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Fourth-Order Emden–Fowler Neutral Differential Equations: Investigating Some Qualitative Properties of Solutions

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Abstract: In this article, we investigate some of the qualitative properties of a class of fourth-order neutral differential equations. We start by obtaining new inequalities and relations between the solution and its corresponding function, as well as with its derivatives. The new relations allow us to improve the monotonic and asymptotic properties of the positive solutions of the studied equation. Then, using an improved approach, we establish new criteria that test the oscillation of all solutions. We also rely on the principle of symmetry between positive and negative solutions to obtain the new criteria. The paper provides illustrative examples that highlight the significance of our findings.

Keywords: oscillatory; non-oscillatory; neutral differential equation; fourth order

MSC: 34C10, 34K11

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1. Introduction

Differential equations (DEs) play a fundamental role in many areas of science and engineering, including physics, chemistry, biology, economics, and computer science. They provide a powerful tool for modeling and analyzing complex phenomena, making predictions, and designing control systems, see [1,2].

Differential equations of neutral type are a kind of DE that arises in various scientific and engineering fields due to their ability to model delay phenomena. They are called "neutral" because they involve derivatives with respect to time and also with respect to a delayed time. The study of neutral DEs has numerous crucial applications in controlling chemical processes, mechanical systems, and electrical circuits. They are also used in modeling population dynamics, epidemiology, and ecological systems. Furthermore, these equations are employed in economics, finance, and game theory. Therefore, neutral DEs offer a potent tool for comprehending complex systems with time delays and have extensive applications in different fields, including science, engineering, and economics, see [3,4].

The oscillation theorem is a crucial outcome in DE theory that characterizes the oscillatory behavior of solutions. According to the theorem, if a solution to a DE oscillates by alternately switching between positive and negative values an infinite number of times, it is considered an oscillation. This theorem has significant applications in diverse fields, including physics, engineering, and economics. For instance, it is used to analyze oscillating systems such as pendulums and vibrating strings, as well as in the study of population dynamics and infectious disease transmission. Additionally, the oscillation theorem has applications in control theory and signal processing, where it is utilized to assess the stability and performance of feedback systems, see [5–7].

There is currently significant interest in obtaining sufficient conditions for the oscillatory behavior of solutions to different types of differential equations. Many researchers have focused on studying oscillatory properties and convergence, particularly for second-order DEs with advanced delay/conditions. Duzrina and Jadlovska [8], Baculikova [9], and Bohner et al. [10] have developed approaches and techniques for optimizing the oscillation parameters of these equations. In addition, Moaaz et al. [11,12] have extended this research to differential equations of the neutral type. Over the past few decades, there has also been extensive study of the oscillation of fourth-order neutral differential equations, as seen in [13–16].

As a result, there are now numerous studies available on the oscillatory properties of various DEs, both in canonical and non-canonical cases. These studies are discussed in works such as [8,10,17,18].

This research is interested in the oscillation behavior of the solutions of the fourth-order quasi-linear neutral DE

$$\left(\mu(\mathrm{s})\left(\Phi'''(\mathrm{s})\right)^{\gamma}\right)' + q(\mathrm{s})x^{\gamma}(\lambda(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0, \tag{1}$$

where $\Phi(\mathrm{s}) = x(\mathrm{s}) + \eta(\mathrm{s})x(\zeta(\mathrm{s}))$. We assume throughout this paper that:

(H_1) γ is the ratio of two positive odd integers;
(H_2) $\zeta, \lambda, \mu \in C^1([\mathrm{s}_0, \infty))$, and $q(\mathrm{s}) \in C([\mathrm{s}_0, \infty))$;
(H_3) $\zeta(\mathrm{s}) \leq \mathrm{s}$, $\lambda(\mathrm{s}) \leq \mathrm{s}$, $\lambda'(\mathrm{s}) > 0$, and $\lim_{\mathrm{s}\to\infty} \zeta(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} \lambda(\mathrm{s}) = \infty$;
(H_4) $\mu(\mathrm{s}) > 0$, $\mu'(\mathrm{s}) \geq 0$, $0 \leq \eta(\mathrm{s}) < \eta_0$, $q(\mathrm{s}) \geq 0$, and

$$\int_{\mathrm{s}_0}^{\mathrm{s}} \frac{1}{\mu^{1/\gamma}(v)} \mathrm{d}v \to \infty \text{ as } \mathrm{s} \to \infty. \tag{2}$$

A function $x \in C^3([\mathrm{s}_x, \infty), \mathbb{R})$, $\mathrm{s}_x \geqslant \mathrm{s}_0$, is said to be a solution of (1), which has the property $\mu(\Phi''')^{\gamma} \in C^1[\mathrm{s}_x, \infty)$, and it satisfies the Equation (1) for all $x \in [\mathrm{s}_x, \infty)$. We consider only those solutions x of (1) that exist on some half-line $[\mathrm{s}_x, \infty)$ and satisfy the condition

$$\sup\{|x(\mathrm{s})| : \mathrm{s} \geqslant \mathrm{S}\} > 0, \text{ for all } \mathrm{S} \geq \mathrm{s}_x.$$

A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be non-oscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Several studies have examined the oscillatory behavior of solutions to various DEs. For example, Tongxing Li and Rogovchenko [19] studied a class of second-order superlinear Emden–Fowler neutral DE

$$\left(\mu(\mathrm{s})(x(\mathrm{s}) + \eta(\mathrm{s})x(\zeta(\mathrm{s})))'\right)' + q(\mathrm{s})|x(\lambda(\mathrm{s}))|^{\gamma-1}x(\lambda(\mathrm{s})) = 0,$$

in the canonical case $\int_{\mathrm{s}_0}^{\infty} 1/\mu(v)\mathrm{d}v = \infty$. Zhang et al. [20] explored the oscillatory behavior of solutions of a delay fourth-order differential equation

$$\left((\mu(\mathrm{s})x'''(\mathrm{s}))^{\gamma}\right)' + q(\mathrm{s})x^{\gamma}(\zeta(\mathrm{s})) = 0,$$

using the Riccatti technique. Grace et al. [21] analyzed the oscillatory behavior of the fourth-order nonlinear DE

$$\left((\mu(\mathrm{s})x'(\mathrm{s}))^{\gamma}\right)''' + q(\mathrm{s})f(x(\zeta(\mathrm{s}))) = 0.$$

Muhib et al. [22] obtained new properties of solutions to the neutral DE

$$\left(\mu(\mathrm{s})(x(\mathrm{s}) + \eta(\mathrm{s})x(\zeta(\mathrm{s})))'''\right)' + q(\mathrm{s})f(\mathrm{s}, x(\lambda(\mathrm{s}))) = 0,$$

in the non-canonical case

$$\int_{s_0}^{\infty} \frac{1}{\mu^{1/\gamma}(v)} dv < \infty. \tag{3}$$

Chatzarakis et al. [23] developed oscillation criteria for a fourth-order nonlinear neutral DE, expressed as

$$\left(\mu(s)\left((x(s)+\eta(s)x(\zeta(s)))'''\right)^{\gamma}\right)' + \int_c^b q(s,v) f(x(\zeta(s,v))) dv = 0,$$

subject to the constraint (2). Dassios and Bazighifan [24] utilized the Riccati transform to demonstrate that the fourth-order nonlinear DE

$$\left(\left(\mu(s)(x(s)+\eta(s)x(\zeta(s)))'''\right)^{\gamma}\right)' + q(s)x^{\gamma}(\lambda(s)) = 0,$$

is nearly oscillatory provided (3) is satisfied.

The objective of this research is to extend the field of study to include neutral DEs of the fourth order. This paper presents novel criteria for examining oscillatory solutions of a quasi-linear fourth-order neutral DE (1). The investigation utilizes the comparison technique and the Riccati method to derive the desired results.

2. Preliminary Results

We begin with some useful lemmas concerning the monotonic properties of the non-oscillatory solutions of the studied equations. For convenience, we assume that

$$\rho'_+(s) := \max\{0,\ \rho'(s)\},\ \tilde{\rho}'_+(s) := \max\{0,\ \tilde{\rho}'(s)\},$$

$$\pi_0(s) := \int_{s_0}^{s} \frac{1}{\mu^{1/\gamma}(v)} dv,\ \pi_i(s) := \int_{s_0}^{s} \pi_{i-1}(v) dv,\ i = 1,2,$$

$$F_{[0]}(s) = F(s) \text{ and } F_{[j]}(s) = F\left(F_{[j-1]}(s)\right),\ \text{for } j = 1,2,\ldots,n,$$

$$\psi_0(s) := \pi_0(s) + \frac{1}{\gamma}\int_{s_1}^{s} \pi_0(v) q(v) \pi_2^{\gamma}(\lambda(v)) \mathrm{B}_0^{\gamma}(\lambda(v),n) dv,$$

$$\psi_i(s) = \int_{s_1}^{s} \psi_{i-1}(v) dv,\ i = 1,2,$$

$$\phi_0(s) = \exp\left(\int_{s_0}^{s} \frac{1}{\psi_0(v)\mu^{1/\gamma}(v)} dv\right),\ \phi_i(s) = \int_{s_0}^{s} \phi_0(v) dv,\ i = 1,2,$$

$$\eta_1(s;n) = \sum_{k=0}^{n}\left(\prod_{i=0}^{2k} \eta\left(\zeta_{[i]}(v)\right)\right)\left[\frac{1}{\eta\left(\zeta_{[2k]}(s)\right)} - 1\right]\frac{\pi_2\left(\zeta_{[2k]}(s)\right)}{\pi_2(s)},$$

$$\widehat{\eta}_1(s;n) = \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1} \frac{1}{\eta\left(\zeta_{[i]}^{-1}(s)\right)}\right)\left[\frac{\pi_2\left(\zeta_{[2k-1]}^{-1}(s)\right)}{\pi_2\left(\zeta_{[2k]}^{-1}(s)\right)} - \frac{1}{\eta\left(\zeta_{[2k]}^{-1}(s)\right)}\right],$$

$$\eta_2(s;n) = \sum_{k=0}^{n}\left(\prod_{i=0}^{2k} \eta\left(\zeta_{[i]}(v)\right)\right)\left[\frac{1}{\eta\left(\zeta_{[2k]}(s)\right)} - 1\right]\frac{\phi_2\left(\zeta_{[2k]}(s)\right)}{\phi_2(s)},$$

and

$$\widehat{\eta}_2(s;n) = \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1} \frac{1}{\eta\left(\zeta_{[i]}^{-1}(s)\right)}\right)\left[\frac{\phi_2\left(\zeta_{[2k-1]}^{-1}(s)\right)}{\phi_2\left(\zeta_{[2k]}^{-1}(s)\right)} - \frac{1}{\eta\left(\zeta_{[2k]}^{-1}(s)\right)}\right].$$

Lemma 1 (See [25], Lemma 13)**.** *Let $y \in \mathbf{C}^n([s_0,\infty),(0,\infty))$, $y^{(i)}(s) > 0$ for $i = 1,2,...,n$, and $y^{(n+1)}(s) \le 0$, eventually. Then, eventually, $y(s)/y'(s) \ge \epsilon s/n$ for every $\epsilon \in (0,1)$.*

Lemma 2 (See [26])**.** *Let γ be a ratio of two odd positive integers. Assume $L > 0$ and K are real numbers. Then*

$$Ku - Lu^{(\gamma+1)/\gamma} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{K^{\gamma+1}}{L^\gamma}. \tag{4}$$

Remark 1. *In what follows, we need only to study the eventually positive solutions of (1), since if x satisfies (1), then $-x$ is also its solution. We begin with the following lemmas.*

Lemma 3 (See [27])**.** *Assume that x is an eventually positive solution of (1), then x satisfies eventually the following cases:*

$$\begin{aligned} C_1 &: \quad \Phi(s) > 0,\ \Phi'(s) > 0,\ \Phi''(s) > 0,\ \Phi'''(s) > 0,\ \left(\mu(s)(\Phi'''(s))^\gamma\right)' < 0, \\ C_2 &: \quad \Phi(s) > 0,\ \Phi'(s) > 0,\ \Phi''(s) < 0,\ \Phi'''(s) > 0, \end{aligned}$$

for $s \geqslant s_1 \geqslant s_0$.

In the following, by Ω, we mean all positive solutions to Equation (1) with $\Phi(s)$ satisfying C_1.

Lemma 4 (See [28], Lemma 1)**.** *Suppose that x is a solution of (1) and is eventually positive. If $\eta_0 < 1$, then,*

$$x(s) > \sum_{k=0}^{n} \left(\prod_{i=0}^{2k} \eta\left(\zeta_{[i]}(s)\right) \right) \left[\frac{\Phi\left(\zeta_{[2k]}(s)\right)}{\eta\left(\zeta_{[2k]}(s)\right)} - \Phi\left(\zeta_{[2k+1]}(s)\right) \right],$$

for any integer $n \ge 0$.

Lemma 5. *Suppose that x is a solution of (1) and is eventually positive. If $\eta_0 > 1$, then,*

$$x(s) > \sum_{k=1}^{n} \left(\prod_{i=1}^{2k-1} \frac{1}{\eta\left(\zeta^{-1}_{[i]}(s)\right)} \right) \left[\Phi\left(\zeta^{-1}_{[2k-1]}(s)\right) - \frac{1}{\eta\left(\zeta^{-1}_{[2k]}(s)\right)} \Phi\left(\zeta^{-1}_{[2k]}(s)\right) \right].$$

Proof. From

$$\Phi(s) = x(s) + \eta(s)x(\zeta(s)),$$

we deduce that

$$\begin{aligned} x(s) &= \frac{1}{\eta(\zeta^{-1}(s))}\left[\Phi\left(\zeta^{-1}(s)\right) - x\left(\zeta^{-1}(s)\right)\right] \\ &= \frac{1}{\eta(\zeta^{-1}(s))}\Phi\left(\zeta^{-1}(s)\right) \\ &\quad - \frac{1}{\eta(\zeta^{-1}(s))} \frac{1}{\eta\left(\zeta^{-1}_{[2]}(s)\right)} \left[\Phi\left(\zeta^{-1}_{[2]}(s)\right) - x\left(\zeta^{-1}_{[2]}(s)\right)\right] \\ &= \frac{1}{\eta(\zeta^{-1}(s))}\Phi\left(\zeta^{-1}(s)\right) - \prod_{i=1}^{2} \frac{1}{\eta\left(\zeta^{-1}_{[i]}(s)\right)} \Phi\left(\zeta^{-1}_{[2]}(s)\right) \\ &\quad + \prod_{i=1}^{3} \frac{1}{\eta\left(\zeta^{-1}_{[i]}(s)\right)} \left[\Phi\left(\zeta^{-1}_{[3]}(s)\right) - x\left(\zeta^{-1}_{[3]}(s)\right)\right]. \end{aligned}$$

By repeating the same technique a number of times, we obtain

$$x(\mathrm{s}) > \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\eta\left(\zeta_{[i]}^{-1}(\mathrm{s})\right)}\right)\left[\Phi\left(\zeta_{[2k-1]}^{-1}(\mathrm{s})\right)-\frac{1}{\eta\left(\zeta_{[2k]}^{-1}(\mathrm{s})\right)}\Phi\left(\zeta_{[2k]}^{-1}(\mathrm{s})\right)\right].$$

Here, the proof ends. □

Lemma 6. *Assume that $x \in \Omega$. Then,*
$(\mathbf{Y}_{1,1})$ $\Phi(\mathrm{s}) \geqslant \mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_2(\mathrm{s})$;
$(\mathbf{Y}_{1,2})$ $\Phi''(\mathrm{s})/\pi_0(\mathrm{s})$, $\Phi'(\mathrm{s})/\pi_1(\mathrm{s})$ *and* $\Phi(\mathrm{s})/\pi_2(\mathrm{s})$ *are decreasing;*
$(\mathbf{Y}_{1,3})$ $\Phi(\mathrm{s}) \geqslant \Phi''(\mathrm{s})\pi_2(\mathrm{s})/\pi_0(\mathrm{s})$.

Proof. $(\mathbf{Y}_{1,1})$ The monotonicity of $\mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})$ implies that

$$\begin{aligned}\Phi''(\mathrm{s}) &\geq \int_{\mathrm{s}_1}^{\mathrm{s}}\mu^{1/\gamma}(v)\Phi'''(v)\frac{1}{\mu^{1/\gamma}(v)}\mathrm{d}v\\ &\geq \mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\int_{\mathrm{s}_1}^{\mathrm{s}}\frac{1}{\mu^{1/\gamma}(v)}\mathrm{d}v\\ &\geq \mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_0(\mathrm{s}).\end{aligned} \tag{5}$$

Integrating twice more from s_1 to s, we obtain

$$\Phi'(\mathrm{s}) \geq \mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_1(\mathrm{s}), \tag{6}$$

and

$$\Phi(\mathrm{s}) \geq \mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_2(\mathrm{s}).$$

$(\mathbf{Y}_{1,2})$ From (5), we obtain

$$\left(\frac{\Phi''(\mathrm{s})}{\pi_0(\mathrm{s})}\right)' = \frac{\mu^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_0(\mathrm{s}) - \Phi''(\mathrm{s})}{\mu^{1/\gamma}(\mathrm{s})\pi_0^2(\mathrm{s})} \leq 0.$$

Since $\Phi''(\mathrm{s})/\pi_0(\mathrm{s})$ is decreasing, then

$$\Phi'(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}}\frac{\Phi''(v)}{\pi_0(v)}\pi_0(v)\mathrm{d}v \geq \frac{\Phi''(\mathrm{s})}{\pi_0(\mathrm{s})}\pi_1(\mathrm{s}). \tag{7}$$

From this we deduce that

$$\left(\frac{\Phi'(\mathrm{s})}{\pi_1(\mathrm{s})}\right)' = \frac{\Phi''(\mathrm{s})\pi_1(\mathrm{s}) - \pi_0(\mathrm{s})\Phi'(\mathrm{s})}{\pi_1^2(\mathrm{s})} \leq 0.$$

Since $\Phi'(\mathrm{s})/\pi_1(\mathrm{s})$ is decreasing, then

$$\Phi(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}}\frac{\Phi'(v)}{\pi_1(v)}\pi_1(v)\mathrm{d}v \geq \frac{\Phi'(\mathrm{s})}{\pi_1(\mathrm{s})}\pi_2(\mathrm{s}). \tag{8}$$

Consequently

$$\left(\frac{\Phi(\mathrm{s})}{\pi_2(\mathrm{s})}\right)' = \frac{\Phi'(\mathrm{s})\pi_2(\mathrm{s}) - \pi_1(\mathrm{s})\Phi(\mathrm{s})}{\pi_2^2(\mathrm{s})} \leq 0.$$

$(\mathbf{Y}_{1,3})$ From (7) and (8), we find

$$\Phi(\mathrm{s}) \geq \frac{\pi_2(\mathrm{s})}{\pi_0(\mathrm{s})}\Phi''(\mathrm{s}).$$

Here, the proof ends. □

Lemma 7. *Suppose that x is a solution of (1) and is eventually positive. Then,*

$$x(\mathrm{s}) > \mathrm{B}_0(\mathrm{s}, n)\Phi(\mathrm{s}),$$

eventually, where

$$\mathrm{B}_0(\mathrm{s}, n) = \begin{cases} \eta_1(\mathrm{s}; n) & \text{for } \eta_0 < 1, \\ \widehat{\eta}_1(\mathrm{s}; n) & \text{for } \eta_0 > \frac{\pi_2(\mathrm{s})}{\pi_2(\zeta(\mathrm{s}))}. \end{cases}$$

Proof. If $\eta_0 < 1$, then, due to the fact that $\Phi(\mathrm{s})$ is increasing and $\zeta_{[2k]}(\mathrm{s}) \geq \zeta_{[2k+1]}(\mathrm{s})$, we have

$$\Phi\Big(\zeta_{[2k]}(\mathrm{s})\Big) \geq \Phi\Big(\zeta_{[2k+1]}(\mathrm{s})\Big),$$

which, along with Lemma 4 implies that

$$\begin{aligned} x(\mathrm{s}) &> \sum_{k=0}^{n}\left(\prod_{i=0}^{2k}\eta\left(\zeta_{[i]}(\mathrm{s})\right)\right)\left[\frac{\Phi\Big(\zeta_{[2k]}(\mathrm{s})\Big)}{\eta\Big(\zeta_{[2k]}(\mathrm{s})\Big)} - \Phi\Big(\zeta_{[2k+1]}(\mathrm{s})\Big)\right] \\ &\geq \sum_{k=0}^{n}\left(\prod_{i=0}^{2k}\eta\left(\zeta_{[i]}(\mathrm{s})\right)\right)\left[\frac{1}{\eta\Big(\zeta_{[2k]}(\mathrm{s})\Big)} - 1\right]\Phi\Big(\zeta_{[2k]}(\mathrm{s})\Big). \end{aligned} \tag{9}$$

Moreover, as $\Phi(\mathrm{s})/\pi_2(\mathrm{s})$ is decreasing and $\zeta_{[2k]}(\mathrm{s}) \leq \mathrm{s}$, we have

$$\frac{\Phi\Big(\zeta_{[2k]}(\mathrm{s})\Big)}{\pi\Big(\zeta_{[2k]}(\mathrm{s})\Big)} \geq \frac{\Phi(\mathrm{s})}{\pi(\mathrm{s})},$$

and

$$\Phi\Big(\zeta_{[2k]}(\mathrm{s})\Big) \geq \frac{\pi\Big(\zeta_{[2k]}(\mathrm{s})\Big)}{\pi(\mathrm{s})}\Phi(\mathrm{s}).$$

Thus, using the above inequality and substituting in (9), we obtain

$$\begin{aligned} x(\mathrm{s}) &> \sum_{k=0}^{n}\left(\prod_{i=0}^{2k}\eta\left(\zeta_{[i]}(\mathrm{s})\right)\right)\left[\frac{1}{\eta\Big(\zeta_{[2k]}(\mathrm{s})\Big)} - 1\right]\frac{\pi\Big(\zeta_{[2k]}(\mathrm{s})\Big)}{\pi(\mathrm{s})}\Phi(\mathrm{s}) \\ &= \eta_1(\mathrm{s}; n)\Phi(\mathrm{s}). \end{aligned}$$

On the other hand, if $\eta_0 > 1$, then $\Phi(\mathrm{s})/\pi_2(\mathrm{s})$ is decreasing and $\zeta^{-1}_{[2k]}(\mathrm{s}) \geq \zeta^{-1}_{[2k-1]}(\mathrm{s})$, implying that

$$\frac{\Phi\Big(\zeta^{-1}_{[2k-1]}(\mathrm{s})\Big)}{\pi_2\Big(\zeta^{-1}_{[2k-1]}(\mathrm{s})\Big)} \geq \frac{\Phi\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big)}{\pi_2\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big)},$$

and

$$\Phi\Big(\zeta^{-1}_{[2k-1]}(\mathrm{s})\Big) \geq \frac{\pi_2\Big(\zeta^{-1}_{[2k-1]}(\mathrm{s})\Big)}{\pi_2\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big)}\Phi\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big).$$

Using Lemma 5, we can conclude that

$$x(\mathrm{s}) > \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\eta\Big(\zeta^{-1}_{[i]}(\mathrm{s})\Big)}\right)\left[\frac{\pi_2\Big(\zeta^{-1}_{[2k-1]}(\mathrm{s})\Big)}{\pi_2\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big)} - \frac{1}{\eta\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big)}\right]\Phi\Big(\zeta^{-1}_{[2k]}(\mathrm{s})\Big).$$

As $\Phi(s)$ is increasing and $\zeta_{[2k]}^{-1}(s) \geq s$, we have

$$\begin{aligned} x(s) &> \sum_{k=1}^{n} \left(\prod_{i=1}^{2k-1} \frac{1}{\eta\left(\zeta_{[i]}^{-1}(s)\right)} \right) \left[\frac{\pi_2\left(\zeta_{[2k-1]}^{-1}(s)\right)}{\pi_2\left(\zeta_{[2k]}^{-1}(s)\right)} - \frac{1}{\eta\left(\zeta_{[2k]}^{-1}(s)\right)} \right] \Phi(s) \\ &= \widehat{\eta}_1(s,n)\Phi(s). \end{aligned}$$

Here, the proof ends. □

Lemma 8. *Assume that* $x \in \Omega$. *Then,*
$(\mathbf{Y}_{2,1})$ $\Phi''(s) \geq \psi_0(s)\mu^{1/\gamma}(s)\Phi'''(s)$;
$(\mathbf{Y}_{2,2})$ $\Phi''(s)/\phi_0(s)$, $\Phi'(s)/\phi_1(s)$, *and* $\Phi(s)/\phi_2(s)$ *are decreasing;*
$(\mathbf{Y}_{2,3})$ $x(s) > B_1(s,n)\Phi(s)$;
$(\mathbf{Y}_{2,4})$ $\Phi(s) \geq \psi_2(s)\mu^{1/\gamma}(s)\Phi'''(s)$,
eventually, where,

$$B_1(s,n) = \begin{cases} \eta_2(s;n) & \text{for } \eta_0 < 1, \\ \widehat{\eta}_2(s;n) & \text{for } \eta_0 > \frac{\phi_2(s)}{\phi_2(\zeta(s))}. \end{cases}$$

Proof. $(\mathbf{Y}_{2,1})$ For convenience, we assume that

$$w(s) = \mu^{1/\gamma}(s)\Phi'''(s).$$

In light of Lemma 7, it can be deduced

$$x(s) > B_0(s,n)\Phi(s) \text{ for } s \geq s_1.$$

Then, (1) becomes

$$(w^{\gamma}(s))' = -q(s)x^{\gamma}(\lambda(s)) \leq -q(s)B_0^{\gamma}(\lambda(s),n)\Phi^{\gamma}(\lambda(s)).$$

Thus, we have

$$\begin{aligned} \left(\Phi''(s) - \pi_0(s)w(s)\right)' &= -\pi_0(s)w'(s) \\ &= -\pi_0(s)\left((w^{\gamma}(s))^{1/\gamma}\right)' \\ &= -\frac{1}{\gamma}\pi_0(s)w^{1-\gamma}(s)(w^{\gamma}(s))' \\ &\leq \frac{1}{\gamma}\pi_0(s)w^{1-\gamma}(s)q(s)B_0^{\gamma}(\lambda(s),n)\Phi^{\gamma}(\lambda(s)). \end{aligned} \quad (10)$$

From Lemma 6, we note that

$$\Phi''(s) - \pi_0(s)w(s) \geq 0.$$

Integrating (10) from s_1 to s, we obtain

$$\Phi''(s) \geq \pi_0(s)w(s) + \frac{1}{\gamma}\int_{s_1}^{s} \pi_0(v)w^{1-\gamma}(v)q(v)B_0^{\gamma}(\lambda(v),n)\Phi^{\gamma}(\lambda(v))dv. \quad (11)$$

Using the facts that $\Phi(s)/\pi_2(s)$ and $w'(s)$ are decreasing, we have

$$\Phi(\lambda(v)) \geq \pi_2(\lambda(s))w(\lambda(s)) \geq \pi_2(\lambda(s))w(s).$$

which with (11) gives

$$\begin{aligned}\Phi''(\mathrm{s}) &\geq \pi_0(\mathrm{s})w(\mathrm{s}) + \frac{1}{\gamma}\int_{\mathrm{s}_1}^{\mathrm{s}} \pi_0(v)q(v)\pi_2^{\gamma}(\lambda(v))\mathrm{B}_0^{\gamma}(\lambda(v),n)w(v)\mathrm{d}v \\ &\geq w(\mathrm{s})\left(\pi_0(\mathrm{s}) + \frac{1}{\gamma}\int_{\mathrm{s}_1}^{\mathrm{s}} \pi_0(v)q(v)\pi_2^{\gamma}(\lambda(v))\mathrm{B}_0^{\gamma}(\lambda(v),n)\mathrm{d}v\right) \\ &= \psi_0(\mathrm{s})w(\mathrm{s}). \end{aligned} \tag{12}$$

($\mathbf{Y}_{2,2}$) Multiplying (12) by

$$\phi_0(\mathrm{s}) = \exp\left(\int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\mathrm{d}v}{\psi_0(v)\mu^{1/\gamma}(v)}\right),$$

we see that

$$\begin{aligned}\left(\frac{\Phi''(\mathrm{s})}{\phi_0(\mathrm{s})}\right)' &= \frac{\phi_0(\mathrm{s})\Phi'''(\mathrm{s}) - \phi_0(\mathrm{s})\frac{1}{\psi_0(\mathrm{s})\mu^{1/\gamma}(\mathrm{s})}\Phi''(\mathrm{s})}{\phi_0^2(\mathrm{s})} \\ &= \frac{\psi_0(\mathrm{s})\mu^{1/\gamma}(\mathrm{s})w(\mathrm{s}) - \Phi''(\mathrm{s})}{\psi_0(\mathrm{s})\mu^{1/\gamma}(\mathrm{s})\phi_0(\mathrm{s})} \leq 0.\end{aligned}$$

Since $\Phi''(\mathrm{s})/\phi_0(\mathrm{s})$ is decreasing, then

$$\Phi'(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\Phi''(v)}{\phi_0(v)}\phi_0(v)\mathrm{d}v \geq \frac{\Phi''(\mathrm{s})}{\phi_0(\mathrm{s})}\phi_1(\mathrm{s}).$$

From this we deduce that

$$\left(\frac{\Phi'(\mathrm{s})}{\phi_1(\mathrm{s})}\right)' = \frac{\Phi''(\mathrm{s})\phi_1(\mathrm{s}) - \phi_0(\mathrm{s})\Phi'(\mathrm{s})}{\phi_1^2(\mathrm{s})} \leq 0.$$

Since $\Phi'(\mathrm{s})/\phi_1(\mathrm{s})$ is decreasing, then

$$\Phi(\mathrm{s}) \geq \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\Phi'(v)}{\phi_1(v)}\phi_1(v)\mathrm{d}v \geq \frac{\Phi'(\mathrm{s})}{\phi_1(\mathrm{s})}\phi_2(\mathrm{s}).$$

Consequently

$$\left(\frac{\Phi(\mathrm{s})}{\phi_2(\mathrm{s})}\right)' = \frac{\Phi'(\mathrm{s})\phi_2(\mathrm{s}) - \phi_1(\mathrm{s})\Phi(\mathrm{s})}{\phi_2^2(\mathrm{s})} \leq 0.$$

($\mathbf{Y}_{2,3}$) Now, as in the proof of Lemma 7, we find

$$x(\mathrm{s}) > \mathrm{B}_1(\mathrm{s},n)\Phi(\mathrm{s}).$$

($\mathbf{Y}_{2,4}$) Integrating (12) from s_1 to s, we have

$$\Phi'(\mathrm{s}) \geq \int_{\mathrm{s}_0}^{\mathrm{s}} \psi_0(\mathrm{s})w(\mathrm{s})\mathrm{d}v \geq w(\mathrm{s})\int_{\mathrm{s}_0}^{\mathrm{s}} \psi_0(\mathrm{s})\mathrm{d}v = w(\mathrm{s})\psi_1(\mathrm{s}).$$

Integrating this inequality from s_1 to s, we arrive at

$$\Phi(\mathrm{s}) \geq \psi_2(\mathrm{s})w(\mathrm{s}).$$

Here, the proof ends. □

3. Oscillatory Theorems

The objective of this section is to use the results obtained in the previous section to develop improved oscillation criteria for Equation (1). The goal is to determine the conditions that guarantee the non-existence of any positive solutions.

Theorem 1. *If there is a $\rho \in C([s_0,\infty),(0,\infty))$ such that*

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left(\rho(v)q(v)B_1^{\gamma}(\lambda(v),n)\frac{\phi_2^{\gamma}(\lambda(v))}{\phi_2^{\gamma}(v)}-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{(\rho'(v))^{\gamma+1}}{(\rho(v)\psi_1(v))^{\gamma}}\right)dv=\infty, \quad (13)$$

and there is a $\widetilde{\rho} \in C([s_0,\infty),(0,\infty))$ such that

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left(\widetilde{\rho}(v)\int_{v}^{\infty}\left(\frac{1}{\mu(u)}\int_{u}^{\infty}Q(v)\left(\frac{\lambda(v)}{v}\right)^{\gamma/\epsilon}dv\right)^{1/\gamma}du-\frac{(\widetilde{\rho}'_{+}(v))^{2}}{4\widetilde{\rho}(v)}\right)dv=\infty \quad (14)$$

hold for some $\epsilon \in (0,1)$, then (1) is oscillatory, where $Q(s)=q(v)(1-\eta(\lambda(v)))^{\gamma}$.

Proof. We define

$$w(s)=\rho(s)\frac{\mu(s)(\Phi'''(s))^{\gamma}}{\Phi^{\gamma}(s)}. \quad (15)$$

Hence, $w(s)>0$. Differentiating (15), we get

$$\begin{aligned} w'(s) &= \rho'(s)\frac{\mu(s)(\Phi'''(s))^{\gamma}}{\Phi^{\gamma}(s)}+\rho(s)\frac{\left(\mu(s)(\Phi'''(s))^{\gamma}\right)'}{\Phi^{\gamma}(s)} \\ &\quad -\gamma\rho(s)\frac{\mu(s)(\Phi'''(s))^{\gamma}\Phi'(s)}{\Phi^{\gamma+1}(s)} \\ &= \frac{\rho'(s)}{\rho(s)}w(s)-\rho(s)\frac{q(s)x^{\gamma}(\lambda(s))}{\Phi^{\gamma}(s)}-\gamma w(s)\frac{\Phi'(s)}{\Phi(s)}. \end{aligned} \quad (16)$$

From $(\mathbf{Y}_{2,3})$, we see that

$$w'(s)\le\frac{\rho'(s)}{\rho(s)}w(s)-\rho(s)q(s)B_1^{\gamma}(\lambda(s),n)\frac{\Phi^{\gamma}(\lambda(s))}{\Phi^{\gamma}(s)}-\gamma w(s)\frac{\Phi'(s)}{\Phi(s)}.$$

From $(\mathbf{Y}_{2,2})$, we obtain

$$\Phi'(s)\ge\mu^{1/\gamma}(s))\Phi'''(s)\psi_1(s). \quad (17)$$

Therefore, (16) can be expressed as

$$\begin{aligned} w'(s) &\le \frac{\rho'(s)}{\rho(s)}w(s)-\rho(s)q(s)B_1^{\gamma}(\lambda(s),n)\frac{\Phi^{\gamma}(\lambda(s))}{\Phi^{\gamma}(s)} \\ &\quad -\gamma\lambda'(s)w(s)\frac{\mu^{1/\gamma}(s)\Phi'''(s)\psi_1(s)}{\Phi(s)} \\ &= \frac{\rho'(s)}{\rho(s)}w(s)-\rho(s)q(s)B_1^{\gamma}(\lambda(s),n)\frac{\Phi^{\gamma}(\lambda(s))}{\Phi^{\gamma}(s)}-\frac{\gamma\psi_1(s)}{\rho^{1/\gamma}(s)}w^{(\gamma+1)/\gamma}(s). \end{aligned} \quad (18)$$

Since $\Phi(s)/\phi_2(s)$ is decreasing, we obtain

$$\frac{\Phi(\lambda(s))}{\phi_2(\lambda(s))}\ge\frac{\Phi(s)}{\phi_2(s)},$$

and

$$\frac{\Phi(\lambda(s))}{\Phi(s)}\ge\frac{\phi_2(\lambda(s))}{\phi_2(s)},$$

which with (18) leads to

$$\begin{aligned} w'(\mathrm{s}) \quad \leq \quad & \frac{\rho'(\mathrm{s})}{\rho(\mathrm{s})} w(\mathrm{s}) - \rho(\mathrm{s}) q(\mathrm{s}) \mathrm{B}_1^\gamma(\lambda(\mathrm{s}), n) \frac{\phi_2^\gamma(\lambda(\mathrm{s}))}{\phi_2^\gamma(\mathrm{s})} \\ & - \frac{\gamma \psi_1(\mathrm{s})}{\rho^{1/\gamma}(\mathrm{s})} w^{(\gamma+1)/\gamma}(\mathrm{s}). \end{aligned}$$

From Lemma 2, with $K = \rho'(\mathrm{s})/\rho(\mathrm{s})$, $L = \gamma\psi_1(\lambda(\mathrm{s}))/\rho^{1/\gamma}(\mathrm{s})$, and $u = w$, we obtain

$$\frac{\rho'(\mathrm{s})}{\rho(\mathrm{s})} w(\mathrm{s}) - \frac{\gamma \psi_1(\mathrm{s})}{\rho^{1/\gamma}(\mathrm{s})} w^{(\gamma+1)/\gamma}(\mathrm{s}) \leq \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\rho'(\mathrm{s}))^{\gamma+1}}{\rho^\gamma(\mathrm{s}) \psi_1^\gamma(\mathrm{s})}.$$

Thus, from (18), we get

$$w'(\mathrm{s}) \leq -\rho(\mathrm{s}) q(\mathrm{s}) \mathrm{B}_1^\gamma(\lambda(\mathrm{s}), n) \frac{\phi_2^\gamma(\lambda(\mathrm{s}))}{\phi_2^\gamma(\mathrm{s})} + \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\rho'(\mathrm{s}))^{\gamma+1}}{(\rho(\mathrm{s}) \psi_1(\mathrm{s}))^\gamma}. \tag{19}$$

Integrating (19) from s_1 to s, we arrive at

$$\int_{\mathrm{s}_1}^{\mathrm{s}} \left(\rho(v) q(v) \mathrm{B}_1^\gamma(\lambda(v), n) \frac{\phi_2^\gamma(\lambda(v))}{\phi_2^\gamma(v)} - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\rho'(v))^{\gamma+1}}{(\rho(v) \psi_1(v))^\gamma} \right) \mathrm{d}v \leq w(\mathrm{s}_1),$$

which contradicts (13).

Assume now that x is a positive solution of (1) with $\Phi(\mathrm{s})$ satisfying C_2. Integrating (1) from s to ∞ and using the fact that $\left(\mu(\Phi''')^\gamma\right)' \leq 0$, we obtain

$$\begin{aligned} \mu(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma \quad &= \quad \int_{\mathrm{s}}^\infty q(v) x^\gamma(\lambda(v)) \mathrm{d}v \\ &\geq \quad \int_{\mathrm{s}}^\infty q(v) (1 - \eta(\lambda(v)))^\gamma \Phi^\gamma(\lambda(v)) \mathrm{d}v. \end{aligned} \tag{20}$$

As $\Phi > 0$, $\Phi' > 0$, and $\Phi'' < 0$, Lemma 1 implies that $\Phi \geq \epsilon \mathrm{s} \Phi'$ for all $\epsilon \in (0,1)$. Integrating this inequality from $\lambda(\mathrm{s})$ to s, we obtain

$$\frac{\Phi(\lambda(\mathrm{s}))}{\Phi(\mathrm{s})} \geq \left(\frac{\lambda(\mathrm{s})}{\mathrm{s}} \right)^{1/\epsilon}.$$

Therefore, (20) becomes

$$\mu(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma \geq \int_{\mathrm{s}}^\infty Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \Phi^\gamma(v) \mathrm{d}v.$$

Since $\Phi'(\mathrm{s}) > 0$, then

$$\mu(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma \geq \Phi^\gamma(\mathrm{s}) \int_{\mathrm{s}}^\infty Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v,$$

or equivalently

$$\Phi'''(\mathrm{s}) \geq \Phi(\mathrm{s}) \left(\frac{1}{\mu(\mathrm{s})} \int_{\mathrm{s}}^\infty Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v \right)^{1/\gamma}.$$

Integrating this inequality from s to ∞, we have

$$\Phi''(\mathrm{s}) \leq -\Phi(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\mu(u)} \int_{u}^{\infty} Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v \right)^{1/\gamma} \mathrm{d}u. \tag{21}$$

Now, define

$$w(\mathrm{s}) := \widetilde{\rho}(\mathrm{s}) \frac{\Phi'(\mathrm{s})}{\Phi(\mathrm{s})}.$$

Then, $w(\mathrm{s}) \geq 0$ for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$ and

$$\begin{aligned} w' &= \widetilde{\rho}'(\mathrm{s}) \frac{\Phi'(\mathrm{s})}{\Phi(\mathrm{s})} + \widetilde{\rho}(\mathrm{s}) \frac{\Phi''(\mathrm{s})}{\Phi(\mathrm{s})} - \widetilde{\rho}(\mathrm{s}) \frac{(\Phi'(\mathrm{s}))^2}{\Phi^2(\mathrm{s})} \\ &= \widetilde{\rho}(\mathrm{s}) \frac{\Phi''(\mathrm{s})}{\Phi(\mathrm{s})} + \frac{\widetilde{\rho}'(\mathrm{s})}{\widetilde{\rho}(\mathrm{s})} w(\mathrm{s}) - \frac{1}{\widetilde{\rho}(\mathrm{s})} w^2(\mathrm{s}). \end{aligned} \tag{22}$$

Hence, by (21), we obtain

$$\begin{aligned} w'(\mathrm{s}) \leq\; & -\widetilde{\rho}(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\mu(u)} \int_{u}^{\infty} Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v \right)^{1/\gamma} \mathrm{d}u \\ & + \frac{\widetilde{\rho}'_{+}(\mathrm{s})}{\widetilde{\rho}(\mathrm{s})} w(\mathrm{s}) - \frac{1}{\widetilde{\rho}(\mathrm{s})} w^2(\mathrm{s}). \end{aligned}$$

Using Lemma 2 with $K = \widetilde{\rho}'_{+}(\mathrm{s})/\widetilde{\rho}(\mathrm{s})$, and $L = 1/\widetilde{\rho}(\mathrm{s})$, we obtain

$$\frac{\widetilde{\rho}'_{+}(\mathrm{s})}{\widetilde{\rho}(\mathrm{s})} w(\mathrm{s}) - \frac{1}{\widetilde{\rho}(\mathrm{s})} w^2(\mathrm{s}) \leq \frac{(\widetilde{\rho}'_{+}(\mathrm{s}))^2}{4\widetilde{\rho}(\mathrm{s})}.$$

Consequently, (22) leads to

$$w'(\mathrm{s}) \leq -\widetilde{\rho}(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \left(\frac{1}{\mu(u)} \int_{u}^{\infty} Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v \right)^{1/\gamma} \mathrm{d}u + \frac{(\widetilde{\rho}'_{+}(\mathrm{s}))^2}{4\widetilde{\rho}(\mathrm{s})}.$$

Integrating this inequality from s_1 to s, we find

$$\int_{\mathrm{s}_1}^{\mathrm{s}} \left(\widetilde{\rho}(v) \int_{v}^{\infty} \left(\frac{1}{\mu(u)} \int_{u}^{\infty} Q(v) \left(\frac{\lambda(v)}{v} \right)^{\gamma/\epsilon} \mathrm{d}v \right)^{1/\gamma} \mathrm{d}u - \frac{(\widetilde{\rho}'_{+}(v))^2}{4\widetilde{\rho}(v)} \right) \mathrm{d}v \leq w(\mathrm{s}_1),$$

which contradicts (14).

Here, the proof ends. □

Theorem 2. *If*

$$\liminf_{\mathrm{s}\to\infty} \int_{\lambda(\mathrm{s})}^{\mathrm{s}} q(v) \mathrm{B}_1^{\gamma}(\lambda(v), n) \psi_2^{\gamma}(\lambda(v)) \mathrm{d}v > \frac{1}{\mathrm{e}}, \tag{23}$$

and (14) holds, then (1) is oscillatory.

Proof. Assume on the contrary that (1) is not oscillatory. Assume that it possesses an eventually positive solution $x(\mathrm{s})$. It follows from Equation (1) that there exist two possible cases as in Lemma 3.

Assume that case $(\mathbf{C}_1)$ holds. From $(\mathbf{Y}_{2,3})$ and $(\mathbf{Y}_{2,4})$, we have

$$x(\mathrm{s}) > \mathrm{B}_1(\mathrm{s}, n) \psi_2(\mathrm{s}) \mu^{1/\gamma}(\mathrm{s}) \Phi'''(\mathrm{s}).$$

Let
$$\omega(s) = \mu(s)\left(\Phi'''(s)\right)^{\gamma}.$$
From (1), we deduce that
$$\omega'(s) + q(s)B_1^{\gamma}(\lambda(s),n)\psi_2^{\gamma}(\lambda(s))\omega(\lambda(s)) \le 0.$$
Using Theorem 1 in [29], we conclude that the equation
$$\omega'(s) + q(s)B_1^{\gamma}(\lambda(s),n)\psi_2^{\gamma}(\lambda(s))\omega(\lambda(s)) = 0, \tag{24}$$
also has a positive solution. It follows from Theorem 2 in [30] that (24) is oscillatory under condition (23), a contradiction.

Assume that case $(\mathbf{C}_2)$ holds. The proof of the case $(\mathbf{C}_2)$ is the same as that of Theorem 1.

Here, the proof ends. □

4. Examples

We illustrate the value of the findings we have obtained through the following examples.

Example 1. *Consider the NDE*
$$\left(s^{-\gamma}\left((x(s)+\eta_0 x(\zeta_0 s))'''\right)^{\gamma}\right)' + \frac{q_0}{s^{4\gamma+1}}x^{\gamma}(\lambda_0 s) = 0,\ s \ge 1, \tag{25}$$
with ζ_0, $\lambda_0 \in (0,1)$ and $q_0 > 0$. By comparing (1) and (25) we see that $\mu(s) = s^{-\gamma}$, $\zeta(s) = \zeta_0 s$, $\lambda(s) = \lambda_0 s$. Then $\pi_0(s) = s^2/2$, $\pi_1(s) = s^3/6$, $\pi_2(s) = s^4/24$, $q(s) = q_0/s^{4\gamma+1}$,
$$\eta_1(s;n) = [1-\eta_0]\sum_{k=0}^{n}\eta_0^{2k}\zeta_0^{8k}(s),$$
$$\widehat{\eta}_1(s;n) = \left[\eta_0\zeta_0^4 - 1\right]\sum_{k=1}^{n}\left(\frac{1}{\eta_0}\right)^{2k-2},$$
$$B_0(s,n) = B_0 = \begin{cases} \eta_1 & \text{for } \eta_0 < 1, \\ \widehat{\eta}_1 & \text{for } \eta_0 > \frac{1}{\zeta_0^4}. \end{cases}$$
$$\begin{aligned} \psi_0(s) &= \pi_0(s) + \frac{1}{\gamma}\int_{s_1}^{s}\pi_0(v)q(v)\pi_2^{\gamma}(\lambda(v))B_0^{\gamma}(\lambda(v),n)\mathrm{d}v \\ &= C_0 s^2,\ C_0 = \left(\frac{1}{2} + \frac{1}{4\gamma}\frac{B_0^{\gamma}\lambda_0^{4\gamma}q_0}{24^{\gamma}}\right), \end{aligned}$$
$$\psi_1(s) = C_0\frac{s^3}{3},\ \psi_2(s) = C_0\frac{s^4}{12},$$
$$\phi_0(s) = \exp\left(\int_{s_0}^{s}\frac{1}{\psi_0(v)\mu^{1/\gamma}(v)}\mathrm{d}v\right) = s^{\frac{1}{C_0}},$$
$$\phi_1(s) = \frac{1}{1+\frac{1}{C_0}}s^{1+\frac{1}{C_0}},$$
$$\phi_2(s) = \frac{1}{\left(1+\frac{1}{C_0}\right)\left(2+\frac{1}{C_0}\right)}s^{2+\frac{1}{C_0}},$$
$$\eta_2(s;n) = \eta_2 = [1-\eta_0]\sum_{k=0}^{n}\eta_0^{2k}\zeta_0^{2k\left(2+\frac{1}{C_0}\right)},$$

$$\widehat{\eta}_2(s;n) = \widehat{\eta}_2 = \left[\eta_0 \zeta_0^{2+\frac{1}{C_0}} - 1\right] \sum_{k=1}^{n} \left(\frac{1}{\eta_0}\right)^{2k-2},$$

and

$$B_1(s,n) = B_1 = \begin{cases} \eta_2 & \text{for } \eta_0 < 1, \\ \widehat{\eta}_2 & \text{for } \eta_0 > \frac{1}{\zeta_0^4}. \end{cases}$$

If we consider the function $\rho(s) = s^{4\gamma}$ and condition (13), we obtain

$$\begin{aligned}
& \limsup_{s\to\infty} \int_{s_0}^{s} \left(\rho(v)q(v)B_1^{\gamma}(\lambda(v),n)\frac{\phi_2^{\gamma}(\lambda(v))}{\phi_2^{\gamma}(v)} - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(\rho'(v))^{\gamma+1}}{(\rho(v)\psi_1(v))^{\gamma}}\right)dv \\
= & \limsup_{s\to\infty} \int_{s_0}^{s} \left(v^{4\gamma}\frac{q_0}{v^{4\gamma+1}}B_1^{\gamma}\lambda_0^{\gamma(2+C_0^{-1})} - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{4^{\gamma+1}v^{4\gamma^2+3\gamma-1}}{v^{4\gamma^2}C_0^{\gamma}\frac{v^{3\gamma}}{3}}\right) \\
= & \limsup_{s\to\infty} \int_{s_0}^{s} \left(q_0 B_1^{\gamma}\lambda_0^{\gamma(2+C_0^{-1})} - \frac{3}{(\gamma+1)^{\gamma+1}}\frac{4^{\gamma+1}}{C_0^{\gamma}}\right)\frac{1}{v} \\
= & \left(q_0 B_1^{\gamma}\lambda_0^{\gamma(2+C_0^{-1})} - \frac{3}{(\gamma+1)^{\gamma+1}}\frac{4^{\gamma+1}}{C_0^{\gamma}}\right)\limsup_{s\to\infty}\ln\frac{s}{s_0} = \infty,
\end{aligned}$$

which is satisfied if the following condition holds

$$q_0 B_1^{\gamma}\lambda_0^{\gamma(2+C_0^{-1})} > \frac{3}{(\gamma+1)^{\gamma+1}}\frac{4^{\gamma+1}}{C_0^{\gamma}}. \tag{26}$$

Similarly, if we consider condition (14) with $\widetilde{\rho}(s) = s$, we obtain

$$\begin{aligned}
& \limsup_{s\to\infty} \int_{s_0}^{s} \left(\widetilde{\rho}(v)\int_{v}^{\infty}\left(\frac{1}{\mu(u)}\int_{u}^{\infty} Q(v)\left(\frac{\lambda(v)}{v}\right)^{\gamma/\epsilon}dv\right)^{1/\gamma}du - \frac{(\widetilde{\rho}'_{+}(v))^2}{4\widetilde{\rho}(v)}\right)dv \\
= & \limsup_{s\to\infty} \int_{s_0}^{s} \left(v\int_{v}^{\infty}\left(u^{\gamma}\int_{u}^{\infty}\frac{q_0}{v^{4\gamma+1}}(1-\eta_0)^{\gamma}\left(\frac{\lambda_0 v}{v}\right)^{\gamma/\epsilon}dv\right)^{1/\gamma}du - \frac{1}{4v}\right)dv \\
= & \limsup_{s\to\infty} \int_{s_0}^{s} \left(\frac{1}{2}\left(\frac{q_0}{4\gamma}(1-\eta_0)^{\gamma}\lambda_0^{\gamma/\epsilon}\right)^{1/\gamma} - \frac{1}{4}\right)\frac{1}{v}dv \\
= & \left(\frac{1}{2}\left(\frac{q_0}{4\gamma}(1-\eta_0)^{\gamma}\lambda_0^{\gamma/\epsilon}\right)^{1/\gamma} - \frac{1}{4}\right)\limsup_{s\to\infty}\ln\frac{s}{s_0} = \infty,
\end{aligned}$$

which is satisfied if the following condition holds

$$\frac{1}{\gamma}q_0(1-\eta_0)^{\gamma}\lambda_0^{\gamma/\epsilon} > 2^{2-\gamma}. \tag{27}$$

Finally, considering condition (23), we obtain

$$\begin{aligned}
& \liminf_{s\to\infty} \int_{\lambda(s)}^{s} q(v)B_1^{\gamma}(\lambda(v),n)\psi_2^{\gamma}(\lambda(v))dv \\
= & \liminf_{s\to\infty} \int_{\lambda_0 s}^{s} \frac{q_0}{v^{4\gamma+1}}B_1^{\gamma}C_0^{\gamma}\frac{v^{4\gamma}}{12}dv \\
= & \frac{1}{12}q_0 B_1^{\gamma}C_0^{\gamma}\liminf_{s\to\infty}\int_{\lambda_0 s}^{s}\frac{1}{v}dv \\
= & \frac{1}{12}q_0 B_1^{\gamma}C_0^{\gamma}\ln\frac{1}{\lambda_0},
\end{aligned}$$

which is satisfied if the following condition holds

$$q_0 B_1^{\gamma} C_0^{\gamma} \ln \frac{1}{\lambda_0} > \frac{12}{e}. \tag{28}$$

The oscillation of DE (25) can be tested by applying the above theorems. Theorem 1 ensures that solutions will oscillate when conditions (26) and (27) are satisfied. Whereas Theorem 2 requires conditions (28) and (27).

Example 2. *Consider the NDE*

$$(x(s) + \eta_0 x(\zeta_0 s))^{(4)} + \frac{q_0}{s^4} x^{\gamma}(\lambda_0 s) = 0, \ s \geq 1, \tag{29}$$

with ζ_0, $\lambda_0 \in (0,1)$ and $q_0 > 0$. By comparing (1) and (29) we see that $\gamma = 1$, $\mu(s) = 1$, $\zeta(s) = \zeta_0 s$, and $\lambda(s) = \lambda_0 s$. Then $\pi_0(s) = s$, $\pi_1(s) = s^2/2$, $\pi_2(s) = s^3/3$, $q(s) = q_0/s^4$,

$$\eta_1(s;n) = [1 - \eta_0] \sum_{k=0}^{n} \eta_0^{2k} \zeta_0^{6k},$$

$$\widehat{\eta}_1(s;n) = \left[\zeta_0^3 \eta_0 - 1\right] \sum_{k=1}^{n} \left(\frac{1}{\eta_0}\right)^{2k-2},$$

$$B_0(s,n) = B_0 = \begin{cases} \eta_1 & \text{for } \eta_0 < 1, \\ \widehat{\eta}_1 & \text{for } \eta_0 > \frac{1}{\zeta_0^3}. \end{cases}$$

$$\psi_0(s) = Cs, \ C = 1 + \frac{q_0 \lambda_0^3 B_0}{3}, \ \psi_1(s) = C\frac{s^2}{2}, \ \psi_2(s) = C\frac{s^3}{6},$$

$$\phi_0(s) = s^{\frac{1}{C}}, \ \phi_1(s) = \frac{s^{1+\frac{1}{C}}}{1+\frac{1}{C}}, \ \phi_2(s) = \frac{s^{2+\frac{1}{C}}}{\left(1+\frac{1}{C}\right)\left(2+\frac{1}{C}\right)},$$

$$\eta_2(s;n) = [1 - \eta_0] \sum_{k=0}^{n} \eta_0^{2k} \zeta_0^{2k\left(2+\frac{1}{C}\right)},$$

and

$$\widehat{\eta}_2(s;n) = \left[\eta_0 \zeta_0^{2+\frac{1}{C}} - 1\right] \sum_{k=1}^{n} \left(\frac{1}{\eta_0}\right)^{2k-2},$$

$$B_1(s,n) = B_1 = \begin{cases} \eta_2 & \text{for } \eta_0 < 1, \\ \widehat{\eta}_2 & \text{for } \eta_0 > \frac{\phi_2(s)}{\phi_2(\zeta(s))}. \end{cases}$$

If we consider the functions $\rho(s) = s^3$, and $\widetilde{\rho}(s) = s$, the Conditions (13), (14), and (23) are satisfied when

$$q_0 \lambda_0^{2+\frac{1}{C}} B_1 > \frac{2}{C},$$

$$q_0 (1 - \eta_0) \lambda_0^{1/\epsilon} > \frac{3}{2},$$

$$q_0 B_1 C \lambda_0^3 \ln \frac{1}{\lambda_0} > \frac{6}{e},$$

respectively. Thus, from Theoerms 1 and 2, we can conclude that (29) is oscillatory.

5. Conclusions

In this study, we investigated the oscillatory behavior of fourth-order neutral DEs. We improved the relationship between the solution and its corresponding function by utilizing the modified monotonic properties of positive solutions and introduced new improved relationships in the main results. Based on the improved monotonic properties, we derived new criteria for oscillation. We provided illustrative examples and notes to demonstrate the significance of our findings. Our results contribute to the theoretical understanding of neutral DEs. Despite some existing research on the study of the oscillatory behavior of fourth-order DEs, this area still offers intriguing analytical points. Extending our results to neutral DEs of a higher order would be an interesting avenue for future research.

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