\left(\alpha(\hbar).\int_{IR}|\ell(s)-f(s)|^2 ds\right)d\hbar}{\kappa}} \\ &= e^{-\frac{\int_{IR}\left(\alpha(\hbar)e^{\nu\int_{IR}\alpha(s)ds}.e^{-\nu\int_{IR}\alpha(s)ds}\int_{IR}|\ell(s)-f(s)|^2 ds\right)d\hbar}{\kappa}} \\ &\geq e^{-\frac{\|\ell-f\|^2_{2,\nu}\int_{IR}\alpha(\hbar)e^{\nu\int_{IR}\alpha(s)ds}d\hbar}{\kappa}} \\ &\geq e^{-\frac{\|\ell-f\|^2_{2,\nu}e^{\nu\int_{IR}\alpha(s)ds}}{\nu M^2\kappa}}. \end{aligned}$$

Thus, we have

$$e^{-\frac{M^2 e^{-\nu\int_{IR}\alpha(s)ds}|(\vartheta\ell)(\hbar)-(\vartheta f)(\hbar)|^2 d\hbar}{\nu\kappa}} \geq e^{-\frac{\|\ell-f\|^2_{2,\nu}}{\kappa}}.$$

This implies that

$$e^{-\frac{M^2\|(\vartheta\ell)-(\vartheta f)\|^2_{2,\nu}}{\nu\kappa}} \geq e^{-\frac{\|\ell-f\|^2_{2,\nu}}{\kappa}}.$$

That is

$$L(\ell, f) d_v((\vartheta\ell), (\vartheta f), \kappa) \geq d_v(\ell, f).$$

Define $\mathcal{J}(s) = \frac{1}{\ln(s)}$ and $\mathcal{S}(s) = \frac{1}{\ln(s^2)}$; then, we have

$$\mathcal{J}(L(\ell, f) d_v((\vartheta\ell), (\vartheta f), \kappa)) \geq \mathcal{S}(F_1(\ell, f)).$$

The defined $\mathcal{J}$ and $\mathcal{S}$ satisfy the remaining conditions of the Theorem 1. Hence, from Theorem 1, the operator ϑ has a unique point. This means that the UIE Equation (29) has a unique solution. □

5. Conclusions

In this manuscript, we have introduced several new types of contractive conditions that ensure the existence of common best proximity points in the framework of FMS. Our examples show that the new contractive conditions generalize the corresponding contractions from the existing literature. The contraction conditions (1), (11), (17) and (23) can be used to demonstrate the presence of solutions to the models of linear and nonlinear dynamic systems, depending on their nature (linear or nonlinear). This paper's study expands on the worthwhile research that was previously published in [7,8,11–13].

Author Contributions: Conceptualization, U.I., I.K.A. and F.J.; methodology, D.A.K.; software, U.I.; validation, I.K.A., F.J. and D.A.K.; formal analysis, U.I.; investigation, U.I. and I.K.A.; resources, U.I.; data curation, F.J.; writing—original draft preparation, U.I.; writing—review and editing, I.K.A. and U.I.; visualization, D.A.K.; supervision, I.K.A.; project administration, U.I.; funding acquisition, D.A.K. and I.K.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available on demand from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
2. Karapinar, E. Interpolative Kannan-Meir-Keeler type contraction. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 611–614. [CrossRef]
3. Karapınar, E.; Fulga, A.; Yesilkaya, S.S. New results on Perov-interpolative contractions of Suzuki type mappings. *J. Funct. Spaces* **2021**, *2021*, 9587604. [CrossRef]
4. Karapınar, E.; Agarwal, R.P. Interpolative Rus-Reich-Ćirić type contractions via simulation functions. Analele ştiinţifice ale Universităţii "Ovidius" Constanţa. *Ser. Mat.* **2019**, *27*, 137–152.
5. Karapınar, E.; Alqahtani, O.; Aydi, H. On interpolative Hardy-Rogers type contractions. *Symmetry* **2018**, *11*, 8. [CrossRef]
6. Altun, I.; Aslantas, M.; Sahin, H. Best proximity point results for p-proximal contractions. *Acta Math. Hung.* **2020**, *162*, 393–402. [CrossRef]
7. Altun, I.; Taşdemir, A. On best proximity points of interpolative proximal contractions. *Quaest. Math.* **2021**, *44*, 1233–1241. [CrossRef]
8. Shahzad, N.; Sadiq Basha, S.; Jeyaraj, R. Common best proximity points: Global optimal solutions. *J. Optim. Theory Appl.* **2011**, *148*, 69–78. [CrossRef]
9. Sadiq Basha, S. Common best proximity points: Global minimal solutions. *TOP* **2013**, *21*, 182–188. [CrossRef]
10. Sadiq Basha, S. Common best proximity points: Global minimization of multi-objective functions. *J. Glob. Optim.* **2012**, *54*, 367–373. [CrossRef]
11. Deep, A.; Batra, R. Common best proximity point theorems under proximal F-weak dominance in complete metric spaces. *J. Anal.* **2023**, 1–17. [CrossRef]
12. Mondal, S.; Dey, L.K. Some common best proximity point theorems in a complete metric space. *Afr. Mat.* **2017** *28*, 85–97. [CrossRef]
13. Shayanpour, H.; Nematizadeh, A. Some results on common best proximity point in fuzzy metric. *Bol. Soc. Parana. Mat.* **2016**, *35*, 177–194. [CrossRef]
14. Roldán López de Hierro, A.F.; Fulga, A.; Karapınar, E.; Shahzad, N. Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces. *Mathematics* **2021**, *9*, 1594. [CrossRef]
15. Zhou, M.; Saleem, N.; Liu, X.; Fulga, A.; Roldán López de Hierro, A.F. A new approach to Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces. *Mathematics* **2021**, *9*, 3001. [CrossRef]

16. Uddin, F.; Ishtiaq, U.; Javed, K.; Aiadi, S.S.; Arshad, M.; Souayah, N.; Mlaiki, N. A New Extension to the Intuitionistic Fuzzy Metric-like Spaces. *Symmetry* **2022**, *14*, 1400. [CrossRef]
17. Saleem, N.; Javed, K.; Uddin, F.; Ishtiaq, U.; Ahmed, K.; Abdeljawad, T.; Alqudah, M.A. Unique Solution of Integral Equations via Intuitionistic Extended Fuzzy b-Metric-Like Spaces. *CMES-Comput. Model. Eng. Sci.* **2023**, *135*, 021031. [CrossRef]
18. Saleem, N.; Ishtiaq, U.; Guran, L.; Bota, M.F. On graphical fuzzy metric spaces with application to fractional differential equations. *Fractal Fract.* **2022**, *6*, 238. [CrossRef]
19. Hussain, A.; Al Sulami, H.; Ishtiaq, U. Some new aspects in the intuitionistic fuzzy and neutrosophic fixed point theory. *J. Funct. Spaces* **2022**, *2022*, 3138740. [CrossRef]
20. Nazam, M.; Aydi, H.; Hussain, A. Generalized interpolative contractions and an application. *J. Math.* **2021**, *2021*, 6461477. [CrossRef]
21. Naseem, T.; Zeb, A.A.; Sohail, M. Reduce Differential Transform Method for Analytical Approximation of Fractional Delay Differential Equation. *Int. J. Emerg. Multidiscip. Math.* **2022**, *1*, 104–123. [CrossRef]