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Best Proximity Point Results for *n*-Cyclic and Regular-*n*-Noncyclic Fisher Quasi-Contractions in Metric Spaces

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Abstract: In this work, we introduce some new concepts such as n-cyclic Fisher quasi-contraction mappings, full-n-noncyclic and regular-n-noncyclic Fisher quasi-contraction mappings in metric spaces. We then generalize the results by Safari-Hafshejani, Amini-Harandi and Fakhar. Meanwhile, we answer the question "under what conditions does a full-n-noncyclic Fisher quasi-contraction mapping have $n(n-1)/2$ unique optimal pairs of fixed points?". Further, to support the main results, we highlight all of the new concepts via non-trivial examples.

Keywords: best proximity point; n-cyclic Fisher quasi-contraction mapping; regular-n-noncyclic and full-n-noncyclic Fisher quasi-contraction mapping

MSC: 47H10; 47H09

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1. Introduction

In 1974, Ćirić [1] proved a fixed point (fp) result for a novel class of contractive mappings, which is called quasi-contraction mapping. In fact, he showed that quasi-contraction is a real generalization of some well-known linear contractions, and his result was an expansion of the Banach contraction principle. Then, he proved the existence and uniqueness of fp for T-orbitally single-valued mappings and F-orbitally multi-valued mappings in complete metric spaces.

A self-mapping S on a metric space Y is named a generalized contraction if nonnegative functions exist $m(i,j)$, $n(i,j)$, $o(i,j)$ and $p(i,j)$ for every $i,j \in Y$ so that

$$\sup_{i,j\in Y}\{m(i,j)+n(i,j)+o(i,j)+2p(i,j)\}<1$$

and

$$d(Si,Sj)\leqslant m(i,j)d(i,j)+n(i,j)d(i,Si)+o(i,j)d(j,Sj)+2p(i,j)[d(i,Sj)+d(j,Si)],$$

and is named a quasi-contraction if there is $0\leqslant\omega<1$ so that

$$d(Si,Sj)\leqslant\omega\max\{d(i,j),d(i,Si),d(j,Sj),d(i,Sj),d(j,Si)\}$$

for all $i,j\in Y$ [1].

In 1977, Rhoades [2] compared various contractive mappings in metric spaces and showed that Ćirić contractive mapping is one of the most total contractive mappings in metric spaces as it contains many different versions of contractions. Thus, many authors became interested in studying quasi-contractions and extended the Ćirić's fp results in various aspects. One of these results was introduced by Fisher [3] as follows:

Theorem 1. *Let (Y,d) be a complete metric space and $S : Y \to Y$ be a continuous mapping. Assume that there exists $m,n \in \mathbb{N}$ and some $\lambda \in [0,1)$ provided that*

$$d(S^m i, S^n j) \leqslant \lambda \max\{d(S^\alpha i, S^\beta j), d(S^\alpha i, S^{\alpha'} i), d(S^\beta j, S^{\beta'} j) : 0 \leq \alpha, \alpha' \leq m \text{ and } 0 \leq \beta, \beta' \leq n\}$$

for all $i,j \in Y$. Then S has a unique fp.

For more details, see [4–6] and references therein.

Although the fp theory is a significant tool for solving fp equations for mappings T defined on a subset A of a metric space (X,d), a non-self mapping $T : A \to B$ does not necessarily have an fp. Hence, one may attempt to find an element x that is, in some sense, closest to Tx. The best approximation theorems and best proximity point (bpp) theorems became famous in this viewpoint. Let (X,d) be a metric space, $\emptyset \neq A, B \subset X$, $d(A,B) = \inf\{d(x,y); x \in A,\ y \in B\}$ and $T : A \to B$ be a non-self mapping. The bpp(s) of T is the set of all points $x \in A$ so that $d(x,Tx) = d(A,B)$. The main goal of the bpp theory is to provide enough conditions that vouch for the existence of such points. Hence, this theory for various mappings has been considered by many researchers (for example, see [7–12]). On the other hand, in 2003, Kirk et al. [13] formulated and defined cyclic mappings as follows:

A mapping $T : A \cup B \to A \cup B$ is said to be cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. Note that if $T(A) \subseteq A$ and $T(B) \subseteq B$, then T is called a noncyclic mapping.

In [7], Eldred et al. proved the existence of an optimal pair of fp(s) of noncyclic mappings. After that, Eldred and Veeremani [8] studied the existence of the bpp of cyclic contraction mappings on uniformly convex Banach spaces. Moreover, Suzuki et al. [9] and Espínola et al. [11] established the existence of the bpp for cyclic contraction mappings in metric spaces by applying the properties: unconditionally Cauchy and weakly unconditionally Cauchy, respectively. In fact, the researchers mentioned above have fused cyclical and noncyclic concepts of mappings with the bpp theory to solve some problems in the approximation and optimization theories. Hence, many authors are working on finding the bpp for cyclic and noncyclic mappings in various spaces in [14–17] and the references therein. Ultimately, Safari-Hafshjani et al. [18] defined a Fisher quasi-contraction and studied the existence of fp(s) and bpp(s) for noncyclic and cyclic Fisher quasi-contraction mappings.

In this work, we define the concepts of n-cyclic Fisher quasi-contraction mappings, as well as full-n-noncyclic and regular-n-noncyclic Fisher quasi-contraction mappings in metric spaces. Next, we generalize the results by Safari-Hafshejan et al. [18] and prove the existence of $n(n-1)/2$, the unique optimal pair of fp(s) for full-n-noncyclic Fisher quasi-contraction mappings.

Let us start with some well-known definitions and notions, which are required in the following sections.

For two nonempty sets, A and B in X, we denote $\delta[A,B]$ by $\delta[A,B] = \sup\{d(x,y) : x \in A, y \in B\}$. Note that $\delta[A,B]$ exhibits symmetry.

Definition 1 ([9,14])**.** *Let A and B be two nonempty subsets of a metric space (X,d). Then,*

1. *The pair (A,B) has the unconditionally Cauchy (UC) property if for two sequences $\{x_n\}$ and $\{x'_n\}$ in A and a sequence $\{y_n\}$ in B, $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x'_n,y_n) = d(A,B)$ implies $\lim_{n\to\infty} d(x_n,x'_n) = 0$.*
2. *The pair (A,B) has the weakly unconditionally Cauchy (WUC) property if for each $\{x_n\} \subseteq A$ and $\epsilon > 0$, there exists $y \in B$ so that $d(x_n,y) \leqslant d(A,B) + \epsilon$ for $n \geqslant n_0$ implies $\{x_n\}$ is Cauchy.*

Proposition 1 ([11])**.** *Let A and B be two nonempty subsets of a metric space (X,d) provided that A is complete and (A,B) has the UC property. Then (A,B) has the WUC property.*

Definition 2 ([19])**.** *The mapping T on $A_1 \cup A_2 \cup \cdots \cup A_n$ is called n-cyclic when $T(A_1) \subseteq A_2$, $T(A_2) \subseteq A_3, \ldots, T(A_n) \subseteq A_1$.*

2. Results of the n-Cyclic Fisher Quasi-Contraction

Inspired by the findings of the study about cyclic Fisher quasi-contractions [18], we introduce the n-cyclic Fisher quasi-contraction as follows:

Definition 3. *Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of a metric space (X,d) and T be n-cyclic mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Point $x^* \in A_1 \cup A_2 \cup \cdots \cup A_n$ is called the bpp for T if there exist $m \in \mathbb{N}$ and $1 \leqslant m \leqslant n$ provided that*

$$x^* \in A_m \quad \text{and} \quad d(x^*, Tx^*) = \begin{cases} d(A_m, A_{m+1}) & 1 \leqslant m \leqslant n-1 \\ d(A_n, A_1) & m = n \end{cases}.$$

It is obvious that if $d(A_m, A_{m+1}) = 0$ or $d(A_n, A_1) = 0$, then the above problem finds an fp of T.

Remark 1. *From now on, whenever the term (A_m, A_{m+1}) is observed, it refers to one of the pairs of consecutive sets like (A_1, A_2), (A_2, A_3), $\ldots$, (A_{n-1}, A_n) and (A_n, A_1).*

Notations. Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of a metric space X, $p, q \in \mathbb{N}$ and T be a n-cyclic mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Then,

$$\mathcal{A}^{x,y}_{pn,qn} = \{T^{in}x, T^{jn-1}y;\ 0 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\} \subseteq A_m,$$
$$\mathcal{B}^{x,y}_{pn,qn} = \{T^{jn+1}x, T^{in}y;\ 0 \leqslant j \leqslant p-1, 0 \leqslant i \leqslant q\} \subseteq A_{m+1},$$

for all $x \in A_m$ and $y \in A_{m+1}$. Moreover, for $x \in A_m$ and $r \in \mathbb{N}$, consider two sets $\mathcal{A}^x_r$ and $\mathcal{B}^x_r$ as follows:

$$\mathcal{A}^x_r = \{x, T^n x, T^{2n} x, T^{3n} x, \ldots, T^{rn} x\} \subseteq A_m,$$
$$\mathcal{B}^x_r = \{Tx, T^{n+1} x, T^{2n+1} x, T^{3n+1} x, \ldots, T^{rn+1} x\} \subseteq A_{m+1}.$$

Definition 4. *Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of a metric space (X,d) and T be n-cyclic mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. The mapping T is called the n-cyclic Fisher quasi-contraction for some $1 \leqslant m \leqslant n$ if there exist $p, q \in \mathbb{N}$ and $0 \leqslant c < 1$ so that*

$$\max\{d(T^{pn}x, T^{qn}y), d(T^{pn+1}x, T^{qn-1}y)\} \leqslant c\delta[\mathcal{A}^{x,y}_{pn,qn}, \mathcal{B}^{x,y}_{pn,qn}] + (1-c)d(A_m, A_{m+1})$$

for all $x \in A_m$ and $y \in A_{m+1}$.

Example 1. *Consider $\mathbb{R}$ with the Euclidean metric, $A_1 = [0,2]$ and $A_2 = [-2,0]$. Assume that $T : A_1 \cup A_2 \to A_1 \cup A_2$ is defined by*

$$Tx = \begin{cases} -x & x \in A_1 \\ -\frac{x}{2} & x \in A_2 \end{cases}.$$

For $p = q = 1$ and $c = \frac{1}{2}$, we have

$$\max\{d(T^2x, T^2y), d(T^3x, T^1y)\} = \frac{|x-y|}{2} \leqslant \frac{1}{2}\delta[\{x, T^2x, Ty\}, \{y, T^2y, Tx\}] + \frac{1}{2}d(A_1, A_2).$$

Thus, T is a 2-cyclic Fisher quasi-contraction for every $x \in A_1$ and $y \in A_2$. Since

$$d(T^2x, T^2y) \leqslant \max\{d(T^2x, T^2y), d(T^3x, T^1y)\},$$

then

$$d(T^2x, T^2y) \leqslant \frac{1}{2}\delta[\{x, T^2x, Ty\}, \{y, T^2y, Tx\}] + \frac{1}{2}d(A_1, A_2).$$

This is the same example from Safari et al. [18] for $n = 2$.

One of the fundamental steps in the literature on fp(s) is to find a Picard iteration sequence. Here, for $x_0 \in A_m$ and $y_0 \in A_{m+1}$, we define our iteration sequence as follows:

$$x_i = \begin{cases} T^{\frac{in}{2}} x_0 \in A_m, & i \text{ is even} \\ T^{\frac{(i-1)n}{2}+1} x_0 \in A_{m+1}, & i \text{ is odd} \end{cases} \quad \text{and} \quad y_i = \begin{cases} T^{\frac{in}{2}} y_0 \in A_{m+1}, & i \text{ is even} \\ T^{\frac{(i+1)n}{2}-1} y_0 \in A_m, & i \text{ is odd} \end{cases}. \tag{1}$$

Lemma 1. *Let $A_1, A_2, \dots, A_n$ be nonempty subsets of a metric space (X, d) and T be the n-cyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$ with the quantities pn, qn, and $0 \leqslant c < 1$. Also, for $x_0 \in A_m$, consider sequence $\{x_i\}$ to be the same as in (1). Then, $\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] = d(T^{sn}x_0, T^{s'n+1}x_0)$ for some $s, s' \in \mathbb{N}$, where $sn < pn$ or $s'n < qn$.*

Proof. For simplicity, assume that $qn \geqslant pn$. Since $\mathcal{A}_r^{x_0}$ and $\mathcal{B}_r^{x_0}$ are finite sets, we have $\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] = \sup\{d(\alpha, \beta); \alpha \in \mathcal{A}_r^{x_0}, \beta \in \mathcal{B}_r^{x_0})\} = d(T^{sn}x_0, T^{s'n+1}x_0)$ for some $s, s' \in \mathbb{N}$. On the contrary, suppose that $p \leqslant s \leqslant r$ and $q \leqslant s' \leqslant r$. Then, by the definition of $\{x_i\}$, $T^{sn-pn}x_0 \in A_m$ and $T^{s'n-qn+1}x_0 \in A_{m+1}$. Thus, we have

$$\begin{aligned} \delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] &= d(T^{sn}x_0, T^{s'n+1}x_0) \\ &= d(T^{pn}(T^{sn-pn}x_0), T^{qn}(T^{s'n-qn+1}x_0)) \\ &\leqslant \max\{d(T^{pn}(T^{sn-pn}x_0), T^{qn}(T^{s'n-qn+1}x_0)), d(T^{pn+1}(T^{sn-pn}x_0), T^{qn-1}(T^{s'n-qn+1}x_0))\} \\ &\leqslant c\delta[\mathcal{A}_{pn,qn}^{T^{(s-p)n}x_0, T^{(s'-q)n+1}x_0}, \mathcal{B}_{pn,qn}^{T^{(s-p)n}x_0, T^{(s'-q)n+1}x_0}] + (1-c)d(A_m, A_{m+1}), \end{aligned}$$

which implies that $\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] \leqslant c\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] + (1-c)d(A_m, A_{m+1})$ and since $0 \leqslant c < 1$, this is impossible. □

Lemma 2. *Assume that all the conditions of Lemma 1 are met. Then, for $k, k' \in \mathbb{N}$ with $k' \geqslant k \geqslant q \geqslant p$, we have*

$$\lim_{k,k' \to \infty} d(x_{2k}, x_{2k'+1}) = d(A_m, A_{m+1}). \tag{2}$$

Proof. From Lemma 1, we have $\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] = d(T^{sn}x_0, T^{s'n+1}x_0)$ for some $s, s' \in \mathbb{N}$, where $sn < pn$ or $s'n < qn$. At the first, we show that $\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}]$ is bounded from above; that is,

$$\delta[\mathcal{A}_r^{x_0}, \mathcal{B}_r^{x_0}] \leqslant M_{x_0}^c, \tag{3}$$

where

$$M_{x_0}^c = \frac{1}{1-c} \max\{d(T^{in+1}x_0, T^{jn+1}x_0); 0 \leqslant i, j \leqslant p+q+1\} + d(A_m, A_{m+1})$$

is the same upper bound. For this, we consider the three following cases.
Case 1: Suppose that $sn < pn$ and $s'n < qn$. Then, we have the following:

$$(1-c)d(T^{sn}x_0, T^{s'n+1}x_0) \leqslant d(T^{sn}x_0, T^{s'n+1}x_0) \leqslant \max\{d(T^{in+1}x_0, T^{jn+1}x_0); 0 \leqslant i, j \leqslant p+q+1\}.$$

Thus,

$$\begin{aligned} d(T^{sn}x_0, T^{s'n+1}x_0) &\leqslant \frac{1}{1-c} \max\{d(T^{sn}x_0, T^{s'n+1}x_0); 0 \leqslant i, j \leqslant p+q+1\} \\ &\leqslant \frac{1}{1-c} \max\{d(T^{sn}x_0, T^{s'n+1}x_0); 0 \leqslant i, j \leqslant p+q+1\} + d(A_m, A_{m+1}), \end{aligned}$$

which implies that $\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] = d(T^{sn}x_0, T^{s'n+1}x_0) \leqslant M_{x_0}^c$.

Case 2: Suppose that $sn < pn$ and $qn \leqslant s'n$. Then, $T^{s'n-qn+1} \in A_{m+1}$ and

$$\begin{aligned} d(T^{sn}x_0, T^{s'n+1}x_0) &\leqslant d(T^{sn}x_0, T^{pn}x_0) + d(T^{pn}x_0, T^{s'n+1}x_0) \\ &= d(T^{sn}x_0, T^{pn}x_0) + d(T^{pn}x_0, T^{qn}(T^{s'n-qn+1}x_0)) \\ &\leqslant d(T^{sn}x_0, T^{pn}x_0) + (c\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] + (1-c)d(A_m, A_{m+1})). \end{aligned}$$

Therefore, $(1-c)\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] \leqslant d(T^{sn}x_0, T^{pn}x_0) + (1-c)d(A_m, A_{m+1})$, which concludes that $\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] = d(T^{sn}x_0, T^{s'n+1}x_0) \leqslant M_{x_0}^c$.

Case 3: Suppose that $pn \leqslant sn$ and $s'n < qn$. Then, $T^{sn-pn}x_0 \in A_m$ and

$$\begin{aligned} d(T^{sn}x_0, T^{s'n+1}x_0) &= d(T^{pn}(T^{sn-pn}x_0), T^{s'n+1}x_0) \\ &\leqslant d(T^{pn}(T^{sn-pn}x_0), T^{qn+1}x_0) + d(T^{qn+1}x_0, T^{s'n+1}x_0) \\ &\leqslant d(T^{qn+1}x_0, T^{s'n+1}x_0) + (c\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] + (1-c)d(A_m, A_{m+1})). \end{aligned}$$

Therefore, $(1-c)\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] \leqslant d(T^{qn+1}x_0, T^{s'n+1}x_0) + (1-c)d(A_m, A_{m+1})$, which concludes that $\delta[\mathcal{A}_r^{x_0},\mathcal{B}_r^{x_0}] = d(T^{sn}x_0, T^{s'n+1}x_0) \leqslant M_{x_0}^c$.

Now, for the optional $\lambda \in \mathbb{N}$ and $k \geqslant q \geqslant p$, we show that

$$\delta[\mathcal{A}_\lambda^{x_{2k}},\mathcal{B}_\lambda^{x_{2k}}] \leqslant c\delta[\mathcal{A}_{\lambda+\frac{q}{2}}^{x_{2k-q}},\mathcal{B}_{\lambda+\frac{q}{2}}^{x_{2k-q}}] + (1-c)d(A_m, A_{m+1}), \tag{4}$$

in which

$$\begin{aligned} \mathcal{A}_{\lambda+\frac{q}{2}}^{x_{2k-q}} &= \{x_{2k-q}, T^n x_{2k-q}, T^{2n}x_{2k-q}, \ldots, T^{(\lambda+\frac{q}{2})n}x_{2k-q}\} = \{x_{2k-q}, x_{2k-q+2}, \ldots, x_{2k+2\lambda}\}, \\ \mathcal{B}_{\lambda+\frac{q}{2}}^{x_{2k-q}} &= \{T^1 x_{2k-q}, T^{n+1}x_{2k-q}, T^{2n+1}x_{2k-q}, \ldots, T^{(\lambda+\frac{q}{2})n+1}x_{2k-q}\} = \{x_{2k-q+1}, x_{2k-q+3}, \ldots, x_{2k+2\lambda+1}\}. \end{aligned}$$

Since $k \geqslant q \geqslant p$, then $x_{2k+2s-2p} \in A_m$ and $x_{2k+2s'-2q+1} \in A_{m+1}$. Also, T is n-cyclic Fisher quasi-contraction mapping. Thus,

$$\begin{aligned} \delta[\mathcal{A}_\lambda^{x_{2k}},\mathcal{B}_\lambda^{x_{2k}}] &= d(T^{sn}x_{2k}, T^{s'n+1}x_{2k}) \\ &= d(T^{pn}x_{2k+2s-2p}, T^{qn}x_{2k+2s'-2q+1}) \\ &\leqslant c\delta[\mathcal{A}_{pn,qn}^{x_{2k+2s-2p},x_{2k+2s'-2q+1}},\mathcal{B}_{pn,qn}^{x_{2k+2s-2p},x_{2k+2s'-2q+1}}] + (1-c)d(A_m, A_{m+1}) \\ &= c\delta[\{T^{in}x_{2k+2s-2p}, T^{jn-1}x_{2k+2s'-2q+1} : 0 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}, \\ &\quad \{T^{jn+1}x_{2k+2s-2p}, T^{in}x_{2k+2s'-2q+1} : 0 \leqslant j \leqslant p-1, 0 \leqslant i \leqslant q\}] + (1-c)d(A_m, A_{m+1}) \\ &= c\delta[\{x_{2k+2s-2p+2i}, x_{2k+2s'-2q+2j} : 0 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}, \\ &\quad \{x_{2k+2s-2p+2j+1}, x_{2k+2s'-2q+2i+1} : 0 \leqslant j \leqslant p-1, 0 \leqslant i \leqslant q\}] + (1-c)d(A_m, A_{m+1}), \end{aligned}$$

which induces that (4) holds.

Also, for $k, k' \in \mathbb{N}$ with $k' \geqslant k \geqslant q \geqslant p$, we show that

$$d(x_{2k}, x_{2k'+1}) \leqslant c\delta[\mathcal{A}_{k'-k+\frac{q}{2}}^{x_{2k-q}},\mathcal{B}_{k'-k+\frac{q}{2}}^{x_{2k-q}}] + (1-c)d(A_m, A_{m+1}), \tag{5}$$

in which

$$\begin{aligned} \mathcal{A}_{k'-k+\frac{q}{2}}^{x_{2k-q}} &= \{x_{2k-q}, T^n x_{2k-q}, T^{2n}x_{2k-q}, \ldots, T^{(k'-k+\frac{q}{2})n}x_{2k-q}\} = \{x_{2k-q}, x_{2k-q+2}, \ldots, x_{2k'}\}, \\ \mathcal{B}_{k'-k+\frac{q}{2}}^{x_{2k-q}} &= \{T^1 x_{2k-q}, T^{n+1}x_{2k-q}, T^{2n+1}x_{2k-q}, \ldots, T^{(k'-k+\frac{q}{2})n+1}x_{2k-q}\} = \{x_{2k-q+1}, x_{2k-q+3}, \ldots, x_{2k'+1}\}. \end{aligned}$$

Since $k' \geqslant k \geqslant q \geqslant p$, then $x_{2k-2p} \in A_m$ and $x_{2k'-2q+1} \in A_{m+1}$. Also, the mapping T is an n-cyclic Fisher quasi-contraction. Thus, we have

$$\begin{aligned}
d(x_{2k}, x_{2k'+1}) &= d(T^{pn}x_{2k-2p}, T^{qn}x_{2k'-2q+1}) \\
&\leqslant c\delta[\mathcal{A}_{pn,qn}^{x_{2k-2p},x_{2k'-2q+1}}, \mathcal{B}_{pn,qn}^{x_{2k-2p},x_{2k'-2q+1}}] + (1-c)d(A_m, A_{m+1}) \\
&= c\delta[\{T^{in}x_{2k-2p}, T^{jn-1}x_{2k'-2q+1} : 0 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}, \\
&\quad \{T^{jn+1}x_{2k-2p}, T^{in}x_{2k'-2q+1} : 0 \leqslant j \leqslant p-1,\ 0 \leqslant i \leqslant q\}] + (1-c)d(A_m, A_{m+1}) \\
&= c\delta[\{x_{2k-2p+2i}, x_{2k'-2q+2j} :: 0 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}, \\
&\quad \{x_{2k-2p+2j+1}, x_{2k'-2q+2i+1} : 0 \leqslant j \leqslant p-1,\ 0 \leqslant i \leqslant q\}] + (1-c)d(A_m, A_{m+1}),
\end{aligned}$$

which induces that (5) holds.

Using (4) and (5), we have

$$\begin{aligned}
d(x_{2k}, x_{2k'+1}) &\leqslant c(c\delta[\mathcal{A}_{k'-k+q}^{x_{2k-2q}}, \mathcal{B}_{k'-k+q}^{x_{2k-2q}}] + (1-c)d(A_m, A_{m+1})) + (1-c)d(A_m, A_{m+1}) \\
&= c^2\delta[\mathcal{A}_{k'-k+q}^{x_{2k-2q}}, \mathcal{B}_{k'-k+q}^{x_{2k-2q}}] + (1-c^2)d(A_m, A_{m+1}).
\end{aligned}$$

Continuing this procedure, we have

$$\begin{aligned}
d(x_{2k}, x_{2k'+1}) &\leqslant c^{[\frac{2k}{q}]}\delta[\mathcal{A}_{k'-k+\frac{q}{2}[\frac{2k}{q}]}^{x_{2k-[\frac{2k}{q}]q}}, \mathcal{B}_{k'-k+\frac{q}{2}[\frac{2k}{q}]}^{x_{2k-[\frac{2k}{q}]q}}] + (1-c^{[\frac{2k}{q}]})d(A_m, A_{m+1}) \\
&\leqslant c^{[\frac{2k}{q}]}\delta[\mathcal{A}_{k'}^{x_0}, \mathcal{B}_{k'}^{x_0}] + (1-c^{[\frac{2k}{q}]})d(A_m, A_{m+1}).
\end{aligned}$$

Using (3), we have

$$d(A_m, A_{m+1}) \leqslant d(x_{2k}, x_{2k'+1}) \leqslant c^{[\frac{2k}{q}]}M_{x_0}^{c} + (1-c^{[\frac{2k}{q}]})d(A_m, A_{m+1}). \tag{6}$$

If $k \to \infty$ in (6), then $k' \to \infty$ and $c^{[\frac{2k}{q}]} \to 0$, and so (2) is established. □

Example 2. *Consider $\mathbb{R}$ with the Euclidean metric, $A_1 = [0,1]$, $A_2 = [2,3)$ and $A_3 = [3,4]$. We define $T : A_1 \cup A_2 \cup A_3 \to A_1 \cup A_2 \cup A_3$ as follows:*

$$Tx = \begin{cases} 2, & x \in A_1 \\ 3, & x \in A_2. \\ -x+4, & x \in A_3 \end{cases}$$

Then, for $p = q = 1$, $c = \frac{1}{2}$ and for any $x \in A_1$ and $y \in A_2$, we have

$$\begin{aligned}
\max\{d(1,2), d(2,1)\} &= \max\{d(T^3x, T^3y), d(T^4x, T^2y)\} \\
&\leqslant \frac{1}{2}\delta[\{x,1\}, \{y,2\}] + (1-\frac{1}{2})d(A_1, A_2) \\
&\leqslant \frac{1}{2}\sup\{d(x,y), d(x,2), d(1,y), d(1,2)\} + \frac{1}{2}.
\end{aligned}$$

Hence, the mapping T is a 3-cyclic Fisher quasi-contraction. On the other hand, by a simple calculation, we have $\{x_{2k}\} = \{1\}$ and $\{x_{2k'+1}\} = \{2\}$. This shows that

$$\lim_{k,k' \to \infty} d(x_{2k}, x_{2k'+1}) = d(A_1, A_2) = 1.$$

Lemma 3. *Consider a metric space (X,d) with the subsets $A_1, A_2, \ldots, A_n \neq \emptyset$ such that A_m and A_{m+1} have the WUC property. Also, suppose that T is an n-cyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Further, for $x_0 \in A_m$, consider the sequence $\{x_i\}$ to be the same as in (1). Then, two sequences $\{x_{2k}\} = \{T^{kn}x_0\}$ and $\{x_{2k+1}\} = \{T^{kn+1}x_0\}$ are Cauchy.*

Proof. Using Lemma 2, for any $\{x_{2k}\} \subseteq A_m$ and for every $\epsilon > 0$, there exists $y \in A_{m+1}$ so that $d(x_{2k}, y) \leqslant d(A_m, A_{m+1}) + \epsilon$ for $n \geqslant n_0$. Since the pair (A_m, A_{m+1}) has the WUC property, then $\{x_{2k}\}$ is a Cauchy sequence. Analogously, $\{x_{2k+1}\}$ is Cauchy. □

Now, we find the bpp for the n-cyclic Fisher quasi-contraction mapping.

Theorem 2. *Assume that T is the n-cyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$ with the quantities pn, qn, $1 \leqslant m \leqslant n$ and $0 \leqslant c < 1$, where $A_1, A_2, \ldots, A_n$ are nonempty subsets of a metric space (X, d) and A_m is complete. If the mapping T is continuous at each point of set*

$$S = \{z \in A_m : z = \lim_{k\to\infty} T^{kn}x \text{ for some } x \in A_m\}$$

and the pair (A_m, A_{m+1}) has the UC property, then

1. *T has at least one bpp $z \in A_m$;*
2. *T^n has at most n fp(s).*

Proof. Using Lemma 3, for any $x_0 \in A_m$, $\{x_{2k}\} = \{T^{kn}x_0\}$ is a Cauchy sequence in A_m. Since A_m is complete, we have $\lim_{k\to\infty} T^{kn}x_0 = z$ for some $z \in A_m$. Thus, $z \in S$. Now, by the continuity of T and d, and using Lemma 2, we have

$$\begin{aligned} d(z, Tz) &= d(\lim_{k\to\infty} T^{kn}x_0, T(\lim_{k\to\infty} T^{kn}x_0)) \\ &= d(\lim_{k\to\infty} T^{kn}x_0, \lim_{k\to\infty} T^{kn+1}x_0) \\ &= \lim_{k\to\infty} d(x_{2k}, x_{2k+1}) \\ &= d(A_m, A_{m+1}). \end{aligned}$$

This displays that $z \in A_m$ is a bpp of T.

Now, we establish T^n has at most n fp(s). Suppose that $\{x_{2k}\} = \{T^{kn}x_0\}$, $\{x'_k\} = \{T^{(k+1)n}x_0\}$ and $\{y_k\} = \{T^{kn+1}x_0\}$. Then, by Lemma 2, we gain

$$\lim_{k\to\infty} d(x_{2k}, y_k) = \lim_{k\to\infty} d(x'_k, y_k) = d(A_m, A_{m+1}).$$

Since the pair (A_m, A_{m+1}) has the UC property, then $\lim_{k\to\infty} d(x_{2k}, x'_k) = 0$. Using the continuity of d, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} d(x_{2k}, x'_k) \\ &= d(\lim_{k\to\infty} x_{2k}, \lim_{k\to\infty} x'_k) \\ &= d(z, T^n(\lim_{k\to\infty} T^{kn}x_0)) \\ &= d(z, T^n z), \end{aligned}$$

which induces that $T^n z = z$; that is, T^n has an fp. Now, we prove this fp is unique. Similar to the above argument, for each $x \in A_m$, assume that $z' \in A_m$ provided that

$$\lim_{k\to\infty} T^{kn}x = z', \quad d(z', Tz') = d(A_m, A_{m+1}) \quad \text{and} \quad T^n z' = z'.$$

Now, without the loss of generality, consider $d(z, Tz') \leqslant d(z', Tz)$. Then, we have

$$\begin{aligned} d(z', Tz) &= d(T^{pn}z', T(T^{qn}z)) \\ &= d(T^{pn}z', T^{qn}(Tz)) \\ &\leqslant \max\{d(T^{pn}z', T^{qn}(Tz)), d(T^{pn+1}z', T^{qn-1}(Tz))\} \\ &\leqslant c\delta[\mathcal{A}^{z',Tz}_{pn,qn}, \mathcal{B}^{z',Tz}_{pn,qn}] + (1-c)d(A_m, A_{m+1}) \\ &= c\delta[\{z', z\}, \{Tz', Tz\}] + (1-c)d(A_m, A_{m+1}) \\ &= c\sup\{d(z', Tz'), d(z', Tz), d(z, Tz'), d(z, Tz)\} + (1-c)d(A_m, A_{m+1}) \\ &\leqslant cd(z', Tz) + (1-c)d(A_m, A_{m+1}), \end{aligned}$$

which implies that $d(z', Tz) = d(A_m, A_{m+1})$. Thus, $d(z', Tz) = d(z, Tz) = d(A_m, A_{m+1})$. Let $z' = \lim_{k\to\infty} T^{kn}x$ and $z = \lim_{k\to\infty} T^{kn}x_0$. Then, we have

$$\lim_{k\to\infty} d(T^{kn}x_0, Tz) = \lim_{k\to\infty} d(T^{kn}x, Tz) = d(A_m, A_{m+1}).$$

Since the pair (A_m, A_{m+1}) has the UC property, we have $z' = z$. Therefore, for each $x \in A_m$, $\{T^{kn}x\}$ converges to z. Since n ordered pairs exist (A_1, A_2),(A_2, A_3),…,(A_n, A_1), then T^n has at most n fp(s). □

Corollary 1. *Assume that all the conditions of Theorem 2 are met. Further, suppose that*

$$\max\{d(T^n u, Tv), d(Tu, T^n v)\} \leqslant c\min\{d(Tu, v), d(u, Tv)\} + (1-c)d(A_m, A_{m+1})$$

for all $u, v \in A_m$. Then, T has a unique bpp $z \in A_m$.

Proof. Let z_1 and z_2 be the bpp(s) of the mapping T. Then, z_1 and z_2 are the fp(s) of the mapping T^n. Now, without loss of generality, assume $d(Tz_1, z_2) \leqslant d(z_1, Tz_2)$. Then,

$$\begin{aligned} d(z_1, Tz_2) &= d(T^n z_1, Tz_2) \\ &\leqslant cd(Tz_1, z_2) + (1-c)d(A_m, A_{m+1}) \\ &\leqslant cd(z_1, Tz_2) + (1-c)d(A_m, A_{m+1}). \end{aligned}$$

Hence, $d(z_1, Tz_2) = d(z_2, Tz_2) = d(A_m, A_{m+1})$. Since (A_m, A_{m+1}) has the UC property, then $z_1 = z_2$. □

Example 3. *Consider d, A_1, A_2, A_3, and T to be the same as in Example 2. Clearly, T has no fp(s). Here, we find the bpp of the mapping T and the fp of the mapping T^3.*

By the definition of the bpp: If $x^* \in A_1$ is the bpp, then $d(x^*, Tx^*) = d(A_1, A_2)$. Thus, $d(x^*, 2) = 1$ and $x^* = 1 \in A_1$ (note that $x^* = 3 \notin A_1$). If $x^* \in A_2$ is the bpp, then $d(x^*, Tx^*) = d(A_2, A_3)$. Thus, $d(x^*, 3) = 0$, which induces that $x^* = 3 \notin A_2$. Similarly, if $x^* \in A_3$ is the bpp, then $x^* = 3 \in A_3$.

Note that all the assumptions of Theorem 2 are held. Thus, we can check the validity of the assertion of this theorem.

1. *By using (A_1, A_2): For $z = 1$ in A_1, $\{x_{2k}\} = \{T^{3k}x_0\} = \{1\}$. Thus, $z = \lim_{k\to\infty} T^{3k}x_0 = 1$ is the bpp of T on A_1. Also, $T^3 1 = 1$ and $z = 1$ are unique fps of T^3.*
2. *By using (A_2, A_3): A_2 is not complete. We cannot apply Theorem 2 to this case.*
3. *By using (A_3, A_1): For $z = 3$ in A_3, $\{x_{2k}\} = \{T^{3k}x_0\} = \{3\}$. Thus, $z = \lim_{k\to\infty} T^{3k}x_0 = 3$ is the bpp of T on A_3. Also, $T^3 3 = 3$ and $z = 3$ are unique fps of T^3.*

Consequently, $z = 1$ and $z = 3$ are bpp(s) for the mapping T. Also, we have $T^3 1 = 1$, $T^3 2 = 2$, and $T^3 3 = 3$. Thus, T^3 has three fp(s).

3. Regular-*n*-Noncyclic and Full-*n*-Noncyclic Fisher Quasi-Contractions

Let $A_1, A_2, \dots, A_n$ be nonempty subsets of a metric space (X, d). A self-mapping T on $A_1 \cup A_2 \cup \cdots \cup A_n$ is called noncyclic if $T(A_i) \subseteq A_i$ for $1 \leqslant i \leqslant n$. Also, the pair $(x'_i, x'_j) \in A_i \times A_j$ for $0 \leqslant i, j \leqslant n$ with $i \neq j$ is denoted as an optimal pair of fp(s) of the noncyclic mapping T if $Tx'_i = x'_i$, $Tx'_j = x'_j$, and $d(x'_i, x'_j) = d(A_i, A_j)$.

It is obvious that if $x_0 \in A_i$, $y_0 \in A_j$ and T is the noncyclic mapping, then $x_{k+1} = Tx_k \in A_i$ and $y_{k+1} = Ty_k \in A_j$ for $k \geq 0$.

Notations. Let $m \in \mathbb{N} \cup \{0\}$. Define the set C^u_m by $C^u_m = \{u, Tu, \dots, T^m u\}$ for $u \in A_1 \cup A_2 \cup \cdots \cup A_n$. Clearly, if $u \in A_i$ for $i = 1, 2, \dots, n$, then $C^u_m \subseteq A_i$.

Now, we define the notion of regular-*n*-noncyclic and full-*n*-noncyclic Fisher quasi-contractions in metric spaces. Then, we obtain the main outcomes of this part.

Definition 5. *Let $A_1, A_2, \dots, A_n$ be nonempty subsets of a metric space (X, d) and T be a noncyclic mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Then, T is said to be*

1. *A regular-n-noncyclic Fisher quasi-contraction if there exist two sets A_i and A_j for $1 \leqslant i, j \leqslant n$ with $i \neq j$ and some $p_i, p_j \in \mathbb{N}$ so that*
$$d(T^{p_i}x, T^{p_j}y) \leqslant c\delta[C^x_{p_i}, C^y_{p_j}] + (1-c)d(A_i, A_j)$$
for each $x \in A_i$ and $y \in A_j$, where $0 \leqslant c < 1$;
2. *A full-n-noncyclic Fisher quasi-contraction if for all A_i and A_j, where $1 \leqslant i, j \leqslant n$ with $i \neq j$, there exist some $p_i, p_j \in \mathbb{N}$ so that*
$$d(T^{p_i}x, T^{p_j}y) \leqslant c\delta[C^x_{p_i}, C^y_{p_j}] + (1-c)d(A_i, A_j)$$
for each $x \in A_i$ and $y \in A_j$, where $0 \leqslant c < 1$.

Lemma 4. *Let $A_1, A_2, \dots, A_n$ be nonempty subsets of a metric space (X, d) and T be a regular-n-noncyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Then,*
$$\delta[C^{x_0}_k, C^{y_0}_l] \leqslant M_{x_0,y_0} \tag{7}$$
for each $k, l \in \mathbb{N}$, where
$$M_{x_0,y_0} = \frac{1}{1-c}\max\{d(T^i x_0, T^j y_0), d(T^i x_0, T^j x_0), d(T^i y_0, T^j y_0); 0 \leqslant i, j \leqslant \max\{p_i, p_j\}\} + d(A_i, A_j).$$

Proof. Since the mapping T is a regular-*n*-noncyclic Fisher quasi-contraction, there exist two sets A_i and A_j for $1 \leqslant i, j \leqslant n$ with $i \neq j$ and some $p_i, p_j \in \mathbb{N}$ such that
$$d(T^{p_i}x, T^{p_j}y) \leqslant c\delta[C^x_{p_i}, C^y_{p_j}] + (1-c)d(A_i, A_j). \tag{8}$$

First, we show that
$$\delta[C^{x_0}_k, C^{y_0}_l] = d(T^r x_0, T^{r'} y_0) \quad \text{where} \quad r < p_i \quad \text{or} \quad r' < p_j. \tag{9}$$

On the contrary, suppose that $\delta[C^{x_0}_k, C^{y_0}_l] = d(T^u x_0, T^v y_0)$, where $p_i \leqslant u \leqslant k$ and $p_j \leqslant v \leqslant l$. Then, $u - p_i \geq 0$ and $v - p_j \geq 0$, and $x_{u-p_i} = T^{u-p_i}x_0$ and $y_{v-p_j} = T^{v-p_j}y_0$, respectively. It follows from (8) that
$$\begin{aligned}
\delta[C^{x_0}_k, C^{y_0}_l] &= d(T^u x_0, T^v y_0) \\
&= d(T^{p_i}T^{u-p_i}x_0, T^{p_j}T^{v-p_j}y_0) \\
&\leqslant c\delta[C^{x_{u-p_i}}_{p_i}, C^{y_{v-p_j}}_{p_j}] + (1-c)d(A_i, A_j) \\
&\leqslant c\delta[C^{x_0}_k, C^{y_0}_l] + (1-c)d(A_i, A_j),
\end{aligned}$$

which, by $c \in [0,1)$, implies that $\delta[C_k^{x_0}, C_l^{y_0}] \leqslant d(A_i, A_j)$. Consequently, $\delta[C_k^{x_0}, C_l^{y_0}] = d(A_i, A_j)$ and (9) holds.

Now, we prove (7) by applying (9) and consider the three following cases.

Case 1: Suppose that $r < p_i$ and $r^{'} < p_j$. Then,

$$\begin{aligned}(1-c)d(T^r x_0, T^{r'} y_0) &\leqslant d(T^r x_0, T^{r'} y_0) \\ &\leqslant \{d(T^i x_0, T^j y_0), d(T^i x_0, T^j x_0), d(T^i y_0, T^j y_0); 0 \leqslant i,j \leqslant \max\{p_i, p_j\}\} \\ &\quad + (1-c)d(A_i, A_j),\end{aligned}$$

which concludes that $d(T^r x_0, T^{r'} y_0) \leqslant M_{x_0, y_0}$. Thus, (7) holds.

Case 2: Assume that $0 \leqslant r < p_i$ and $p_j \leqslant r' \leqslant l$. Then,

$$\begin{aligned}\delta[C_k^{x_0}, C_l^{y_0}] &= d(T^r x_0, T^{r'} y_0) \\ &\leqslant d(T^r x_0, T^{p_i} x_0) + d(T^{p_i} x_0, T^{r'} y_0) \\ &\leqslant d(T^r x_0, T^{p_i} x_0) + d(T^{p_i} x_0, T^{p_j} T^{r'-p_j} y_0) \\ &\leqslant d(T^r x_0, T^{p_i} x_0) + (c\delta[C_k^{x_0}, C_l^{y_0}] + (1-c)d(A_i, A_j)),\end{aligned}$$

which implies that $(1-c)\delta[C_k^{x_0}, C_l^{y_0}] \leqslant d(T^r x_0, T^{p_i} x_0) + (1-c)d(A_i, A_j)$. Thus, (7) holds.

Case 3: Similarly, if $p_i \leqslant r \leqslant k$ and $r' < p_j$, then (7) holds. □

Lemma 5. *Assume that all the conditions of Lemma 4 are met. Further, suppose that $x_{t+1} = Tx_t$ and $y_{t'+1} = Ty_{t'}$ for $t, t' \in \mathbb{N} \cup \{0\}$. Then,*

$$\lim_{t,t' \to \infty} d(x_t, y_{t'}) = d(A_i, A_j). \tag{10}$$

Proof. Since $t, t' \to \infty$, without loss of generality, we can suppose that $t, t' \geqslant \max\{p_i, p_j\}$. Hence, $t - p_i \geq 0$ and $t' - p_j \geq 0$. On the other hand, $\delta[C_k^{x_t}, C_l^{y_{t'}}] = sup\{d(x,y) : x \in C_k^{x_t}, y \in C_l^{y_{t'}}\}$. Thus, for $0 \leqslant r' \leqslant k$ and $0 \leqslant s' \leqslant l$, we obtain

$$\begin{aligned}\delta[C_k^{x_t}, C_l^{y_{t'}}] &= d(T^{r'} x_t, T^{s'} y_{t'}) \\ &= d(T^{p_i + r'} x_{t-p_i}, T^{p_j + s'} y_{t'-p_j}) \\ &= d(T^{p_i} T^{r'} x_{t-p_i}, T^{p_j} T^{s'} y_{t'-p_j}) \\ &\leqslant c\delta[C_{k+p_i}^{x_{t-p_i}}, C_{l+p_j}^{y_{t'-p_j}}] + (1-c)d(A_i, A_j).\end{aligned}$$

Consequently,

$$\delta[C_k^{x_t}, C_l^{y_{t'}}] \leqslant c\delta[C_{k+p_i}^{x_{t-p_i}}, C_{l+p_j}^{y_{t'-p_j}}] + (1-c)d(A_i, A_j). \tag{11}$$

Now, by using (11) and by putting $k = l = 0$, we have

$$\begin{aligned}d(x_t, y_{t'}) = \delta[C_0^{x_t}, C_0^{y_{t'}}] &\leqslant c\delta[C_{p_i}^{x_{t-p_i}}, C_{p_j}^{y_{t'-p_j}}] + (1-c)d(A_i, A_j) \\ &\leqslant c(c\delta[C_{2p_i}^{x_{t-2p_i}}, C_{2p_j}^{y_{t'-2p_j}}] + (1-c)d(A_i, A_j)) + (1-c)d(A_i, A_j) \\ &= c^2\delta[C_{2p_i}^{x_{t-2p_i}}, C_{2p_j}^{y_{t'-2p_j}}] + (1-c^2)d(A_i, A_j) \\ &\leqslant c^2(c\delta[C_{3p_i}^{x_{t-3p_i}}, C_{3p_j}^{y_{t'-3p_j}}] + (1-c)d(A_i, A_j)) + (1-c^2)d(A_i, A_j) \\ &= c^3\delta[C_{3p_i}^{x_{t-3p_i}}, C_{3p_j}^{y_{t'-3p_j}}] + (1-c^3)d(A_i, A_j)\end{aligned}$$

for $t, t' \geqslant \max\{2p_i, 2p_j\}$.

Continuing this process, using Lemma 4 and setting $\alpha(t,t') = \min\{[\frac{t}{p_i}], [\frac{t'}{p_j}]\}$, we have

$$\begin{aligned} d(A_i, A_j) &\leqslant d(x_t, y_{t'}) \\ &\leqslant c^{\alpha(t,t')}\delta[C_{\alpha(t,t').p_i}^{x_{t-\alpha(t,t').p_i}}, C_{\alpha(t,t').p_j}^{y_{t'-\alpha(t,t').p_j}}] + (1 - c^{\alpha(t,t')})d(A_i, A_j) \\ &\leqslant c^{\alpha(t,t')}\delta[C_t^{x_0}, C_{t'}^{y_0}] + (1 - c^{\alpha(t,t')})d(A_i, A_j) \\ &\leqslant c^{\alpha(t,t')}M_{x_0,y_0} + (1 - c^{\alpha(t,t')})d(A_i, A_j). \end{aligned}$$

Now, by taking the limit as $t, t' \to \infty$, (10) is established. □

Lemma 6. *Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of a metric space (X, d) and T be a regular-n-noncyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Further, suppose that (A_i, A_j) has the WUC property, and for $x_0 \in A_i$, consider $x_{t+1} = Tx_t$ for any $t \geqslant 0$. Then, the sequence $\{x_t\}$ is Cauchy.*

Proof. By Lemma 5, $\{x_t\}$ is Cauchy. □

Theorem 3. *Let $A_1, A_2, \ldots, A_n \neq \emptyset$ be subsets of (X, d) and T be a regular-n-noncyclic Fisher quasi-contraction mapping on $A_1 \cup \cdots \cup A_n$. Also, let A_i and A_j for $1 \leqslant i, j \leqslant n$ be complete subsets of X such that (A_i, A_j) and (A_j, A_i) have the UC property. Further, assume that $T : A_i \to A_i$ and $T : A_j \to A_j$ are continuous. Then, T has a unique optimal pair of fp(s) (x_i^*, y_i^*) provided that $\{T^n x_0\}$ and $\{T^n y_0\}$ converge to x_i^* and y_j^* for each $x_0 \in A_i$ and $y_0 \in A_j$, respectively.*

Proof. From Lemma 6, $\{T^n x_0\}$ is Cauchy. Since A_i is complete, $\{T^n x_0\}$ converges to a certain $x_i^* \in A_i$. Since T is the continuous mapping on A_i, we deduce that $Tx_i^* = x_i^*$; that is, $x_i^* \in A_i$ is an fp of T. For uniqueness, assume that $x_i^{**} \in A_i$ is another fp of T. Also, let $y_0 \in A_j$. Using Lemma 5, we have

$$\lim_{n\to\infty} d(x_i^*, T^n y_0) = \lim_{n\to\infty} d(T^n x_i^*, T^n y_0) = d(A_i, A_j) = \lim_{n\to\infty} d(T^n x_i^{**}, T^n y_0) = \lim_{n\to\infty} d(x_i^{**}, T^n y_0).$$

Since the pair (A_i, A_j) has the UC property, then $x_i^* = x_i^{**}$. Similarly, T has a unique fp $y_j^* \in A_j$ such that $\{T^n y_0\}$ converges to a certain $y_i^* \in A_i$. Also, by Lemma 5, we have

$$d(x_i^*, y_j^*) = \lim_{n\to\infty} d(T^n x_i^*, T^n y_j^*) = d(A_i, A_j).$$

Hence, $(x_i^*, y_j^*) \in A_i \times A_j$ is a unique optimal pair of fp of T. □

As an application, in the following corollary, we show that a full-n-noncyclic Fisher quasi-contraction mapping has $n(n-1)/2$ unique optimal pairs of fixed points.

Corollary 2. *Let $A_1, A_2, \ldots, A_n$ be nonempty and complete subsets of a metric space (X, d) and T be a full-n-noncyclic Fisher quasi-contraction mapping on $A_1 \cup A_2 \cup \cdots \cup A_n$. Also, assume that the pairs (A_i, A_j) and (A_j, A_i) have the UC property for each $1 \leqslant i, j \leqslant n$ with $i \neq j$. Further, suppose that the mapping $T : A_i \to A_i$ is continuous. Then, T has $\frac{n(n-1)}{2}$ unique optimal pairs of fp(s) (x_i^*, y_i^*) provided that $\{T^n x_0\}$ and $\{T^n y_0\}$ converge to x_i^* and y_j^* for each $x_0 \in A_i$ and $y_0 \in A_j$, respectively.*

Proof. By Theorem 3, (x_i^*, y_j^*) is a unique optimal pair of fp of T for some $i, j = 1, \ldots, n$ with $i \neq j$. Since there exist $\frac{n(n-1)}{2}$ cases of the different pairs (A_i, A_j) for any i, j, then the assertion holds. □

Example 4. *Consider $\mathbb{R}^2$ with the metric $d((\alpha, \beta), (\gamma, \delta)) = \sqrt{(\alpha - \gamma)^2 + (\beta - \delta)^2}$ for each $(\alpha, \beta), (\gamma, \delta) \in \mathbb{R}^2$. Also, assume that $A_1 = \{(a, 0) : 1 \leqslant a \leqslant 2\}$, $A_2 = \{(0, b) : 1 \leqslant$*

$b \leqslant 2\}$ and $A_3 = \{(a,0) : -2 \leqslant a \leqslant -1\}$ are three subsets of $\mathbb{R}^2$. Moreover, suppose that $T : A_1 \cup A_2 \cup A_3 \to A_1 \cup A_2 \cup A_3$ is defined as follows:

$$Tx = \begin{cases} (1,0), & x \in A_1 \\ (0,1), & x \in A_2 \\ (-2,0), & x \in A_3 \end{cases}.$$

Clearly, T is a noncyclic mapping. For the pair (A_1, A_2), let $p_1 = p_2 = 1$ and c be a fixed number in $[0,1)$. For each $x \in A_1$ and $y \in A_2$, we have $\delta[C_1^x, C_1^y] \geqslant \sqrt{2}$. Thus, we conclude that

$$d(T^1x, T^1y) = d((1,0),(0,1)) = \sqrt{2} \leqslant c\delta[C_1^x, C_1^y] + \sqrt{2}(1-c);$$

that is, the mapping T is a regular-3-noncyclic Fisher quasi-contraction. Further, since

$$\lim_{t,t' \to \infty} d(x_t, y_{t'}) = d((1,0),(-1,0)) = d(A_1, A_2) = \sqrt{2},$$

the assertion of Lemma 5 holds. Furthermore,

$$T(1,0) = (1,0), \quad T(0,1) = (0,1) \quad \text{and} \quad d((1,0),(0,1)) = d(A_1, A_2).$$

Thus, ((1,0),(0,1)) is an optimal pair of fp of T. Moreover, look at the pair (A_1, A_3). For every $p_1, p_2 \in \mathbb{N}$ and $x = (1,0) \in A_1$ and $y = (-1,0) \in A_3$, we obtain

$$d(T^{p_1}x, T^{p_2}y) = d((1,0),(-2,0)) = 3 \nleqslant c\delta[C_{p_1}^x, C_{p_2}^y] + 2(1-c).$$

Note that $\delta[C_{p_1}^x, C_{p_2}^y] = \delta[C_{p_1}^{(1,0)}, C_{p_2}^{(-1,0)}] = \delta[d((1,0),(-1,0)), d((1,0),(-2,0))] = 3$. So, T is not the full-3-noncyclic Fisher quasi-contraction mapping. In addition,

$$\lim_{t,t' \to \infty} d(x_t, y_{t'}) = d((1,0),(-2,0)) = 3 \neq 2 = d(A_1, A_3).$$

Example 5. *Let $(\mathbb{R}^2, d)$ be the same metric space in Example 4. Also, suppose that $A_1 = \{(a,0) : 1 \leqslant a \leqslant 2\}$, $A_2 = \{(a,0) : -2 \leqslant a \leqslant -1\}$, $A_3 = \{(0,b) : 1 \leqslant b \leqslant 2\}$ and $A_4 = \{(0,b) : -2 \leqslant b \leqslant -1\}$ are four arbitrary subsets of $\mathbb{R}^2$. We define a mapping T on $A_1 \cup A_2 \cup A_3 \cup A_4$ as follows:*

$$Tx = \begin{cases} (1,0), & x \in A_1 \\ (-1,0), & x \in A_2 \\ (0,1), & x \in A_3 \\ (0,-1), & x \in A_4 \end{cases}.$$

For $p = q = 1$ and for each $x \in A_1$ and $y \in A_2$, we have

$$C_1^x = \{x, (1,0)\}, \quad C_1^y = \{y, (-1,0)\}, \quad \delta[C_1^x, C_1^y] = 4 \quad \text{and} \quad d(A_1, A_2) = 2.$$

Thus, there exists a $0 \leqslant c < 1$ so that

$$d(Tx, Ty) = d((1,0),(-1,0)) = 2 \leqslant c\delta[C_1^x, C_1^y] + (1-c)d(A_1, A_2).$$

Also,

$$\lim_{t,t' \to \infty} d(x_t, y_{t'}) = d((1,0),(-1,0)) = d(A_1, A_2) = 2.$$

Similarly, we can use the above discussion for all $1 \leqslant i, j \leqslant 4$ *with* $i \neq j$*. Hence, mapping* T *is a full-*4*-noncyclic Fisher quasi-contraction. In addition,* T *has six unique optimal pairs of fp(s) as follows:*

$$((1,0),(0,1)), \quad ((1,0),(-1,0)), \quad ((1,0),(0,-1))$$
$$((0,1),(-1,0)), \quad ((-1,0),(0,-1)), \quad ((0,1),(0,-1)).$$

4. Conclusions

In the present paper, we introduced the concepts of n-cyclic Fisher quasi-contraction mappings, as well as full-n-noncyclic and regular-n-noncyclic Fisher quasi-contraction mappings in metric spaces. Then we stated and proved several bpp theorems regarding these contractions. Moreover, we solved an open problem about the number of optimal pairs of fp(s) for full-n-noncyclic Fisher quasi-contraction mappings. Due to the generalization of this paper, unlike the other articles, we found more than one bpp. In future studies, readers may concentrate on specific aspects of these points. For example, they could explore the optimum of a bpp or discuss the unique conditions of these points. Also, they may obtain similar results in various metric spaces.

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