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Symmetry and Asymmetry in Moment, Functional Equations, and Optimization Problems

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Abstract: The purpose of this work is to provide applications of real, complex, and functional analysis to moment, interpolation, functional equations, and optimization problems. Firstly, the existence of the unique solution for a two-dimensional full Markov moment problem is characterized on the upper half-plane. The issue of the unknown form of nonnegative polynomials on $\mathbb{R} \times \mathbb{R}_+$ in terms of sums of squares is solved using polynomial approximation by special nonnegative polynomials, which are expressible in terms of sums of squares. The main new element is the proof of Theorem 1, based only on measure theory and on a previous approximation-type result. Secondly, the previous construction of a polynomial solution is completed for an interpolation problem with a finite number of moment conditions, pointing out a method of determining the coefficients of the solution in terms of the given moments. Here, one uses methods of symmetric matrix theory. Thirdly, a functional equation having nontrivial solution (defined implicitly) and a consequence are discussed. Inequalities, the implicit function theorem, and elements of holomorphic functions theory are applied. Fourthly, the constrained optimization of the modulus of some elementary functions of one complex variable is studied. The primary aim of this work is to point out the importance of symmetry in the areas mentioned above.

Keywords: polynomial approximation; moment problem; symmetric matrix; self-adjoint operator; implicitly defined function; holomorphic solution

MSC: 41A10; 46A22; 26B10; 47B15; 30A10

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1. Introduction

The classical moment problem is an interpolation problem with the positivity condition on the solution. Namely, given a sequence $(y_j)_{j\geq 0}$ of real numbers, one studies the existence, the uniqueness, and, eventually, the construction of a nondecreasing real-valued function $\sigma(t)$ $(t \geq 0)$, which verifies the moment conditions $\int_0^\infty t^j d\sigma = y_j$ $(j = 0, 1, 2, \ldots)$. This is the original formulation of the moment problem on $[0, \infty)$, as in the works of T.J. Stieltjes [1], recalled by N.I. Akhiezer in [2]. If such a function σ does exist, the sequence $(y_k)_{k\geq 0}$ is called a Stieltjes moment sequence. In the Markov moment problem, other than the interpolation conditions, a sandwich condition on the solution is imposed as well. Going back to the problem formulated by T.J. Stieltjes, this is a one-dimensional moment problem on an unbounded interval. Specifically, it is an interpolation problem with the condition on the positivity of the measure $d\sigma$. The numbers y_j, $j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ are called the moments of the measure $d\sigma$. The moment problem is an inverse problem: one is looking for an unknown measure, starting from its given moments. The following notations are used:

$$\mathbb{N} = \{0, 1, 2, \ldots\},\ \mathbb{R}_+ = [0, +\infty).$$

$$\varphi_j(t) := \varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n},\ j = (j_1, \ldots, j_n) \in \mathbb{N}^n,$$

$$t = (t_1, \ldots, t_n) \in F,\ n \in \mathbb{N},\ n \geq 1.$$

For a set $F = F_1 \times \cdots \times F_n \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and functions $f_l : F_l \to \mathbb{R}$, $l = 1, \ldots, n$, denote

$$f_1 \bigotimes \cdots \bigotimes f_n : F \to \mathbb{R},\ \left(f_1 \bigotimes \cdots \bigotimes f_n\right)(t_1, \ldots, t_n) := f_1(t_1) \cdots f_n(t_n).$$

In general, F is a closed bounded or unbounded subset in $\mathbb{R}^n$, $\mathcal{P} = \mathbb{R}[t_1, \ldots, t_n]$ is the real vector space of all polynomials with real coefficients, and $\mathcal{P}_+(F)$ denotes the convex cone of all polynomials $p \in \mathcal{P}$, taking nonnegative values at all points of F. If X is an ordered vector space, one denotes by X_+ the positive cone of X. The open unit disc in the complex plane is denoted by U, and $\mathbb{T}$ is its boundary, the unit circle. If Y is a Banach lattice and $(y_j)_{j \in \mathbb{N}^n}$ is a given sequence of elements in Y, by a solution for the interpolation problem

$$T(\varphi_j) = y_j, \quad j \in \mathbb{N}^n, \tag{1}$$

One means a linear operator T which verifies (1), mapping a Banach lattice X containing the space of polynomials and the space $C_c(F)$ of all real-valued continuous and compactly supported functions defined on F into the space Y. In most cases, when $Y = \mathbb{R}$, one has a scalar-valued solution. When Y is a function or operator Banach lattice, one requires the order completeness of Y. The reason is to permit application of the Hahn–Banach-type extension results of linear operators having Y as codomain. For general knowledge on the moment problem and related areas, see monographs [2–5]. Basic results in real and complex analysis published in [6] are applied in the present study. Further knowledge in analysis and functional analysis, accompanied by applications, can be found in [7–11]. For more general results than those of Section 3.2 below, related to self-adjoint operators and Hankel moment matrices, see [12]. In the case of the classical moment problem, other than the interpolation conditions (1), the positivity of the solution is imposed: $x \geq 0$ in $X \Longrightarrow T(x) \geq 0$ in Y. If $Y = \mathbb{R}$, this positivity condition implies the representation of the linear positive functional T by means of a positive Borel regular measure [6] on F. A variant of the Markov moment problem appearing in the present article consists of requirements (1) and (2) on the solution, where (2) is as follows:

$$T_1(x) \leq T(x) \leq T_2(x) \quad \forall x \in X_+. \tag{2}$$

Basic earlier results on the classical moment problem have been published in references [13–15]. Articles [16,17] provide solutions to the moment problem on special compact subsets of $\mathbb{R}^n$. The expression of polynomials taking positive values on these compact subsets in terms of special positive polynomials follows as well. In [18], an operator valued moment problem is solved. Article [19] applies extension theorems with two constraints on the linear extension under attention in the Markov moment problem. The codomain space is an order-complete vector lattice. In the articles [20–22], the study of the moment problem on semi-algebraic compact subsets [17] is strongly improved and generalized. Moreover, in [22], the author solves a moment problem on an unbounded semi-algebraic subset. The very recent article [23] applies methods of operator theory to study the stability in some truncated moment problems. Recall that, for $n \geq 2$, there exist nonnegative polynomials on $\mathbb{R}^n$ which are not sums of squares. Hence, in this case, moment problems cannot be solved directly in terms of quadratic forms. An exception is the case pointed out by M. Marshall [24], who found and proved the explicit form of nonnegative polynomials on a strip in terms of sums of squares. This is not a problem in the case $n = 1$, since any nonnegative polynomial on $\mathbb{R}$ is a sum of two squares of polynomials. A similar well-known result is valid for a nonnegative polynomial on the nonnegative semiaxis (see Theorem 2 below). In [25,26], some main results are proven. Namely, in [26], the authors prove that, for $n \geq 2$, there exist moment determinate measures ν on $\mathbb{R}^n$, such that the polynomials are not dense in $L^2_\nu(\mathbb{R}^n)$. New checkable sufficient conditions for determinacy of some usual important measures are proven in [27]. The articles [28,29] solve optimization problems related to the truncated moment problem. In [30], the author constructs a solution for the full moment problem, as a limit of solutions for truncated moment problems. Articles [31,32] provide interesting approximation results, not necessarily referring to the moment problem. References [4,5,18,19,33–37] are devoted to, or contain significant results on, the Markov moment problem. Existence, uniqueness, or construction of the solutions of some Markov moment problems are under attention. Finally, the paper [38] refers to some functional and operatorial equations, whose study is completed in the present article. The unknown function is defined implicitly, and its explicit solution is difficult or impossible to find. The complex case was also considered in [38]. The references illustrate the connection of the moment problem and of functional equations with other research areas, such as operator theory, approximation, optimization, algebra, real and complex analysis, geometric functional analysis, and inverse problems. The first aim of this paper is to apply and give a new proof for the previous polynomial approximation and Markov moment problem results for concrete spaces. The motivation is that that, since the explicit form of nonnegative polynomials is not known, one must approximate any nonnegative function of the positive cone of the space $L^1_\nu(F)$, where ν is a positive moment determinate measure with special nonnegative polynomials. Such polynomials are expressible in terms of sums of squares. Then, passing to the limit, the proof of Theorem 2 from below follows. Namely, in Section 3.1, a two-dimensional full Markov moment problem on $F := \mathbb{R} \times \mathbb{R}_+$ is solved via this method. Unlike the recently reviewed results in [36,37], which refer to the vector-valued moment problem (or to operator-valued moment problems), here, the focus is on the classical scalar Markov moment problem. In this classical case, the linear solution is represented by function $h \in L^\infty_{d\nu}(\mathbb{R} \times \mathbb{R}_+)$, and the inequalities $h_1 \leq h \leq h_2$ hold almost everywhere in $\mathbb{R} \times \mathbb{R}_+$. Therefore, measure theory results for representing the linear form solution are also applied. The existence and the uniqueness of the solutions are derived. This classical case is important since, here, the quadratic forms appearing in Theorem 2, point (b) have real coefficients. Thus, their signature can be determined by means of computational algebraic methods. Theorem 2 follows from Theorem 1, whose proof is new, more complete, and simple, but is omitted in [36,37]. The second purpose is to complete the construction of a polynomial solution for the reduced interpolation problem (1), involving only a finite number of conditions (1) (when $j_l \leq d,\ l = 1, \ldots, n$, for a fixed positive integer d). This is carried out in Section 3.2. Then, in Section 3.3, a concrete functional equation whose solution is defined implicitly is discussed. Special care is accorded to the analyticity of the solutions (see [38] and Theorem 4, discussed in Section 3.3). A related inequality which is valid for a large class of self-adjoint operators is derived in Theorem 5. In Section 3.4, optimization of the modulus of the function

$$\psi(z) = \frac{1}{2}(z + 1/z), \text{ on the circular annulus } \{z \in \mathbb{C}; r \leq |z| \leq R\}, \quad 0 < r < R < \infty,$$

is studied. The points where the extreme values are attained are also determined. All the spaces and linear operators/functionals are considered over the real field unless another specification is mentioned. A connection of the function ψ with the previous section is briefly outlined. In the end, optimization problems of the function $\psi_\alpha(z) := \psi(z^\alpha) = (1/2)(z^\alpha + z^{-\alpha})$, $z \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{R}$, on the circular annulus

$$\{z \in \mathbb{C}; r \leq |z| \leq R\}, \quad 0 < r < R < 1,$$

are deduced. The rest of the paper is organized as follows. Section 2 summarizes the basic methods applied in this work. Section 3 is devoted to the results and their motivations, while Section 4 (Discussion) and Section 5 (Conclusions) conclude the paper.

2. Methods

The methods applied in this article can be summarized as follows:

(I) Evaluation of the consequences of polynomial approximation on unbounded closed subsets $F \subseteq \mathbb{R}^n$, $n \geq 1$, by means of special polynomials, and their applications to the characterization of the existence and uniqueness of the solution of the full Markov moment problem on $L^1_\nu(F)$, where ν is a moment determinate measure on F. See [27,36,37] and the refences therein for details and general type results. To prove the applied approximation result of Theorem 1 and the related previous results from [36,37], measure theory and a fundamental theorem in functional analysis were applied. Among other results, Fubini's theorem and Haviland theorem [13] were used, as well as an extension of linear positive functionals and operators from a majorizing subspace to the entire domain-ordered vector space [8]. This is a Hahn–Banach-type result. For much more general theorems on the moment problem, deduced from extension of linear operator-type results and giving necessary and sufficient conditions on the existence of the constrained solution, see [19]. Such earlier results do not use any approximation theorem on unbounded subsets of $\mathbb{R}^n$; only polynomial approximation on compact subsets [16] are applied (see also [17]). On the other hand, in references [36,37], polynomial approximation on unbounded subsets is studied as well.

(II) Decomposition of $\mathbb{R}^{d+1}$ ($d \in \mathbb{N} \setminus \{0\}$) as direct sum of orthogonal eigenspaces associated with the symmetric Hankel moment matrix. Such a result also holds true in infinite dimensional separable Hilbert spaces H, for compact self-adjoint operators from H to H (see [7]). This method led to the polynomial solution of the reduced interpolation problem solved in Section 3.2. See [12] for recently published deep results in operator theory, most of them referring to self-adjoint operators defined on proper vector subspaces of a Hilbert space and associated with Hankel moment matrices.

(III) Applying results of analysis over the real field [6,10] and elements of complex analysis [6] for solving functional equations when the unknown holomorphic function is defined implicitly, by means of a given holomorphic function with natural properties. In the present work, this is one of the subjects which is carefully focused on. The analyticity of the involved given or unknown functions plays a significant role. All these analytic functions apply the intersection of their domain with the real axis into the real axis.

(IV) Functional calculus for self-adjoint operators [7,8].

(V) Using known inequalities from which new ones are derived. Almost all the results involve this.

(VI) Application of the maximum modulus principle of holomorphic functions for determining the extreme values and the points where they are attained, for the modulus of *Joukowski's* function and for a related elementary function in a closed circular annulus not containing the origin. The corresponding result for fractional powers of the complex variable z is also deduced.

(VII) Measure theory [6,10,27].

3. Results

3.1. Solving Full Scalar-Valued Markov Moment Problems on Unbounded Subsets

These results start with a new detailed proof for one of the previous results on polynomial approximation on unbounded subsets. Here, any Stone–Weierstrass uniform approximation on compact subsets is used. Then, the solution for a full Markov moment problem on $\mathbb{R} \times \mathbb{R}_+$ is derived.

As seen in the Introduction, in the classical moment problem, the positivity of the solution or/and sandwich conditions on the positive cone of the domain space have been studied (see also the Introduction and the references on the moment problem). If a full moment problem on an unbounded subset is under investigation, then the uniqueness of the solution makes sense as well. Next, some consequences of a few results from [37] are proven, where the key point consists of polynomial approximation on unbounded subsets. In the sequel, a two-dimensional Markov moment problem is investigated. Recall the notion of a moment determinate measure ν on a closed subset F of $\mathbb{R}^n$. The positive

Borel measure ν on F is moment-determinate if it is uniquely determinate by its classical moments (which are assumed to be finite). In other words, ν is moment determinate on F if, for any other measure μ, for which

$$\int_F t^j d\mu = \int_F t^j d\nu \quad \forall j \in \mathbb{N}^n,$$

$\mu = \nu$ are present as measures (that is, $\int_F \varphi d\mu = \int_F \varphi d\nu$. for any real-valued continuous compactly supported function φ defined on F) (see [2,4,12,25–27] and many other sources on this subject). This section starts with a new proof for one of the previous polynomial approximation results. Let $d\nu_1 = f_1(t_1)dt_1$, (with $f_1 \in L^1_{dt_1}(\mathbb{R}_+)$) be a positive moment-determinate measure on $\mathbb{R}_+$, with finite moments of all orders, and $d\nu_2 = f_2(t_2)dt_2$, (with $f_2 \in L^1_{dt_2}(\mathbb{R}_+)$) be a positive moment-determinate measure on $\mathbb{R}_+$, with finite moments of all orders. On $\mathbb{R} \times \mathbb{R}_+$, consider the product measure $\nu = \nu_1 \times \nu_2$. Unlike the previous proof of such a result, which used Bernstein polynomials in several variables, measure theory plays the central role in the proof of the next theorem.

Theorem 1. *Any nonnegative function $f \in L^1_\nu(\mathbb{R} \times \mathbb{R}_+)$ can be approximated in $L^1_\nu(\mathbb{R} \times \mathbb{R}_+)$ by a sequence of special nonnegative polynomials $(p_m)_{m\in\mathbb{N}}$, where each p_m is a finite sum of polynomials $p_{m,1} \otimes p_{m_2}$, with $p_{m_1} \in \mathcal{P}_+(\mathbb{R})$, $p_{m_2} \in \mathcal{P}_+(\mathbb{R}_+)$.*

Proof. Any nonnegative function $f \in L^1_\nu(\mathbb{R} \times \mathbb{R}_+)$ can be approximated in $L^1_\nu(\mathbb{R} \times \mathbb{R}_+)$ by a sequence of simple functions, each of which is a finite sum of terms of the form $a\chi_{[a_1,b_1)\times[a_2,b_2)} = a\chi_{[a_1,b_1)} \otimes \chi_{[a_2,b_2)}$, $a \in \mathbb{R}_+$. Consider the cell-decomposition of an open subset of $\mathbb{R} \times \mathbb{R}_+$ as a union of disjoint rectangles $[a_1, b_1) \times [a_2, b_2)$. Since any such rectangle is the union of the compact rectangles of the form $[a_1, b_1 - \varepsilon_k] \times [a_2, b_2 - \varepsilon_k]$, $\varepsilon_k \downarrow 0$, it is sufficient to approximate any function of the form $\chi_{[a_1,c_1]\times[a_2,c_2]}$ by $q_m \otimes r_m$, with $q_m \in \mathcal{P}_+(\mathbb{R})$, $r_m \in \mathcal{P}_+(\mathbb{R}_+)$, in the space $L^1_\nu(\mathbb{R} \times \mathbb{R}_+)$. A simple measure theory argument ensures the existence of a decreasing sequence $(h_m)_m$ of continuous nonnegative functions on $\mathbb{R}$, $h_m \downarrow \chi_{[a_1,c_1]}$, $h_m(t_1) = 1\ \forall t_1 \in [a_{1,}, c_1]$, $h_m(t_1) = 0\ \forall t_1 \in \mathbb{R} \setminus [a_1 - \varepsilon_m, c_1 + \varepsilon_m]$, $\varepsilon_m \downarrow 0$. The convergence $h_m \downarrow \chi_{[a_1,c_1]}$ holds pointwise on $\mathbb{R}$. For each $m \in \mathbb{N}$, application of one of the results from [36,37] leads to the existence of a polynomial $q_m \in \mathcal{P}_+(\mathbb{R})$, $q_m(t_1) \ge h_m(t_1) \ge 0$ for all $t_1 \in \mathbb{R}$, such that

$$\int_{\mathbb{R}} (q_m(t_1) - h_m(t_1))d\nu_1 \to 0, \qquad m \to \infty.$$

This results in $0 \le q_m(t_1) - \chi_{[a_1,c_1]}(t_1) = (q_m - h_m)(t_1) + \left(h_m - \chi_{[a_1,c_1]}\right)(t_1)$ for all $t_1 \in \mathbb{R}$; hence,

$$\int_{\mathbb{R}} \left(q_m - \chi_{[a_1,c_1]}\right) d\nu_1 \to 0,\ \ m \to \infty.$$

The conclusion is that $q_m \to \chi_{[a_1,c_1]}$ in $L^1_{\nu_1}(\mathbb{R})$, $q_m \ge \chi_{[a_1,c_1]} \ge 0$ on $\mathbb{R}$. Proceeding in the same way, one infers the existence of a sequence of polynomials $r_m \to \chi_{[a_2,c_2]}$, $m \to \infty$, the convergence holding in $L^1_{\nu_2}(\mathbb{R}_+)$. Moreover, $r_m(t_2) \ge \chi_{[a_2,c_2]}(t_2) \ge 0$ for all $t_2 \in \mathbb{R}_+$ The above considerations and Fubini's theorem yield

$$\iint_{\mathbb{R}\times\mathbb{R}_+} q_m \otimes r_m d\nu = \int_{\mathbb{R}} q_m d\nu_1 \cdot \int_{\mathbb{R}_+} r_m d\nu_2 \to \int_{\mathbb{R}} \chi_{[a_1,c_1]} d\nu_1 \cdot \int_{\mathbb{R}_+} \chi_{[a_2,c_2]} d\nu_2 =$$
$$\iint_{\mathbb{R}\times\mathbb{R}_+} \left(\chi_{[a_1,\times c_1]} \otimes \chi_{[a_2,c_2]}\right) d\nu = \iint_{\mathbb{R}\times\mathbb{R}_+} \chi_{[a_1,c_1]\times[a_2,c_2]} d\nu, \quad m \to \infty.$$

In other words, also using the fact that $q_m \bigotimes r_m \geq \chi_{[a_1,c_1]} \bigotimes \chi_{[a_2,c_2]}$, the preceding convergence can be written as

$$\left\| q_m \otimes r_m - \chi_{[a_1,c_1]} \otimes \chi_{[a_2,c_2]} \right\|_{L^1_\nu(\mathbb{R}\times\mathbb{R}_+)} =$$
$$\iint_{\mathbb{R}\times\mathbb{R}_+} \left| q_m \otimes r_m - \chi_{[a_1,c_1]} \otimes \chi_{[a_2,c_2]} \right| d\nu =$$
$$\iint_{\mathbb{R}\times\mathbb{R}_+} \left(q_m \otimes r_m - \chi_{[a_1,c_1]\times[a_2,c_2]} \right) d\nu \to 0, \quad m \to \infty.$$

Thus, $q_m \bigotimes r_m \to \chi_{[a_1,c_1]} \bigotimes \chi_{[a_2,c_2]} = \chi_{[a_1,c_1]\times[a_2,c_2]}$, as $m \to \infty$, in $L^1_\nu(\mathbb{R}\times\mathbb{R}_+)$. The conclusion follows. This ends the proof. □

The present proof of Theorem 1 seems to be simpler than using Bernstein polynomials or other uniform approximation results of continuous functions by means of polynomials on compact subsets. On the other hand, Theorem 1 solves only an approximation on the Cartesian product $\mathbb{R}\times\mathbb{R}_+$. One of the reasons for considering this case was that the form of nonnegative polynomials on $\mathbb{R}$ and on $\mathbb{R}_+$, in terms of sums of squares, is simpler compared with that of nonnegative polynomials on any other closed subset of $\mathbb{R}$. As a possible generalization of Theorem 1, it makes sense to consider the following problem. If $F = F_1 \times \cdots \times F_n,\ n \geq 2$, where $F_i \subseteq \mathbb{R}$ is a closed subset, and on F_i a moment-determinate measure ν_i is given, $i = 1, \ldots, n$, considering the product-measure $\nu := \nu_1 \times \cdots \times \nu_n$, the space $L^1_\nu(F)$ and a nonnegative function f from $L^1_\nu(F)$, can one approximate f by finite sums of polynomials $p_1 \bigotimes \cdots \bigotimes p_n$, with p_i nonnegative polynomial on $F_i,\ i = 1, \ldots, n$? In this problem, even in the case when F_i is bounded (i.e., it is compact), it makes sense to also consider the case when F_i is not an interval. For example, this is the case when F_i is the spectrum of a symmetric matrix or of a self-adjoint operator. Clearly, any positive regular Borel measure on a nonempty compact subset of $\mathbb{R}^n$ is moment-determinate, due to the Weierstrass uniform approximation theorem of continuous functions by polynomials on compact subsets.

The purpose of the next result is to show how Theorem 1 can be applied to solve a two-dimensional Markov moment problem in terms of quadratic forms. Namely, such a problem is solved on the upper half-plane $\mathbb{R}\times\mathbb{R}_+$.

Theorem 2. *Let* $d\nu_1(t) := e^{-at_1^2}dt_1,\ d\nu_2(t) := e^{-bt_2}dt_2,\ a, b > 0,\ d\nu := d\nu_1 \times d\nu_2,\ t = (t_1, t_2) \in \mathbb{R}\times\mathbb{R}_+$, *and* $(y_j)_{j\in\mathbb{N}^2}$ *be a given sequence of real numbers. Let* h_1, h_2 *be functions from* $L^\infty_{d\nu}(\mathbb{R}\times\mathbb{R}_+)$, *such that*

$$0 \leq h_1(t_1, t_2) \leq h_2(t_1, t_2) \textit{ almost everywhere in } \mathbb{R}\times\mathbb{R}_+.$$

The following statements are equivalent:

(a) *There exists a unique* $h \in L^\infty_{d\nu}(\mathbb{R}\times\mathbb{R}_+)$ *which satisfies the conditions* $h_1 \leq h \leq h_2$ *almost everywhere in* $\mathbb{R}\times\mathbb{R}_+$, *with*

$$\iint_{\mathbb{R}\times\mathbb{R}_+} t_1^{j_1} t_2^{j_2} h(t_1, t_2) d\nu = y_{(j_1,j_2)}, \quad j := (j_1, j_2) \in \mathbb{N}^2$$

(b) *For any finite subset* $J_0 \subset \mathbb{N}^2$, *and any* $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, *the following implication holds true:*

$$\sum_{j\in J_0} \alpha_j \varphi_j \in \mathcal{P}_+(\mathbb{R}\times\mathbb{R}_+) \Longrightarrow \sum_{j\in J_0} \alpha_j \iint\limits_{\mathbb{R}\times\mathbb{R}_+} t^j h_1(t_1, t_2) d\nu \leq \sum_{j\in J_0} \alpha_j y_j, \ \ t^j = t_1^{j_1} t_2^{j_2};$$

for any finite subsets $J_k \subset \mathbb{N},\ k = 1,2,$ *and any* $\{\alpha_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, *the following inequalities hold:*

$$l \in \quad \{0,1\} := 0 \leq \sum_{i_1,j_1 \in J_1} \left(\left(\sum_{i_2,j_2 \in J_2} \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2} \iint_{\mathbb{R}\times\mathbb{R}_+} t_1^{i_1+j_1} t_2^{i_2+j_2+l} h_1(t_1,t_2)d\nu \right) \right),$$
$$\sum_{i_1,j_1 \in J_1} \left(\left(\sum_{i_2,j_2 \in J_2} \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2} y_{i_1+j_1,\ i_2+j_2+l} \right) \right)$$
$$\leq \sum_{i_1,j_1 \in J_1} \left(\left(\sum_{i_2,j_2 \in J_2} \alpha_{i_1}\alpha_{j_1}\alpha_{i_2}\alpha_{j_2} \iint_{\mathbb{R}\times\mathbb{R}_+} t_1^{i_1+j_1} t_2^{i_2+j_2+l} h_2(t_1,t_2)d\nu \right) \right).$$

Proof. The measure $d\nu_1$ is moment-determinate on $\mathbb{R}$ and $d\nu_2$ is moment-determinate on $\mathbb{R}_+$, due to the corresponding results proven in [27]. In the sequel, Theorem 1 is applied. The convex cone of all sums of polynomials of the form $p_1 \otimes p_2$, with p_1 taking nonnegative values on the entire real axis and p_2 taking nonnegative values on $\mathbb{R}_+$, is dense in the positive cone of $L^1_{d\nu}(\mathbb{R}\times\mathbb{R}_+)$, according to Theorem 1 proven above. Condition (b) says that the hypothesis (b) of Theorem 4 from [36] is accomplished. Indeed, any nonnegative polynomial on $\mathbb{R}$ is a sum of two squares of polynomials (with real coefficients) [2], and any nonnegative polynomial on $\mathbb{R}_+$ has the form $p(t) = q^2(t) + tr^2(t),\ t \in \mathbb{R}_+$, for some polynomials q, r with real coefficients. Next, apply the theorem invoked above, where $\mathcal{P}_{++}(\mathbb{R}\times\mathbb{R}_+)$ stands for the cone of all finite sums of polynomials of the form $p_1 \otimes p_2,\ p_1 \in \mathcal{P}_+(\mathbb{R}),\ p_2 \in \mathcal{P}_+(\mathbb{R}_+)$ (see also the references above for the proofs and details). Since the limit of a finite sum of convergent sequences equals the sum of their limits, the conclusion follows via all these results, passing to the limit as $m \to \infty$. Thus, $(b) \Longrightarrow (a)$ is proven. The converse implication is obvious. This ends the proof. □

Example 1. In Theorem 2, one can take $h_1(t_1,t_2) = t_1^2 t_2 e^{-t_1^2 - t_2}, h_2(t_1,t_2) = e^{-2},\ (t_1,t_2) \in \mathbb{R}\times\mathbb{R}_+$.

The comments following Theorem 1 make sense of the problem of solving full Markov moment problems on Cartesian products of closed intervals endowed with moment-determinate measures in quadratic forms. Thus, Theorem 2 can be generalized as well. On the other hand, in Theorem 2, if conditions mentioned at point (b) are satisfied, then the conclusion (a) follows, without giving any information about explicit expression of the solution h in terms of elementary functions. Only the inequalities for $h_1 \leq h \leq h_2$ are obtained almost everywhere in $\mathbb{R}\times\mathbb{R}_+$ and, of course, the moment interpolation conditions are satisfied by h, although the explicit expressions for h_1 and h_2 in terms of exponential function are known.

3.2. Constructing a Polynomial Solution for a Reduced Interpolation Problem

The next theorem completes and solves results from [36] on the polynomial solution of the interpolation problem (1), formulated for a limited (finite) number of conditions $(j \leq d)$.

Theorem 3. *If* $[a,b] \subset \mathbb{R}$ *is a compact interval,* $d \in \mathbb{N},\ d \geq 1,\ y = (y_0, \ldots, y_d)$ *an arbitrary given vector in* $\mathbb{R}^{d+1}$ *, then there exists a polynomial solution* p *with real coefficients, of degree* d*, for the interpolation problem*

$$\int_{[a,b]} t^j p(t)dt = y_j,\ \ j \in \{0,1,\ldots,d\}.$$

The coefficients of the solution p with respect to the eigenvectors and eigenvalues of the Hankel matrix $M_d = \left(m_{j+l}\right)_{j,l=0}^{d}$ *are given by equality (7) written below, where* m_{j+l} *is defined by (3),* $j, l = 0, 1, \ldots, d$.

Proof. To simplify the notation, consider the case $n = 1$. Condition (1) should be satisfied for all $j \in \{0, 1, \ldots, d\}$. Similar considerations, arguments, and computations can be performed for $n \geq 2$, $j_l \leq d$, $l = 1, \ldots, n$. One looks for a polynomial solution

$$p(t) = \sum_{l=0}^{d} a_l t^l = \sum_{l=0}^{d} a_l \varphi_l(t), \;\; a_l \in \mathbb{R}, \;\; \sum_{l=0}^{d} a_l^2 > 0, \quad t \in K,$$

such that

$$\int_{[a,b]} t^j p(t) dt = y_j, \quad j \in \{0, 1, \ldots, d\}.$$

The following linear system in the unknowns a_l, $l \in \{0, 1, \ldots, d\}$ should be solved:

$$\sum_{l=0}^{d} a_l \int_{[a,b]} t^{j+l} dt = \sum_{l=0}^{d} a_l m_{j+l} = y_j, \;\; m_{j+l} := \int_{[a,b]} t^{j+l} dt, \quad j, l = 0, 1, \ldots, d. \tag{3}$$

The square symmetric matrix of this system is

$$M_d := \left(m_{j+l}\right)_{j,l=0}^{d} \tag{4}$$

System (3) may be written as

$$M_d \begin{pmatrix} a_0 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix}. \tag{5}$$

The matrix M_d is positive (as a linear symmetric operator) and invertible, since

$$\sum_{j,l-0}^{d} m_{j+l} \lambda_j \lambda_l = \int_{[a,b]} \left(\sum_{j,l=0}^{d} \lambda_j \lambda_l t^{j+l} \right) dt = \int_{[a,b]} \left(\sum_{j=0}^{d} \lambda_j t^j \right)^2 dt > 0,$$

for all $\lambda := (\lambda_0, \ldots, \lambda_d) \neq (0, \ldots, 0)$. The last strict inequality holds, since the square of the polynomial appearing under the integral sign is positive, except for a finite number of the roots of that polynomial. If the polynomial appearing in the last integral would be null, this would contradict the assumption $\lambda \neq 0$. Since the boundary S_d of the unit ball B_{d+1} in $\mathbb{R}^{d+1}$ is closed and bounded (i.e., it is compact), there exists $r > 0$ such that

$$\sum_{j,l-0}^{d} m_{j+l} \lambda_j \lambda_l \geq r,$$

for all vectors $(\lambda_0, \ldots, \lambda_d)$, with $\sum_{l=0}^{d} \lambda_l^2 = 1$. Evaluations (inequalities) for the Euclidean norm $\|a_0, \ldots, a_d\|_2$ of the vector of the unknown coefficients have been also pointed out in [36]. Namely, from (5), one infers that

$$\begin{pmatrix} a_0 \\ \vdots \\ a_d \end{pmatrix} = M_d^{-1} \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix} \Longrightarrow$$
$$\|a_0, \ldots, a_d\|_2 \leq M_d^{-1} \cdot \|y_0, \ldots, y_d\|_2 = \left\| 1/\alpha_{min,d} \right\| \cdot \|y_0, \ldots, y_d\|_2.$$

Here, $\alpha_{min,d}$ is the smallest (positive) eigenvalue of the positive definite matrix M_d; hence, $1/\alpha_{min,d}$ is the greatest eigenvalue of M_d^{-1}. However, paper [36] does not provide any method for determining all the coefficients $a_0, \ldots, a_d$ of the polynomial solution p regarding the moments $y_j,\ j \in \{0, 1, \ldots d\}$. In the sequel, this problem is solved without computing (determining) the elements of the matrix M_d^{-1}. As is well known [7], if $\mu_0, \ldots, \mu_d$ are the (note necessarily distinct) positive eigenvalues of the matrix M_d, and $\{f_0, \ldots, f_d\}$ is the orthonormal basis of $\mathbb{R}^{d+1}$ formed by the corresponding eigenvectors, one can write

$$x = \sum_{l=0}^{d} < x, f_l > f_l, \quad M_d x = \sum_{l=0}^{d} \langle x, f_l \rangle \mu_l f_l, \quad x \in \mathbb{R}^{d+1}.$$

Let $\{e_0, \ldots, e_d\}$ be the canonical Hilbert base in $\mathbb{R}^{d+1}$. Then, for $a = (a_0, \ldots, a_d)$, one has

$$a = (a_0, \ldots, a_d) = \sum_{l=0}^{d} a_l e_l = \sum_{l=0}^{d} < a, f_l > f_l \tag{6}$$

The system (5) can be written as

$$\sum_{l=0}^{d} \langle a, f_l \rangle \mu_l f_l = \sum_{l=0}^{d} \langle y, f_l \rangle f_l, \ \ y := \sum_{j=0}^{d} y_j e_j$$

Hence, the coefficients $\langle a, f_l \rangle,\ l = 0, 1, \ldots, d$ of the vector

$$a = (a_0, \ldots, a_d) = \textstyle\sum_{l=0}^{d} a_l e_l \text{ in the base } \{f_0, \ldots, f_d\},$$

are given by the equalities

$$\langle a, f_l \rangle = \mu_l^{-1} \langle y, f_l \rangle\ ,\ \ l = 0, 1, \ldots, d. \tag{7}$$

Of note, the right-hand side numbers in (7) can be expressed only in terms of the moments $y_l,\ l = 0, 1, \ldots, d$. Indeed, $\mu_l,\ l = 0, 1, \ldots, d$ are the (positive) eigenvalues of the Hankel matrix M_d defined by (4), which is known and does not depend on y or on a. The vectors f_l are the eigenvectors of the matrix $M_d = \left(m_{j+l}\right)_{j,l=0}^{d}$. Hence, according to (7) and (6), the unknown vector a can be found starting from the numbers $y_l,\ l = 0, 1, \ldots, d$. This ends the proof. □

Remark 1. Recall that any sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in l_2$ defines a holomorphic function f in U, by means of the power series having α_n as coefficients, as was recently recalled in [12]:

$$f(z) := \sum_{n=0}^{\infty} \alpha_n z^n, \quad |z| < 1. \tag{8}$$

The convergence of this power series holds uniformly in any closed disc of radius $r < 1$. Indeed, one can write:

$$|z| \leq r \Longrightarrow \sum_{n=0}^{\infty} |\alpha_n||z|^n \leq \left(\sum_{n=0}^{\infty} |\alpha_n|^2\right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{1/2} =$$
$$\|\alpha_2\| \cdot \frac{1}{\sqrt{1 - |z|^2}} \leq \|\alpha_2\| \cdot \frac{1}{\sqrt{1 - r^2}}.$$

Remark 2. Any sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in l_1$ defines a holomorphic function as written in (8), which makes sense for $|z| = 1$. The convergence of the power series (8) is absolutely and uniformly on $\overline{U}$; hence, f is also continuous in $\overline{U}$. Conversely, for any holomorphic function f in U, defined and continuous in $\overline{U}$, whose expansion (8) converges absolutely and uniformly on $\overline{U}$, the coefficients of the Taylor expansion (8) form a sequence of the space l_1. Indeed, $|z| = 1 \Longrightarrow \sum_{n=0}^{\infty}|a_n| = \sum_{n=0}^{\infty}|a_n z^n| < \infty$. Such a function is $f(z) := \sum_{n=1}^{\infty} z^n/n^{\alpha},\ \alpha > 1,\ |z| \leq 1$.

Theorem 3 provides the coefficients of the polynomial solution only for the truncated moment problem defined by $d+1$ interpolation conditions. An interesting problem might be that of finding the coefficients of the analytic solution $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ satisfying the full problem of the interpolation conditions $\int_{[a,b]} t^j f(t)dt = y_j,\ j \in \mathbb{N}$, with $(a_n)_{n=0}^{\infty} \in l_2$, under the assumption $[a,b] \subset (-1,1)$. With the notations from above, a related problem is when the infinite matrix $M := \left(m_{j+l}\right)_{j,l=0}^{\infty}$ defines an invertible operator acting on l_2. Since the operator is self-adjoint and positive, this condition on M is true if and only if $\inf_{h \in l_2,\ \|h\|=1} \langle Mh, h\rangle = \inf_{d \geq 2} \alpha_{min,d} > 0$. Another problem could be, given an interval $[a,b] \subseteq [-1,1]$, look for an analytic function f given by (8), $(a_n)_{n=0}^{\infty} \in l_1$, satisfying the conditions of Remark 2, such that $\int_{[a,b]} t^j f(t)dt = y_j,\ j \in \mathbb{N}$. Such problems seem to belong to operator theory or could be related to optimization problems in the finite dimensional space $\mathbb{R}^{d+1}$, $d \in \mathbb{N},\ d \geq 2$, passing to the limit as $d \to \infty$.

3.3. On a Class of Analytic Functional Equations

In the sequel, the existence, and properties of the nontrivial holomorphic solution $\widetilde{f}$ of the equation

$$\widetilde{g}(z) = \widetilde{g}\left(\widetilde{f}(z)\right), \tag{9}$$

are studied, where $\widetilde{g}$ is a given holomorphic function defined on a convex domain Ω, such that $\widetilde{g}(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$ satisfies some conditions, while the nontrivial solution, the function

$$\widetilde{f},\ \widetilde{f}(z) \neq z \text{ for all } z \in \Omega \setminus \{x_0\},\ \widetilde{f}(x_0) = x_0, \text{ where } x_0 \in \Omega \cap \mathbb{R}$$

is the unique minimum or maximum point for the restriction of $\widetilde{g}$ to an open interval contained in $\Omega \cap \mathbb{R}$ Moreover, the restriction of $\widetilde{f}$ to $\Omega \cap \mathbb{R}$ is decreasing and $\widetilde{f}'(x_0) = -1$. One such concrete functional is Equation (9), whose solution is approximated locally, in the neighborhood of the point $x_0 = 0$.

Remark 3. Define $H(z,w) := \widetilde{g}(z) - \widetilde{g}(w),\ z, w \in \mathbb{C}$; then, for $w_0 \neq 0$, one has

$$H_w(z_0, w_0) = -\widetilde{g}'(w_0) \neq 0.$$

Thus, the implicit function theorem can be used for the existence of a unique solution h defined in a neighborhood V_0 of z_0, such that $\widetilde{g}(h(z)) = \widetilde{g}(z)$ for all z in V_0. Since $h(z) = z,\ z \in V_0$ is a solution, from the uniqueness of the solution, it follows that the identity mapping is the unique solution. However, this is the trivial solution, which is not of interest. Thus, if g is holomorphic in a region of the complex plane and the other conditions in the statement of Theorem 4 below are satisfied, for finding a nontrivial solution f, the only chance is to look for it in the neighborhood of the point $z_0 = x_0 = 0$ at which the first derivative of g equals zero. For $(z_0, w_0) = (0,0)$, one has $\widetilde{g}'(0) = 0$, so that the implicit function theorem and the uniqueness of the local solution are no longer working. In this case, the proof from [38] works. This last method does not use the implicit function theorem. Only the continuity of f and the properties $f(0) = 0,\ f$ are decreasing, and the analyticity of $\widetilde{g}$ is applied to deduce the complex differentiability of $\widetilde{f}$ at $z_0 = x_0 = 0$.

Theorem 4. *Consider the following functional equation:*

$$h(x) := x - log(1+x) = h(f(x)) = f(x) - log(1+f(x)),\ \ x \in (-1,\infty).$$

Then, there exists a unique continuous solution f satisfying the equation $h(x) = h(fx))$ for all $x \in (-1,+\infty)$, with the following additional properties:

(i) *f is decreasing on $(-1,\infty)$, and one has*

$$\lim_{x\downarrow -1} f(x) = \infty,\ \ \lim_{x\uparrow\infty} f(x) = -1;$$

(ii) *$x_0 = 0$ is the unique fixed point of f;*

(iii) *one has $f^{-1} = f$ on $(-1,\infty)$;*

(iv) *there is a complex neighborhood D of 0 and a holomorphic extension $\widetilde{f}$ of $f, \widetilde{f} : D \to \mathbb{C}$, satisfying the equation*

$$\widetilde{h}\left(\widetilde{f}(z)\right) = \widetilde{h}(z),\ \ z \in D,\ \widetilde{f}'(0) = -1, \text{where } \widetilde{h}(z) := z - \log(1+z);$$

(v) *in a disc of sufficiently small radius $\varepsilon > 0$, one has*

$$\widetilde{f}(z) \approx -z(1+2z), \qquad |z| < \varepsilon;$$

(vi) *for sufficiently small $\varepsilon > 0$, the following inequality holds:*

$$f(x) > -x(1+2x),\ \ \forall x \in\ [-\varepsilon, 0). \tag{10}$$

Proof. The real-valued function h is continuous on $(-1,+\infty)$, decreasing on $(-1,0)$ from $+\infty$ to zero, and is increasing on $(0,+\infty)$ from zero to $+\infty$. Applying general-type results from [38], the conclusions stated at points (i)–(iv) follow for a sufficiently small complex neighborhood D of zero contained in the open unit disc U. To prove (v), for sufficiently small $\varepsilon > 0,\ \varepsilon < 1$, one can write the equation $\widetilde{h}\left(\widetilde{f}(z)\right) = \widetilde{h}(z),\ |z| < \varepsilon$ as

$$\tilde{f}(z) - z = log\left(\frac{1+\tilde{f}(z)}{1+z}\right) = log\left(1 + \frac{\tilde{f}(z)-z}{1+z}\right) =$$
$$\int_0^{\frac{\tilde{f}(z)-z}{1+z}} \frac{1}{1+w}dw = \int_0^{\frac{\tilde{f}(z)-z}{1+z}} \left(1 - w + w^2 + \cdots + (-1)^n w^n + \cdots\right)dw =$$
$$\frac{\tilde{f}(z)-z}{1+z} - \frac{1}{2}\cdot\left(\frac{\tilde{f}(z)-z}{1+z}\right)^2 + \frac{1}{3}\cdot\left(\frac{\tilde{f}(z)-z}{1+z}\right)^3 - \cdots.$$

Observe that, in the above remarks, for small $\varepsilon > 0$, one has $\left|\frac{\widetilde{f}(z)-z}{1+z}\right| < 1\ \forall z$ with $|z| \le \varepsilon$, since $\frac{\widetilde{f}(z)-z}{1+z} \to 0$ as $\varepsilon \to 0$. Hence, if one denotes $w_1 = w_1(z) = \frac{\widetilde{f}(z)-z}{1+z}$, under the above-mentioned conditions, one can write

$$log(1+w_1) = \int_0^{w_1} \frac{1}{1+w}dw = \int_0^{w_1} (1 - w + w^2 + \cdots + (-1)^n w^n + \cdots)dw =$$
$$w_1 - \frac{1}{2}\cdot w_1^2 + \frac{1}{3}\cdot w_1^3 - \cdots\ .$$

The last integral can be computed on any C^1 path of ends 0 and w_1, whose image is contained in the open unit disc U, according to Cauchy theorem for the holomorphic function $w \mapsto 1/(1+w),\ |w| < 1$. Such a path is the line segment joining the origin 0 with $w_1 = w_1(z)$. Its parameterization is $w(t) = tw_1,\ t \in [0,1]$. Integration term by term is allowed due to the uniform convergence of the geometric series $\sum_{n=0}^{\infty}(-w)^n$, for w in the closed disc of radius $|w_1|$, namely, in the disc

$$\{w;\ |w| \le |w_1| < 1\}.$$

Since $\widetilde{f}(z) \neq z,\ \widetilde{f}(z) - z \approx 0$ for $|z| < \varepsilon,\ z \neq 0$, dividing by $\widetilde{f}(z) - z$ and neglecting the powers $n \ge 2$ of $\widetilde{f}(z) - z,$ which are very close to zero, one derives that

$$1 \approx \frac{1}{1+z} - \frac{1}{2} \cdot \frac{\widetilde{f}(z) - z}{(1+z)^2}.$$

This can be rewritten as

$$2(1+z)^2 - 2(1+z) + \widetilde{f}(z) - z \approx 0,$$

which is equivalent to

$$\widetilde{f}(z) \approx z - 2z(1+z) = -z(1+2z).$$

Thus, (v) is proven. Next, write the computational result in the proof of point (v) for real

$$z = x \in (-1, 0),$$

x close to zero. For such numbers x, one has

$$f(x) - x > 0,\ 1 + x > 0, 1 > \frac{f(x) - x}{1+x} > 0,$$

and

$$(x) - x = \frac{f(x) - x}{1+x} - \frac{1}{2} \cdot \left(\frac{f(x) - x}{1+x}\right)^2 + \frac{1}{3} \cdot \left(\frac{f(x) - x}{1+x}\right)^3 - \frac{1}{4}\left(\frac{f(x) - x}{1+x}\right)^4 + \cdots =$$
$$\frac{f(x) - x}{1+x} - \frac{1}{2} \cdot \left(\frac{f(x) - x}{1+x}\right)^2 + \left(\frac{f(x) - x}{1+x}\right)^3 \left(\frac{1}{3} - \frac{1}{4} \cdot \frac{f(x) - x}{1+x}\right) + r(x).$$

Here, apply the well-known evaluation of the sum of the involved (alternate) Leibniz-type series, whose general term is

$$\frac{(-1)^n}{n+1} \left(\frac{f(x) - x}{1+x}\right)^{n+1}.$$

Namely,

$$\frac{1}{3} - \frac{1}{4} \cdot \frac{f(x) - x}{1+x} > 0,$$

and the rest $r(x)$ is the sum of positive numbers of the form

$$\left(\frac{f(x) - x}{1+x}\right)^{2k+1} \left(\frac{1}{2k+1} - \frac{1}{2k+2} \cdot \frac{f(x) - x}{1+x}\right).$$

The conclusion is $r(x) > 0$, and

$$f(x) - x > \frac{f(x) - x}{1+x} - \frac{1}{2} := \left(\frac{f(x) - x}{1+x}\right)^2.$$

The last inequality is equivalent to

$$1 > \frac{1}{1+x} - \frac{1}{2} \cdot \frac{f(x) - x}{(1+x)^2}, x \in [-\varepsilon, 0).$$

This inequality holds true for sufficiently small $\varepsilon > 0,\ \varepsilon < 1$. Equality occurs in the last inequality if and only if $x = 0$. As seen in the proof of point (v), where the equality sign is replaced by the inequality one, the conclusion is

$$2(1+x)^2 - 2(1+x) > -f(x) + x,$$

$$f(x) > x - 2x(1+x) = -x(1+2x),\ \ x \in\ [-\varepsilon, 0).$$

Thus, (10) is proven. This ends the proof. □

As in the case of Theorem 2, in Theorem 4, there is information on the behavior of the unknown implicitly defined function f, but not on its expression in terms of elementary functions, although the given function h is an elementary analytic function. Only local approximation, inequalities, $f^{-1} = f$, and other main properties of f are available.

Remark 4. Note the following similarity between Sections 3.1 and 3.3. In Theorem 4, the main properties of the function f mentioned above are deduced, but an exact analytic expression for f seems to be impossible to be found. This is a quasi-general remark on the functions defined implicitly. Only the local approximation provided by Theorem 4, and points (v) and (vi), represent a simple way of making an idea in this respect. Similarly, in Theorem 2, a characterization of the existence and uniqueness of the function h with properties mentioned at point (a) is proven. However, a formula for the exact expression of h in terms of the moments is not easily obtained (this is an inverse problem).

Remark 5. From points (i), (ii) of Theorem 4, one already knows that $x \in (-1, 0)$ is equivalent to $f(x) > 0$. Therefore, the assertion (vi) of the same theorem is interesting for $1 + 2x \geq 1 - 2\varepsilon > 0$, that is, for $\varepsilon < 1/2$.

Recall that, if H is an arbitrary real or complex Hilbert space of dimension ≥ 2, and if $\mathcal{A}$ denotes the real vector space of all self-adjoint operators acting on H, then the natural order relation on $\mathcal{A}$ is defined by

$$U, V \in \mathcal{A},\ \ U \leq V \Longleftrightarrow\ \langle Uh, h\rangle\ \leq\ \langle Vh, h\rangle \text{ for all } h \in H.$$

Endowed with this order relation and the usual operatorial norm, $\mathcal{A}$ is an ordered Banach space which is not a lattice (if $dim(H) \geq 2$). If f is a continuous real-valued function on the spectrum $\sigma(-A)$, then one can denote by $f(-A)$ the corresponding self-adjoint operator obtained via functional calculus.

Theorem 5. *Let H be an arbitrary real or complex Hilbert space and $\varepsilon \in (0, 1/2]$ be a sufficiently small number, such that the inequality proven at point (vi) of Theorem 4 holds true. Let A be a positive self-adjoint operator from H to itself, such that $\|A\| \leq \varepsilon \leq 1/2$. Then,*

$$f(-A) \geq A(I - 2A). \tag{11}$$

Proof Let t be an arbitrary real number in the spectrum $\sigma(A) \subseteq [0, \varepsilon]$ of the self-adjoint positive operator A, with $\|A\| \leq \varepsilon$. Then, $x := -t \in [-\varepsilon, 0]$. For $-t \in [-\varepsilon, 0)$; according to Theorem 4, point (vi), one has

$$f(-t) > t(1-2t).$$

Since $f(0) = 0$, this results in $f(-t) \geq t(1-2t)$ for all $-t \in [-\varepsilon, 0]$. By means of functional calculus for continuous functions on the spectrum $\sigma(-A) = -\sigma(A) \subseteq [-\varepsilon, 0]$ of the self-adjoint operator $-A$, one infers that $f(-A) \geq A(I - 2A)$. This ends the proof. □

With the notations and conditions on ε mentioned in Theorems 4 and 5, the following consequence of Theorem 5 follows.

Corollary 1 *If* $n \geq 2$ is an integer, then (11) holds for any symmetric $n \times n$ matrix A with real entries, whose eigenvalues are all contained in the interval $[0, \varepsilon]$.

3.4. On Some Optimization Problems for the Modulus of the Complex Joukowski Function

Let $K_{r,R} := \{z \in \mathbb{C}; r \leq |z| \leq R\}$, where $0 < r < R < \infty$. In any point of $K_{r,R}$, the Joukowski function

$$\psi(z) := \frac{1}{2}(z + 1/z),$$

makes sense. Actually, $\psi \in H(\mathbb{C} \setminus \{0\})$. Next, two optimization-type theorems related to this function are proven, and a consequence is derived.

Theorem 6. *The following inequalities and respective equalities hold:*

$$M := \max_{z \in K_{r,R}} |\psi(z)| = \max_{r \leq |z| \leq R} \left| \frac{1}{2}\left(z + \frac{1}{z}\right) \right| = \frac{1}{2} max\left\{R + \frac{1}{R}, r + \frac{1}{r}\right\} = max\{\psi(R), \psi(r)\}. \tag{12}$$

Moreover, one has the following:

(a) $0 < r < R \leq 1 \Longrightarrow M = \frac{1}{2}\left(r + \frac{1}{r}\right) = \psi(r)$;

(b) $1 \leq r < R \Longrightarrow M = \frac{1}{2}\left(R + \frac{1}{R}\right) = \psi(R) = |\psi(-R)|$; *and*

(c) $0 < r < 1 < R \Longrightarrow$*(12) is the only available information. If* $0 < r < 1$ *and* $R = \frac{1}{r}$ *, then*

$$M = \frac{1}{2}\left(r + \frac{1}{r}\right) = \frac{1}{2}\left(R + \frac{1}{R}\right).$$

Proof. The equality (12) follows from the maximum modulus property [6] for the holomorphic function ψ, followed by the computational conclusion from below. The maximum is attained at a point located on the boundary $\partial K = \{z; |z| = R\} \cup \{z; |z| = r\}$. These lead to

$$M = max\left\{M_R := \max_{|z|=R}|\psi(z)|,\ M_r := \max_{|z|=r}|\psi(z)|\right\}.$$

The same computations determine the maximum points on the circles of radiuses R, respective of radius r. Namely, one finds

$$|z| = R \Longleftrightarrow z = Re^{i\theta}, \theta \in [0, 2\pi) \Longrightarrow$$
$$\left|z + \frac{1}{z}\right| = \left|z + \frac{\bar{z}}{z\bar{z}}\right| = \left|Re^{i\theta} + \frac{Re^{-i\theta}}{R^2}\right| =$$
$$\left|R(cos(\theta) + isin(\theta)) + \frac{1}{R}(cos(\theta) - isin(\theta))\right| =$$
$$\left|\left(R + \frac{1}{R}\right)cos(\theta) + i\left(R - \frac{1}{R}\right)sin(\theta)\right| =$$
$$\left(\left(R + \frac{1}{R}\right)^2 cos^2(\theta) + \left(R - \frac{1}{R}\right)^2 sin^2(\theta)\right)^{1/2} =$$
$$\left(R^2 + \frac{1}{R^2} + 2\left(cos^2(\theta) - sin^2(\theta)\right)\right)^{1/2} \leq \left(R^2 + \frac{1}{R^2} + 2\right)^{1/2} = R + \frac{1}{R}.$$

Of note, equality occurs in the last inequality if and only if $sin^2(\theta) = 0$, which means $cos^2(\theta) = 1$. Thus, the maximum value on $[0, 2\pi)$ is attained at $\theta = 0$ and $\theta = \pi$. The first conclusion is

$$M_R := \max_{|z|=R}\left|\frac{1}{2}\left(z + \frac{1}{z}\right)\right| = \frac{1}{2}\left(R + \frac{1}{R}\right),$$

and this maximum is attained at $z_1 = R$ and at $z_2 = -R$. Repeating the same calculations on the circle $|z| = r$, one obtains:

$$M_r := \max_{|z|=r}\left|\frac{1}{2}\left(z+\frac{1}{z}\right)\right| = \frac{1}{2}\left(r+\frac{1}{r}\right) = \psi(r).$$

Hence, (12) is proven. Next, one must compare M_R with M_r. Namely, to prove (a), assume that $0 < r < R \leq 1$. Then, one computes the difference

$$M_R - M_r = \frac{1}{2}\left(R - r + \left(\frac{1}{R} - \frac{1}{r}\right)\right) = \frac{1}{2}(R-r)\left(1 - \frac{1}{Rr}\right). \quad (13)$$

This results in $M_R - M_r > 0 \iff Rr > 1$, $M_R - M_r < 0 \iff Rr < 1$, $M_R - M_r = 0 \iff Rr = 1$. If $0 < r < R \leq 1$, then $Rr < 1$; hence, $M_R < M_r$, that is, $M = M_r$. Similarly,

$$1 \leq r\langle R \Longrightarrow Rr\rangle 1 \iff M_R > M_r \Longrightarrow M = M_R.$$

Clearly, $Rr = 1$ if and only if $M_R = M_r$. According to (13), or simply because $r = 1/R$, this implies

$$\psi(R) = \psi\left(\frac{1}{R}\right) = \psi(r).$$

If $0 < r < 1 < R$ and $R \neq \frac{1}{r}$, then one cannot derive any conclusion regarding the signature of $Rr - 1$, and one has $\psi(R) \neq \psi(r)$. Using (13), this means one cannot decide the signature of $M_R - M_r$. In this case, the conclusion remains the equality written in (12). The proof is complete. □

Corollary 2. *If* $0 < r < R < 1$, *then* $\underset{z \in K_{r,1}}{max}|\psi(z)| = \frac{1}{2}\left(r + \frac{1}{r}\right) = \psi(r)$.

Proof. One applies Theorem 6 (a) for $R = 1$. □

Next, the minimum value m of the modulus is discussed for the same function ψ, $\psi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, on the subset $K_{R,r}$, also determining the corresponding minimum points.

Theorem 7. *With the above notations, the following statements hold:*

(i) *If* $[r, R] \subset (0, 1)$, *then* $m = \frac{1}{2}\left(\frac{!}{R} - R\right) = \psi(iR) = \psi(-iR)$;

(ii) *If* $[r, R] \subset (1, +\infty)$, then $m = \frac{1}{2}\left(r - \frac{1}{r}\right) = \psi(ir) = \psi(-ir)$;

(iii) *If* $1 \in [r, R]$, *then* $m = 0 = \psi(i) = \psi(-i)$.

Proof. For $z = \rho e^{i\theta}$, $\rho > 0$, $\theta \in [0, 2\pi)$, following the computation from the proof of Theorem 6, one finds

$$\left|z + \frac{1}{z}\right| = \left(\rho^2 + \frac{1}{\rho^2} + 2\left(cos^2(\theta) - sin^2(\theta)\right)\right)^{1/2} \geq$$
$$\left(\rho^2 + \frac{1}{\rho^2} - 2\right)^{1/2} = \left|\rho - \frac{1}{\rho}\right|, \quad \forall\theta \in [0, 2\pi).$$

Observe that equality occurs in the last inequality if and only if $cos^2(\theta) = 0$ (that is, $sin^2(\theta) = 1$), which is equivalent to $\theta \in \{\pi/2, 3\pi/2\}$. Thus,

$$\mathrm{m} = \min_{\rho \in [r,R]}\left(\min_{\theta \in [0,2\pi)}\left|\psi\left(\rho e^{i\theta}\right)\right|\right) = \frac{1}{2}\min_{\rho \in [r,R]}\left|\rho - \frac{1}{\rho}\right|. \quad (14)$$

If $\rho \in [r, R] \subset (0, 1)$, then $\left|\rho - \frac{1}{\rho}\right| = \frac{1}{\rho} - \rho$ and $\min_{\rho \in [r,R]} \left|\rho - \frac{1}{\rho}\right| = \min_{\rho \in [r,R]} \left(\frac{1}{\rho} - \rho\right) = \frac{1}{R} - R$, since the function $g(\rho) = \frac{1}{\rho} - \rho$ is decreasing on $(0, +\infty)$. Hence, $m = \frac{1}{2}\left(\frac{!}{R} - R\right)$ and the minimum value is attained for

$$\rho = R, \quad \theta \in \{\pi/2, 3\pi/2\}.$$

This means the minimum points are iR and $-iR$. Thus, the assertion stated at point (i) is proven. The point (ii) follows via similar reasoning, with the remarks that $\left|\rho - \frac{1}{\rho}\right| = \rho - \frac{1}{\rho} = -g(\rho)$ for $\rho \in [r, R] \subset (1, +\infty)$ and $-g$ is increasing. The minimum points correspond to $\rho = r,\ \theta \in \{\pi/2, 3\pi/2\}$. Thus, (ii) is proven. To prove (iii), observe that the global minimum value zero for m from (14) is attained if and only if $\rho = 1$. The condition $\theta \in \{\pi/2, 3\pi/2\}$ remains from the optimization in the variable θ, so that, in this case, the only minimum points are i and $-i$. This concludes the proof. □

Remark 6. Now, note the connection of the function ψ with the functional equations appearing in the preceding section. As is well-known and easy to prove directly, the unique nontrivial solution f of the functional equation $\psi(f(z)) = \psi(z),\ z \neq 0$, is

$$f(z) = \frac{1}{z}, \quad z \neq 0.$$

Starting from the function Ψ and an arbitrary sequence $(a_n)_{n \geq 0} \in l_1$, define

$$a_{-n} := a_n, \quad n \in \mathbb{N},$$

$$\psi(z) := \sum_{n \in \mathbb{Z}} a_n z^n, \quad z \in \mathbb{T}.$$

Then, clearly,

$$\psi(z) = a_0 + \sum_{n \geq 1} a_n \left(z^n + \frac{1}{z^n}\right), \quad z \in \mathbb{T},$$

verifies $\psi(z) = \psi\left(\frac{1}{z}\right)$.

From the last two theorems, one derives the following consequence involving fractional power of the variable z.

Corollary 3. *Let $\alpha \in (0, \infty)$ and $\psi_\alpha(z) := \psi(z^\alpha) = (1/2)(z^\alpha + z^{-\alpha}),\ z \in \mathbb{C} \setminus \{0\}$. Then, the following hold:*

(i) $0 < r < R \leq 1 \Longrightarrow \max_{r \leq |z| \leq R} \frac{1}{2}\left|z^\alpha + \frac{1}{z^\alpha}\right| = \frac{1}{2}\left(r^\alpha + \frac{1}{r^\alpha}\right)$;

(ii) $0 < r < R < 1 \Longrightarrow \min_{r \leq |z| \leq R} \frac{1}{2}\left|z^\alpha + \frac{1}{z^\alpha}\right| = \frac{1}{2}\left(\frac{1}{R^\alpha} - R^\alpha\right)$.

Proof. The following equalities hold:

$$z = \rho e^{i\theta}, \;\; w := z^\alpha := e^{\alpha \log(z)} = e^{\alpha(log(\rho) + i\theta)} = \rho^\alpha := e^{i\alpha\theta}.$$

This implies

$$|w| = |z^\alpha| = \rho^\alpha = |z|^\alpha.$$

This results in

$$r \leq |z| \leq R \Longleftrightarrow r^\alpha \leq |w| \leq R^\alpha.$$

Using Theorem 6, assertion (a), these further yield

$$\max_{r\le|z|\le R}\frac{1}{2}\left|z^{\alpha}+\frac{1}{z^{\alpha}}\right|=\max_{r^{\alpha}\le|w|\le R^{\alpha}}\frac{1}{2}\left|w+\frac{1}{w}\right|=\frac{1}{2}\left(r^{\alpha}+\frac{1}{r^{\alpha}}\right).$$

Thus, the implication (i) is proven. The assertion (ii) follows by means of Theorem 7, point (i). This ends the proof. □

It seems that condition $\alpha\in(0,\infty)$ is not necessary, since $\psi_{-\alpha}(z)=\psi_{\alpha}(z)$ for all $\alpha\in\mathbb{R}$, $z\neq 0$. An interesting problem could be that of applying the results of Section 3.4 to normal operators having their spectrum in a circular anulus $K_{r,R}$.

4. Discussion

The present article provides new, or improved and completed, versions of previous results on polynomial approximations on unbounded closed subsets, Markov moment problems on such subsets, polynomial solutions for reduced interpolation problems, functional equations, and optimization of the functions ψ and ψ_{α} in circular annuluses. In the first part, a scalar version of previous theorems on the existence and the uniqueness of the solution for the full Markov moment problem is pointed out (see Theorems 1 and 2). Such theorems use polynomial approximation of any function from $\left(L^1_{d\nu}\right)_+$ by special nonnegative polynomials, which are expressible in terms of sums of squares, in the space $L^1_{d\nu}$. Here, ν is a product $\nu_1\times\nu_2$, with ν_1 being a moment-determinate measure on $\mathbb{R}$ and ν_2 a moment-determinate measure on $\mathbb{R}_+$. On the other hand, at the end of the paper, it is pointed out that, in the simplest case of the function $\psi(x)=x+1/x,\ x>0$, the graph of ψ is not symmetric with respect to the vertical line of equation $x=1$, passing through the minimum point 1 of ψ located in the interval $(0,\infty)$. However, in the complex analysis framework, considering the meromorphic function

$$\Psi(z)=a_0+\sum_{n\ge1}a_n\left(z^n+\frac{1}{z^n}\right),\quad z\in\mathbb{T},$$

note the symmetry of its coefficients (see Remark 6). Additionally, for $z\in\mathbb{T}$, one has $\Psi(z)=\Psi(1/z)=\Psi(\bar{z})$ and $\bar{z}$ is the symmetric of z with respect to the real axis. If all the coefficients $a_n,\ n\in\mathbb{N}$ are real numbers, $(a_n)_{n\ge0}\in l_1$, one finds that $\Psi(z)\in\mathbb{R}$ for all $z\in\mathbb{T}$. Part of the previous results is the basis for the new ones. Theorem 3 provides all the Fourier coefficients of the unknown polynomial solution in the orthonormal base defined by the eigenvectors of the Hankel matrix M_d in terms of the given moments, unlike the previous result on this topic [36]. In addition to the results pointed out in the Abstract, in Theorem 5, an inequality valid for any positive self-adjoint operator with sufficiently small norm is deduced from Theorem 4.

5. Conclusions

Sections 3.2–3.4 are directly related to notions and/or results involving symmetry. To name a few of them, in Section 3.2, the matrix M_d is a special symmetric matrix with real entries. In Section 3.3, the graph of the unknown function f is symmetric with respect to the line of equation $y=x$, since f is its own inverse for the operation of composition of functions. In Section 3.4, the symmetry of the coefficients of the meromorphic function Ψ from Remark 6 has already been discussed. As a common aspect of Sections 3.1 and 3.3, in both these sections, one can prove significant properties of the solutions, without knowing their expressions in terms of elementary functions, although the given functions are elementary (see Theorem 2, Example 1, and Theorem 4).

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