

Article

Inverse Graphs in m -Polar Fuzzy Environments and Their Application in Robotics Manufacturing Allocation Problems with New Techniques of Resolvability

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Abstract: The inverse in crisp graph theory is a well-known topic. However, the inverse concept for fuzzy graphs has recently been created, and its numerous characteristics are being examined. Each node and edge in m -polar fuzzy graphs (m PFG) include m components, which are interlinked through a minimum relationship. However, if one wants to maximize the relationship between nodes and edges, then the m -polar fuzzy graph concept is inappropriate. Considering everything we wish to obtain here, we present an inverse graph under an m -polar fuzzy environment. An inverse m PFG is one in which each component's membership value (MV) is greater than or equal to that of each component of the incidence nodes. This is in contrast to an m PFG, where each component's MV is less than or equal to the MV of each component's incidence nodes. An inverse m PFG's characteristics and some of its isomorphic features are introduced. The α -cut concept is also studied here. Here, we also define the composition and decomposition of an inverse m PFG uniquely with a proper explanation. The connectivity concept, that is, the strength of connectedness, cut nodes, bridges, etc., is also developed on an inverse m PF environment, and some of the properties of this concept are also discussed in detail. Lastly, a real-life application based on the robotics manufacturing allocation problem is solved with the help of an inverse m PFG.

Keywords: graphs; inverse m -polar fuzzy graph; connectivity; strength of connectedness; cut nodes; blocks; robotics



Citation: Alanazi, A.M.; Muhiuddin, G.; Mahapatra, T.; Bassfar, Z.; Pal, M. Inverse Graphs in m -Polar Fuzzy Environments and Their Application in Robotics Manufacturing Allocation Problems with New Techniques of Resolvability. *Symmetry* **2023**, *15*, 1387. <https://doi.org/10.3390/sym15071387>

Academic Editor: Zhenbing Zeng

Received: 9 June 2023

Revised: 26 June 2023

Accepted: 29 June 2023

Published: 9 July 2023



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1. Introduction

1.1. Research Background

The interconnection of a graph can be presented with the help of a graph. Its basis was laid by the well-known Swiss mathematician Euler in 1736 when he mentioned the answer to the seven-bridge problem. There are programs of graph principles for unique regions or networks like electric-powered, PC, road, and picture-capturing networks, etc. A widely known topic in combinatorial optimization and discrete areas is the allocation problem. Some related problems include wiring, printed circuit board issues, load issues, resource task issues, frequency allocation issues, various scheduling issues, and computer register mapping. In technical development, fuzzy graph (FG) theory is important. Many rule-based expert systems for engineers have been created from FG theory. In 1975, Rosenfeld [1] studied relations on fuzzy sets and coined the term fuzzy relations. Graph theory is an essential part of connectivity in some fields of geometry, algebra, topology, number theory, computer science, operations research, and optimization. The only other characteristic of FGs is that their edge MV is less than the minimum of their end node MVs. The

literature review below discusses many works based on the allocation problem through graphs and their different variations. However, in real life, there are many problems where vertices and edges cannot be defined in a specific way. Sometimes, they are inverse m -polar fuzzy sets in nature, and their interlinkage can be suitably described by an inverse m -polar fuzzy graph.

1.2. Review of the Literature and Related Works

Zadeh [2] introduced fuzzy sets to deal with ambiguity and fuzziness. Since then, the fuzzy set hypothesis has been investigated in several sequences. Zhang [3,4] introduced bipolar fuzzy sets in 1994. These sets have a wide range of applications in both mathematical and practical situations. Data from numerous sources worldwide have been used to tackle several problems in the real world. This approach for collecting data is a prime example of multipolarity. FGs or bipolar FGs cannot adequately structure this type of polarity. The notion of the m -polar fuzzy set (m PFS) is imposed on graphs to elaborate the various origins in order to simplify the issue. Chen et al. [5] introduced the set concept in an m -dimension fuzzy environment as an extension of bipolar fuzzy sets. Utilizing Zadeh's procedure, Kauffman [6] fostered a strategy of fuzzy graph theory in 1973. Then, Rosenfeld [1] gave another explanation of a fuzzy graph. Several fuzzy graph concepts have been investigated in [7–11]. An inverse fuzzy graph was first introduced by Borzooei et al. [12]. Then, Poulik and Ghorai [13] studied the concept of an inverse fuzzy mixed graph. Ghorai and Pal introduced for the first time the m PFG, and then they [14,15] introduced m -polar FPGs, dual m -polar FPGs and faces, and m -polar FPGs' isomorphic features. Then, Mahapatra and Pal introduced the fuzzy coloring of an m PFG [16]. They continued their work on this topic in [17,18]. In Akram et al. [19–21], several features of graphs in an m -polar fuzzy environment were examined.

1.3. Motivation of the Work

In real life, many problems have been solved using experimental data that come from different origins or sources. In many situations, it is observed that such data contain multiple attributes for a particular piece of information, i.e., multipolarity. These types of multipolar data cannot be structured well by the conception of intuitionistic or bipolar fuzzy models. For example, consider the allocation problem of some robotic manufacturing on some parameters, such as the following:

- (i) Communication system;
- (ii) Pollution-free zone;
- (iii) Worker availability.

For the nodes and the edges, the parameters may be the following:

- (i) Transportation costs;
- (ii) Availability of labor;
- (iii) The number of workers who service both locations.

In this problem, we consider more than one component for each node and edge, so it is impossible to model this type of situation using a fuzzy set, as there is a single component for this concept. Again, we cannot apply a bipolar or intuitionistic fuzzy graph model, as each edge or node has just two components. Thus, the m PFG models give more flexibility than the fuzzy model or other varieties of fuzzy models. Also, it is very interesting to develop and analyze such types of inverse m PFGs with examples and related theorems.

1.4. Framework of This Study

The structure of this article is as follows: Section 2 contains studies on the preliminary material. In Section 3, we carefully examine a novel idea known as an “inverse 3-polar fuzzy graph”, along with its many features. Section 3.1 properly describes the composition and decomposition of an inverse m -polar fuzzy graph. We introduce a generalized notion of connection on inverse m -polar fuzzy graphs in Section 4 and provide a detailed description

of it. In this section, we look at inverse m PF cut nodes and bridges, as well as some of their intriguing features. Then, inverse m PFG is used to address a real-world application based on a production allocation problem in Section 5. Section 6 gives a comparative study of the proposed work. In Section 7, the advantages and limitations of the proposed work are discussed. The work is summarized in Section 8.

1.5. Notations and Symbols

Table 1 gives some notations and abbreviations which are used in the remaining part of the article.

Table 1. Table of Abbreviations.

Full Name	Abbreviated Form
Fuzzy Graph	FG
m -polar Fuzzy Graph	m PFG
Underlying Crisp Graph	UCG
Membership Value	MV
m -polar Fuzzy Set	m PFS
Strength-of-connectedness	SC

2. Preliminaries

Here, we shortly revisit a few definitions related to m PFG, like strong m PFG, complete m PFG, and path in m PFG.

Throughout this article, $p_s : [0, 1]^m \rightarrow [0, 1]$ indicates the s th material of projection mapping, and $s = 1, 2, \dots, m$ is denoted by $s = 1(1)m$.

Definition 1 ([14]). Take $G = (\tilde{V}, \sigma, \mu)$ to be an m PFG of the UCG $H^* = (\tilde{V}, \tilde{E})$, where σ and μ map from $\tilde{V}$ to $[0, 1]^m$ and $\tilde{V} \times \tilde{V}$ to $[0, 1]^m$, respectively. Here, σ and μ represent an m PFS of $\tilde{V}$ and $\tilde{V} \times \tilde{V}$, respectively, which maintain the relation $p_s \circ \mu(x, w) \leq \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$ for every $s = 1(1)m$ and $(x, w) \in \tilde{V} \times \tilde{V}$ as well as $\mu(x, w) = \mathbf{0}$ for every $(x, w) \in (\tilde{V} \times \tilde{V} - \tilde{E})$.

Definition 2 ([15]). $G = (V, \sigma, \mu)$ is assigned as a complete m PFG if $p_s \circ \mu(x, w) = \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$, for every $x, w \in V$ and $s = 1(1)m$.

Definition 3 ([14]). $G = (V, \sigma, \mu)$ is assigned as an m PF strong graph if

$$p_s \circ \mu(x, w) = \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\},$$

for every $(x, w) \in E$ and $s = 1(1)m$.

Definition 4 ([20]). Let $G = (V, \sigma, \mu)$ be an m PFG and $P : x_1, x_2, \dots, x_k$ be a path in G . $S(P)$ denotes the strength of P , which is defined as $S(P) = (\min_{1 \leq s < j \leq k} p_1 \circ \mu(x_s, x_j), \min_{1 \leq s < j \leq k} p_2 \circ \mu(x_s, x_j), \dots, \min_{1 \leq s < j \leq k} p_m \circ \mu(x_s, x_j)) = (\mu_1^n(x_s, x_j), \mu_2^n(x_s, x_j), \dots, \mu_m^n(x_s, x_j))$.

The strength of the connectedness of a path in-between x_1 and x_k is denoted by $\text{CONN}_G(x_1, x_k)$ and is given as follows:

$$\text{CONN}_G(x_1, x_k) = (p_1 \circ \mu(x_s, x_j)^\infty, p_2 \circ \mu(x_s, x_j)^\infty, \dots, p_m \circ \mu(x_s, x_j)^\infty),$$

where $(p_i \circ \mu(x_s, x_j)^\infty) = \max_{n \in \mathbb{N}} (\mu_i^n(x_s, x_j)), i = 1, 2, \dots, m$.

Definition 5 ([16]). For an mPFG, an edge (x, w) , $x, w \in V$ is considered independently strong in $G = (V, \sigma, \mu)$ if $\frac{1}{2}\{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\} \leq p_s \circ \mu(x, w)$, $s = 1(1)m$. If not, it is perceived as weak on its own. The s th component of the strength of an edge (x, w) is defined as

$$p_s \circ I(x, w) = \frac{p_s \circ \mu(x, w)}{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)}, \quad s = 1(1)m.$$

Definition 6. Take $G = (V, \sigma, \mu)$ and $G' = (V', \sigma', \mu')$ as two mPFGs of the UCG $G^* = (V, E)$ and $G'^* = (V', E')$, respectively. An isomorphism is a bijection of G and G' , that is, $g : V \rightarrow V'$, and satisfies

$$p_s \circ \sigma(x) = p_s \circ \sigma'(g(x)) \text{ and } p_s \circ \mu(x, w) = p_s \circ \mu'(g(x), g(w))$$

$\forall x, w \in V$ and every $s = 1(1)m$. G is then said to be isomorphic with G' .

3. Inverse m -Polar Fuzzy Graph

In this section, we discuss a new idea of mPFGs, which have nodes and edges along with an MV such that they fulfill a specific criterion.

Definition 7. Let $G = (V, \sigma, \mu)$ be an mPFG of the UCG $G^* = (\tilde{V}, \tilde{E})$, where σ and μ map from $\tilde{V}$ to $[0, 1]^m$ and $\tilde{V} \times \tilde{V}$ to $[0, 1]^m$, respectively. Here, σ and μ represent an mPFS of $\tilde{V}$ and $\tilde{V} \times \tilde{V}$, respectively, which maintain the relation $p_s \circ \mu(x, w) \geq \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$ for every $s = 1(1)m$ and $(x, w) \in \tilde{V} \times \tilde{V}$ as well as $\mu(x, w) = \mathbf{0}$ for every $(x, w) \in (\tilde{V} \times \tilde{V} - \tilde{E})$.

Example 1. The above definition is depicted through an example, which is given in Figure 1.

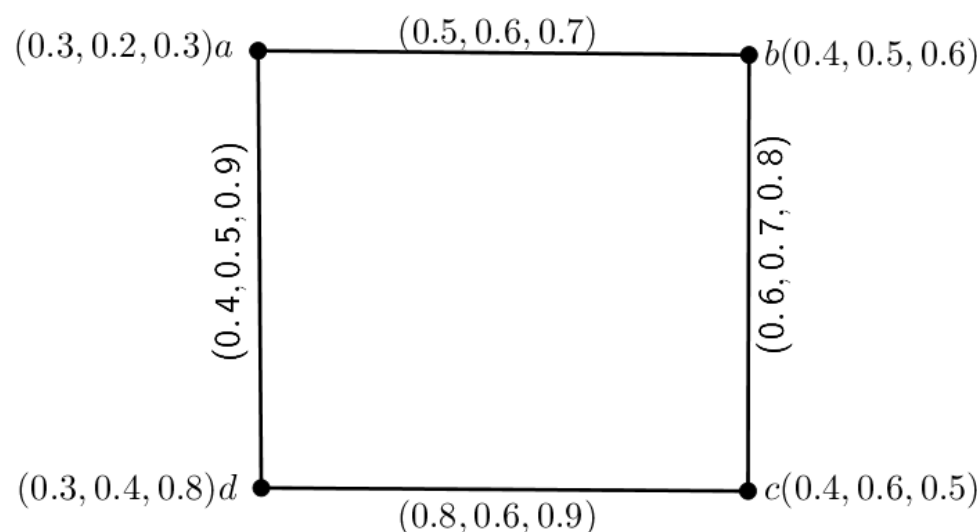


Figure 1. An illustration of inverse 3-polar fuzzy graph.

Definition 8. Let $G = (V, \sigma, \mu)$ be an inverse mPFG, having UCG $G^* = (V, E)$. Then, G is called a complete inverse mPFG if $E = V \times V$ and $\forall s = 1(1)m$, $p_s \circ \mu(b, d) > \{p_s \circ \sigma(b) \wedge p_s \circ \sigma(d)\} \forall (b, d) \in V \times V$.

Example 2. The above definition is illustrated through an example, which is given in Figure 2.

Definition 9. Let $G = (V, \sigma, \mu)$ be an inverse mPFG. Then, the complement of G is indicated by $G^c = (V, \sigma^c, \mu^c)$, which is conferred by $p_s \circ \sigma^c(x) = p_s \circ \sigma(x)$ and $p_s \circ \mu^c(x, w) = 1 - p_s \circ \mu(x, w) + (p_s \circ \sigma(x) \wedge p_s \circ \sigma(w))$, for $s = 1(1)m$.

Example 3. The complement of the inverse 3PFG of Figure 1 is shown in Figure 3.

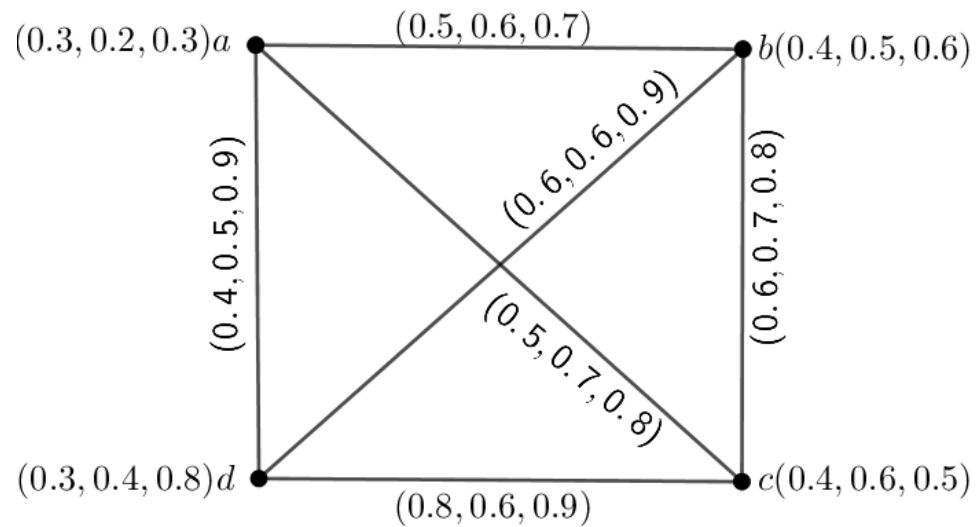


Figure 2. Complete inverse 3-polar fuzzy graph.

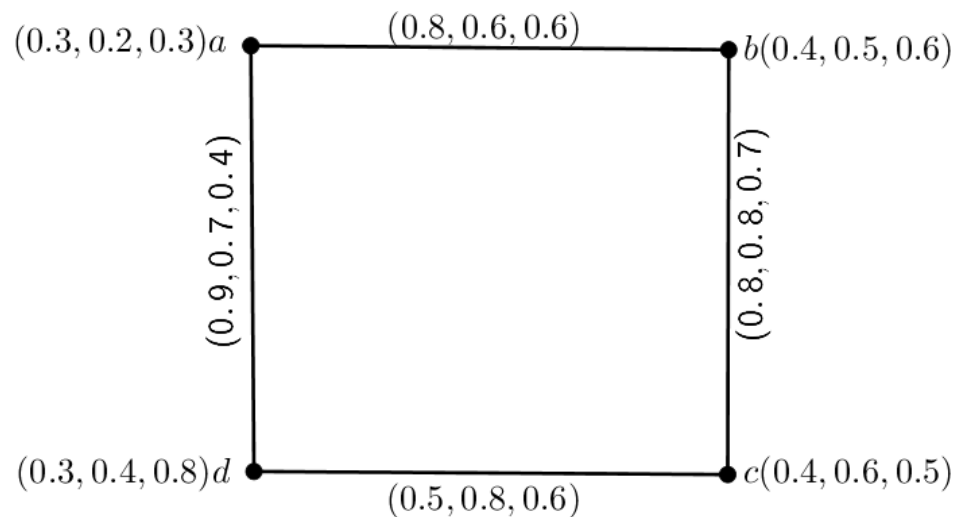


Figure 3. Complement of 3PFG of Figure 1.

Note 1: It is clear from the above discussion that the union of inverse mPFG and complement inverse mPFG does not form a complete inverse mPFG.

Definition 10. Suppose $G = (V, \sigma, \mu)$ and $G' = (V', \sigma', \mu')$ are the inverse mPFGs of the UCG $G^* = (V, E)$ and $G'^* = (V', E')$, respectively. An isomorphism is a bijection between G and G' , that is, $g : V \rightarrow V'$, and satisfies

$$p_s \circ \sigma(x) = p_s \circ \sigma'(g(x)) \text{ and } p_s \circ \mu(x, w) = p_s \circ \mu'(g(x), g(w))$$

$\forall x, w \in V$ and for every $s = 1(1)m$. Then, G is said to be isomorphic to G' .

Definition 11. Let $G = (V, \sigma, \mu)$ and $G' = (V', \sigma', \mu')$ be two inverse mPFGs. Then, G' is called a partial inverse mPFG of G if $V' \subseteq V$ and $p_s \circ \sigma'(x) \leq p_s \circ \sigma(x)$, for all $a \in V'$ as well as $p_s \circ \mu'(x, w) \leq p_s \circ \mu(x, w)$, for all $(x, w) \in E'$.

Definition 12. Let $G = (\tilde{V}, \sigma, \mu)$ and $G' = (V', \sigma', \mu')$ be two inverse mPFGs. Then, G' is called an inverse mPF subgraph of G if $V' \subseteq V$ and $p_s \circ \sigma'(x) = p_s \circ \sigma(x)$, for all $x \in V'$ as well as $p_s \circ \mu'(x, w) = p_s \circ \mu(x, w)$, for all $(x, w) \in E'$.

Definition 13. Let $G = (V, \sigma, \mu)$ be an inverse mPFG. Let $a \in V$. Then, the order of x is indicated by $d(a)$ and conferred by $p_s \circ d(x) = \sum_{(x,w) \in E} p_s \circ \mu(x,w)$, for $s = 1(1)m$. The total degree of the vertex x is indicated by $d_T(x)$ and is conferred by $p_s \circ d_T(x) = p_s \circ d(x) + p_s \circ \sigma(x)$, for $s = 1(1)m$. The degree of G is indicated by $d(G)$ and is conferred by $p_s \circ d(G) = \sum_{(x) \in V} p_s \circ \sigma(x)$, for $s = 1(1)m$. The size of G is indicated by $S(G)$ and conferred by $p_s \circ S(G) = \sum_{(x,w) \in E} p_s \circ \mu(x,w)$, for $s = 1(1)m$.

Definition 14. Let $G = (V, \sigma, \mu)$ be an inverse mPFG. Then, α -cut, $\alpha \in [0, 1]$, is indicated as G_α and is defined by $\sigma_\alpha = \{a \in V : p_s \circ \sigma(a) \geq \alpha, i = 1, 2, \dots, m\}$ and $\mu_\alpha = \{(x, w) \in E : p_s \circ \sigma(x, w) \geq \alpha, i = 1, 2, \dots, m\}$.

In ordinary mPFG theory, we know that every α -cut, $\alpha \in [0, 1]$, is a crisp graph. But, in our proposed model, this is not generally true; it is possible that not every α -cut, $\alpha \in [0, 1]$, of an inverse mPFG is a crisp graph. This is illustrated through Example 4.

Example 4. Let us consider $\alpha = (0.25, 0.25, 0.25)$. Then, the α -cut of the mPFG of Figure 1 is shown in Figure 4.

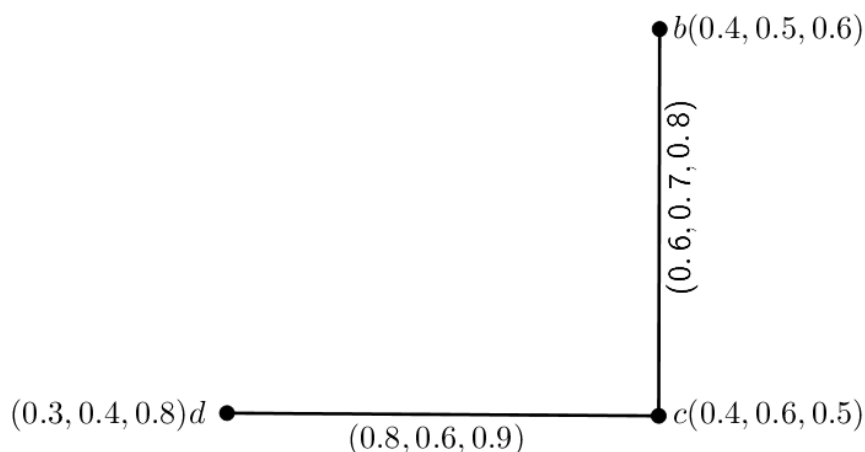


Figure 4. The 0.25-cut 3PFG of Figure 1.

Theorem 1. Let $G = (V, \sigma, \mu)$ be an inverse mPFG of an UCG and $\alpha \in [0, 1]$. If for any $a \in V$, $p_s \circ \sigma(a) \geq \alpha$ and $p_s \circ \mu(x, w) \geq \alpha$, for every $(x, w) \in E$ and for each $s = 1(1)m$, then G_α is a crisp graph.

Proof. Suppose $p_s \circ \mu(x, w) \geq \alpha$ for each $s = 1(1)m$. Then, we must have $p_s \circ \sigma(x) \geq \alpha$ and $p_s \circ \sigma(w) \leq \alpha$, for each $s = 1(1)m$, or $p_s \circ \sigma(x) \leq \alpha$ and $p_s \circ \sigma(w) \geq \alpha$, for each $s = 1(1)m$. It is clear for case 1.

For the second case, if $p_s \circ \sigma(x) \leq \alpha$ and $p_s \circ \sigma(w) \geq \alpha$, for $s = 1(1)m$, then we obtain $p_s \circ \mu(x, w) \leq \alpha$, for $s = 1(1)m$, which is a contradiction.

Hence, G_α is a crisp graph. $\square$

Theorem 2. Let $G = (V, \sigma, \mu)$ be an inverse mPFG of an UCG. If there exists a $\beta \in [0, 1]$ and an edge (x, w) such that $p_s \circ \mu(x, w) \geq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$, then G_β is not a crisp graph.

Proof. If possible, let $\forall \alpha \in [0, 1]$ G_α be a crisp graph. Then, we must have G_α as an mPFG. Then, from the definition of mPFG, we have $\forall s = 1(1)m$, $p_s \circ \mu(x, w) \leq \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\} \forall (x, w) \in \widetilde{V \times V}$. Again, from the definition of an inverse mPFG, we obtain $\forall s = 1(1)m$, $p_s \circ \mu(x, w) \geq \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\} \forall (x, w) \in \widetilde{V \times V}$.

Combining both of them, we obtain, $\forall s = 1(1)m$, $p_s \circ \mu(x, w) = \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\} \forall (x, w) \in \widetilde{V \times V}$, which is a contradiction, hence the theorem. $\square$

3.1. Decomposition and Composition of Inverse m -Polar Fuzzy Graph

3.1.1. Decomposition

By collecting each component from the inverse m -polar fuzzy graph, we can construct the m number of fuzzy graphs. An inverse m -polar fuzzy graph, G , is used for this purpose. The inverse m -polar fuzzy graph G can be created by building the graph G_i and edges using the i th component of the membership values of the nodes in the fuzzy graph $G_1, G_2, \dots, G_m$. The following example is used to demonstrate this concept.

Example 5. The above definition is depicted through an example, which is given in Figure 5.

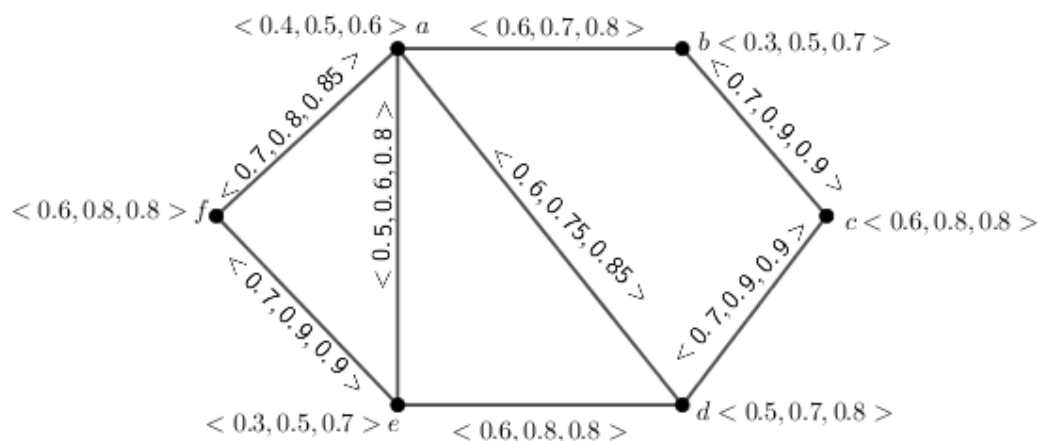


Figure 5. An Inverse 3-polar fuzzy graph illustrates the decomposition.

Here, we consider the inverse 3PFG shown in Figure 6. Taking the first component for each node and edge of G , we obtain an inverse fuzzy graph G_1 . Taking the second component for each node and edge of G , we obtain an inverse fuzzy graph G_2 . Finally, taking the third component for each node and edge of G , we obtain an inverse fuzzy graph G_3 . Hence, we obtain three inverse fuzzy graphs from G . All those inverse fuzzy graphs are shown in Figure 6.

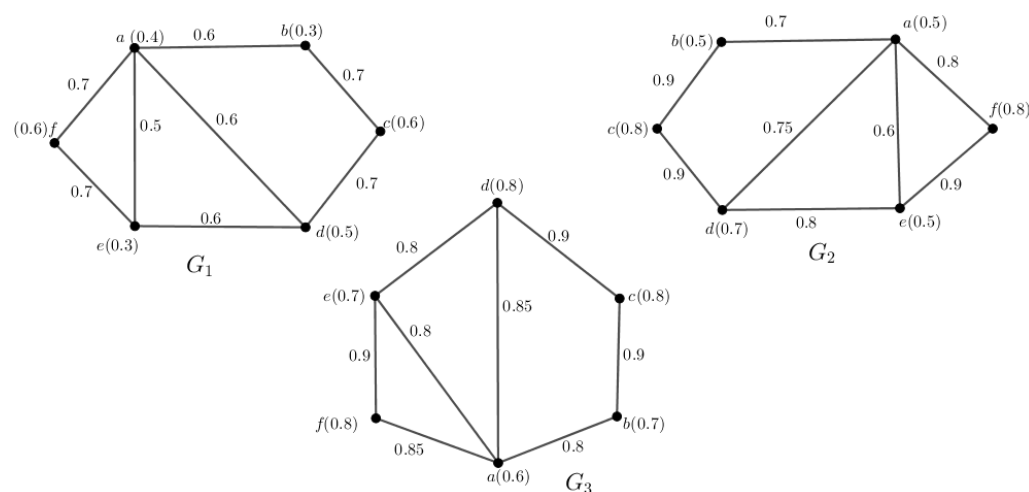


Figure 6. Inverse fuzzy graphs G_1, G_2, G_3 obtained by decomposition of 3PFG of Figure 5.

3.1.2. Composition

Think about the $G_1, G_2, \dots, G_m$, where m is the number of inverse fuzzy graphs. Their corresponding underlying crisp graphs $G_1^*, G_2^*, \dots, G_m^*$ are isomorphic. We can suppose that there are m spaces for m objects to be filled because we have m fuzzy graphs. The $m!$ technique can be used for this purpose. Thus, for $m!$ integers, we can obtain inverse m PFs. The following example illustrates this idea.

Example 6. To illustrate the above discussion, we consider three inverse fuzzy graphs G_1, G_2, G_3 , the crisp graphs of which are isomorphic, as shown in Figure 7.

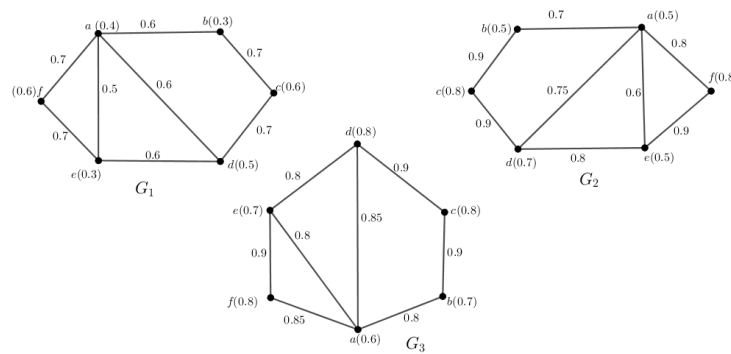


Figure 7. Inverse fuzzy graphs G_1, G_2, G_3 whose crisp graphs are isomorphic.

To convert the graph into an inverse 3PFG, we can imagine that there are three places that would be filled by three objects. This can be achieved in the following way:

$$\begin{aligned} H_1 &= G_1 G_2 G_3 \\ H_2 &= G_2 G_3 G_1 \\ H_3 &= G_3 G_1 G_2 \\ H_4 &= G_1 G_3 G_2 \\ H_5 &= G_2 G_1 G_3 \\ H_6 &= G_3 G_2 G_1 \end{aligned}$$

Hence, we obtain six inverse 3PFGs, which are shown in Figures 8–13.

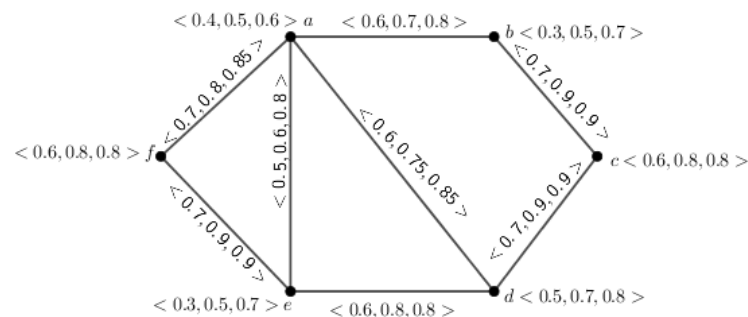


Figure 8. Inverse 3-polar fuzzy graph H_1 .

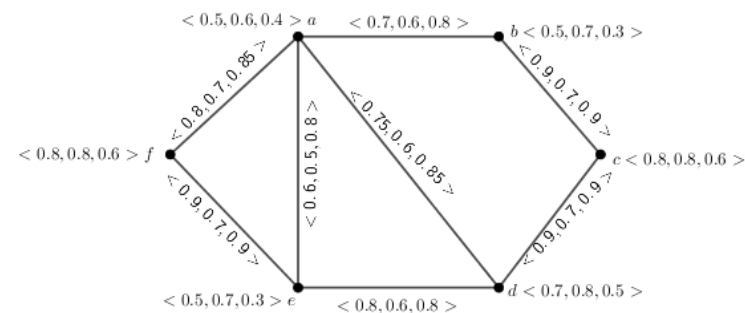
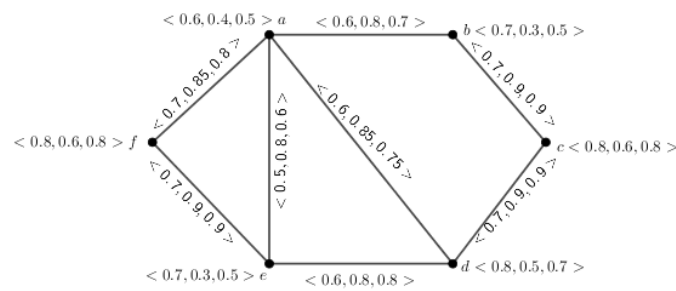
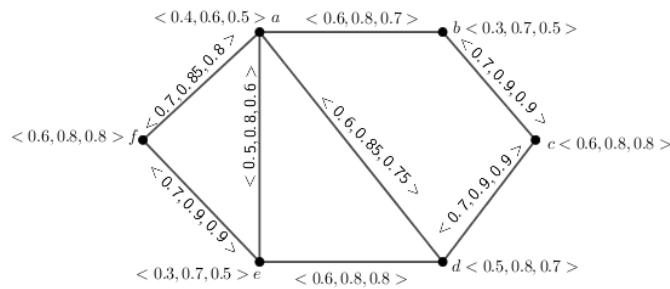
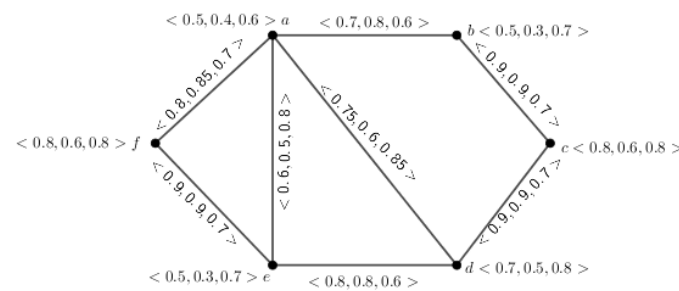
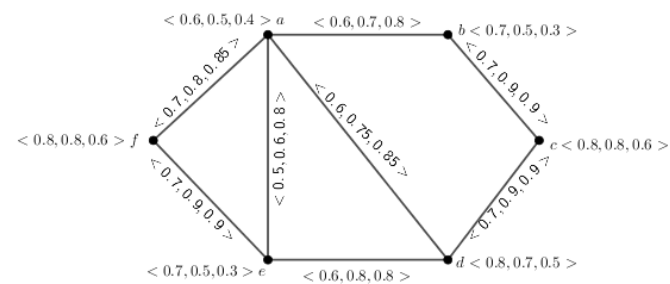


Figure 9. Inverse 3-polar fuzzy graph H_2 .

Figure 10. Inverse 3-polar fuzzy graph H_3 .Figure 11. Inverse 3-polar fuzzy graph H_4 .Figure 12. Inverse 3-polar fuzzy graph H_5 .Figure 13. Inverse 3-polar fuzzy graph H_6 .

4. Connectivity in Inverse m -Polar Fuzzy Graphs

Definition 15. Let $G = (V, \sigma, \mu)$ be an inverse m PFG. Then, a path P is an alternating sequence of distinct nodes $u_0, u_1, u_2, \dots, u_n$ such that $p_s \circ \mu(u_j, u_{j+1}) > 0$, for $s = 1(1)m$ and $j = 0(1)n$. The strength of a path is the minimum of the MV of each component that is $p_s \circ S(P_{(x,w)}) = \min_{d \in P} p_s \circ \sigma(d)$, for $s = 1(1)m$, for the path x to w . The strength of connectedness (SC) between two nodes, for example, x, w , is given as $p_s \circ \sigma_G^\infty(x, w) = \sup_{P_{(x,w)}} p_s \circ S(P_{(x,w)})$, for $s = 1(1)m$.

Note 2: A path between x and w is denoted as $P_{(x,w)}$.

Note 3: As $p_s \circ S(P_{(x,w)}) = \min_{d \in P} p_s \circ \sigma(d)$, for $s = 1(1)m$, we have $p_s \circ \sigma_G^\infty(x, w) \leq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Theorem 3. Let $G = (V, \sigma, \mu)$ be an inverse mPFG. If any two nodes x, w are connected through an edge, then $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Proof. To prove this theorem, we consider two cases.

Case 1: Let $P_{(x,w)}$ be the only path between x and w . Then, we can easily obtain that $p_s \circ \sigma_G^\infty(x, w) = p_s \circ S(P) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Case 2: Let there exist more than one path between x and w . Let $P'_{(x,w)}$ be another path between x and w . Suppose $d \in P'_{(x,w)}$.

Case 2.1: Let $p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Then, $p_s \circ S(P') \leq p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Hence, $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Case 2.2: Let $p_s \circ \sigma(d) \geq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Therefore, $p_s \circ S(P') = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Hence, $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Combining all these together, we can say that if any two nodes x, w are connected through an edge, then $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. $\square$

Definition 16. Let $G = (V, \sigma, \mu)$ be a connected inverse mPFG and $c \in V$ be any node. Then, c is called an inverse fuzzy cut node if there exist two distinct nodes d, e such that for $s = 1(1)m$,

$$p_s \circ \sigma_{G'}^\infty(d, e) < p_s \circ \sigma_G^\infty(d, e),$$

where $G' = G - \{c\}$. If G does not consist of any inverse mPF cut nodes, then it is said to be made of inverse mPF blocks.

Example 7. The above definition is depicted through an example in which we consider an inverse 3PFG shown in Figure 14.

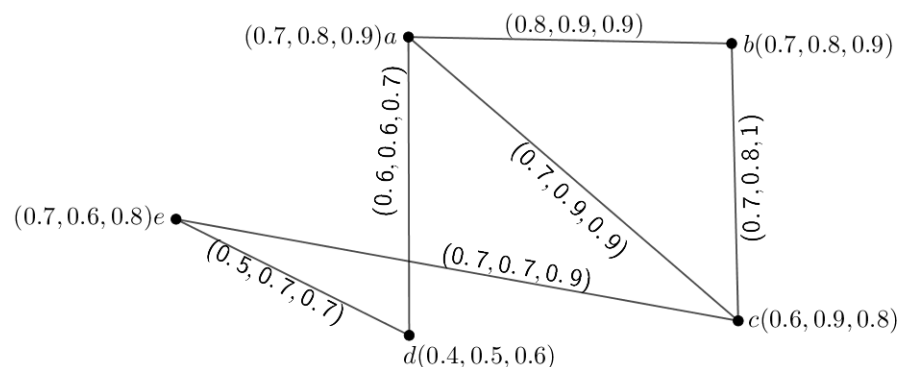


Figure 14. Inverse 3-polar fuzzy graph G to describe Definition 16.

Here, if we want to calculate the strength of connectedness between a and e , then we see that there are three paths from a to e . They are namely $a - b - c - e$, $a - d - e$, $a - c - e$. Now, the SC between a and e is $(0.6, 0.6, 0.8)$. If we delete the node c that supposes $G' = G - \{c\}$, then the SC between a and e in G' is $(0.4, 0.5, 0.6)$. Therefore, SC is decreased in G' . Hence, c is an inverse 3PF cut node.

Theorem 4. Let $G = (V, \sigma, \mu)$ be a complete inverse mPFG. Then, G is an inverse mPF block.

Proof. From Theorem 3, we have that if any two nodes x, w are connected through an edge, then $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Therefore, the removal of any nodes d other than x, w does not change the connectedness between x, w . Hence, d is not an inverse mPF cut node.

Again, in a complete inverse m PF G , $E = V \times V$, every two nodes are connected by an edge. So, no nodes are inverse m PF cut nodes of G . Therefore, G is an inverse m PF block.

The opposite of the aforementioned theorem might not be true, as shown by an example. $\square$

Example 8. The above theorem is depicted by an example for which we consider an inverse 3PFG shown in Figure 15.

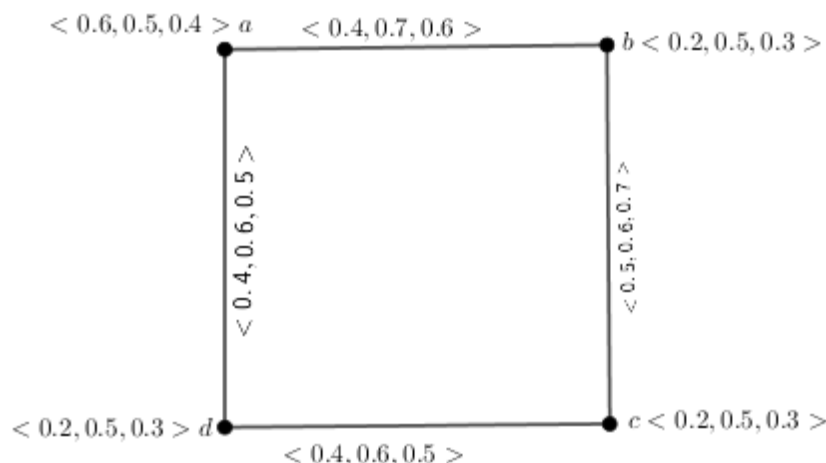


Figure 15. Inverse 3-polar fuzzy graph G to describe Example 8.

Here, we see that for the SC between any two nodes $x, w \in V$, we have $p_s \circ \sigma_G^\infty(x, w) = p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. So, the SC does not decrease if we remove any nodes in V . Hence, G has no 3PF cut nodes. Therefore, it is a 3PF block. Again, clearly, it is not an inverse complete 3PFG.

Definition 17. Let $G = (V, \sigma, \mu)$ be a connected inverse m PF G . Then, (u, v) is called an inverse m PF bridge if there exists two distinct nodes d, e such that for $s = 1(1)m$,

$$p_s \circ \sigma_{G'}^\infty(d, e) < p_s \circ \sigma_G^\infty(d, e),$$

where $G' = G - \{(u, v)\}$.

Example 9. The above definition is depicted through an example for which we consider an inverse 3PFG, shown in Figure 14.

We want to calculate the SC between a and e . We assume that there are three paths from a to e . These are $a - b - c - e$, $a - d - e$, and $a - c - e$. Now, the SC between a and e is $(0.6, 0.6, 0.8)$. If we delete the edge (c, e) , that is, if we suppose $G' = G - \{(c, e)\}$, then the SC between a and e in G' is $(0.4, 0.5, 0.6)$. This shows that SC is decreased in G' . Hence, (c, e) is an inverse 3PF bridge.

Theorem 5. Let $G = (V, \sigma, \mu)$ be a connected inverse m PF G and $(x, w) \in E$. Then, (x, w) is an inverse m PF bridge iff either (x, w) is a crisp bridge or for every path P between b and c of $G' = G - \{(x, w)\}$ there exists $d \in P$ such that $p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$.

Proof. Let (x, w) be an inverse m PF bridge and $G' = G - \{(x, w)\}$. Since (x, w) is an inverse m PF bridge, there thus exist two distinct nodes d, e such that for $s = 1(1)m$,

$$p_s \circ \sigma_{G'}^\infty(d, e) < p_s \circ \sigma_G^\infty(d, e).$$

If $p_s \circ \sigma_{G'}^\infty(d, e) = 0$, for $s = 1(1)m$, then d and e are not connected by any path in G' . Therefore, (x, w) is a crisp bridge.

If $p_s \circ \sigma_{G'}^\infty(d, e) > 0$ and $p_s \circ \sigma_{G'}^\infty(d, e) < p_s \circ \sigma_G^\infty(d, e)$, for $s = 1(1)m$, then (x, w) are connected by at least one path in G' . If possible, let u be such that $p_s \circ \sigma(u) \geq p_s \circ$

$\sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Then, $p_s \circ S(P') \geq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. So, $p_s \circ \sigma_{G'}^\infty(d, e) \geq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Since (x, w) is an inverse m PF bridge, therefore, it is an edge of every strong path from x to w . Hence, $p_s \circ \sigma_{G'}^\infty(d, e) \leq p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. So, $p_s \circ \sigma_{G'}^\infty(d, e) \geq p_s \circ \sigma_G^\infty(d, e)$, for $s = 1(1)m$, which is a contradiction. Hence, for every path P in between x and w of $G' = G - \{(x, w)\}$, there exists $d \in P$ such that $p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. $\square$

Theorem 6. Let $G = (V, \sigma, \mu)$ be an inverse m PFG having UCG $G^* = (V, E)$ and $(x, w) \in E$. Then, (x, w) is an inverse m PF bridge in G iff $p_s \circ \sigma_{G'}^\infty(x, w) < p_s \circ \sigma_G^\infty(x, w)$, for $s = 1(1)m$, where $G' = G - \{(x, w)\}$.

Proof. Let (x, w) be an inverse m PF bridge. Then, from Theorem 5, we find that either (x, w) is crisp bridge or in every path P between x and w in G' there exists a node d such that $p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)$, for $s = 1(1)m$. Hence, $p_s \circ S(P) \leq p_s \circ \sigma(d) < p_s \circ \sigma(x) \wedge p_s \circ \sigma(w) = p_s \circ \sigma_G^\infty(x, w)$, for $s = 1(1)m$. Therefore, $p_s \circ \sigma_{G'}^\infty(x, w) < p_s \circ \sigma_G^\infty(x, w)$, for $s = 1(1)m$, where $G' = G - \{(x, w)\}$.

Conversely, $p_s \circ \sigma_{G'}^\infty(x, w) < p_s \circ \sigma_G^\infty(x, w)$, for $s = 1(1)m$, where $G' = G - \{(x, w)\}$. Then, it is easy to verify that (x, w) is an inverse m PF bridge in G . $\square$

5. Application

In many real linked graphical systems where the vertices and edges are both part of an inverse m -polar fuzzy information, the inverse m PFG is a crucial mathematical structure that represents the information. In this section, we attempt to resolve a specific type of allocation problem utilizing a cut node in inverse m PFG.

5.1. Model Construction

Robotics is an important issue for every human body nowadays. The engineering field of robotics deals with the creation, design, production, and use of robots. The goal of the area of robotics is to develop smart machines that can help people in a number of ways. There are many different types of robotics. They boost productivity because robots are programmed to carry out repetitive activities endlessly, but the human brain is not. Robots are used in industries to carry out boring, redundant jobs, freeing up workers to take on more difficult duties and even pick up new abilities.

Here, five towns $a_1, a_2, \dots, a_5$ are regarded as nodes. If there is a common worker between two nodes, there will be an edge between them. The allocation problem is solved using an inverse 3PFG $G = (V, \sigma, \mu)$. In this model, the first component is the communication system, the second is the pollution-free zone, and the third is the worker's availability. This is an inverse 3PFG model. Here, every advantage is based on the factors of transportation costs, labor resources, and workers who service both locations. Therefore, the MVs of each node's components are listed in Table 2.

Table 2. Vertex membership values of G .

Node	Membership Values
a_1	(0.4, 0.5, 0.3)
a_2	(0.5, 0.4, 0.2)
a_3	(0.3, 0.2, 0.1)
a_4	(0.4, 0.5, 0.3)
a_5	(0.7, 0.5, 0.2)
a_6	(0.3, 0.5, 0.1)
a_7	(0.6, 0.5, 0.4)
a_8	(0.7, 0.6, 0.2)

An m PFG is symmetric if and only if $\mu(a, b) = \mu(b, a)$, where $\mu(a, b)$ denotes the MV of the edge (a, b) . The 3PFS is $E = \{(a_1, a_2), (a_1, a_6), (a_1, a_8), (a_2, a_3), (a_3, a_4), (a_4, a_5), (a_4, a_7), (a_5, a_7), (a_5, a_6)\}$. The MVs of the edges are taken into account based on factors such as

transportation costs, the availability of labour, and the number of workers who are working in both locations. The MVs of edges are taken into consideration in Table 3, for instance.

Table 3. Edge membership values of G .

Edge	Membership Values
(a_1, a_2)	$(0.3, 0.4, 0.2)$
(a_1, a_6)	$(0.3, 0.3, 0.1)$
(a_1, a_8)	$(0.3, 0.3, 0.2)$
(a_2, a_3)	$(0.2, 0.2, 0.1)$
(a_2, a_8)	$(0.5, 0.4, 0.2)$
(a_3, a_4)	$(0.3, 0.2, 0.1)$
(a_3, a_7)	$(0.3, 0.1, 0.1)$
(a_4, a_5)	$(0.4, 0.5, 0.1)$
(a_4, a_7)	$(0.4, 0.4, 0.2)$
(a_5, a_6)	$(0.2, 0.2, 0.1)$
(a_5, a_7)	$(0.6, 0.4, 0.2)$
(a_6, a_7)	$(0.3, 0.4, 0.1)$
(a_7, a_8)	$(0.5, 0.4, 0.2)$

The model inverse 3PFG is given in Figure 16.

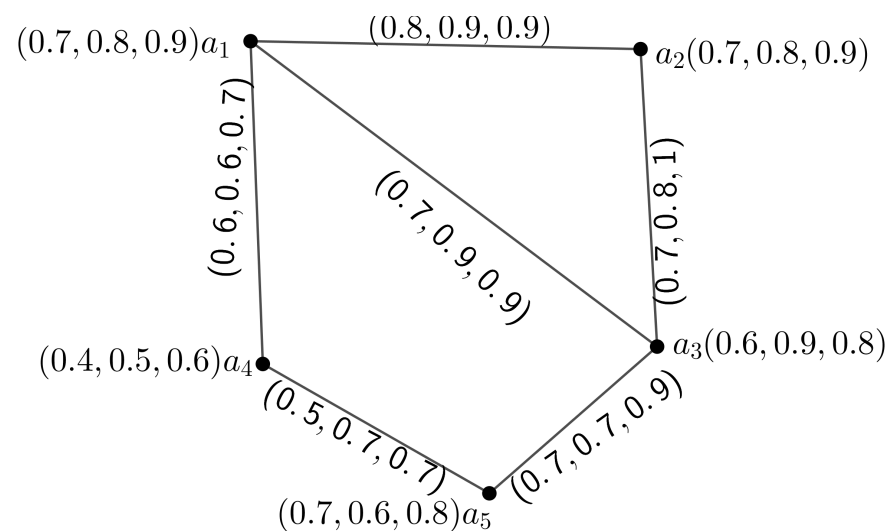


Figure 16. Model inverse 3-polar fuzzy graph G .

Here, an algorithm is given in Algorithm 1 to find the best suitable location for robotics manufacturing allocation.

Algorithm 1: An algorithm to find the best suitable location for robotic manufacturing allocation.

Input: An inverse m PFG $G = (V, \sigma, \mu)$.

Output: Find the best suitable location for robotic manufacturing allocation.

Step 1: Put the membership value of vertices $a_j, j = 1, 2, \dots, n$.

Step 2: Put the membership value of edges which satisfied

$$p_i \circ \mu(a_j, a_k) \geq \inf\{p_i \circ \sigma(a_j), p_i \circ \sigma(a_k)\}, i = 1, 2, \dots, m.$$

Step 3: Find the path between any two nodes of G .

Step 4: Find the SC between any path of G .

Step 5: Find the cut-vertex of G .

5.2. Illustration of Membership Values

The paths between the vertices a_1 and a_5 in G are $P_{(a_1,a_5)}^1 : a_1 - a_2 - a_3 - a_5$, $P_{(a_1,a_5)}^2 : a_1 - a_3 - a_5$, and $P_{(a_1,a_5)}^3 : a_1 - a_4 - a_5$, and their respective strengths using the formula $p_s \circ S(P_{(u,v)}^j) = \min_{w \in P} \{p_s \circ \sigma(w)\}$, for $i = 1(1)3$ (where j indicates the path number), are $S(P_{(a_1,a_5)}^1) = (0.6, 0.6, 0.8)$, $S(P_{(a_1,a_5)}^2) = (0.6, 0.6, 0.8)$, and $S(P_{(a_1,a_5)}^3) = (0.4, 0.5, 0.6)$. Therefore, the SC between the vertices a_1 and a_5 is $\sigma_G^\infty(a_1, a_5) = (0.6, 0.6, 0.8)$.

Similarly, the paths between the vertices a_1 and a_4 in G are $P_{(a_1,a_4)}^1 : a_1 - a_2 - a_3 - a_5 - a_4$, $P_{(a_1,a_4)}^2 : a_1 - a_3 - a_5 - a_4$, and $P_{(a_1,a_4)}^3 : a_1 - a_4$, and their respective strengths are $S(P_{(a_1,a_4)}^1) = (0.4, 0.5, 0.6)$, $S(P_{(a_1,a_4)}^2) = (0.4, 0.5, 0.6)$, and $S(P_{(a_1,a_4)}^3) = (0.4, 0.5, 0.6)$. Therefore, the SC between the vertices a_1 and a_4 is $\sigma_G^\infty(a_1, a_4) = (0.4, 0.5, 0.6)$. So, the SCs of all pairs of vertices are given in Table 4.

Table 4. SC of each pair of vertices of G .

Pair of Vertices	SC
(a_1, a_5)	$(0.6, 0.6, 0.8)$
(a_1, a_4)	$(0.4, 0.5, 0.6)$
(a_1, a_3)	$(0.6, 0.8, 0.8)$
(a_1, a_2)	$(0.7, 0.8, 0.9)$
(a_2, a_5)	$(0.6, 0.6, 0.8)$
(a_2, a_4)	$(0.4, 0.5, 0.6)$
(a_2, a_3)	$(0.6, 0.8, 0.8)$
(a_3, a_5)	$(0.6, 0.6, 0.8)$
(a_3, a_4)	$(0.4, 0.5, 0.6)$
(a_4, a_5)	$(0.4, 0.5, 0.6)$

Now, if we delete the vertex a_3 in G , then we have a different figure, say $G' = G - \{a_3\}$, which is shown in Figure 2. Thereafter, we determined the SC in between each pair of vertices in G' in the following way:

The path between the vertices a_1 and a_5 in G' is $a_1 - a_4 - a_5$, with an SC $\sigma_{G'}^\infty(a_1, a_5) = (0.4, 0.5, 0.6)$. Similarly, the SCs of the other pairs in G' are $\sigma_{G'}^\infty(a_2, a_5) = (0.4, 0.5, 0.6)$ and $\sigma_{G'}^\infty(a_4, a_5) = (0.4, 0.5, 0.6)$. Clearly, we can see that the SC in G is decreased in G' . Hence, a_3 is an inverse 3PF cut node.

5.3. Decision Making

We may conclude that town a_3 is the best-suited location to develop a robotics manufacturing facility among all the towns taken into consideration in our suggested model, since it is the only cut node in the model's inverse 3PFG G . Through the description above, we may infer that the cut node in the inverse m PFG truly matters in this kind of allocation problem. In addition, we acknowledge that cut nodes in inverse m PFGs are more suitable for allocation problems than cut nodes in inverse FGs.

6. Comparative Study

At first, Borzooei et al. [12] introduced an inverse graph for fuzzy graph theory. In their model, they only considered a single component for each node as well as edges. Next, Poulik and Ghorai [13] studied inverse graphs for the mixed fuzzy graph. They considered directed as well as undirected edges in a fuzzy graph. They considered only a single component as a membership value of a vertex and edge. All of them studied inverse graphs for fuzzy graph theory. So, none of the results discussed earlier are applicable when the model is considered in another environment, like in inverse m -polar fuzzy sets. This is why the proposed model in this paper plays a significant role in obtaining better results in such situations.

7. Advantages and Limitations of the Proposed Work

The following are some of the advantages of the proposed work:

- (i) Anyone can analyze the membership values in a multi-polar inverse fuzzy environment in a certain way.
- (ii) Many important definitions and theorems are presented in this study that are very useful.
- (iii) A real application of an inverse m -polar fuzzy model using the concept of SC is described, through which we solve an allocation problem called robotic manufacturing allocation.

Some of the limitations of this study are given as follows:

- (i) This work mainly focuses on the SC concept in inverse m -polar fuzzy environments.
- (ii) If the membership values are available as the interval-valued m -polar fuzzy environments, then the proposed method is not applicable.
- (iii) Negative membership values are also not investigated in the proposed study.

8. Conclusions

The inverse m PFG and some of its fascinating facts have been introduced in this article. A few inverse m PFG complements and isomorphic qualities have been introduced. Here, the idea of the alpha-cut has also been researched. Here, we also provide a highly specialized definition for the composition and decomposition of an inverse m PFG. The idea of connectivity, which includes connection strength, cut nodes, bridges, etc., is likewise established in an inverted m PFG context. Finally, using inverse m PFG, a real-world application based on a production allocation problem is resolved. We shall extend our investigation on inverse graphs to different kinds of fuzzy models so that we can solve a lot of real-life problems with their help.

Author Contributions: Conceptualization, G.M. and M.P.; methodology, T.M. and A.M.A.; software, A.M.A and Z.B.; validation, T.M.; investigation, G.M.; data curation, T.M., A.M.A. and Z.B.; visualization, M.P. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for funding this work through research group no. S-0140-1443.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the reviewers for their valuable comments and suggestions to improve the quality of the article. The authors extend their appreciation to the Deanship of Scientific Research at University of Tabuk for funding this work through research group no. S-0140-1443.

Conflicts of Interest: The authors declare no conflicts of interest.

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