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The Modified-Lomax Distribution: Properties, Estimation Methods, and Application

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Abstract: This paper introduces a flexible three-parameter extension of the Lomax model called the odd Lomax–Lomax (OLxLx) distribution. The OLxLx distribution can provide left-skewed, symmetrical, right-skewed, and reversed-J shaped densities and increasing, constant, unimodal, and decreasing hazard rate shapes. Some mathematical properties of the introduced model are derived. The OLxLx density can be expressed as mixture of Lomax densities. The OLxLx parameters are estimated by using eight estimation methods and their performance is explored by using detailed simulation studies. The partial and overall ranks of the mean relative errors, absolute biases, and mean square errors of different estimators are presented to choose the best estimation method. The flexibility and applicability of the OLxLx distribution is shown using real-life medicine data, illustrating the superior fit of the OLxLx distribution over other competing Lomax distributions. The OLxLX distribution outperforms some rival Lomax distributions including the Kumaraswamy–Lomax, McDonald–Lomax, Weibull–Lomax, transmuted Weibull–Lomax, exponentiated-Lomax, Lomax–Weibull, modified Kies–Lomax, Burr X Lomax, beta exponentiated-Lomax, odd exponentiated half-logistic Lomax, and transmuted-Lomax distributions, among others.

Keywords: estimation methods; Lomax distribution; simulation; odd Lomax generator; generating function; statistical model; Cramér-von Mises estimation

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1. Introduction

The choice of appropriate distributions to be used on real-life data plays a fundamental role in improving the power, efficiency, and sensitivity of statistical tests. This is so because appropriate distributions lead to a good fit of the data. Therefore, good knowledge of the appropriate distribution to be used for a specific data set is essential.

Probability distributions are very important in data analysis. They can be used to model wide range of data shapes in applied fields. The Lomax (Lx) (also known as Pareto II) distribution has many applications in several applied areas such as income and wealth inequality, biological sciences, lifetime and reliability, engineering, and actuarial sciences. The Lx distribution has been applied in modeling real-life data in income and wealth, firm size, and reliability and life testing (see [1–4]). Chahkandi and Ganjali [5] showed that the Lx distribution belongs to decreasing hazard rate (HR) family. More information about the Lx distribution can be explored in [6–8].

The procedure of adding new shape parameters for generalizing classical distributions is a well-known technique in the statistical literature. Hence, there are several extensions of the Lx distribution which are developed using well-known families to improve its flexibility and applicability in modeling different types of data. For example, Gupta et al. [9] introduced the exponentiated-Lomax, the Marshall–Olkin Lomax was proposed by [10], the Kumaraswamy–Lomax by [11], the Weibull–Lomax was studied by [8], the exponentiated half logistic-Lomax was introduced by [12], the Fréchet Topp–Leone Lomax was proposed by [13], and the generalized linear failure rate Lomax by [14].

Recently, developing new generators of distributions by adding one or more extra shape parameters to the well-known distributions has received great interest among statisticians. One of the most notable families is called the odd Lomax-G (OLx-G) family, which was introduced by Cordeiro et al. [15].

In this paper, a new flexible three-parameter Lx extension called the odd Lomax–Lomax (OLxLx) distribution is introduced to improve the flexibility of the Lx distribution for modeling real-life data. The OLxLx distribution is generated by replacing the baseline one-parameter Lx distribution in the OLx-G family. The OLxLx distribution can provide better fit than fourteen existing competing Lx extensions which contain four or five parameters. The three-parameter OLxLx distribution provides constant, increasing, unimodal, decreasing, and reversed-J shaped failure rate functions, as well as left-skewed, symmetrical, right-skewed, and reversed-J shaped densities. Another important goal of this paper is to explore the estimation of the OLxLx parameters via several classical methods including the maximum likelihood, Cramér–von Mises, Anderson–Darling, least-squares, right-tail Anderson–Darling, weighted least-squares, maximum product of spacings, and percentiles based estimation methods. Furthermore, the performance of these estimation approaches is investigated using detailed simulation results based on the average values of mean square error (MSE), absolute biases ($|BIAS|$), and mean relative errors (MRE) of the estimates. Additionally, the three measures, MSE, $|BIAS|$, and MRE, are ordered based on partial and overall ranks in order to compare the performances of the introduced estimators as well as to determine the best estimation approach for the unknown parameters of the OLxLx distribution.

The rest of the paper is outlined as follows. The OLxLx distribution is presented in Section 2. In Section 3, the basic properties of the OLxLx distribution are determined. Eight estimation methods are explored in Section 4. Detailed simulations for the studied estimation methods are presented in Section 5. The applicability of the OLxLx model is studied using real-life data in Section 6. Finally, the conclusions are provided in Section 7.

2. The OLxLx Distribution

In this section, we define the OLxLx distribution based on the OLx-G family [15].

The cumulative distribution function (CDF) of the OLx-G family is defined (for $x > 0$) by

$$F(x;a,b,\phi) = a\,b^a \int_0^{\frac{G(x;\phi)}{1-G(x;\phi)}} (b+t)^{-a-1}\,dt = 1 - b^a \left[b + \frac{G(x;\phi)}{1-G(x;\phi)}\right]^{-a}. \tag{1}$$

The probability density function (PDF) of the OLx-G family reduces to

$$f(x;a,b,\phi) = \frac{a\,b^a g(x;\phi)}{[1-G(x;\phi)]^2}\left[b + \frac{G(x;\phi)}{1-G(x;\phi)}\right]^{-a-1}, \tag{2}$$

where $g(x;\phi) = dG(x;\phi)/dx$. The OLx-G family reduces to the Marshall-Olkin-G (MO-G) family (Marshall and Olkin, [16]) for $a = 1$.

The HR function (HRF) of the OLx-G follows as

$$h(x;a,b,\phi) = \frac{a\,\tau(x;\phi)}{b + (1-b)G(x;\phi)}.$$

The quantile function (QF) of the OLx-G takes the form

$$Q(u) = F^{-1}(u) = Q_G\left\{\frac{b - b(1-u)^{\frac{1}{a}}}{b + (1-b)(1-u)^{\frac{1}{a}}}\right\}, \tag{3}$$

where $Q_G(u) = G^{-1}(u)$ is the baseline QF of any G distribution and $u \in (0,1)$.

The one-parameter Lx distribution with shape parameter θ is specified by the CDF

$$G(x;\theta) = 1 - (1+x)^{-\theta},\ x > 0\ \theta > 0. \tag{4}$$

The PDF of the Lx distribution reduces to

$$g(x;\theta) = \theta(1+x)^{-\theta-1},\ x > 0\ \theta > 0. \tag{5}$$

By inserting (4) in (1), the CDF of the OLxLx distribution follows as

$$F(x;\theta,a,b) = 1 - b^a\Big[b + (1+x)^\theta - 1\Big]^{-a},\ x > 0\ a,b,\theta > 0. \tag{6}$$

The PDF of the OLxLx distribution follows, by inserting (4) and (5) in Equation (2), as

$$f(x;\theta,a,b) = a\,\theta\,b^a(1+x)^{\theta-1}\Big[b + (1+x)^\theta - 1\Big]^{-a-1},\ x > 0\ a,b,\theta > 0. \tag{7}$$

The survival function (SF) and HRF of the OLxLx distribution are given by

$$S(x;\theta,a,b) = b^a\Big[b + (1+x)^\theta - 1\Big]^{-a}, \tag{8}$$

$$h(x;\theta,a,b) = a\,\theta\,(1+x)^{\theta-1}\Big[b + (1+x)^\theta - 1\Big]^{-1}. \tag{9}$$

The two-parameter Marshall–Olkin Lomax (TPMOLx) distribution follows from (7) as a special case of the OLxLx distribution for $a = 1$. Possible shapes for the PDF and HRF of the OLxLx distribution are given in Figures 1 and 2. The plots in these figures illustrate that the OLxLx distribution can provide symmetrical, right-skewed, left-skewed, and J shaped densities. Furthermore, the OLxLx HRF can be constant, increasing, decreasing, and unimodal shaped.

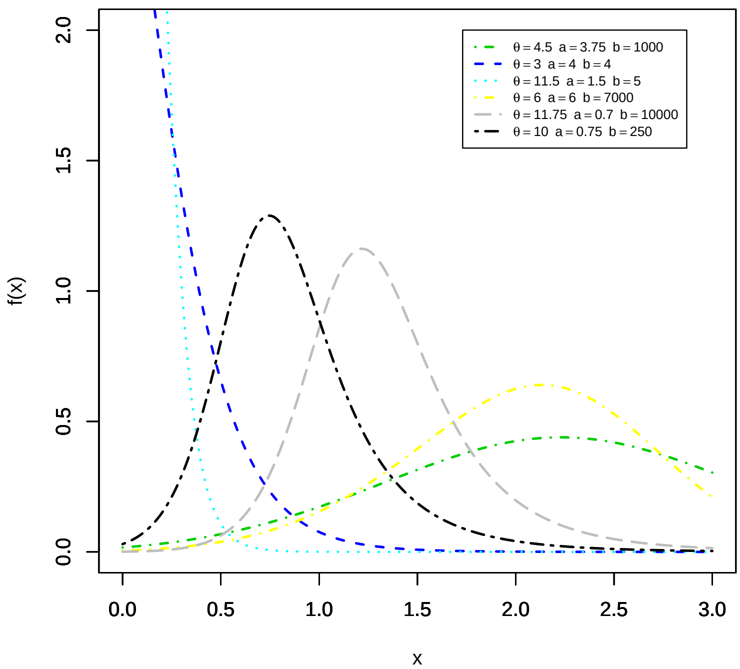

Figure 1. Possible PDF curves of the OLxLx distribution for several parametric values.

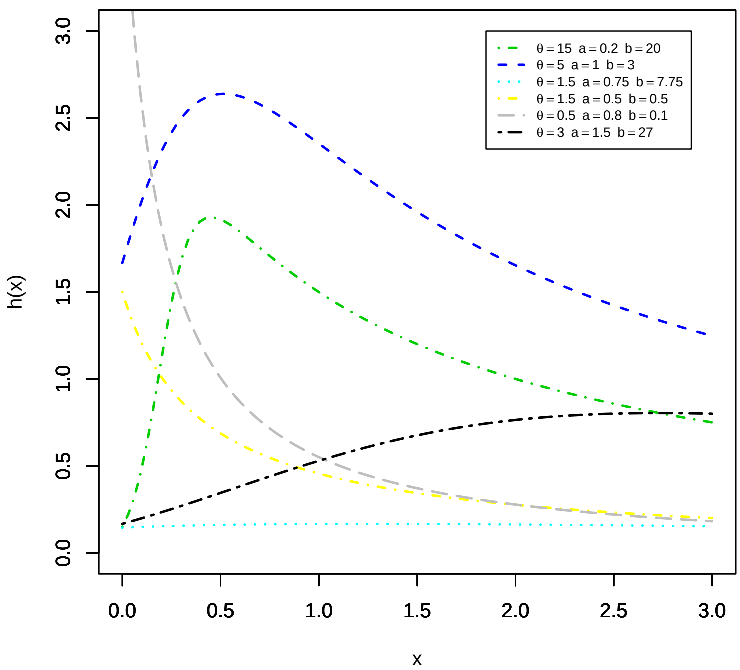

Figure 2. Possible HRF curves of the OLxLx distribution for several parametric values.

3. Mathematical Properties

This section provides some statistical properties of the newly introduced model.

3.1. Linear Expansion

A useful linear mixture representation for the PDF (7) of the OLxLx distribution is derived. Cordeiro et al. [15] provided a linear representation of the OLx-G CDF (2) as follows

$$f(x) = \sum_{i,j=0}^{\infty} v_{i,j}\, h_{i+j+1}(x), \tag{10}$$

where

$$v_{i,j} = \frac{(-1)^j\, a}{(i+j+1)\, b^{i+1}} \binom{-a-1}{k} \binom{-i-2}{j},$$

and $h_\delta(x) = \delta\, g(x)\, G(x)^{\delta-1}$ denotes the exponentiated-G (Exp-G) family density with power parameter $\delta > 0$.

By inserting (4) and (5) in Equation (10), we obtain the linear representation of the OLxLx PDF in terms of exponentiated-Lomax (ExLx) model as follows

$$f(x) = \sum_{i,j=0}^{\infty} v_{i,j}\, (i+j+1)\theta(1+x)^{-\theta-1}\Big[1-(1+x)^{-\theta}\Big]^{i+j}. \tag{11}$$

For $|z| < 1$, the binomial power series holds

$$(1-z)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k. \tag{12}$$

Applying the power series (12) to (11) yields

$$f(x) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} (-1)^k \binom{i+j}{k} v_{i,j}\, (i+j+1)\theta(1+x)^{-\theta(k+1)-1}. \tag{13}$$

Equation (13) can be simplified in the following form

$$f(x) = \sum_{k=0}^{i+j} \gamma_k \theta(k+1)(1+x)^{-\theta(k+1)-1} = \sum_{k=0}^{i+j} \gamma_k g_{\theta(k+1)}(x), \tag{14}$$

where $g_{\theta(k+1)}(x)$ is the Lx PDF with parameter $\theta(k+1)$ and γ_k is the constant term which is defined by

$$\gamma_k = \sum_{i,j=0}^{\infty} \frac{(-1)^{j+k}\, a}{(k+1)\, b^{i+1}} \binom{-a-1}{k} \binom{-i-2}{j} \binom{i+j}{k}.$$

Hence, the OLxLx PDF can be expressed in terms of a linear combination of Lx densities. Then, several mathematical quantities of the OLxLx model are determined from those quantities of the Lx distribution. Equation (14) is the main result of this section.

3.2. Quantile Function

The QF of the OLxLx distribution follows by determining the inverse function of its CDF (6) as

$$Q(u) = \left\{1 - \frac{b - b(1-u)^{\frac{1}{a}}}{b + (1-b)(1-u)^{\frac{1}{a}}}\right\}^{-1/\theta} - 1, \quad 0 < u < 1. \tag{15}$$

The first three quartiles of the OLxLx distribution are obtained directly by setting $u = 0.25, 0.5$, and 0.75, respectively, in Equation (15).

Let $u \sim \mathrm{U}(0,1)$, then the QF can be adopted in generating random data from the OLxLx distribution as follows:

$$x_i = \left\{1 - \frac{b - b(1-u_i)^{\frac{1}{a}}}{b + (1-b)(1-u_i)^{\frac{1}{a}}}\right\}^{-1/\theta} - 1, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

3.3. Moments

The rth moments of the OLxLx distribution take the form

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f(x)\, dx = \sum_{k=0}^{i+j} \gamma_k \int_0^{\infty} x^r g_{\theta(k+1)}(x) dx.$$

Hence, the rth moments of the OLxLx distribution can be determined, from the moments of Lx distribution, as

$$\mu'_r = \sum_{k=0}^{i+j} \gamma_k \frac{\theta(k+1)\Gamma(r+1)\Gamma(\theta(k+1)-r)}{\Gamma(\theta(k+1)+1)}. \tag{16}$$

The first four moments of the OLxLx distribution can be obtained simply by setting $r = 1, 2, 3$, and 4 in (16).

The moment-generating function of the OLxLx distribution follows as

$$M(t) = \sum_{k=0}^{i+j} \gamma_k \frac{t^r}{r!} \int_0^{\infty} x^r g_{\theta(k+1)}(x) dx = \sum_{k=0}^{i+j} \gamma_k \frac{t^r}{r!} \frac{\theta(k+1)\Gamma(r+1)\Gamma(\theta(k+1)-r)}{\Gamma(\theta(k+1)+1)}.$$

The characteristic function of the OLxLx distribution follows from the previous equation with $t = it$.

3.4. Incomplete Moments

The sth incomplete moment of the OLxLx distribution can be defined as

$$\Psi_s(t) = \int_0^t x^s f(x)dx = \sum_{k=0}^{i+j} \gamma_k \int_0^t x^s g_{\theta(k+1)}(x)dx$$
$$= \sum_{k=0}^{i+j} \gamma_k \theta(k+1)(-1)^{1-s} B_{-t}(s+1, -\theta(k+1)),$$

where $B_z(a,b) = \int_0^z y^{a-1}(1-y)^{b-1}\,dy$.

The first incomplete moment, $\Psi_1(t)$, has some important applications, where it can be used to calculate some useful quantities such as the Bonferroni, $L(p) = \Psi_1(t)/\mu_1'$, and Lorenz curves, $B(p) = \Psi_1(x_p)/(p\mu_1')$, where x_p is evaluated numerically using Equation (16) for a given probability p. These curves have their importance in insurance, economics, engineering, demography, and medicine. Additionally, the $\Psi_1(t)$ is useful in calculating the mean residual life (MRL), $m_1(t) = [1 - \Psi_1(t)]/S(t) - t$, and mean waiting time, $M_1(t) = t - \Psi_1(t)/F(t)$.

3.5. Order Statistics

Cordeiro et al. [15] provided a simple formula for the PDF of the ith order statistic, say $X_{i:n}$, of the OLx-G family as a linear combination of Exp-G densities. This PDF takes the form

$$f_{i:n}(x) = \sum_{r,s=0}^{\infty} \delta_{r,s}\, h_{s+r+1}(x), \tag{17}$$

where

$$\delta_{r,s} = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \frac{(-1)^{k+s} a\, b^{-r-1}}{\mathrm{B}(i, n-i+1)} \binom{n-i}{j} \binom{-r-2}{s} \times \binom{j+i-1}{k} \binom{-(k+1)\alpha - 1}{r},$$

and $h_{s+r+1}(x)$ is the Exp-G density with power parameter $s+r+1$.

Then, by inserting Equations (4) and (5) in (17), the PDF of $X_{i:n}$ for the OLxLx distribution follows as

$$f_{i:n}(x) = \sum_{r,s=0}^{\infty} \delta_{r,s}\,(s+r+1)\theta(1+x)^{-\theta-1}\left[1-(1+x)^{-\theta}\right]^{s+r}. \tag{18}$$

Applying the power series (12) to (18), we obtain the PDF of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{r,s=0}^{\infty} \sum_{m=0}^{s+r} \delta_{r,s} \binom{s+r}{m} (-1)^m (s+r+1)\theta(1+x)^{-\theta(m+1)-1} = \sum_{m=0}^{s+r} \vartheta_m\, g_{\theta(m+1)}(x), \tag{19}$$

where $g_{\theta(m+1)}(x)$ is the PDF of the Lomax distribution with parameter $\theta(m+1)$ and

$$\vartheta_m = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i-1} \sum_{r,s=0}^{\infty} \frac{(-1)^{k+s+m} a\, b^{-r-1}(s+r+1)}{(m+1)\,\mathrm{B}(i, n-i+1)} \binom{n-i}{j} \binom{-r-2}{s} \times \binom{j+i-1}{k} \binom{-(k+1)\alpha-1}{r} \binom{s+r}{m}.$$

Based on Equation (19), some mathematical properties of $X_{i:n}$ follow from those Lomax properties, such as the qth moment of $X_{i:n}$.

The qth moment of $X_{i:n}$ of the OLxLx distribution follows as

$$E\left(X_{i:n}^q\right) = \sum_{m=0}^{s+r} \vartheta_m \frac{\theta(m+1)\Gamma(q+1)\Gamma(\theta(m+1)-q)}{\Gamma(\theta(m+1)+1)}. \tag{20}$$

The joint PDF of the ith and sth order statistics is defined by

$$f_{i,s:n}(x,y) = \frac{n!\, f(x)\, f(y)}{(i-1)!\,(n-s)!\,(s-i-1)!}[F(x)]^{i-1}\,[1-F(y)]^{n-s}\,[F(y)-F(x)]^{s-i-1},\ 1 \leqslant i < s \leqslant\ n.$$

Using Equations (4) and (5), the joint PDF of two-order statistics from the OLxLx distribution has the form

$$f_{i,s:n}(x,y) = \frac{n!\,\theta^2\,(1+x)^{-\theta-1}(1+y)^{-\theta(n-s+1)-1}}{(i-1)!\,(s-i-1)!\,(n-s)!}\left[1-(1+x)^{-\theta}\right]^{i-1}\left[(1+y)^{-\theta})-(1+x)^{-\theta})\right]^{s-i-1}.$$

4. Estimation Methods

This section presents different estimation approaches of the OLxLx parameters such as the maximum likelihood estimates (MLEs), maximum product of spacings estimates (MPSEs), Anderson–Darling estimates (ADEs), Cramér–von Mises estimates (CVMEs), least-squares estimates (LSEs), right-tail Anderson–Darling estimates (RADEs), weighted least squares estimates (WLSEs), and percentiles estimates (PCEs).

4.1. Maximum Likelihood

Consider a random sample of size n, say $x_1, x_2, \ldots, x_n$, from the PDF (7), then the log-likelihood function, ℓ, reduces to

$$\ell = n\log\left(a\,\theta\,b^{-a}\right) + (\theta-1)\sum_{i=1}^{n}\log(1+x_i) - (a+1)\sum_{i=1}^{n}\log\left[b+(1+x_i)^{\theta}-1\right]. \tag{21}$$

Differentiating Equation (21) with respect to θ, a, and b and equating to zero, yield the following equations

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n}\log(1+x_i) - (a+1)\sum_{i=1}^{n}\frac{(1+x_i)^{\theta}\log(1+x_i)}{b+(1+x_i)^{\theta}-1} + \frac{n}{\theta} = 0,$$

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\log\left[b+(1+x_i)^{\theta}-1\right] = 0$$

and

$$\frac{\partial\ell}{\partial b} = \sum_{i=1}^{n}\frac{-a-1}{b+(1+x_i)^{\theta}-1} - \frac{a\,n}{b} = 0.$$

Solving the three previous equations gives the MLEs of the OLxLx parameters. These three equations cannot be solved explicitly. Hence, the numerical techniques can be adopted to maximize the log-likelihood function to obtain the MLEs via several software such as the Mathematics, R, Mathcad, and SAS.

4.2. Least-Squares and Weighted Least-Squares

Consider the order statistics of a random sample of size n from the OLxLx distribution denoted by $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$. Then, the LSEs of the OLxLx parameters are obtained by minimizing the equation:

$$LS = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a} - \frac{i}{n+1}\right\}^2,$$

Additionally, the OLxLx parameters can be estimated using the LSEs by solving the following three equations (for $q = 1,2,3$):

$$\sum_{i=1}^{n}\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}-\frac{i}{n+1}\right\}\xi_{q}(x_{i:n})=0,$$

where

$$\xi_2(x_{i:n}) = \frac{\partial}{\partial\theta}F(x_{i:n}) = a\,b^{a}\,(1+x_{i:n})^{\theta}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a-1}\log(1+x_{i:n}), \tag{22}$$

$$\xi_1(x_{i:n}) = \frac{\partial}{\partial a}F(x_{i:n}) = b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}\left\{\log\left[b-1+(1+x_{i:n})^{\theta}\right]-\log(b)\right\} \tag{23}$$

and

$$\xi_3(x_{i:n}) = \frac{\partial}{\partial b}F(x_{i:n}) = a\,b^{a-1}\left[1-(1+x_{i:n})^{\theta}\right]\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a-1}. \tag{24}$$

The WLSEs of the parameters of the OLxLx distribution are calculated by minimizing the following equation:

$$WL=\sum_{i=1}^{n}\phi(i,n)\left[F(x_{i:n})-\frac{i}{n+1}\right]^{2}=\sum_{i=1}^{n}\phi(i,n)\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}-\frac{i}{n+1}\right\}^{2},$$

where $\phi(i,n)=\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}$. Additionally, the OLxLx parameters can be estimated using the WLSEs by solving the following equations:

$$\sum_{i=1}^{n}\phi(i,n)\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}-\frac{i}{n+1}\right\}\xi_{q}(x_{i:n})=0,$$

where $\xi_q(x_{i:n})$ are defined in (22)–(24) for $q=1,2,3$.

4.3. Cramér–von Mises and Percentiles

The CVMEs of OLxLx parameters can be determined by minimizing the equation:

$$CV=\frac{1}{12\,n}+\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{2\,i-1}{2\,n}\right]^{2}=\frac{1}{12\,n}+\sum_{i=1}^{n}\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}-\frac{2\,i-1}{2\,n}\right\}^{2},$$

or by solving the following three equations (for $q=1,2,3$):

$$\sum_{i=1}^{n}\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}-\frac{2\,i-1}{2\,n}\right\}\xi_{q}(x_{i:n})=0,$$

where $\xi_q(x_{i:n})$ are defined (for $q=1,2,3$) in (22)–(24).

The percentiles approach is originally introduced by [17,18]. If $u_i=i/(1+n)$ is an unbiased estimator of $F(x_{i:n})$. Hence, the PCEs of the OLxLx parameters can be obtained by minimizing the following function:

$$PC(\theta,a,b)=\sum_{i=1}^{n}\left(x_{i:n}-\left\{1-\frac{b-b(1-u_i)^{\frac{1}{a}}}{b+(1-b)(1-u_i)^{\frac{1}{a}}}\right\}^{-1/\theta}-1\right)^{2},$$

with respect to θ, a, and b.

4.4. Anderson–Darling and Right-Tail Anderson–Darling

The ADEs of the parameters of the OLxLx distribution are obtained by minimizing:

$$\begin{aligned}AD&=-n-\frac{1}{n}\sum_{i=1}^{n}(2\,i-1)[\log F(x_{i:n})+\log S(x_{i:n})].\\&=-n-\frac{1}{n}\sum_{i=1}^{n}(2\,i-1)\left(\log\left\{1-b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}\right\}+\log\left\{b^{a}\left[b-1+(1+x_{i:n})^{\theta}\right]^{-a}\right\}\right).\end{aligned}$$

Additionally, the ADEs can also be determined by solving the following equations (for $q = 1, 2, 3$):

$$\sum_{i=1}^{n} (2\,i-1)\left[\frac{\xi_q(x_{i:n})}{F(x_{i:n})} - \frac{\xi_q(x_{n+1-i:n})}{S(x_{n+1-i:n})}\right] = 0,$$

where $\xi_q(x_{i:n})$ are defined (for $q = 1, 2, 3$) in (22)–(24).

4.5. Maximum Product of Spacings

Consider the uniform spacings, say $D_i = F(x_{i:n}) - F(x_{i-1:n})$, of a random sample of size n from the OLxLx distribution, where $\sum_{i=1}^{n+1} D_i = 1$, $F(x_{0:n}) = 0$, and $F(x_{n+1:n} = 1)$. The MPSEs of the OLxLx parameters are determined by maximizing

$$G(\theta, a, b) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i),$$

with respect to θ, a, and b. Furthermore, the MPSEs of the OLxLx parameters are also obtained by solving

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i} [\xi_q(x_{i:n}) - \xi_q(x_{i-1:n})] = 0,$$

where $\xi_q(x_{i:n})$ are defined (for $q = 1, 2, 3$) in (22)–(24).

5. Simulation Results

This section presents detailed simulations to compare the behavior and performance of the different estimates of the OLxLx parameters with respect to their: MSE, MSE $= \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi})^2$, biases, $|BIAS| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}|$, and MRE , MRE $= \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}|/\boldsymbol{\psi}$, where $\boldsymbol{\psi} = (\theta, a, b)'$.

Different sample sizes, $n = \{20, 70, 150, 250, 400\}$, are generated for some parametric values for the OLxLx parameters, $\theta = \{0.25, 0.50, 0.75, 1.50, 2.0, 3.0\}$, $a = \{0.50, 0.75, 1.50, 2.0, 3.0\}$ and $b = \{0.25, 0.50, 0.75, 1.50, 2.0\}$. We generate $N = 10{,}000$ random samples from the OLxLx distribution using its QF (15) and to estimate the measures of MSE, $|BIAS|$, and MRE using the R program.

The simulation results for the different eight estimation methods including MSE, $|BIAS|$, and MRE for the OLxLx parameters are listed in Tables 1–8. These tables also report the rank of the three simulation measures (MSE, $|BIAS|$, and MRE) using partial and overall ranks which are calculated for each sample size and parameters combination. Note that, the superscript numbers in these tables refer to the ranks in each row. In conclusion, all parameter estimates, from the eight estimation methods, of the OLxLx distribution are quite good and close to their true values. Moreover, the values of MSE, $|BIAS|$ and MRE are decreased in all parameter combinations. Additionally, the eight methods have the consistency property, where the MSE and MRE are decreased as n increases, for different parameters combinations. Table 9 shows the partial and overall ranks of the eight estimators based on the ranks of their MSE, $|BIAS|$, and MRE. The performance ordering of the eight estimators, based on Table 9, is MPSEs, ADEs, MLEs, WLSEs, PCEs, RADEs, LSE, and CVMEs. Finally, based on simulation results and estimators' ranks, it is concluded that the MPSEs outperform other studied estimators with an overall score of 80.5. Hence, we can confirm the superiority of MPSEs, ADEs, and MLEs for estimating the parameters of the OLxLx distribution.

Table 1. Simulation results of the OLxLx distribution for $\psi = (\theta = 0.50, a = 0.35, b = 0.75)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	0.43143 $^{\{4\}}$	0.50890 $^{\{7\}}$	0.45418 $^{\{5\}}$	0.50575 $^{\{6\}}$	0.42194 $^{\{3\}}$	1.14419 $^{\{8\}}$	0.38930 $^{\{2\}}$	0.38625 $^{\{1\}}$
		$\hat{a}$	0.16811 $^{\{1\}}$	0.22023 $^{\{6\}}$	0.21715 $^{\{4\}}$	0.21717 $^{\{5\}}$	0.23105 $^{\{7\}}$	0.32886 $^{\{8\}}$	0.19369 $^{\{2\}}$	0.19767 $^{\{3\}}$
		$\hat{b}$	0.43118 $^{\{4\}}$	0.44652 $^{\{6\}}$	0.44074 $^{\{5\}}$	0.42916 $^{\{3\}}$	0.45088 $^{\{7\}}$	0.46119 $^{\{8\}}$	0.41615 $^{\{2\}}$	0.41456 $^{\{1\}}$
	MSE	$\hat{\theta}$	0.48740 $^{\{4\}}$	0.90300 $^{\{7\}}$	0.64177 $^{\{5\}}$	0.79482 $^{\{6\}}$	0.44170 $^{\{3\}}$	7.06738 $^{\{8\}}$	0.43607 $^{\{2\}}$	0.40702 $^{\{1\}}$
		$\hat{a}$	0.06393 $^{\{1\}}$	0.11739 $^{\{6\}}$	0.11187 $^{\{4\}}$	0.14413 $^{\{7\}}$	0.11451 $^{\{5\}}$	0.54764 $^{\{8\}}$	0.07457 $^{\{2\}}$	0.10771 $^{\{3\}}$
		$\hat{b}$	0.33375 $^{\{7\}}$	0.33738 $^{\{8\}}$	0.32119 $^{\{4\}}$	0.33373 $^{\{6\}}$	0.32510 $^{\{5\}}$	0.31242 $^{\{3\}}$	0.28809 $^{\{2\}}$	0.27864 $^{\{1\}}$
	MRE	$\hat{\theta}$	0.95873 $^{\{4\}}$	1.13089 $^{\{7\}}$	1.00930 $^{\{5\}}$	1.12388 $^{\{6\}}$	0.93764 $^{\{3\}}$	2.54263 $^{\{8\}}$	0.86511 $^{\{2\}}$	0.85833 $^{\{1\}}$
		$\hat{a}$	0.48032 $^{\{1\}}$	0.62923 $^{\{6\}}$	0.62043 $^{\{4\}}$	0.62049 $^{\{5\}}$	0.66015 $^{\{7\}}$	0.93960 $^{\{8\}}$	0.55341 $^{\{2\}}$	0.56477 $^{\{3\}}$
		$\hat{b}$	0.57490 $^{\{4\}}$	0.59536 $^{\{6\}}$	0.58765 $^{\{5\}}$	0.57221 $^{\{3\}}$	0.60118 $^{\{7\}}$	0.61492 $^{\{8\}}$	0.55487 $^{\{2\}}$	0.55274 $^{\{1\}}$
		$\sum Ranks$	**30** $^{\{3\}}$	**59** $^{\{7\}}$	**41** $^{\{4\}}$	**47** $^{\{5.5\}}$	**47** $^{\{5.5\}}$	**67** $^{\{8\}}$	**18** $^{\{2\}}$	**15** $^{\{1\}}$
50	|BIAS|	$\hat{\theta}$	0.24922 $^{\{3\}}$	0.30069 $^{\{7\}}$	0.27029 $^{\{5\}}$	0.28416 $^{\{6\}}$	0.24979 $^{\{4\}}$	0.46826 $^{\{8\}}$	0.24049 $^{\{2\}}$	0.23431 $^{\{1\}}$
		$\hat{a}$	0.13839 $^{\{1\}}$	0.16465 $^{\{4\}}$	0.16884 $^{\{6\}}$	0.15607 $^{\{2\}}$	0.19783 $^{\{7\}}$	0.22878 $^{\{8\}}$	0.15976 $^{\{3\}}$	0.16480 $^{\{5\}}$
		$\hat{b}$	0.30610 $^{\{1\}}$	0.34174 $^{\{7\}}$	0.33185 $^{\{6\}}$	0.32444 $^{\{4\}}$	0.32865 $^{\{5\}}$	0.36275 $^{\{8\}}$	0.31743 $^{\{3\}}$	0.31700 $^{\{2\}}$
	MSE	$\hat{\theta}$	0.11764 $^{\{4\}}$	0.21149 $^{\{7\}}$	0.15814 $^{\{5\}}$	0.19123 $^{\{6\}}$	0.10876 $^{\{2\}}$	0.74438 $^{\{8\}}$	0.11731 $^{\{3\}}$	0.10023 $^{\{1\}}$
		$\hat{a}$	0.04000 $^{\{1\}}$	0.05259 $^{\{5\}}$	0.05696 $^{\{6\}}$	0.04891 $^{\{4\}}$	0.07823 $^{\{7\}}$	0.07836 $^{\{8\}}$	0.04875 $^{\{3\}}$	0.04690 $^{\{2\}}$
		$\hat{b}$	0.15046 $^{\{2\}}$	0.17574 $^{\{7\}}$	0.16500 $^{\{6\}}$	0.15967 $^{\{5\}}$	0.15601 $^{\{4\}}$	0.17832 $^{\{8\}}$	0.15186 $^{\{3\}}$	0.14091 $^{\{1\}}$
	MRE	$\hat{\theta}$	0.55383 $^{\{3\}}$	0.66820 $^{\{7\}}$	0.60064 $^{\{5\}}$	0.63147 $^{\{6\}}$	0.55510 $^{\{4\}}$	1.04059 $^{\{8\}}$	0.53441 $^{\{2\}}$	0.52068 $^{\{1\}}$
		$\hat{a}$	0.39539 $^{\{1\}}$	0.47043 $^{\{4\}}$	0.48239 $^{\{6\}}$	0.44592 $^{\{2\}}$	0.56523 $^{\{7\}}$	0.65365 $^{\{8\}}$	0.45646 $^{\{3\}}$	0.47086 $^{\{5\}}$
		$\hat{b}$	0.40814 $^{\{1\}}$	0.45565 $^{\{7\}}$	0.44247 $^{\{6\}}$	0.43258 $^{\{4\}}$	0.43820 $^{\{5\}}$	0.48367 $^{\{8\}}$	0.42324 $^{\{3\}}$	0.42266 $^{\{2\}}$
		$\sum Ranks$	**17** $^{\{1\}}$	**55** $^{\{7\}}$	**51** $^{\{6\}}$	**39** $^{\{4\}}$	**45** $^{\{5\}}$	**72** $^{\{8\}}$	**25** $^{\{3\}}$	**20** $^{\{2\}}$
80	|BIAS|	$\hat{\theta}$	0.20012 $^{\{3\}}$	0.23963 $^{\{6\}}$	0.20857 $^{\{5\}}$	0.23990 $^{\{7\}}$	0.20504 $^{\{4\}}$	0.31514 $^{\{8\}}$	0.19291 $^{\{1\}}$	0.19882 $^{\{2\}}$
		$\hat{a}$	0.13054 $^{\{1\}}$	0.14640 $^{\{5\}}$	0.14307 $^{\{4\}}$	0.14271 $^{\{3\}}$	0.18082 $^{\{7\}}$	0.21396 $^{\{8\}}$	0.13863 $^{\{2\}}$	0.15053 $^{\{6\}}$
		$\hat{b}$	0.26096 $^{\{1\}}$	0.29772 $^{\{7\}}$	0.27839 $^{\{4\}}$	0.28953 $^{\{6\}}$	0.28327 $^{\{5\}}$	0.31801 $^{\{8\}}$	0.26596 $^{\{2\}}$	0.27666 $^{\{3\}}$
	MSE	$\hat{\theta}$	0.07122 $^{\{4\}}$	0.11386 $^{\{6\}}$	0.08207 $^{\{5\}}$	0.12124 $^{\{7\}}$	0.06509 $^{\{3\}}$	0.22388 $^{\{8\}}$	0.06488 $^{\{2\}}$	0.06421 $^{\{1\}}$
		$\hat{a}$	0.03683 $^{\{1\}}$	0.04178 $^{\{4\}}$	0.04247 $^{\{6\}}$	0.03994 $^{\{3\}}$	0.06692 $^{\{8\}}$	0.06676 $^{\{7\}}$	0.03850 $^{\{2\}}$	0.04205 $^{\{5\}}$
		$\hat{b}$	0.10839 $^{\{3\}}$	0.12880 $^{\{7\}}$	0.11396 $^{\{4\}}$	0.12433 $^{\{6\}}$	0.11453 $^{\{5\}}$	0.13487 $^{\{8\}}$	0.10288 $^{\{1\}}$	0.10711 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.44471 $^{\{3\}}$	0.53251 $^{\{6\}}$	0.46348 $^{\{5\}}$	0.53311 $^{\{7\}}$	0.45564 $^{\{4\}}$	0.70030 $^{\{8\}}$	0.42868 $^{\{1\}}$	0.44182 $^{\{2\}}$
		$\hat{a}$	0.37298 $^{\{1\}}$	0.41830 $^{\{5\}}$	0.40877 $^{\{4\}}$	0.40773 $^{\{3\}}$	0.51663 $^{\{7\}}$	0.61130 $^{\{8\}}$	0.39609 $^{\{2\}}$	0.43008 $^{\{6\}}$
		$\hat{b}$	0.34795 $^{\{1\}}$	0.39697 $^{\{7\}}$	0.37119 $^{\{4\}}$	0.38604 $^{\{6\}}$	0.37769 $^{\{5\}}$	0.42401 $^{\{8\}}$	0.35461 $^{\{2\}}$	0.36888 $^{\{3\}}$
		$\sum Ranks$	**18** $^{\{2\}}$	**53** $^{\{7\}}$	**41** $^{\{4\}}$	**48** $^{\{5.5\}}$	**48** $^{\{5.5\}}$	**71** $^{\{8\}}$	**15** $^{\{1\}}$	**30** $^{\{3\}}$
150	|BIAS|	$\hat{\theta}$	0.16273 $^{\{4\}}$	0.19231 $^{\{7\}}$	0.16351 $^{\{5\}}$	0.18302 $^{\{6\}}$	0.15429 $^{\{1\}}$	0.23966 $^{\{8\}}$	0.15803 $^{\{3\}}$	0.15529 $^{\{2\}}$
		$\hat{a}$	0.11087 $^{\{1\}}$	0.13220 $^{\{6\}}$	0.11867 $^{\{2\}}$	0.12187 $^{\{4\}}$	0.14822 $^{\{7\}}$	0.19296 $^{\{8\}}$	0.11979 $^{\{3\}}$	0.12714 $^{\{5\}}$
		$\hat{b}$	0.21620 $^{\{1\}}$	0.25102 $^{\{7\}}$	0.22574 $^{\{5\}}$	0.23872 $^{\{6\}}$	0.22446 $^{\{4\}}$	0.27365 $^{\{8\}}$	0.22272 $^{\{2\}}$	0.22311 $^{\{3\}}$
	MSE	$\hat{\theta}$	0.04671 $^{\{5\}}$	0.06282 $^{\{7\}}$	0.04419 $^{\{4\}}$	0.05780 $^{\{6\}}$	0.03618 $^{\{1\}}$	0.10279 $^{\{8\}}$	0.04005 $^{\{3\}}$	0.03675 $^{\{2\}}$
		$\hat{a}$	0.02807 $^{\{1\}}$	0.03685 $^{\{6\}}$	0.03090 $^{\{3\}}$	0.03039 $^{\{2\}}$	0.05099 $^{\{7\}}$	0.05963 $^{\{8\}}$	0.03119 $^{\{4\}}$	0.03472 $^{\{5\}}$
		$\hat{b}$	0.07960 $^{\{5\}}$	0.08897 $^{\{7\}}$	0.07454 $^{\{3\}}$	0.08172 $^{\{6\}}$	0.07718 $^{\{4\}}$	0.10248 $^{\{8\}}$	0.07238 $^{\{1\}}$	0.07240 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.36162 $^{\{4\}}$	0.42736 $^{\{7\}}$	0.36336 $^{\{5\}}$	0.40670 $^{\{6\}}$	0.34287 $^{\{1\}}$	0.53259 $^{\{8\}}$	0.35117 $^{\{3\}}$	0.34509 $^{\{2\}}$
		$\hat{a}$	0.31677 $^{\{1\}}$	0.37771 $^{\{6\}}$	0.33907 $^{\{2\}}$	0.34820 $^{\{4\}}$	0.42348 $^{\{7\}}$	0.55130 $^{\{8\}}$	0.34224 $^{\{3\}}$	0.36326 $^{\{5\}}$
		$\hat{b}$	0.28826 $^{\{1\}}$	0.33469 $^{\{7\}}$	0.30098 $^{\{5\}}$	0.31829 $^{\{6\}}$	0.29929 $^{\{4\}}$	0.36486 $^{\{8\}}$	0.29696 $^{\{2\}}$	0.29748 $^{\{3\}}$
		$\sum Ranks$	**23** $^{\{1\}}$	**60** $^{\{7\}}$	**34** $^{\{4\}}$	**46** $^{\{6\}}$	**36** $^{\{5\}}$	**72** $^{\{8\}}$	**24** $^{\{2\}}$	**29** $^{\{3\}}$
500	|BIAS|	$\hat{\theta}$	0.11344 $^{\{5\}}$	0.11794 $^{\{7\}}$	0.09984 $^{\{4\}}$	0.11716 $^{\{6\}}$	0.08691 $^{\{1\}}$	0.14311 $^{\{8\}}$	0.09587 $^{\{2\}}$	0.09661 $^{\{3\}}$
		$\hat{a}$	0.08020 $^{\{5\}}$	0.08307 $^{\{7\}}$	0.07528 $^{\{3\}}$	0.08213 $^{\{6\}}$	0.07615 $^{\{4\}}$	0.13761 $^{\{8\}}$	0.07372 $^{\{1\}}$	0.07424 $^{\{2\}}$
		$\hat{b}$	0.15325 $^{\{5\}}$	0.15940 $^{\{7\}}$	0.14296 $^{\{4\}}$	0.15871 $^{\{6\}}$	0.12712 $^{\{1\}}$	0.18755 $^{\{8\}}$	0.13843 $^{\{3\}}$	0.13831 $^{\{2\}}$
	MSE	$\hat{\theta}$	0.03127 $^{\{7\}}$	0.02143 $^{\{6\}}$	0.01592 $^{\{4\}}$	0.02116 $^{\{5\}}$	0.01346 $^{\{1\}}$	0.03133 $^{\{8\}}$	0.01503 $^{\{3\}}$	0.01467 $^{\{2\}}$
		$\hat{a}$	0.01839 $^{\{7\}}$	0.01664 $^{\{5\}}$	0.01471 $^{\{2\}}$	0.01620 $^{\{4\}}$	0.01797 $^{\{6\}}$	0.04030 $^{\{8\}}$	0.01486 $^{\{3\}}$	0.01435 $^{\{1\}}$
		$\hat{b}$	0.05617 $^{\{7\}}$	0.03919 $^{\{6\}}$	0.03289 $^{\{4\}}$	0.03844 $^{\{5\}}$	0.03085 $^{\{1\}}$	0.05619 $^{\{8\}}$	0.03213 $^{\{3\}}$	0.03129 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.25208 $^{\{5\}}$	0.26209 $^{\{7\}}$	0.22187 $^{\{4\}}$	0.26035 $^{\{6\}}$	0.19312 $^{\{1\}}$	0.31803 $^{\{8\}}$	0.21305 $^{\{2\}}$	0.21470 $^{\{3\}}$
		$\hat{a}$	0.22914 $^{\{5\}}$	0.23734 $^{\{7\}}$	0.21510 $^{\{3\}}$	0.23466 $^{\{6\}}$	0.21757 $^{\{4\}}$	0.39317 $^{\{8\}}$	0.21062 $^{\{1\}}$	0.21212 $^{\{2\}}$
		$\hat{b}$	0.20434 $^{\{5\}}$	0.21253 $^{\{7\}}$	0.19062 $^{\{4\}}$	0.21161 $^{\{6\}}$	0.16949 $^{\{1\}}$	0.25006 $^{\{8\}}$	0.18457 $^{\{3\}}$	0.18442 $^{\{2\}}$
		$\sum Ranks$	**51** $^{\{6\}}$	**59** $^{\{7\}}$	**32** $^{\{4\}}$	**50** $^{\{5\}}$	**20** $^{\{2\}}$	**72** $^{\{8\}}$	**21** $^{\{3\}}$	**19** $^{\{1\}}$

Table 2. Simulation results of the OLxLx distribution for $\psi = (\theta = 0.50, a = 1.25, b = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	0.21949 $^{\{8\}}$	0.19925 $^{\{5\}}$	0.19179 $^{\{3\}}$	0.21102 $^{\{7\}}$	0.16233 $^{\{1\}}$	0.18503 $^{\{2\}}$	0.19534 $^{\{4\}}$	0.20818 $^{\{6\}}$
		$\hat{a}$	2.09115 $^{\{7\}}$	1.97412 $^{\{6\}}$	1.74062 $^{\{4\}}$	2.14426 $^{\{8\}}$	1.64389 $^{\{3\}}$	1.56467 $^{\{2\}}$	1.47604 $^{\{1\}}$	1.90650 $^{\{5\}}$
		$\hat{b}$	1.29950 $^{\{8\}}$	1.10917 $^{\{5\}}$	1.07834 $^{\{4\}}$	1.23157 $^{\{7\}}$	0.90590 $^{\{1\}}$	0.96122 $^{\{2\}}$	1.06118 $^{\{3\}}$	1.16826 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.06780 $^{\{3\}}$	0.08081 $^{\{6\}}$	0.06732 $^{\{2\}}$	0.07986 $^{\{5\}}$	0.03580 $^{\{1\}}$	0.09352 $^{\{8\}}$	0.07208 $^{\{4\}}$	0.08746 $^{\{7\}}$
		$\hat{a}$	12.16351 $^{\{6\}}$	11.36830 $^{\{5\}}$	9.17222 $^{\{4\}}$	13.40949 $^{\{8\}}$	9.00936 $^{\{3\}}$	7.10955 $^{\{2\}}$	6.58475 $^{\{1\}}$	12.68988 $^{\{7\}}$
		$\hat{b}$	2.60241 $^{\{7\}}$	2.34379 $^{\{5\}}$	2.16066 $^{\{4\}}$	2.83554 $^{\{8\}}$	1.22953 $^{\{1\}}$	1.46427 $^{\{2\}}$	1.84796 $^{\{3\}}$	2.51992 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.48776 $^{\{8\}}$	0.44278 $^{\{5\}}$	0.42620 $^{\{3\}}$	0.46894 $^{\{7\}}$	0.36073 $^{\{1\}}$	0.41118 $^{\{2\}}$	0.43409 $^{\{4\}}$	0.46262 $^{\{6\}}$
		$\hat{a}$	1.67292 $^{\{7\}}$	1.57930 $^{\{6\}}$	1.39250 $^{\{4\}}$	1.71541 $^{\{8\}}$	1.31511 $^{\{3\}}$	1.25174 $^{\{2\}}$	1.18083 $^{\{1\}}$	1.52520 $^{\{5\}}$
		$\hat{b}$	1.03960 $^{\{8\}}$	0.88733 $^{\{5\}}$	0.86267 $^{\{4\}}$	0.98526 $^{\{7\}}$	0.72472 $^{\{1\}}$	0.76897 $^{\{2\}}$	0.84894 $^{\{3\}}$	0.93460 $^{\{6\}}$
		$\sum Ranks$	**62** $^{\{7\}}$	**48** $^{\{5\}}$	**32** $^{\{4\}}$	**65** $^{\{8\}}$	**15** $^{\{1\}}$	**24** $^{\{2.5\}}$	**24** $^{\{2.5\}}$	**54** $^{\{6\}}$
50	|BIAS|	$\hat{\theta}$	0.18161 $^{\{7\}}$	0.17412 $^{\{5\}}$	0.16664 $^{\{4\}}$	0.17867 $^{\{6\}}$	0.13817 $^{\{1\}}$	0.16364 $^{\{3\}}$	0.16208 $^{\{2\}}$	0.18797 $^{\{8\}}$
		$\hat{a}$	0.92482 $^{\{6\}}$	1.05999 $^{\{7\}}$	0.84781 $^{\{4\}}$	1.11104 $^{\{8\}}$	0.70657 $^{\{1\}}$	0.77302 $^{\{3\}}$	0.75802 $^{\{2\}}$	0.91179 $^{\{5\}}$
		$\hat{b}$	0.99912 $^{\{8\}}$	0.94217 $^{\{5\}}$	0.90300 $^{\{4\}}$	0.99165 $^{\{6\}}$	0.74384 $^{\{1\}}$	0.84587 $^{\{2\}}$	0.87275 $^{\{3\}}$	0.99883 $^{\{7\}}$
	MSE	$\hat{\theta}$	0.05049 $^{\{7\}}$	0.04355 $^{\{3\}}$	0.04846 $^{\{6\}}$	0.04641 $^{\{5\}}$	0.02871 $^{\{1\}}$	0.04557 $^{\{4\}}$	0.03809 $^{\{2\}}$	0.06756 $^{\{8\}}$
		$\hat{a}$	2.20945 $^{\{5\}}$	3.15930 $^{\{7\}}$	1.81270 $^{\{4\}}$	3.46078 $^{\{8\}}$	1.38539 $^{\{2\}}$	1.43871 $^{\{3\}}$	1.32728 $^{\{1\}}$	2.31052 $^{\{6\}}$
		$\hat{b}$	1.49793 $^{\{7\}}$	1.26478 $^{\{5\}}$	1.22443 $^{\{4\}}$	1.49444 $^{\{6\}}$	0.85984 $^{\{1\}}$	1.03484 $^{\{2\}}$	1.12340 $^{\{3\}}$	1.53687 $^{\{8\}}$
	MRE	$\hat{\theta}$	0.40359 $^{\{7\}}$	0.38694 $^{\{5\}}$	0.37031 $^{\{4\}}$	0.39703 $^{\{6\}}$	0.30704 $^{\{1\}}$	0.36365 $^{\{3\}}$	0.36018 $^{\{2\}}$	0.41771 $^{\{8\}}$
		$\hat{a}$	0.73986 $^{\{6\}}$	0.84799 $^{\{7\}}$	0.67825 $^{\{4\}}$	0.88883 $^{\{8\}}$	0.56526 $^{\{1\}}$	0.61842 $^{\{3\}}$	0.60642 $^{\{2\}}$	0.72944 $^{\{5\}}$
		$\hat{b}$	0.79930 $^{\{8\}}$	0.75373 $^{\{5\}}$	0.72240 $^{\{4\}}$	0.79332 $^{\{6\}}$	0.59508 $^{\{1\}}$	0.67670 $^{\{2\}}$	0.69820 $^{\{3\}}$	0.79907 $^{\{7\}}$
		$\sum Ranks$	**61** $^{\{7\}}$	**49** $^{\{5\}}$	**38** $^{\{4\}}$	**59** $^{\{6\}}$	**10** $^{\{1\}}$	**25** $^{\{3\}}$	**20** $^{\{2\}}$	**62** $^{\{8\}}$

Table 2. *Cont.*

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
		$\hat{\theta}$	0.16341 $^{\{7\}}$	0.15965 $^{\{5\}}$	0.14775 $^{\{2\}}$	0.16089 $^{\{6\}}$	0.12364 $^{\{1\}}$	0.15516 $^{\{4\}}$	0.14996 $^{\{3\}}$	0.16752 $^{\{8\}}$
	|BIAS|	$\hat{a}$	0.64810 $^{\{4\}}$	0.82504 $^{\{7\}}$	0.65653 $^{\{5\}}$	0.84366 $^{\{8\}}$	0.50865 $^{\{1\}}$	0.59025 $^{\{2\}}$	0.60458 $^{\{3\}}$	0.66922 $^{\{6\}}$
		$\hat{b}$	0.88765 $^{\{7\}}$	0.86424 $^{\{5\}}$	0.80193 $^{\{2\}}$	0.87459 $^{\{6\}}$	0.65107 $^{\{1\}}$	0.81125 $^{\{4\}}$	0.80479 $^{\{3\}}$	0.89089 $^{\{8\}}$
		$\hat{\theta}$	0.04447 $^{\{7\}}$	0.03609 $^{\{4\}}$	0.03294 $^{\{2\}}$	0.03948 $^{\{6\}}$	0.02554 $^{\{1\}}$	0.03859 $^{\{5\}}$	0.03375 $^{\{3\}}$	0.04752 $^{\{8\}}$
80	**MSE**	$\hat{a}$	0.86206 $^{\{4\}}$	1.69600 $^{\{7\}}$	0.93102 $^{\{5\}}$	1.77778 $^{\{8\}}$	0.56451 $^{\{1\}}$	0.67174 $^{\{2\}}$	0.70256 $^{\{3\}}$	0.98616 $^{\{6\}}$
		$\hat{b}$	1.27346 $^{\{8\}}$	1.04974 $^{\{5\}}$	0.95637 $^{\{2\}}$	1.09232 $^{\{6\}}$	0.71614 $^{\{1\}}$	0.97790 $^{\{4\}}$	0.96516 $^{\{3\}}$	1.22160 $^{\{7\}}$
		$\hat{\theta}$	0.36314 $^{\{7\}}$	0.35478 $^{\{5\}}$	0.32833 $^{\{2\}}$	0.35753 $^{\{6\}}$	0.27476 $^{\{1\}}$	0.34481 $^{\{4\}}$	0.33324 $^{\{3\}}$	0.37227 $^{\{8\}}$
	MRE	$\hat{a}$	0.51848 $^{\{4\}}$	0.66003 $^{\{7\}}$	0.52522 $^{\{5\}}$	0.67493 $^{\{8\}}$	0.40692 $^{\{1\}}$	0.47220 $^{\{2\}}$	0.48366 $^{\{3\}}$	0.53537 $^{\{6\}}$
		$\hat{b}$	0.71012 $^{\{7\}}$	0.69139 $^{\{5\}}$	0.64154 $^{\{2\}}$	0.69967 $^{\{6\}}$	0.52085 $^{\{1\}}$	0.64900 $^{\{4\}}$	0.64383 $^{\{3\}}$	0.71271 $^{\{8\}}$
		$\sum$ *Ranks*	**55** $^{\{6\}}$	**50** $^{\{5\}}$	**27** $^{\{2.5\}}$	**60** $^{\{7\}}$	**9** $^{\{1\}}$	**31** $^{\{4\}}$	**27** $^{\{2.5\}}$	**65** $^{\{8\}}$
		$\hat{\theta}$	0.13588 $^{\{5\}}$	0.13885 $^{\{6\}}$	0.12818 $^{\{2\}}$	0.14479 $^{\{8\}}$	0.10243 $^{\{1\}}$	0.13578 $^{\{4\}}$	0.12883 $^{\{3\}}$	0.14377 $^{\{7\}}$
	|BIAS|	$\hat{a}$	0.45347 $^{\{3\}}$	0.58543 $^{\{7\}}$	0.47960 $^{\{5\}}$	0.59179 $^{\{8\}}$	0.36740 $^{\{1\}}$	0.44209 $^{\{2\}}$	0.45952 $^{\{4\}}$	0.49967 $^{\{6\}}$
		$\hat{b}$	0.72562 $^{\{5\}}$	0.75145 $^{\{6\}}$	0.68986 $^{\{3\}}$	0.78845 $^{\{8\}}$	0.53838 $^{\{1\}}$	0.71491 $^{\{4\}}$	0.68771 $^{\{2\}}$	0.77618 $^{\{7\}}$
		$\hat{\theta}$	0.03436 $^{\{7\}}$	0.02852 $^{\{4\}}$	0.02634 $^{\{2\}}$	0.03154 $^{\{6\}}$	0.01997 $^{\{1\}}$	0.02941 $^{\{5\}}$	0.02705 $^{\{3\}}$	0.03465 $^{\{8\}}$
150	**MSE**	$\hat{a}$	0.32956 $^{\{3\}}$	0.64980 $^{\{7\}}$	0.38309 $^{\{5\}}$	0.66539 $^{\{8\}}$	0.25686 $^{\{1\}}$	0.30722 $^{\{2\}}$	0.34073 $^{\{4\}}$	0.40687 $^{\{6\}}$
		$\hat{b}$	0.93536 $^{\{7\}}$	0.81780 $^{\{5\}}$	0.74193 $^{\{2\}}$	0.90770 $^{\{6\}}$	0.55021 $^{\{1\}}$	0.79826 $^{\{4\}}$	0.74809 $^{\{3\}}$	0.96131 $^{\{8\}}$
		$\hat{\theta}$	0.30196 $^{\{5\}}$	0.30857 $^{\{6\}}$	0.28484 $^{\{2\}}$	0.32177 $^{\{8\}}$	0.22762 $^{\{1\}}$	0.30172 $^{\{4\}}$	0.28628 $^{\{3\}}$	0.31950 $^{\{7\}}$
	MRE	$\hat{a}$	0.36277 $^{\{3\}}$	0.46835 $^{\{7\}}$	0.38368 $^{\{5\}}$	0.47343 $^{\{8\}}$	0.29392 $^{\{1\}}$	0.35367 $^{\{2\}}$	0.36762 $^{\{4\}}$	0.39973 $^{\{6\}}$
		$\hat{b}$	0.58049 $^{\{5\}}$	0.60116 $^{\{6\}}$	0.55189 $^{\{3\}}$	0.63076 $^{\{8\}}$	0.43071 $^{\{1\}}$	0.57193 $^{\{4\}}$	0.55017 $^{\{2\}}$	0.62094 $^{\{7\}}$
		$\sum$ *Ranks*	**43** $^{\{5\}}$	**54** $^{\{6\}}$	**29** $^{\{3\}}$	**68** $^{\{8\}}$	**9** $^{\{1\}}$	**31** $^{\{4\}}$	**28** $^{\{2\}}$	**62** $^{\{7\}}$
		$\hat{\theta}$	0.08456 $^{\{2\}}$	0.10033 $^{\{7\}}$	0.08575 $^{\{4\}}$	0.10465 $^{\{8\}}$	0.05337 $^{\{1\}}$	0.09235 $^{\{5\}}$	0.08526 $^{\{3\}}$	0.09916 $^{\{6\}}$
	|BIAS|	$\hat{a}$	0.25224 $^{\{2\}}$	0.34542 $^{\{8\}}$	0.27767 $^{\{4\}}$	0.34530 $^{\{7\}}$	0.17300 $^{\{1\}}$	0.27022 $^{\{3\}}$	0.27780 $^{\{5\}}$	0.31236 $^{\{6\}}$
		$\hat{b}$	0.44718 $^{\{2\}}$	0.54785 $^{\{7\}}$	0.46267 $^{\{4\}}$	0.56902 $^{\{8\}}$	0.27597 $^{\{1\}}$	0.49613 $^{\{5\}}$	0.46033 $^{\{3\}}$	0.54337 $^{\{6\}}$
		$\hat{\theta}$	0.01854 $^{\{6\}}$	0.01787 $^{\{5\}}$	0.01480 $^{\{3\}}$	0.01940 $^{\{8\}}$	0.00822 $^{\{1\}}$	0.01722 $^{\{4\}}$	0.01453 $^{\{2\}}$	0.01863 $^{\{7\}}$
500	**MSE**	$\hat{a}$	0.10120 $^{\{2\}}$	0.18080 $^{\{8\}}$	0.11915 $^{\{5\}}$	0.17940 $^{\{7\}}$	0.06804 $^{\{1\}}$	0.11007 $^{\{3\}}$	0.11800 $^{\{4\}}$	0.14716 $^{\{6\}}$
		$\hat{b}$	0.47263 $^{\{4\}}$	0.50966 $^{\{6\}}$	0.41210 $^{\{3\}}$	0.54875 $^{\{8\}}$	0.22827 $^{\{1\}}$	0.47478 $^{\{5\}}$	0.40250 $^{\{2\}}$	0.53834 $^{\{7\}}$
		$\hat{\theta}$	0.18790 $^{\{2\}}$	0.22295 $^{\{7\}}$	0.19055 $^{\{4\}}$	0.23256 $^{\{8\}}$	0.11861 $^{\{1\}}$	0.20522 $^{\{5\}}$	0.18947 $^{\{3\}}$	0.22035 $^{\{6\}}$
	MRE	$\hat{a}$	0.20179 $^{\{2\}}$	0.27633 $^{\{8\}}$	0.22214 $^{\{4\}}$	0.27624 $^{\{7\}}$	0.13840 $^{\{1\}}$	0.21617 $^{\{3\}}$	0.22224 $^{\{5\}}$	0.24989 $^{\{6\}}$
		$\hat{b}$	0.35774 $^{\{2\}}$	0.43828 $^{\{7\}}$	0.37014 $^{\{4\}}$	0.45521 $^{\{8\}}$	0.22078 $^{\{1\}}$	0.39690 $^{\{5\}}$	0.36827 $^{\{3\}}$	0.43470 $^{\{6\}}$
		$\sum$ *Ranks*	**24** $^{\{2\}}$	**63** $^{\{7\}}$	**35** $^{\{4\}}$	**69** $^{\{8\}}$	**9** $^{\{1\}}$	**38** $^{\{5\}}$	**30** $^{\{3\}}$	**56** $^{\{6\}}$

Table 3. Simulation results of the OLxLx distribution for $\psi = (\theta = 0.75, a = 1.25, b = 0.75)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
		$\hat{\theta}$	0.23131 $^{\{8\}}$	0.19347 $^{\{4\}}$	0.18824 $^{\{3\}}$	0.22044 $^{\{7\}}$	0.16818 $^{\{1\}}$	0.17595 $^{\{2\}}$	0.20078 $^{\{5\}}$	0.20678 $^{\{6\}}$
	|BIAS|	$\hat{a}$	1.81723 $^{\{8\}}$	1.70582 $^{\{5\}}$	1.46058 $^{\{3\}}$	1.81229 $^{\{7\}}$	1.44213 $^{\{2\}}$	1.64385 $^{\{4\}}$	1.31400 $^{\{1\}}$	1.74351 $^{\{6\}}$
		$\hat{b}$	0.76512 $^{\{8\}}$	0.60703 $^{\{4\}}$	0.59499 $^{\{3\}}$	0.69872 $^{\{7\}}$	0.54571 $^{\{1\}}$	0.55394 $^{\{2\}}$	0.62013 $^{\{5\}}$	0.65619 $^{\{6\}}$
		$\hat{\theta}$	0.08605 $^{\{7\}}$	0.05951 $^{\{4\}}$	0.05349 $^{\{3\}}$	0.08083 $^{\{6\}}$	0.04142 $^{\{1\}}$	0.05206 $^{\{2\}}$	0.06219 $^{\{5\}}$	0.09781 $^{\{8\}}$
20	**MSE**	$\hat{a}$	7.14691 $^{\{6\}}$	6.94228 $^{\{5\}}$	5.20980 $^{\{2\}}$	7.97822 $^{\{7\}}$	5.33075 $^{\{3\}}$	6.92205 $^{\{4\}}$	4.22640 $^{\{1\}}$	8.75147 $^{\{8\}}$
		$\hat{b}$	0.88641 $^{\{8\}}$	0.55017 $^{\{4\}}$	0.52287 $^{\{3\}}$	0.80224 $^{\{7\}}$	0.43406 $^{\{1\}}$	0.44311 $^{\{2\}}$	0.58111 $^{\{5\}}$	0.73213 $^{\{6\}}$
		$\hat{\theta}$	0.51403 $^{\{8\}}$	0.42993 $^{\{4\}}$	0.41831 $^{\{3\}}$	0.48988 $^{\{7\}}$	0.37374 $^{\{1\}}$	0.39100 $^{\{2\}}$	0.44617 $^{\{5\}}$	0.45951 $^{\{6\}}$
	MRE	$\hat{a}$	1.45379 $^{\{8\}}$	1.36466 $^{\{5\}}$	1.16846 $^{\{3\}}$	1.44983 $^{\{7\}}$	1.15371 $^{\{2\}}$	1.31508 $^{\{4\}}$	1.05120 $^{\{1\}}$	1.39481 $^{\{6\}}$
		$\hat{b}$	1.02016 $^{\{8\}}$	0.80937 $^{\{4\}}$	0.79332 $^{\{3\}}$	0.93163 $^{\{7\}}$	0.72762 $^{\{1\}}$	0.73858 $^{\{2\}}$	0.82684 $^{\{5\}}$	0.87492 $^{\{6\}}$
		$\sum$ *Ranks*	**69** $^{\{8\}}$	**39** $^{\{5\}}$	**26** $^{\{3\}}$	**62** $^{\{7\}}$	**13** $^{\{1\}}$	**24** $^{\{2\}}$	**33** $^{\{4\}}$	**58** $^{\{6\}}$
		$\hat{\theta}$	0.20336 $^{\{8\}}$	0.17387 $^{\{5\}}$	0.16611 $^{\{3\}}$	0.18323 $^{\{7\}}$	0.14716 $^{\{1\}}$	0.15965 $^{\{2\}}$	0.16669 $^{\{4\}}$	0.17399 $^{\{6\}}$
	|BIAS|	$\hat{a}$	0.91965 $^{\{6\}}$	1.08478 $^{\{7\}}$	0.87526 $^{\{4\}}$	1.13228 $^{\{8\}}$	0.74843 $^{\{1\}}$	0.86259 $^{\{3\}}$	0.76077 $^{\{2\}}$	0.88666 $^{\{5\}}$
		$\hat{b}$	0.62457 $^{\{8\}}$	0.53928 $^{\{5\}}$	0.51687 $^{\{4\}}$	0.58090 $^{\{7\}}$	0.45684 $^{\{1\}}$	0.49977 $^{\{2\}}$	0.50851 $^{\{3\}}$	0.53975 $^{\{6\}}$
		$\hat{\theta}$	0.07056 $^{\{8\}}$	0.04380 $^{\{4\}}$	0.04168 $^{\{3\}}$	0.05042 $^{\{6\}}$	0.03576 $^{\{2\}}$	0.03446 $^{\{1\}}$	0.04381 $^{\{5\}}$	0.05071 $^{\{7\}}$
50	**MSE**	$\hat{a}$	1.73599 $^{\{3\}}$	2.84070 $^{\{7\}}$	1.74071 $^{\{4\}}$	3.07360 $^{\{8\}}$	1.33295 $^{\{2\}}$	1.96437 $^{\{5\}}$	1.23641 $^{\{1\}}$	2.02244 $^{\{6\}}$
		$\hat{b}$	0.61996 $^{\{8\}}$	0.39510 $^{\{5\}}$	0.38074 $^{\{3\}}$	0.46786 $^{\{7\}}$	0.33254 $^{\{1\}}$	0.34245 $^{\{2\}}$	0.38451 $^{\{4\}}$	0.42983 $^{\{6\}}$
		$\hat{\theta}$	0.45192 $^{\{8\}}$	0.38637 $^{\{5\}}$	0.36913 $^{\{3\}}$	0.40718 $^{\{7\}}$	0.32702 $^{\{1\}}$	0.35477 $^{\{2\}}$	0.37042 $^{\{4\}}$	0.38665 $^{\{6\}}$
	MRE	$\hat{a}$	0.73572 $^{\{6\}}$	0.86782 $^{\{7\}}$	0.70021 $^{\{4\}}$	0.90582 $^{\{8\}}$	0.59874 $^{\{1\}}$	0.69007 $^{\{3\}}$	0.60862 $^{\{2\}}$	0.70933 $^{\{5\}}$
		$\hat{b}$	0.83276 $^{\{8\}}$	0.71905 $^{\{5\}}$	0.68916 $^{\{4\}}$	0.77453 $^{\{7\}}$	0.60912 $^{\{1\}}$	0.66636 $^{\{2\}}$	0.67802 $^{\{3\}}$	0.71967 $^{\{6\}}$
		$\sum$ *Ranks*	**63** $^{\{7\}}$	**50** $^{\{5\}}$	**32** $^{\{4\}}$	**65** $^{\{8\}}$	**11** $^{\{1\}}$	**22** $^{\{2\}}$	**28** $^{\{3\}}$	**53** $^{\{6\}}$
		$\hat{\theta}$	0.17526 $^{\{8\}}$	0.15990 $^{\{6\}}$	0.15312 $^{\{4\}}$	0.16485 $^{\{7\}}$	0.13028 $^{\{1\}}$	0.15080 $^{\{2\}}$	0.15235 $^{\{3\}}$	0.15787 $^{\{5\}}$
	|BIAS|	$\hat{a}$	0.68462 $^{\{5\}}$	0.83446 $^{\{7\}}$	0.69720 $^{\{6\}}$	0.83998 $^{\{8\}}$	0.53486 $^{\{1\}}$	0.64435 $^{\{3\}}$	0.61539 $^{\{2\}}$	0.67009 $^{\{4\}}$
		$\hat{b}$	0.53124 $^{\{8\}}$	0.49960 $^{\{6\}}$	0.47421 $^{\{4\}}$	0.51831 $^{\{7\}}$	0.39954 $^{\{1\}}$	0.46878 $^{\{3\}}$	0.46279 $^{\{2\}}$	0.48684 $^{\{5\}}$
		$\hat{\theta}$	0.05866 $^{\{8\}}$	0.03826 $^{\{4\}}$	0.03752 $^{\{3\}}$	0.04160 $^{\{7\}}$	0.03042 $^{\{1\}}$	0.03207 $^{\{2\}}$	0.03841 $^{\{5\}}$	0.03945 $^{\{6\}}$
80	**MSE**	$\hat{a}$	0.89397 $^{\{4\}}$	1.56642 $^{\{7\}}$	1.04020 $^{\{6\}}$	1.58796 $^{\{8\}}$	0.59038 $^{\{1\}}$	0.86476 $^{\{3\}}$	0.71468 $^{\{2\}}$	1.00968 $^{\{5\}}$
		$\hat{b}$	0.49576 $^{\{8\}}$	0.34945 $^{\{6\}}$	0.33879 $^{\{4\}}$	0.38039 $^{\{7\}}$	0.27410 $^{\{1\}}$	0.30581 $^{\{2\}}$	0.33056 $^{\{3\}}$	0.34608 $^{\{5\}}$
		$\hat{\theta}$	0.38947 $^{\{8\}}$	0.35533 $^{\{6\}}$	0.34027 $^{\{4\}}$	0.36634 $^{\{7\}}$	0.28950 $^{\{1\}}$	0.33512 $^{\{2\}}$	0.33856 $^{\{3\}}$	0.35082 $^{\{5\}}$
	MRE	$\hat{a}$	0.54770 $^{\{5\}}$	0.66757 $^{\{7\}}$	0.55776 $^{\{6\}}$	0.67199 $^{\{8\}}$	0.42789 $^{\{1\}}$	0.51548 $^{\{3\}}$	0.49232 $^{\{2\}}$	0.53607 $^{\{4\}}$
		$\hat{b}$	0.70833 $^{\{8\}}$	0.66614 $^{\{6\}}$	0.63228 $^{\{4\}}$	0.69108 $^{\{7\}}$	0.53272 $^{\{1\}}$	0.62504 $^{\{3\}}$	0.61705 $^{\{2\}}$	0.64912 $^{\{5\}}$
		$\sum$ *Ranks*	**62** $^{\{7\}}$	**55** $^{\{6\}}$	**41** $^{\{4\}}$	**66** $^{\{8\}}$	**9** $^{\{1\}}$	**23** $^{\{2\}}$	**24** $^{\{3\}}$	**44** $^{\{5\}}$

Table 3. *Cont.*

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
150	|BIAS|	$\hat{\theta}$	0.15124 {8}	0.14042 {6}	0.13267 {3}	0.14267 {7}	0.11206 {1}	0.13651 {5}	0.13449 {4}	0.13228 {2}
		$\hat{a}$	0.47732 {4}	0.60202 {8}	0.50525 {6}	0.59714 {7}	0.38293 {1}	0.46043 {3}	0.46029 {2}	0.49195 {5}
		$\hat{b}$	0.45132 {8}	0.44019 {6}	0.40865 {2}	0.44673 {7}	0.33799 {1}	0.42368 {5}	0.40870 {3}	0.41391 {4}
	MSE	$\hat{\theta}$	0.04902 {8}	0.03189 {5}	0.03116 {4}	0.03371 {7}	0.02628 {1}	0.02882 {3}	0.03259 {6}	0.02804 {2}
		$\hat{a}$	0.37974 {4}	0.70390 {7}	0.45079 {6}	0.70633 {8}	0.27144 {1}	0.34252 {2}	0.34810 {3}	0.40653 {5}
		$\hat{b}$	0.39695 {8}	0.29003 {6}	0.27268 {4}	0.30653 {7}	0.22430 {1}	0.26986 {3}	0.27967 {5}	0.26222 {2}
	MRE	$\hat{\theta}$	0.33610 {8}	0.31205 {6}	0.29483 {3}	0.31705 {7}	0.24903 {1}	0.30337 {5}	0.29886 {4}	0.29395 {2}
		$\hat{a}$	0.38185 {4}	0.48161 {8}	0.40420 {6}	0.47771 {7}	0.30634 {1}	0.36834 {3}	0.36823 {2}	0.39356 {5}
		$\hat{b}$	0.60176 {8}	0.58691 {6}	0.54487 {2}	0.59564 {7}	0.45066 {1}	0.56490 {5}	0.54493 {3}	0.55188 {4}
		$\sum$ *Ranks*	**60** {7}	**58** {6}	**36** {5}	**64** {8}	**9** {1}	**34** {4}	**32** {3}	**31** {2}
500	|BIAS|	$\hat{\theta}$	0.08437 {4}	0.09794 {7}	0.08291 {3}	0.09950 {8}	0.06040 {1}	0.09353 {6}	0.08118 {2}	0.09108 {5}
		$\hat{a}$	0.25668 {2}	0.34439 {8}	0.28098 {4}	0.34214 {7}	0.19507 {1}	0.28700 {5}	0.26743 {3}	0.29363 {6}
		$\hat{b}$	0.25392 {3}	0.31025 {7}	0.25828 {4}	0.31555 {8}	0.18344 {1}	0.29413 {6}	0.25194 {2}	0.28723 {5}
	MSE	$\hat{\theta}$	0.02140 {8}	0.01919 {6}	0.01567 {3}	0.01993 {7}	0.00991 {1}	0.01663 {5}	0.01546 {2}	0.01597 {4}
		$\hat{a}$	0.10848 {2}	0.18653 {8}	0.12608 {5}	0.18316 {7}	0.07553 {1}	0.12370 {4}	0.11417 {3}	0.13194 {6}
		$\hat{b}$	0.17186 {6}	0.17618 {7}	0.13883 {3}	0.18449 {8}	0.08770 {1}	0.15746 {5}	0.13607 {2}	0.15036 {4}
	MRE	$\hat{\theta}$	0.18748 {4}	0.21764 {7}	0.18424 {3}	0.22111 {8}	0.13422 {1}	0.20785 {6}	0.18040 {2}	0.20239 {5}
		$\hat{a}$	0.20534 {2}	0.27551 {8}	0.22479 {4}	0.27371 {7}	0.15605 {1}	0.22960 {5}	0.21395 {3}	0.23491 {6}
		$\hat{b}$	0.33856 {3}	0.41367 {7}	0.34437 {4}	0.42074 {8}	0.24459 {1}	0.39218 {6}	0.33592 {2}	0.38297 {5}
		$\sum$ *Ranks*	**34** {4}	**65** {7}	**33** {3}	**68** {8}	**9** {1}	**48** {6}	**21** {2}	**46** {5}

Table 4. Simulation results of the OLxLx distribution for $\psi = (\theta = 0.75, a = 2.25, b = 0.75)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	0.23968 {6}	0.23147 {5}	0.22102 {4}	0.24444 {7}	0.17893 {1}	0.20942 {2}	0.21395 {3}	0.25818 {8}
		$\hat{a}$	2.91735 {5}	3.28627 {7}	2.74133 {4}	3.39951 {8}	2.62260 {3}	2.54481 {2}	2.41761 {1}	3.26759 {6}
		$\hat{b}$	1.45806 {3}	1.66848 {6}	1.51042 {5}	1.82930 {7}	1.16339 {1}	1.25652 {2}	1.47694 {4}	1.94695 {8}
	MSE	$\hat{\theta}$	0.12823 {2}	0.14443 {5}	0.14332 {4}	0.14974 {6}	0.10791 {1}	0.19768 {8}	0.14236 {3}	0.19134 {7}
		$\hat{a}$	20.87416 {5}	24.51158 {6}	17.47100 {3}	26.70102 {7}	17.54837 {4}	14.06944 {2}	13.88508 {1}	28.76561 {8}
		$\hat{b}$	7.28694 {4}	10.45758 {6}	7.61466 {5}	10.59888 {7}	3.93918 {1}	4.97696 {2}	7.07777 {3}	12.39200 {8}
	MRE	$\hat{\theta}$	0.53262 {6}	0.51438 {5}	0.49115 {4}	0.54320 {7}	0.39763 {1}	0.46537 {2}	0.47544 {3}	0.57374 {8}
		$\hat{a}$	1.16694 {5}	1.31451 {7}	1.09653 {4}	1.35981 {8}	1.04904 {3}	1.01792 {2}	0.96704 {1}	1.30704 {6}
		$\hat{b}$	1.94408 {3}	2.22463 {6}	2.01389 {5}	2.43906 {7}	1.55119 {1}	1.67536 {2}	1.96926 {4}	2.59593 {8}
		$\sum$ *Ranks*	**39** {5}	**53** {6}	**38** {4}	**64** {7}	**16** {1}	**24** {3}	**23** {2}	**67** {8}
50	|BIAS|	$\hat{\theta}$	0.16705 {7}	0.15814 {5}	0.13978 {4}	0.16373 {6}	0.10665 {1}	0.11584 {2}	0.12648 {3}	0.16772 {8}
		$\hat{a}$	1.30738 {2}	1.91965 {7}	1.49445 {5}	1.97249 {8}	1.29091 {1}	1.33338 {3}	1.35028 {4}	1.56424 {6}
		$\hat{b}$	0.74298 {3}	1.07521 {6}	0.91501 {5}	1.12159 {7}	0.67854 {1}	0.73584 {2}	0.82934 {4}	1.17632 {8}
	MSE	$\hat{\theta}$	0.06862 {5}	0.07277 {6}	0.06800 {4}	0.07772 {7}	0.03328 {1}	0.04708 {2}	0.04802 {3}	0.09340 {8}
		$\hat{a}$	4.09243 {4}	8.49402 {7}	4.95803 {5}	9.18748 {8}	4.07534 {3}	3.77715 {1}	3.86803 {2}	6.17941 {6}
		$\hat{b}$	1.92597 {3}	3.53331 {6}	2.81227 {5}	3.88463 {7}	1.42982 {1}	1.85209 {2}	2.21586 {4}	4.81919 {8}
	MRE	$\hat{\theta}$	0.37122 {7}	0.35142 {5}	0.31063 {4}	0.36385 {6}	0.23701 {1}	0.25742 {2}	0.28107 {3}	0.37270 {8}
		$\hat{a}$	0.52295 {2}	0.76786 {7}	0.59778 {5}	0.78900 {8}	0.51637 {1}	0.53335 {3}	0.54011 {4}	0.62570 {6}
		$\hat{b}$	0.99064 {3}	1.43362 {6}	1.22001 {5}	1.49546 {7}	0.90472 {1}	0.98112 {2}	1.10579 {4}	1.56842 {8}
		$\sum$ *Ranks*	**36** {4}	**55** {6}	**42** {5}	**64** {7}	**11** {1}	**19** {2}	**31** {3}	**66** {8}
80	|BIAS|	$\hat{\theta}$	0.13526 {8}	0.12115 {5}	0.09503 {4}	0.12680 {7}	0.07186 {1}	0.08149 {2}	0.09389 {3}	0.12401 {6}
		$\hat{a}$	0.93572 {2}	1.39353 {7}	1.09376 {5}	1.46736 {8}	0.83871 {1}	0.94471 {3}	0.98484 {4}	1.15441 {6}
		$\hat{b}$	0.51086 {2}	0.80630 {6}	0.62649 {5}	0.85715 {8}	0.45598 {1}	0.51701 {3}	0.61257 {4}	0.84097 {7}
	MSE	$\hat{\theta}$	0.04921 {6}	0.04683 {5}	0.03177 {4}	0.04956 {7}	0.01573 {1}	0.02220 {2}	0.02939 {3}	0.06172 {8}
		$\hat{a}$	1.98654 {4}	4.29261 {7}	2.51870 {5}	4.90159 {8}	1.49529 {1}	1.78467 {2}	1.82646 {3}	2.93187 {6}
		$\hat{b}$	0.89594 {2}	2.15718 {6}	1.38144 {5}	2.31932 {7}	0.67688 {1}	0.90584 {3}	1.29609 {4}	2.79171 {8}
	MRE	$\hat{\theta}$	0.30058 {8}	0.26922 {5}	0.21118 {4}	0.28179 {7}	0.15969 {1}	0.18109 {2}	0.20865 {3}	0.27557 {6}
		$\hat{a}$	0.37429 {2}	0.55741 {7}	0.43750 {5}	0.58695 {8}	0.33548 {1}	0.37788 {3}	0.39394 {4}	0.46176 {6}
		$\hat{b}$	0.68114 {2}	1.07507 {6}	0.83532 {5}	1.14287 {8}	0.60797 {1}	0.68935 {3}	0.81676 {4}	1.12130 {7}
		$\sum$ *Ranks*	**36** {4}	**54** {6}	**42** {5}	**68** {8}	**9** {1}	**23** {2}	**32** {3}	**60** {7}
150	|BIAS|	$\hat{\theta}$	0.10901 {8}	0.07948 {7}	0.05681 {4}	0.07720 {6}	0.04561 {1}	0.04971 {2}	0.05503 {3}	0.07192 {5}
		$\hat{a}$	0.61630 {2}	0.95944 {7}	0.73640 {5}	0.97608 {8}	0.56206 {1}	0.63122 {3}	0.68043 {4}	0.75896 {6}
		$\hat{b}$	0.31885 {3}	0.53323 {8}	0.37083 {5}	0.51832 {7}	0.28853 {1}	0.31577 {2}	0.35870 {4}	0.48517 {6}
	MSE	$\hat{\theta}$	0.03620 {8}	0.02271 {7}	0.01019 {4}	0.01957 {6}	0.00555 {1}	0.00686 {2}	0.00901 {3}	0.01852 {5}
		$\hat{a}$	0.69440 {2}	1.84387 {7}	1.00571 {5}	1.89282 {8}	0.60456 {1}	0.69777 {3}	0.81534 {4}	1.02964 {6}
		$\hat{b}$	0.24442 {2}	1.01467 {8}	0.45326 {5}	0.88502 {6}	0.23513 {1}	0.28800 {3}	0.40393 {4}	0.88843 {7}
	MRE	$\hat{\theta}$	0.24225 {8}	0.17662 {7}	0.12625 {4}	0.17155 {6}	0.10136 {1}	0.11047 {2}	0.12228 {3}	0.15983 {5}
		$\hat{a}$	0.24652 {2}	0.38378 {7}	0.29456 {5}	0.39043 {8}	0.22482 {1}	0.25249 {3}	0.27217 {4}	0.30358 {6}
		$\hat{b}$	0.42513 {3}	0.71097 {8}	0.49443 {5}	0.69110 {7}	0.38470 {1}	0.42103 {2}	0.47826 {4}	0.64690 {6}
		$\sum$ *Ranks*	**38** {4}	**66** {8}	**42** {5}	**62** {7}	**9** {1}	**22** {2}	**33** {3}	**52** {6}
500	|BIAS|	$\hat{\theta}$	0.09296 {8}	0.03261 {6}	0.02553 {4}	0.03274 {7}	0.01972 {1}	0.02302 {2}	0.02446 {3}	0.02908 {5}
		$\hat{a}$	0.33940 {3}	0.47663 {7}	0.37463 {5}	0.47751 {8}	0.24731 {1}	0.32140 {2}	0.35821 {4}	0.39312 {6}
		$\hat{b}$	0.18648 {5}	0.21740 {7}	0.16329 {4}	0.21798 {8}	0.12073 {1}	0.14478 {2}	0.15642 {3}	0.19021 {6}
	MSE	$\hat{\theta}$	0.03431 {8}	0.00237 {6}	0.00121 {4}	0.00241 {7}	0.00076 {1}	0.00095 {2}	0.00106 {3}	0.00173 {5}
		$\hat{a}$	0.17417 {3}	0.37906 {7}	0.23114 {5}	0.38001 {8}	0.13540 {1}	0.16907 {2}	0.20571 {4}	0.25308 {6}
		$\hat{b}$	0.06637 {5}	0.10772 {7}	0.05106 {4}	0.10957 {8}	0.03013 {1}	0.03878 {2}	0.04416 {3}	0.07921 {6}
	MRE	$\hat{\theta}$	0.20659 {8}	0.07248 {6}	0.05672 {4}	0.07275 {7}	0.04382 {1}	0.05115 {2}	0.05436 {3}	0.06462 {5}
		$\hat{a}$	0.13576 {3}	0.19065 {7}	0.14985 {5}	0.19100 {8}	0.09893 {1}	0.12856 {2}	0.14328 {4}	0.15725 {6}
		$\hat{b}$	0.24864 {5}	0.28986 {7}	0.21771 {4}	0.29065 {8}	0.16097 {1}	0.19304 {2}	0.20856 {3}	0.25362 {6}
		$\sum$ *Ranks*	**48** {5}	**60** {7}	**39** {4}	**69** {8}	**9** {1}	**18** {2}	**30** {3}	**51** {6}

Table 5. Simulation results of the OLxLx distribution for $\psi = (\theta = 1.5, a = 2.25, b = 2.5)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	0.26440 $^{\{4\}}$	0.27140 $^{\{5\}}$	0.27404 $^{\{6\}}$	0.28624 $^{\{7\}}$	0.21756 $^{\{1\}}$	0.24609 $^{\{2\}}$	0.25505 $^{\{3\}}$	0.29296 $^{\{8\}}$
		$\hat{a}$	3.71244 $^{\{5\}}$	4.32795 $^{\{7\}}$	3.28525 $^{\{4\}}$	4.50763 $^{\{8\}}$	2.95481 $^{\{3\}}$	2.73069 $^{\{1\}}$	2.75979 $^{\{2\}}$	3.82620 $^{\{6\}}$
		$\hat{b}$	10.78117 $^{\{6\}}$	9.69086 $^{\{5\}}$	9.57232 $^{\{4\}}$	12.12504 $^{\{8\}}$	6.27003 $^{\{1\}}$	7.24078 $^{\{2\}}$	8.67848 $^{\{3\}}$	11.98645 $^{\{7\}}$
	MSE	$\hat{\theta}$	0.17705 $^{\{8\}}$	0.15639 $^{\{4\}}$	0.16293 $^{\{5\}}$	0.17487 $^{\{6\}}$	0.11991 $^{\{1\}}$	0.14341 $^{\{2\}}$	0.15146 $^{\{3\}}$	0.17661 $^{\{7\}}$
		$\hat{a}$	52.93173 $^{\{5\}}$	65.36489 $^{\{7\}}$	39.51464 $^{\{4\}}$	71.26457 $^{\{8\}}$	36.51284 $^{\{3\}}$	22.02327 $^{\{1\}}$	27.77793 $^{\{2\}}$	61.75642 $^{\{6\}}$
		$\hat{b}$	495.06002 $^{\{5\}}$	694.26802 $^{\{6\}}$	402.32698 $^{\{4\}}$	999.80169 $^{\{8\}}$	141.10150 $^{\{1\}}$	192.83668 $^{\{2\}}$	283.71288 $^{\{3\}}$	787.63885 $^{\{7\}}$
	MRE	$\hat{\theta}$	0.58755 $^{\{4\}}$	0.60311 $^{\{5\}}$	0.60898 $^{\{6\}}$	0.63610 $^{\{7\}}$	0.48346 $^{\{1\}}$	0.54687 $^{\{2\}}$	0.56679 $^{\{3\}}$	0.65103 $^{\{8\}}$
		$\hat{a}$	1.48498 $^{\{5\}}$	1.73118 $^{\{7\}}$	1.31410 $^{\{4\}}$	1.80305 $^{\{8\}}$	1.18192 $^{\{3\}}$	1.09228 $^{\{1\}}$	1.10392 $^{\{2\}}$	1.53048 $^{\{6\}}$
		$\hat{b}$	4.31247 $^{\{6\}}$	3.87635 $^{\{5\}}$	3.82893 $^{\{4\}}$	4.85002 $^{\{8\}}$	2.50801 $^{\{1\}}$	2.89631 $^{\{2\}}$	3.47139 $^{\{3\}}$	4.79458 $^{\{7\}}$
		$\sum$ *Ranks*	**48** $^{\{5\}}$	**51** $^{\{6\}}$	**41** $^{\{4\}}$	**68** $^{\{8\}}$	**15** $^{\{1.5\}}$	**15** $^{\{1.5\}}$	**24** $^{\{3\}}$	**62** $^{\{7\}}$
50	|BIAS|	$\hat{\theta}$	0.14771 $^{\{2\}}$	0.21454 $^{\{8\}}$	0.18365 $^{\{5\}}$	0.21389 $^{\{7\}}$	0.13210 $^{\{1\}}$	0.15862 $^{\{3\}}$	0.16607 $^{\{4\}}$	0.21055 $^{\{6\}}$
		$\hat{a}$	1.36063 $^{\{4\}}$	1.92830 $^{\{7\}}$	1.52590 $^{\{5\}}$	2.06722 $^{\{8\}}$	1.18536 $^{\{1\}}$	1.32585 $^{\{3\}}$	1.30724 $^{\{2\}}$	1.58710 $^{\{6\}}$
		$\hat{b}$	4.63894 $^{\{3\}}$	6.39807 $^{\{6\}}$	5.52336 $^{\{5\}}$	6.84825 $^{\{7\}}$	3.63832 $^{\{1\}}$	4.30041 $^{\{2\}}$	4.93046 $^{\{4\}}$	7.04501 $^{\{8\}}$
	MSE	$\hat{\theta}$	0.06982 $^{\{2\}}$	0.11457 $^{\{6\}}$	0.09479 $^{\{5\}}$	0.11565 $^{\{8\}}$	0.05058 $^{\{1\}}$	0.07876 $^{\{3\}}$	0.08433 $^{\{4\}}$	0.11493 $^{\{7\}}$
		$\hat{a}$	4.91841 $^{\{4\}}$	10.07173 $^{\{7\}}$	5.66180 $^{\{5\}}$	12.86291 $^{\{8\}}$	3.97461 $^{\{3\}}$	3.60574 $^{\{1\}}$	3.71836 $^{\{2\}}$	6.04345 $^{\{6\}}$
		$\hat{b}$	93.50937 $^{\{3\}}$	133.55169 $^{\{6\}}$	109.88900 $^{\{5\}}$	155.23946 $^{\{7\}}$	49.53378 $^{\{1\}}$	69.82118 $^{\{2\}}$	93.84751 $^{\{4\}}$	170.36460 $^{\{8\}}$
	MRE	$\hat{\theta}$	0.32825 $^{\{2\}}$	0.47676 $^{\{8\}}$	0.40811 $^{\{5\}}$	0.47532 $^{\{7\}}$	0.29355 $^{\{1\}}$	0.35249 $^{\{3\}}$	0.36904 $^{\{4\}}$	0.46788 $^{\{6\}}$
		$\hat{a}$	0.54425 $^{\{4\}}$	0.77132 $^{\{7\}}$	0.61036 $^{\{5\}}$	0.82689 $^{\{8\}}$	0.47415 $^{\{1\}}$	0.53034 $^{\{3\}}$	0.52290 $^{\{2\}}$	0.63484 $^{\{6\}}$
		$\hat{b}$	1.85558 $^{\{3\}}$	2.55923 $^{\{6\}}$	2.20935 $^{\{5\}}$	2.73930 $^{\{7\}}$	1.45533 $^{\{1\}}$	1.72016 $^{\{2\}}$	1.97218 $^{\{4\}}$	2.81800 $^{\{8\}}$
		$\sum$ *Ranks*	**27** $^{\{3\}}$	**61** $^{\{6.5\}}$	**45** $^{\{5\}}$	**67** $^{\{8\}}$	**11** $^{\{1\}}$	**22** $^{\{2\}}$	**30** $^{\{4\}}$	**61** $^{\{6.5\}}$
80	|BIAS|	$\hat{\theta}$	0.10736 $^{\{2\}}$	0.17533 $^{\{8\}}$	0.13566 $^{\{5\}}$	0.17409 $^{\{7\}}$	0.09045 $^{\{1\}}$	0.11837 $^{\{3\}}$	0.12209 $^{\{4\}}$	0.16353 $^{\{6\}}$
		$\hat{a}$	0.93766 $^{\{2\}}$	1.34715 $^{\{7\}}$	1.11537 $^{\{5\}}$	1.42250 $^{\{8\}}$	0.82833 $^{\{1\}}$	0.95822 $^{\{3\}}$	0.96822 $^{\{4\}}$	1.16783 $^{\{6\}}$
		$\hat{b}$	3.08584 $^{\{2\}}$	5.12435 $^{\{6\}}$	3.95277 $^{\{5\}}$	5.25137 $^{\{8\}}$	2.38867 $^{\{1\}}$	3.19872 $^{\{3\}}$	3.48486 $^{\{4\}}$	5.18981 $^{\{7\}}$
	MSE	$\hat{\theta}$	0.04925 $^{\{2\}}$	0.08678 $^{\{8\}}$	0.05990 $^{\{5\}}$	0.08664 $^{\{7\}}$	0.02617 $^{\{1\}}$	0.05108 $^{\{3\}}$	0.05196 $^{\{4\}}$	0.07994 $^{\{6\}}$
		$\hat{a}$	1.69040 $^{\{4\}}$	3.48281 $^{\{7\}}$	2.45678 $^{\{5\}}$	4.36749 $^{\{8\}}$	1.29237 $^{\{1\}}$	1.62218 $^{\{2\}}$	1.65889 $^{\{3\}}$	2.56983 $^{\{6\}}$
		$\hat{b}$	47.37410 $^{\{3\}}$	91.33746 $^{\{6\}}$	65.23730 $^{\{5\}}$	97.89241 $^{\{7\}}$	23.27946 $^{\{1\}}$	44.24889 $^{\{2\}}$	52.08591 $^{\{4\}}$	101.41735 $^{\{8\}}$
	MRE	$\hat{\theta}$	0.23857 $^{\{2\}}$	0.38962 $^{\{8\}}$	0.30146 $^{\{5\}}$	0.38687 $^{\{7\}}$	0.20101 $^{\{1\}}$	0.26305 $^{\{3\}}$	0.27130 $^{\{4\}}$	0.36341 $^{\{6\}}$
		$\hat{a}$	0.37506 $^{\{2\}}$	0.53886 $^{\{7\}}$	0.44615 $^{\{5\}}$	0.56900 $^{\{8\}}$	0.33133 $^{\{1\}}$	0.38329 $^{\{3\}}$	0.38729 $^{\{4\}}$	0.46713 $^{\{6\}}$
		$\hat{b}$	1.23434 $^{\{2\}}$	2.04974 $^{\{6\}}$	1.58111 $^{\{5\}}$	2.10055 $^{\{8\}}$	0.95547 $^{\{1\}}$	1.27949 $^{\{3\}}$	1.39394 $^{\{4\}}$	2.07592 $^{\{7\}}$
		$\sum$ *Ranks*	**21** $^{\{2\}}$	**63** $^{\{7\}}$	**45** $^{\{5\}}$	**68** $^{\{8\}}$	**9** $^{\{1\}}$	**25** $^{\{3\}}$	**35** $^{\{4\}}$	**58** $^{\{6\}}$
150	|BIAS|	$\hat{\theta}$	0.06372 $^{\{2\}}$	0.12053 $^{\{8\}}$	0.08563 $^{\{5\}}$	0.11880 $^{\{7\}}$	0.05138 $^{\{1\}}$	0.06648 $^{\{3\}}$	0.07851 $^{\{4\}}$	0.10481 $^{\{6\}}$
		$\hat{a}$	0.63776 $^{\{2\}}$	0.98124 $^{\{8\}}$	0.77219 $^{\{5\}}$	0.96167 $^{\{7\}}$	0.53420 $^{\{1\}}$	0.65510 $^{\{3\}}$	0.71271 $^{\{4\}}$	0.81705 $^{\{6\}}$
		$\hat{b}$	1.70773 $^{\{2\}}$	3.41183 $^{\{7\}}$	2.31767 $^{\{5\}}$	3.42894 $^{\{8\}}$	1.29135 $^{\{1\}}$	1.71092 $^{\{3\}}$	2.13844 $^{\{4\}}$	3.13376 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.01571 $^{\{2\}}$	0.04865 $^{\{8\}}$	0.02657 $^{\{5\}}$	0.04830 $^{\{7\}}$	0.00898 $^{\{1\}}$	0.01606 $^{\{3\}}$	0.02321 $^{\{4\}}$	0.03890 $^{\{6\}}$
		$\hat{a}$	0.70234 $^{\{2\}}$	1.69995 $^{\{8\}}$	0.99417 $^{\{5\}}$	1.62977 $^{\{7\}}$	0.56126 $^{\{1\}}$	0.72199 $^{\{3\}}$	0.85644 $^{\{4\}}$	1.12625 $^{\{6\}}$
		$\hat{b}$	14.13687 $^{\{3\}}$	47.08032 $^{\{7\}}$	24.08381 $^{\{5\}}$	49.03672 $^{\{8\}}$	7.61965 $^{\{1\}}$	13.55833 $^{\{2\}}$	21.23868 $^{\{4\}}$	43.27343 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.14159 $^{\{2\}}$	0.26784 $^{\{8\}}$	0.19029 $^{\{5\}}$	0.26399 $^{\{7\}}$	0.11419 $^{\{1\}}$	0.14773 $^{\{3\}}$	0.17446 $^{\{4\}}$	0.23292 $^{\{6\}}$
		$\hat{a}$	0.25510 $^{\{2\}}$	0.39250 $^{\{8\}}$	0.30888 $^{\{5\}}$	0.38467 $^{\{7\}}$	0.21368 $^{\{1\}}$	0.26204 $^{\{3\}}$	0.28508 $^{\{4\}}$	0.32682 $^{\{6\}}$
		$\hat{b}$	0.68309 $^{\{2\}}$	1.36473 $^{\{7\}}$	0.92707 $^{\{5\}}$	1.37158 $^{\{8\}}$	0.51654 $^{\{1\}}$	0.68437 $^{\{3\}}$	0.85538 $^{\{4\}}$	1.25350 $^{\{6\}}$
		$\sum$ *Ranks*	**19** $^{\{2\}}$	**69** $^{\{8\}}$	**45** $^{\{5\}}$	**66** $^{\{7\}}$	**9** $^{\{1\}}$	**26** $^{\{3\}}$	**36** $^{\{4\}}$	**54** $^{\{6\}}$
500	|BIAS|	$\hat{\theta}$	0.02611 $^{\{2\}}$	0.04740 $^{\{8\}}$	0.03164 $^{\{5\}}$	0.04637 $^{\{7\}}$	0.01577 $^{\{1\}}$	0.02615 $^{\{3\}}$	0.03087 $^{\{4\}}$	0.03989 $^{\{6\}}$
		$\hat{a}$	0.31961 $^{\{2\}}$	0.51418 $^{\{8\}}$	0.38211 $^{\{5\}}$	0.50490 $^{\{7\}}$	0.20047 $^{\{1\}}$	0.32053 $^{\{3\}}$	0.37145 $^{\{4\}}$	0.43745 $^{\{6\}}$
		$\hat{b}$	0.62810 $^{\{3\}}$	1.22797 $^{\{8\}}$	0.77588 $^{\{5\}}$	1.19235 $^{\{7\}}$	0.34998 $^{\{1\}}$	0.62185 $^{\{2\}}$	0.75559 $^{\{4\}}$	1.04335 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.00126 $^{\{3\}}$	0.00681 $^{\{8\}}$	0.00200 $^{\{5\}}$	0.00651 $^{\{7\}}$	0.00091 $^{\{1\}}$	0.00121 $^{\{2\}}$	0.00186 $^{\{4\}}$	0.00419 $^{\{6\}}$
		$\hat{a}$	0.16168 $^{\{2\}}$	0.43448 $^{\{8\}}$	0.23739 $^{\{5\}}$	0.42697 $^{\{7\}}$	0.11103 $^{\{1\}}$	0.16197 $^{\{3\}}$	0.22183 $^{\{4\}}$	0.31511 $^{\{6\}}$
		$\hat{b}$	0.80855 $^{\{3\}}$	5.85144 $^{\{8\}}$	1.37521 $^{\{5\}}$	5.31883 $^{\{7\}}$	0.57506 $^{\{1\}}$	0.73520 $^{\{2\}}$	1.25554 $^{\{4\}}$	3.78249 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.05803 $^{\{2\}}$	0.10534 $^{\{8\}}$	0.07030 $^{\{5\}}$	0.10304 $^{\{7\}}$	0.03505 $^{\{1\}}$	0.05812 $^{\{3\}}$	0.06860 $^{\{4\}}$	0.08865 $^{\{6\}}$
		$\hat{a}$	0.12784 $^{\{2\}}$	0.20567 $^{\{8\}}$	0.15284 $^{\{5\}}$	0.20196 $^{\{7\}}$	0.08019 $^{\{1\}}$	0.12821 $^{\{3\}}$	0.14858 $^{\{4\}}$	0.17498 $^{\{6\}}$
		$\hat{b}$	0.25124 $^{\{3\}}$	0.49119 $^{\{8\}}$	0.31035 $^{\{5\}}$	0.47694 $^{\{7\}}$	0.13999 $^{\{1\}}$	0.24874 $^{\{2\}}$	0.30223 $^{\{4\}}$	0.41734 $^{\{6\}}$
		$\sum$ *Ranks*	**22** $^{\{2\}}$	**72** $^{\{8\}}$	**45** $^{\{5\}}$	**63** $^{\{7\}}$	**9** $^{\{1\}}$	**23** $^{\{3\}}$	**36** $^{\{4\}}$	**54** $^{\{6\}}$

Table 6. Simulation results of the OLxLx distribution for $\psi = (\theta = 1.5, a = 2.25, b = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	0.23878 $^{\{4\}}$	0.24396 $^{\{5\}}$	0.24627 $^{\{6\}}$	0.26625 $^{\{7\}}$	0.20949 $^{\{1\}}$	0.21562 $^{\{2\}}$	0.21892 $^{\{3\}}$	0.27273 $^{\{8\}}$
		$\hat{a}$	3.26900 $^{\{5\}}$	3.78271 $^{\{7\}}$	3.12508 $^{\{4\}}$	3.96325 $^{\{8\}}$	2.94905 $^{\{3\}}$	2.63540 $^{\{2\}}$	2.62525 $^{\{1\}}$	3.40534 $^{\{6\}}$
		$\hat{b}$	3.17062 $^{\{4\}}$	3.30235 $^{\{6\}}$	3.25808 $^{\{5\}}$	4.08908 $^{\{7\}}$	2.46608 $^{\{1\}}$	2.49712 $^{\{2\}}$	2.88089 $^{\{3\}}$	4.11162 $^{\{8\}}$
	MSE	$\hat{\theta}$	0.14494 $^{\{4\}}$	0.13885 $^{\{2\}}$	0.15604 $^{\{5\}}$	0.15729 $^{\{6\}}$	0.20220 $^{\{8\}}$	0.14180 $^{\{3\}}$	0.13783 $^{\{1\}}$	0.16791 $^{\{7\}}$
		$\hat{a}$	31.52626 $^{\{5\}}$	39.33780 $^{\{7\}}$	28.02026 $^{\{4\}}$	45.05061 $^{\{8\}}$	27.73351 $^{\{3\}}$	17.33411 $^{\{1\}}$	20.61758 $^{\{2\}}$	38.53511 $^{\{6\}}$
		$\hat{b}$	35.86071 $^{\{4\}}$	42.53578 $^{\{6\}}$	38.15151 $^{\{5\}}$	75.54361 $^{\{8\}}$	25.62682 $^{\{2\}}$	20.55147 $^{\{1\}}$	29.72342 $^{\{3\}}$	58.14463 $^{\{7\}}$
	MRE	$\hat{\theta}$	0.53063 $^{\{4\}}$	0.54214 $^{\{5\}}$	0.54727 $^{\{6\}}$	0.59167 $^{\{7\}}$	0.46553 $^{\{1\}}$	0.47915 $^{\{2\}}$	0.48649 $^{\{3\}}$	0.60608 $^{\{8\}}$
		$\hat{a}$	1.30760 $^{\{5\}}$	1.51308 $^{\{7\}}$	1.25003 $^{\{4\}}$	1.58530 $^{\{8\}}$	1.17962 $^{\{3\}}$	1.05416 $^{\{2\}}$	1.05010 $^{\{1\}}$	1.36214 $^{\{6\}}$
		$\hat{b}$	2.53650 $^{\{4\}}$	2.64188 $^{\{6\}}$	2.60647 $^{\{5\}}$	3.27127 $^{\{7\}}$	1.97286 $^{\{1\}}$	1.99769 $^{\{2\}}$	2.30471 $^{\{3\}}$	3.28929 $^{\{8\}}$
		$\sum$ *Ranks*	**39** $^{\{4\}}$	**51** $^{\{6\}}$	**44** $^{\{5\}}$	**66** $^{\{8\}}$	**23** $^{\{3\}}$	**17** $^{\{1\}}$	**20** $^{\{2\}}$	**64** $^{\{7\}}$
50	|BIAS|	$\hat{\theta}$	0.14636 $^{\{4\}}$	0.18289 $^{\{6\}}$	0.15546 $^{\{5\}}$	0.18599 $^{\{8\}}$	0.11457 $^{\{1\}}$	0.13984 $^{\{3\}}$	0.13544 $^{\{2\}}$	0.18408 $^{\{7\}}$
		$\hat{a}$	1.40142 $^{\{4\}}$	1.91400 $^{\{7\}}$	1.49214 $^{\{5\}}$	2.01377 $^{\{8\}}$	1.22101 $^{\{1\}}$	1.27504 $^{\{2\}}$	1.32233 $^{\{3\}}$	1.52051 $^{\{6\}}$
		$\hat{b}$	1.54273 $^{\{2\}}$	2.24324 $^{\{6\}}$	1.92478 $^{\{5\}}$	2.45685 $^{\{7\}}$	1.33734 $^{\{1\}}$	1.56097 $^{\{3\}}$	1.67967 $^{\{4\}}$	2.46450 $^{\{8\}}$
	MSE	$\hat{\theta}$	0.07482 $^{\{3\}}$	0.08933 $^{\{6\}}$	0.08058 $^{\{4\}}$	0.09458 $^{\{7\}}$	0.03882 $^{\{1\}}$	0.08081 $^{\{5\}}$	0.06196 $^{\{2\}}$	0.09814 $^{\{8\}}$
		$\hat{a}$	5.24588 $^{\{4\}}$	9.16209 $^{\{7\}}$	5.50591 $^{\{5\}}$	11.09046 $^{\{8\}}$	3.87622 $^{\{2\}}$	3.44698 $^{\{1\}}$	3.98643 $^{\{3\}}$	5.69719 $^{\{6\}}$
		$\hat{b}$	9.61286 $^{\{3\}}$	15.48749 $^{\{6\}}$	13.75485 $^{\{5\}}$	19.90902 $^{\{7\}}$	6.22504 $^{\{1\}}$	9.33753 $^{\{2\}}$	10.79651 $^{\{4\}}$	20.73653 $^{\{8\}}$
	MRE	$\hat{\theta}$	0.32526 $^{\{4\}}$	0.40641 $^{\{6\}}$	0.34546 $^{\{5\}}$	0.41330 $^{\{8\}}$	0.25461 $^{\{1\}}$	0.31075 $^{\{3\}}$	0.30098 $^{\{2\}}$	0.40907 $^{\{7\}}$
		$\hat{a}$	0.56057 $^{\{4\}}$	0.76560 $^{\{7\}}$	0.59686 $^{\{5\}}$	0.80551 $^{\{8\}}$	0.48840 $^{\{1\}}$	0.51002 $^{\{2\}}$	0.52893 $^{\{3\}}$	0.60820 $^{\{6\}}$
		$\hat{b}$	1.23418 $^{\{2\}}$	1.79459 $^{\{6\}}$	1.53982 $^{\{5\}}$	1.96548 $^{\{7\}}$	1.06987 $^{\{1\}}$	1.24878 $^{\{3\}}$	1.34373 $^{\{4\}}$	1.97160 $^{\{8\}}$
		$\sum$ *Ranks*	**30** $^{\{4\}}$	**57** $^{\{6\}}$	**44** $^{\{5\}}$	**68** $^{\{8\}}$	**10** $^{\{1\}}$	**24** $^{\{2\}}$	**27** $^{\{3\}}$	**64** $^{\{7\}}$

Table 6. *Cont.*

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
80	|BIAS|	$\hat{\theta}$	0.11286 {5}	0.14127 {7}	0.11133 {4}	0.14524 {8}	0.07938 {1}	0.09739 {2}	0.09775 {3}	0.14033 {6}
		$\hat{a}$	0.93519 {2}	1.38362 {7}	1.08576 {5}	1.41838 {8}	0.83483 {1}	0.94620 {3}	0.95028 {4}	1.09694 {6}
		$\hat{b}$	1.00720 {2}	1.72331 {6}	1.34372 {5}	1.80902 {7}	0.91205 {1}	1.11050 {3}	1.17674 {4}	1.82842 {8}
	MSE	$\hat{\theta}$	0.03957 {3}	0.06172 {6}	0.04451 {5}	0.06508 {7}	0.01989 {1}	0.04105 {4}	0.03414 {2}	0.06562 {8}
		$\hat{a}$	1.99894 {4}	4.37866 {7}	2.42338 {6}	4.65549 {8}	1.46965 {1}	1.71354 {3}	1.67412 {2}	2.30403 {5}
		$\hat{b}$	4.14404 {2}	10.30272 {6}	7.14115 {5}	11.27000 {7}	3.02250 {1}	5.25501 {3}	5.48062 {4}	12.92332 {8}
	MRE	$\hat{\theta}$	0.25081 {5}	0.31394 {7}	0.24740 {4}	0.32276 {8}	0.17641 {1}	0.21641 {2}	0.21723 {3}	0.31184 {6}
		$\hat{a}$	0.37408 {2}	0.55345 {7}	0.43431 {5}	0.56735 {8}	0.33393 {1}	0.37848 {3}	0.38011 {4}	0.43878 {6}
		$\hat{b}$	0.80576 {2}	1.37865 {6}	1.07498 {5}	1.44721 {7}	0.72964 {1}	0.88840 {3}	0.94139 {4}	1.46274 {8}
		$\sum$ *Ranks*	**27** {3}	**59** {6}	**44** {5}	**68** {8}	**9** {1}	**26** {2}	**30** {4}	**61** {7}
150	|BIAS|	$\hat{\theta}$	0.07916 {5}	0.09234 {7}	0.06598 {4}	0.09739 {8}	0.04744 {1}	0.05370 {2}	0.06188 {3}	0.08371 {6}
		$\hat{a}$	0.60417 {2}	0.93455 {7}	0.71725 {5}	0.94703 {8}	0.55438 {1}	0.62816 {3}	0.67554 {4}	0.78638 {6}
		$\hat{b}$	0.58639 {2}	1.10967 {7}	0.77242 {5}	1.18687 {8}	0.53433 {1}	0.60694 {3}	0.72470 {4}	1.04052 {6}
	MSE	$\hat{\theta}$	0.02172 {5}	0.03049 {7}	0.01572 {4}	0.03319 {8}	0.00646 {1}	0.01076 {2}	0.01303 {3}	0.02561 {6}
		$\hat{a}$	0.63314 {2}	1.69239 {8}	0.90962 {5}	1.69186 {7}	0.59297 {1}	0.68376 {3}	0.77597 {4}	1.10317 {6}
		$\hat{b}$	1.17626 {2}	4.85779 {7}	2.47806 {5}	5.43586 {8}	0.95156 {1}	1.50187 {3}	1.98384 {4}	4.50230 {6}
	MRE	$\hat{\theta}$	0.17592 {5}	0.20520 {7}	0.14662 {4}	0.21642 {8}	0.10542 {1}	0.11933 {2}	0.13751 {3}	0.18603 {6}
		$\hat{a}$	0.24167 {2}	0.37382 {7}	0.28690 {5}	0.37881 {8}	0.22175 {1}	0.25126 {3}	0.27022 {4}	0.31455 {6}
		$\hat{b}$	0.46911 {2}	0.88773 {7}	0.61793 {5}	0.94949 {8}	0.42746 {1}	0.48555 {3}	0.57976 {4}	0.83241 {6}
		$\sum$ *Ranks*	**27** {3}	**64** {7}	**42** {5}	**71** {8}	**9** {1}	**24** {2}	**33** {4}	**54** {6}
500	|BIAS|	$\hat{\theta}$	0.05679 {8}	0.03777 {7}	0.02722 {4}	0.03579 {6}	0.01914 {1}	0.02327 {2}	0.02636 {3}	0.03366 {5}
		$\hat{a}$	0.32101 {3}	0.48260 {8}	0.37000 {5}	0.47775 {7}	0.23830 {1}	0.31328 {2}	0.35561 {4}	0.41255 {6}
		$\hat{b}$	0.26678 {3}	0.43749 {8}	0.30681 {5}	0.41468 {7}	0.20536 {1}	0.25571 {2}	0.29599 {4}	0.39159 {6}
	MSE	$\hat{\theta}$	0.01701 {8}	0.00347 {7}	0.00139 {4}	0.00293 {6}	0.00080 {1}	0.00094 {2}	0.00126 {3}	0.00243 {5}
		$\hat{a}$	0.15952 {3}	0.37621 {8}	0.22276 {5}	0.37201 {7}	0.12676 {1}	0.15874 {2}	0.20217 {4}	0.27346 {6}
		$\hat{b}$	0.12177 {3}	0.51414 {8}	0.18753 {5}	0.42653 {7}	0.09799 {1}	0.11775 {2}	0.16554 {4}	0.36393 {6}
	MRE	$\hat{\theta}$	0.12620 {8}	0.08394 {7}	0.06048 {4}	0.07954 {6}	0.04254 {1}	0.05171 {2}	0.05858 {3}	0.07479 {5}
		$\hat{a}$	0.12840 {3}	0.19304 {8}	0.14800 {5}	0.19110 {7}	0.09532 {1}	0.12531 {2}	0.14224 {4}	0.16502 {6}
		$\hat{b}$	0.21342 {3}	0.35000 {8}	0.24545 {5}	0.33175 {7}	0.16429 {1}	0.20457 {2}	0.23679 {4}	0.31327 {6}
		$\sum$ *Ranks*	**42** {4.5}	**69** {8}	**42** {4.5}	**60** {7}	**9** {1}	**18** {2}	**33** {3}	**51** {6}

Table 7. Simulation results of the OLxLx distribution for $\psi = (\theta = 3, a = 0.35, b = 1.25)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	|BIAS|	$\hat{\theta}$	2.16925 {1}	2.93594 {7}	2.66136 {6}	2.60921 {5}	2.42171 {3}	5.47792 {8}	2.38732 {2}	2.50875 {4}
		$\hat{a}$	0.18321 {1}	0.21043 {3}	0.22429 {7}	0.21662 {6}	0.21391 {5}	0.31140 {8}	0.19373 {2}	0.21197 {4}
		$\hat{b}$	0.60782 {1}	0.67376 {7}	0.66541 {4}	0.67088 {6}	0.63670 {2}	0.68196 {8}	0.65902 {3}	0.66558 {5}
	MSE	$\hat{\theta}$	10.85598 {1}	24.69643 {7}	18.47048 {5}	20.01900 {6}	14.04590 {3}	159.55140 {8}	13.76235 {2}	15.86948 {4}
		$\hat{a}$	0.09480 {3}	0.09669 {4}	0.11056 {6}	0.14195 {7}	0.09051 {2}	0.45840 {8}	0.06728 {1}	0.09828 {5}
		$\hat{b}$	0.60870 {1}	0.69410 {7}	0.66142 {3}	0.71312 {8}	0.62071 {2}	0.66654 {4}	0.68228 {6}	0.67693 {5}
	MRE	$\hat{\theta}$	0.72308 {1}	0.97865 {7}	0.88712 {6}	0.86974 {5}	0.80724 {3}	1.82597 {8}	0.79577 {2}	0.83625 {4}
		$\hat{a}$	0.52346 {1}	0.60124 {3}	0.64084 {7}	0.61892 {6}	0.61116 {5}	0.88972 {8}	0.55350 {2}	0.60564 {4}
		$\hat{b}$	0.48625 {1}	0.53901 {7}	0.53233 {4}	0.53671 {6}	0.50936 {2}	0.54557 {8}	0.52722 {3}	0.53246 {5}
		$\sum$ *Ranks*	**11** {1}	**52** {6}	**48** {5}	**55** {7}	**27** {3}	**68** {8}	**23** {2}	**40** {4}
50	|BIAS|	$\hat{\theta}$	1.43580 {1}	1.72642 {7}	1.61271 {5}	1.72443 {6}	1.49943 {3}	2.43544 {8}	1.46905 {2}	1.55709 {4}
		$\hat{a}$	0.15206 {1}	0.16951 {5}	0.16564 {3}	0.16597 {4}	0.18985 {7}	0.23094 {8}	0.16332 {2}	0.17880 {6}
		$\hat{b}$	0.45734 {1}	0.50889 {6}	0.49411 {4}	0.51160 {7}	0.49309 {3}	0.54503 {8}	0.48060 {2}	0.50088 {5}
	MSE	$\hat{\theta}$	3.69183 {2}	5.71382 {6}	4.74933 {5}	6.00416 {7}	4.05320 {3}	15.66070 {8}	3.64201 {1}	4.20995 {4}
		$\hat{a}$	0.05291 {5}	0.05025 {4}	0.05016 {3}	0.04714 {1}	0.07438 {7}	0.10380 {8}	0.04788 {2}	0.05407 {6}
		$\hat{b}$	0.31544 {1}	0.36158 {6}	0.34701 {3}	0.37169 {7}	0.35066 {5}	0.39350 {8}	0.32195 {2}	0.35031 {4}
	MRE	$\hat{\theta}$	0.47860 {1}	0.57547 {7}	0.53757 {5}	0.57481 {6}	0.49981 {3}	0.81181 {8}	0.48968 {2}	0.51903 {4}
		$\hat{a}$	0.43447 {1}	0.48433 {5}	0.47324 {3}	0.47421 {4}	0.54242 {7}	0.65983 {8}	0.46664 {2}	0.51085 {6}
		$\hat{b}$	0.36587 {1}	0.40711 {6}	0.39529 {4}	0.40928 {7}	0.39447 {3}	0.43602 {8}	0.38448 {2}	0.40071 {5}
		$\sum$ *Ranks*	**14** {1}	**52** {7}	**35** {3}	**49** {6}	**41** {4}	**72** {8}	**17** {2}	**44** {5}
80	|BIAS|	$\hat{\theta}$	1.15981 {1}	1.42027 {7}	1.26742 {4}	1.41779 {6}	1.19915 {2}	1.82079 {8}	1.22139 {3}	1.30153 {5}
		$\hat{a}$	0.13680 {1}	0.15057 {5}	0.14291 {3}	0.14507 {4}	0.16722 {7}	0.20904 {8}	0.14132 {2}	0.15617 {6}
		$\hat{b}$	0.38571 {1}	0.44556 {7}	0.42099 {3}	0.44360 {6}	0.42296 {4}	0.48472 {8}	0.41287 {2}	0.43161 {5}
	MSE	$\hat{\theta}$	2.20214 {1}	3.29629 {6}	2.56883 {4}	3.48002 {7}	2.48390 {3}	6.35040 {8}	2.37524 {2}	2.72452 {5}
		$\hat{a}$	0.04546 {6}	0.04275 {4}	0.04002 {3}	0.03971 {2}	0.06221 {7}	0.06820 {8}	0.03963 {1}	0.04461 {5}
		$\hat{b}$	0.22545 {1}	0.27583 {6}	0.25042 {3}	0.27846 {7}	0.26824 {5}	0.31303 {8}	0.24099 {2}	0.26293 {4}
	MRE	$\hat{\theta}$	0.38660 {1}	0.47342 {7}	0.42247 {4}	0.47260 {6}	0.39972 {2}	0.60693 {8}	0.40713 {3}	0.43384 {5}
		$\hat{a}$	0.39085 {1}	0.43020 {5}	0.40831 {3}	0.41449 {4}	0.47776 {7}	0.59727 {8}	0.40378 {2}	0.44620 {6}
		$\hat{b}$	0.30857 {1}	0.35645 {7}	0.33679 {3}	0.35488 {6}	0.33837 {4}	0.38778 {8}	0.33030 {2}	0.34529 {5}
		$\sum$ *Ranks*	**14** {1}	**54** {7}	**30** {3}	**48** {6}	**41** {4}	**72** {8}	**19** {2}	**46** {5}
150	|BIAS|	$\hat{\theta}$	0.91952 {2}	1.08389 {6}	0.96903 {4}	1.11611 {7}	0.87261 {1}	1.34497 {8}	0.96065 {3}	1.00373 {5}
		$\hat{a}$	0.11404 {2}	0.12596 {5}	0.11293 {1}	0.12288 {4}	0.13600 {7}	0.18020 {8}	0.11614 {3}	0.13069 {6}
		$\hat{b}$	0.31682 {1}	0.35727 {6}	0.32919 {2}	0.36643 {7}	0.33085 {3}	0.40149 {8}	0.33237 {4}	0.34956 {5}
	MSE	$\hat{\theta}$	1.34756 {1}	1.80477 {6}	1.46867 {4}	1.94623 {7}	1.41290 {2}	2.95790 {8}	1.41340 {3}	1.52554 {5}
		$\hat{a}$	0.03498 {5}	0.03441 {4}	0.02859 {1}	0.03061 {3}	0.04929 {7}	0.05550 {8}	0.02993 {2}	0.03588 {6}
		$\hat{b}$	0.15822 {1}	0.18617 {6}	0.16318 {2}	0.19213 {7}	0.18311 {5}	0.22968 {8}	0.16451 {3}	0.18110 {4}
	MRE	$\hat{\theta}$	0.30651 {2}	0.36130 {6}	0.32301 {4}	0.37204 {7}	0.29087 {1}	0.44832 {8}	0.32022 {3}	0.33458 {5}
		$\hat{a}$	0.32583 {2}	0.35989 {5}	0.32267 {1}	0.35108 {4}	0.38858 {7}	0.51485 {8}	0.33184 {3}	0.37341 {6}
		$\hat{b}$	0.25346 {1}	0.28581 {6}	0.26335 {2}	0.29314 {7}	0.26468 {3}	0.32119 {8}	0.26590 {4}	0.27965 {5}
		$\sum$ *Ranks*	**17** {1}	**50** {6}	**21** {2}	**53** {7}	**36** {4}	**72** {8}	**28** {3}	**47** {5}

Table 7. *Cont.*

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
500	**|BIAS|**	$\hat{\theta}$	0.53401 $^{\{2\}}$	0.66504 $^{\{7\}}$	0.56192 $^{\{3\}}$	0.64836 $^{\{6\}}$	0.37758 $^{\{1\}}$	0.83816 $^{\{8\}}$	0.56954 $^{\{4\}}$	0.57252 $^{\{5\}}$
		$\hat{a}$	0.06631 $^{\{4\}}$	0.07516 $^{\{7\}}$	0.06322 $^{\{2\}}$	0.07168 $^{\{6\}}$	0.05798 $^{\{1\}}$	0.11978 $^{\{8\}}$	0.06453 $^{\{3\}}$	0.06821 $^{\{5\}}$
		$\hat{b}$	0.19082 $^{\{2\}}$	0.22489 $^{\{7\}}$	0.19470 $^{\{3\}}$	0.22013 $^{\{6\}}$	0.15947 $^{\{1\}}$	0.26919 $^{\{8\}}$	0.19781 $^{\{5\}}$	0.19732 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.48497 $^{\{2\}}$	0.70558 $^{\{7\}}$	0.51350 $^{\{3\}}$	0.65778 $^{\{6\}}$	0.44060 $^{\{1\}}$	1.09130 $^{\{8\}}$	0.51757 $^{\{4\}}$	0.53459 $^{\{5\}}$
		$\hat{a}$	0.01557 $^{\{7\}}$	0.01503 $^{\{5\}}$	0.01087 $^{\{1\}}$	0.01350 $^{\{4\}}$	0.01530 $^{\{6\}}$	0.03260 $^{\{8\}}$	0.01112 $^{\{2\}}$	0.01319 $^{\{3\}}$
		$\hat{b}$	0.06634 $^{\{4\}}$	0.08493 $^{\{7\}}$	0.06483 $^{\{2\}}$	0.07996 $^{\{6\}}$	0.06322 $^{\{1\}}$	0.12275 $^{\{8\}}$	0.06602 $^{\{3\}}$	0.06826 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.17800 $^{\{2\}}$	0.22168 $^{\{7\}}$	0.18731 $^{\{3\}}$	0.21612 $^{\{6\}}$	0.12586 $^{\{1\}}$	0.27939 $^{\{8\}}$	0.18985 $^{\{4\}}$	0.19084 $^{\{5\}}$
		$\hat{a}$	0.18945 $^{\{4\}}$	0.21474 $^{\{7\}}$	0.18063 $^{\{2\}}$	0.20479 $^{\{6\}}$	0.16566 $^{\{1\}}$	0.34222 $^{\{8\}}$	0.18438 $^{\{3\}}$	0.19487 $^{\{5\}}$
		$\hat{b}$	0.15266 $^{\{2\}}$	0.17991 $^{\{7\}}$	0.15576 $^{\{3\}}$	0.17611 $^{\{6\}}$	0.12758 $^{\{1\}}$	0.21535 $^{\{8\}}$	0.15825 $^{\{5\}}$	0.15785 $^{\{4\}}$
		$\sum$ *Ranks*	**29** $^{\{3\}}$	**61** $^{\{7\}}$	**22** $^{\{2\}}$	**52** $^{\{6\}}$	**14** $^{\{1\}}$	**72** $^{\{8\}}$	**33** $^{\{4\}}$	**41** $^{\{5\}}$

Table 8. Simulation results of the OLxLx distribution for $\psi = (\theta = 3, a = 0.35, b = 0.75)^{\intercal}$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
20	**|BIAS|**	$\hat{\theta}$	2.48720 $^{\{3\}}$	3.06324 $^{\{7\}}$	2.75433 $^{\{5\}}$	3.00137 $^{\{6\}}$	2.44203 $^{\{2\}}$	7.54395 $^{\{8\}}$	2.37436 $^{\{1\}}$	2.48865 $^{\{4\}}$
		$\hat{a}$	0.19124 $^{\{1\}}$	0.22830 $^{\{5\}}$	0.23001 $^{\{6\}}$	0.23692 $^{\{7\}}$	0.22546 $^{\{4\}}$	0.40522 $^{\{8\}}$	0.20611 $^{\{2\}}$	0.20961 $^{\{3\}}$
		$\hat{b}$	0.42225 $^{\{2\}}$	0.44929 $^{\{7\}}$	0.44205 $^{\{6\}}$	0.43128 $^{\{5\}}$	0.42696 $^{\{3\}}$	0.46317 $^{\{8\}}$	0.41740 $^{\{1\}}$	0.43085 $^{\{4\}}$
	MSE	$\hat{\theta}$	14.62421 $^{\{2\}}$	30.37985 $^{\{7\}}$	20.54163 $^{\{5\}}$	26.32017 $^{\{6\}}$	15.27398 $^{\{3\}}$	301.54150 $^{\{8\}}$	14.38511 $^{\{1\}}$	16.76935 $^{\{4\}}$
		$\hat{a}$	0.10133 $^{\{2\}}$	0.13993 $^{\{6\}}$	0.13388 $^{\{5\}}$	0.18106 $^{\{7\}}$	0.11484 $^{\{4\}}$	0.97360 $^{\{8\}}$	0.09416 $^{\{1\}}$	0.11425 $^{\{3\}}$
		$\hat{b}$	0.31608 $^{\{4\}}$	0.34173 $^{\{8\}}$	0.32763 $^{\{6\}}$	0.33408 $^{\{7\}}$	0.29191 $^{\{1\}}$	0.31728 $^{\{5\}}$	0.29234 $^{\{2\}}$	0.30552 $^{\{3\}}$
	MRE	$\hat{\theta}$	0.82907 $^{\{3\}}$	1.02108 $^{\{7\}}$	0.91811 $^{\{5\}}$	1.00046 $^{\{6\}}$	0.81401 $^{\{2\}}$	2.51465 $^{\{8\}}$	0.79145 $^{\{1\}}$	0.82955 $^{\{4\}}$
		$\hat{a}$	0.54641 $^{\{1\}}$	0.65227 $^{\{5\}}$	0.65716 $^{\{6\}}$	0.67692 $^{\{7\}}$	0.64419 $^{\{4\}}$	1.15778 $^{\{8\}}$	0.58887 $^{\{2\}}$	0.59888 $^{\{3\}}$
		$\hat{b}$	0.56300 $^{\{2\}}$	0.59905 $^{\{7\}}$	0.58940 $^{\{6\}}$	0.57504 $^{\{5\}}$	0.56927 $^{\{3\}}$	0.61757 $^{\{8\}}$	0.55653 $^{\{1\}}$	0.57446 $^{\{4\}}$
		$\sum$ *Ranks*	**20** $^{\{2\}}$	**59** $^{\{7\}}$	**50** $^{\{5\}}$	**56** $^{\{6\}}$	**26** $^{\{3\}}$	**69** $^{\{8\}}$	**12** $^{\{1\}}$	**32** $^{\{4\}}$
50	**|BIAS|**	$\hat{\theta}$	1.54547 $^{\{3\}}$	1.91976 $^{\{7\}}$	1.74951 $^{\{5\}}$	1.89441 $^{\{6\}}$	1.46656 $^{\{1\}}$	3.02331 $^{\{8\}}$	1.55242 $^{\{4\}}$	1.54501 $^{\{2\}}$
		$\hat{a}$	0.13904 $^{\{1\}}$	0.17217 $^{\{6\}}$	0.16781 $^{\{5\}}$	0.16263 $^{\{4\}}$	0.17808 $^{\{7\}}$	0.25633 $^{\{8\}}$	0.15985 $^{\{2\}}$	0.16024 $^{\{3\}}$
		$\hat{b}$	0.29325 $^{\{1\}}$	0.33999 $^{\{7\}}$	0.32841 $^{\{5\}}$	0.33068 $^{\{6\}}$	0.31018 $^{\{2\}}$	0.36505 $^{\{8\}}$	0.31128 $^{\{3\}}$	0.31283 $^{\{4\}}$
	MSE	$\hat{\theta}$	4.78792 $^{\{4\}}$	8.58045 $^{\{7\}}$	6.39022 $^{\{5\}}$	8.41641 $^{\{6\}}$	4.18580 $^{\{1\}}$	28.78420 $^{\{8\}}$	4.75600 $^{\{3\}}$	4.33195 $^{\{2\}}$
		$\hat{a}$	0.04368 $^{\{1\}}$	0.06186 $^{\{6\}}$	0.05843 $^{\{5\}}$	0.05817 $^{\{4\}}$	0.06416 $^{\{7\}}$	0.20710 $^{\{8\}}$	0.04981 $^{\{3\}}$	0.04489 $^{\{2\}}$
		$\hat{b}$	0.13888 $^{\{1\}}$	0.17521 $^{\{7\}}$	0.16120 $^{\{5\}}$	0.16927 $^{\{6\}}$	0.14429 $^{\{4\}}$	0.17926 $^{\{8\}}$	0.14315 $^{\{3\}}$	0.14036 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.51516 $^{\{3\}}$	0.63992 $^{\{7\}}$	0.58317 $^{\{5\}}$	0.63147 $^{\{6\}}$	0.48885 $^{\{1\}}$	1.00777 $^{\{8\}}$	0.51747 $^{\{4\}}$	0.51500 $^{\{2\}}$
		$\hat{a}$	0.39725 $^{\{1\}}$	0.49193 $^{\{6\}}$	0.47946 $^{\{5\}}$	0.46467 $^{\{4\}}$	0.50879 $^{\{7\}}$	0.73236 $^{\{8\}}$	0.45673 $^{\{2\}}$	0.45783 $^{\{3\}}$
		$\hat{b}$	0.39100 $^{\{1\}}$	0.45332 $^{\{7\}}$	0.43788 $^{\{5\}}$	0.44090 $^{\{6\}}$	0.41358 $^{\{2\}}$	0.48673 $^{\{8\}}$	0.41504 $^{\{3\}}$	0.41711 $^{\{4\}}$
		$\sum$ *Ranks*	**16** $^{\{1\}}$	**60** $^{\{7\}}$	**45** $^{\{5\}}$	**48** $^{\{6\}}$	**32** $^{\{4\}}$	**72** $^{\{8\}}$	**27** $^{\{3\}}$	**24** $^{\{2\}}$
80	**|BIAS|**	$\hat{\theta}$	1.23790 $^{\{2\}}$	1.57486 $^{\{7\}}$	1.41474 $^{\{5\}}$	1.56031 $^{\{6\}}$	1.18013 $^{\{1\}}$	2.11711 $^{\{8\}}$	1.31380 $^{\{3\}}$	1.32478 $^{\{4\}}$
		$\hat{a}$	0.12428 $^{\{1\}}$	0.15053 $^{\{6\}}$	0.14720 $^{\{4\}}$	0.14466 $^{\{3\}}$	0.15719 $^{\{7\}}$	0.22721 $^{\{8\}}$	0.14208 $^{\{2\}}$	0.14914 $^{\{5\}}$
		$\hat{b}$	0.24883 $^{\{1\}}$	0.29575 $^{\{7\}}$	0.28226 $^{\{5\}}$	0.28837 $^{\{6\}}$	0.26380 $^{\{2\}}$	0.32077 $^{\{8\}}$	0.27070 $^{\{3\}}$	0.27489 $^{\{4\}}$
	MSE	$\hat{\theta}$	2.83986 $^{\{2\}}$	5.02446 $^{\{7\}}$	3.71901 $^{\{5\}}$	4.95456 $^{\{6\}}$	2.55790 $^{\{1\}}$	9.79230 $^{\{8\}}$	3.01410 $^{\{4\}}$	2.87589 $^{\{3\}}$
		$\hat{a}$	0.03804 $^{\{1\}}$	0.04579 $^{\{6\}}$	0.04540 $^{\{5\}}$	0.04411 $^{\{4\}}$	0.05242 $^{\{7\}}$	0.11060 $^{\{8\}}$	0.04037 $^{\{2\}}$	0.04123 $^{\{3\}}$
		$\hat{b}$	0.09823 $^{\{1\}}$	0.12706 $^{\{7\}}$	0.11575 $^{\{5\}}$	0.12213 $^{\{6\}}$	0.10486 $^{\{2\}}$	0.13633 $^{\{8\}}$	0.10589 $^{\{4\}}$	0.10544 $^{\{3\}}$
	MRE	$\hat{\theta}$	0.41263 $^{\{2\}}$	0.52495 $^{\{7\}}$	0.47158 $^{\{5\}}$	0.52010 $^{\{6\}}$	0.39338 $^{\{1\}}$	0.70570 $^{\{8\}}$	0.43793 $^{\{3\}}$	0.44159 $^{\{4\}}$
		$\hat{a}$	0.35508 $^{\{1\}}$	0.43008 $^{\{6\}}$	0.42057 $^{\{4\}}$	0.41332 $^{\{3\}}$	0.44912 $^{\{7\}}$	0.64916 $^{\{8\}}$	0.40595 $^{\{2\}}$	0.42611 $^{\{5\}}$
		$\hat{b}$	0.33177 $^{\{1\}}$	0.39433 $^{\{7\}}$	0.37635 $^{\{5\}}$	0.38449 $^{\{6\}}$	0.35173 $^{\{2\}}$	0.42769 $^{\{8\}}$	0.36093 $^{\{3\}}$	0.36652 $^{\{4\}}$
		$\sum$ *Ranks*	**12** $^{\{1\}}$	**60** $^{\{7\}}$	**43** $^{\{5\}}$	**46** $^{\{6\}}$	**30** $^{\{3\}}$	**72** $^{\{8\}}$	**26** $^{\{2\}}$	**35** $^{\{4\}}$
150	**|BIAS|**	$\hat{\theta}$	0.90893 $^{\{2\}}$	1.21687 $^{\{6\}}$	1.09374 $^{\{5\}}$	1.22086 $^{\{7\}}$	0.84860 $^{\{1\}}$	1.55320 $^{\{8\}}$	1.03239 $^{\{3\}}$	1.03274 $^{\{4\}}$
		$\hat{a}$	0.10555 $^{\{1\}}$	0.12632 $^{\{7\}}$	0.12053 $^{\{3\}}$	0.12220 $^{\{4\}}$	0.12367 $^{\{5\}}$	0.19198 $^{\{8\}}$	0.11983 $^{\{2\}}$	0.12606 $^{\{6\}}$
		$\hat{b}$	0.19238 $^{\{1\}}$	0.24083 $^{\{7\}}$	0.22794 $^{\{5\}}$	0.23816 $^{\{6\}}$	0.20015 $^{\{2\}}$	0.27141 $^{\{8\}}$	0.21894 $^{\{3\}}$	0.22323 $^{\{4\}}$
	MSE	$\hat{\theta}$	1.44104 $^{\{2\}}$	2.50083 $^{\{6\}}$	1.98446 $^{\{5\}}$	2.56440 $^{\{7\}}$	1.42755 $^{\{1\}}$	4.16810 $^{\{8\}}$	1.67202 $^{\{4\}}$	1.63049 $^{\{3\}}$
		$\hat{a}$	0.03267 $^{\{4\}}$	0.03341 $^{\{6\}}$	0.03258 $^{\{3\}}$	0.03122 $^{\{1\}}$	0.03716 $^{\{7\}}$	0.05960 $^{\{8\}}$	0.03216 $^{\{2\}}$	0.03312 $^{\{5\}}$
		$\hat{b}$	0.05926 $^{\{1\}}$	0.08241 $^{\{7\}}$	0.07624 $^{\{5\}}$	0.08151 $^{\{6\}}$	0.06531 $^{\{2\}}$	0.10141 $^{\{8\}}$	0.07047 $^{\{3\}}$	0.07211 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.30298 $^{\{2\}}$	0.40562 $^{\{6\}}$	0.36458 $^{\{5\}}$	0.40695 $^{\{7\}}$	0.28287 $^{\{1\}}$	0.51773 $^{\{8\}}$	0.34413 $^{\{3\}}$	0.34425 $^{\{4\}}$
		$\hat{a}$	0.30157 $^{\{1\}}$	0.36093 $^{\{7\}}$	0.34436 $^{\{3\}}$	0.34915 $^{\{4\}}$	0.35335 $^{\{5\}}$	0.54851 $^{\{8\}}$	0.34236 $^{\{2\}}$	0.36019 $^{\{6\}}$
		$\hat{b}$	0.25650 $^{\{1\}}$	0.32111 $^{\{7\}}$	0.30391 $^{\{5\}}$	0.31755 $^{\{6\}}$	0.26686 $^{\{2\}}$	0.36189 $^{\{8\}}$	0.29193 $^{\{3\}}$	0.29765 $^{\{4\}}$
		$\sum$ *Ranks*	**15** $^{\{1\}}$	**59** $^{\{7\}}$	**39** $^{\{4\}}$	**48** $^{\{6\}}$	**26** $^{\{3\}}$	**72** $^{\{8\}}$	**25** $^{\{2\}}$	**40** $^{\{5\}}$
500	**|BIAS|**	$\hat{\theta}$	0.48952 $^{\{2\}}$	0.78993 $^{\{7\}}$	0.65694 $^{\{5\}}$	0.77312 $^{\{6\}}$	0.32286 $^{\{1\}}$	0.95551 $^{\{8\}}$	0.62809 $^{\{3\}}$	0.64043 $^{\{4\}}$
		$\hat{a}$	0.06694 $^{\{2\}}$	0.08481 $^{\{7\}}$	0.07377 $^{\{4\}}$	0.08041 $^{\{6\}}$	0.04931 $^{\{1\}}$	0.13680 $^{\{8\}}$	0.07221 $^{\{3\}}$	0.07594 $^{\{5\}}$
		$\hat{b}$	0.11060 $^{\{2\}}$	0.16042 $^{\{7\}}$	0.13999 $^{\{5\}}$	0.15685 $^{\{6\}}$	0.08912 $^{\{1\}}$	0.18694 $^{\{8\}}$	0.13544 $^{\{3\}}$	0.13926 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.45542 $^{\{2\}}$	0.96442 $^{\{7\}}$	0.69129 $^{\{5\}}$	0.92500 $^{\{6\}}$	0.38409 $^{\{1\}}$	1.39480 $^{\{8\}}$	0.64144 $^{\{3\}}$	0.65317 $^{\{4\}}$
		$\hat{a}$	0.02200 $^{\{7\}}$	0.01755 $^{\{6\}}$	0.01430 $^{\{3\}}$	0.01540 $^{\{4\}}$	0.00992 $^{\{1\}}$	0.03960 $^{\{8\}}$	0.01393 $^{\{2\}}$	0.01546 $^{\{5\}}$
		$\hat{b}$	0.02167 $^{\{2\}}$	0.03958 $^{\{7\}}$	0.03183 $^{\{4\}}$	0.03779 $^{\{6\}}$	0.01981 $^{\{1\}}$	0.05605 $^{\{8\}}$	0.03040 $^{\{3\}}$	0.03208 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.16317 $^{\{2\}}$	0.26331 $^{\{7\}}$	0.21898 $^{\{5\}}$	0.25771 $^{\{6\}}$	0.10762 $^{\{1\}}$	0.31850 $^{\{8\}}$	0.20936 $^{\{3\}}$	0.21348 $^{\{4\}}$
		$\hat{a}$	0.19124 $^{\{2\}}$	0.24231 $^{\{7\}}$	0.21078 $^{\{4\}}$	0.22973 $^{\{6\}}$	0.14087 $^{\{1\}}$	0.39084 $^{\{8\}}$	0.20632 $^{\{3\}}$	0.21698 $^{\{5\}}$
		$\hat{b}$	0.14747 $^{\{2\}}$	0.21389 $^{\{7\}}$	0.18665 $^{\{5\}}$	0.20914 $^{\{6\}}$	0.11882 $^{\{1\}}$	0.24925 $^{\{8\}}$	0.18059 $^{\{3\}}$	0.18568 $^{\{4\}}$
		$\sum$ *Ranks*	**23** $^{\{2\}}$	**62** $^{\{7\}}$	**40** $^{\{4.5\}}$	**52** $^{\{6\}}$	**9** $^{\{1\}}$	**72** $^{\{8\}}$	**26** $^{\{3\}}$	**40** $^{\{4.5\}}$

Table 9. Partial and overall ranks of different estimation methods of the OLxLx parameters under various parametric values of ψ.

ψ			n	MLEs	LSEs	WLSEs	CVMEs	MPSEs	PCEs	ADEs	RADEs
θ	a	b									
0.50	0.35	0.75	20	3	7	4	5.5	5.5	8	2	1
			50	1	7	6	4	5	8	3	2
			80	2	7	4	5.5	5.5	8	1	3
			150	1	7	4	6	5	8	2	3
			500	6	7	4	5	2	8	3	1
0.50	1.25	1.25	20	7	5	4	8	1	2.5	2.5	6
			50	7	5	4	6	1	3	2	8
			80	6	5	2.5	7	1	4	2.5	8
			150	5	6	3	8	1	4	2	7
			500	2	7	4	8	1	5	3	6
0.75	1.25	0.75	20	8	5	3	7	1	2	4	6
			50	7	5	4	8	1	2	3	6
			80	7	6	4	8	1	2	3	5
			150	7	6	5	8	1	4	3	2
			500	4	7	3	8	1	6	2	5
0.75	2.25	0.75	20	5	6	4	7	1	3	2	8
			50	4	6	5	7	1	2	3	8
			80	4	6	5	8	1	2	3	7
			150	4	8	5	7	1	2	3	6
			500	5	7	4	8	1	2	3	6
1.5	2.25	2.5	20	5	6	4	8	1.5	1.5	3	7
			50	3	6.5	5	8	1	2	4	6.5
			80	2	7	5	8	1	3	4	6
			150	2	8	5	7	1	3	4	6
			500	2	8	5	7	1	3	4	6
1.5	2.25	1.25	20	4	6	5	8	3	1	2	7
			50	4	6	5	8	1	2	3	7
			80	3	6	5	8	1	2	4	7
			150	3	7	5	8	1	2	4	6
			500	4.5	8	4.5	7	1	2	3	6
3	0.35	1.25	20	1	6	5	7	3	8	2	4
			50	1	7	3	6	4	8	2	5
			80	1	7	3	6	4	8	2	5
			150	1	6	2	7	4	8	3	5
			500	3	7	2	6	1	8	4	5
3	0.35	0.75	20	2	7	5	6	3	8	1	4
			50	1	7	5	6	4	8	3	2
			80	1	7	5	6	3	8	2	4
			150	1	7	4	6	3	8	2	5
			500	2	7	4.5	6	1	8	3	4.5
$\sum$ *Ranks*				**141.5**	**261.5**	**168.5**	**278**	**80.5**	**187**	**111**	**212**
Overall Rank				**3**	**7**	**4**	**8**	**1**	**5**	**2**	**6**

6. Real-Life Data Application

In this section, the flexibility and applicability of the OLxLx distribution are explored by fitting real-life data from medical field. The data consists of 128 bladder cancer patients, and it represents and refers to the remission times (in months) [19]. The data are: 2.09, 0.08, 3.48, 6.94, 4.87, 8.66, 23.63, 13.11, 0.20, 3.52, 2.23, 4.98, 9.02, 6.97, 13.29, 2.26, 0.40, 3.57, 7.09, 5.06, 7.66, 9.22, 25.74, 13.80, 0.50, 3.64, 2.46, 5.09, 7.26, 14.24, 9.47, 25.82, 2.54, 0.51, 3.70, 7.28, 5.17, 9.74, 26.31, 14.76, 0.81, 3.82, 2.62, 2.69, 5.32, 10.06, 7.32, 14.77, 2.64, 32.15, 3.88, 7.39, 5.32, 10.34, 34.26, 14.83, 0.90, 4.18, 5.34, 2.69, 7.59, 10.66, 15.96, 36.66, 1.05, 7.62, 4.23, 5.41, 16.62, 10.75, 1.19, 2.75, 43.01, 5.41, 7.63, 4.26, 46.12, 1.26, 17.12, 4.33, 5.49, 2.83, 11.25, 79.05, 17.14, 1.35, 5.62, 2.87, 7.87, 17.36, 11.64, 1.40, 4.34, 3.02, 5.71, 11.79, 7.93, 18.10, 4.40, 1.46, 5.85, 11.98, 8.26, 19.13, 3.25, 1.76, 4.50, 8.37, 6.25, 2.02, 12.02, 4.51, 3.31, 6.54, 12.03, 8.53, 20.28, 3.36, 2.02, 6.76, 21.73, 12.07, 2.07, 6.93, 3.36, 22.69, 8.65, 12.63.

The proposed OLxLx distribution is compared with some competing Lx extensions using some discrimination criteria, including minus maximized log-likelihood ($-\ell$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–Darling (A^*), Cramér–von Mises (W^*), and Kolmogorov–Smirnov (KS) statistics with its p-value (KS

p-value). The competing distributions of the OLxLx distribution include the Weibull–Lomax (WLx) [8], complementary generalized-transmuted Poisson–Lomax (CGTPLx) [20], Lomax–Weibull (LxW) [21], transmuted Weibull–Lomax (TWLx) [22], exponentiated-Lomax (ExLx) [23], Kumaraswamy–Lomax (KwLx) and McDonald–Lomax (McLx) [11], modified Kies–Lomax (MKLx) [24], Burr X Lomax (BXLx) [25], beta exponentiated-Lomax (BExLx) [23], odd exponentiated half-logistic Lomax (OEHLLx) [26], transmuted-Lomax (TLx) [27], TPMOLx (special case), and Lx distributions.

The maximum likelihood (ML) estimates of the parameters of the competing models and their SEs (standard errors) are listed in Table 10 for the cancer data. Additionally, the goodness-of-fit measures are displayed in Table 10. The findings in Table 10 show that the OLxLx distribution provides better fits as compared to the WLx, CGTPLx, LxW, TWLx, ExLx, KwLx, MKLx, BXLx, BExLx, McLx, OEHLLx, TLx, TPMOLx, and Lx models.

Table 10. The ML estimates, SEs, and fitting measures for bladder cancer data.

Distribution	Estimates	SEs	$-\ell$	AIC	BIC	W^*	A^*	KS	KS p-Value
OLxLx	$\hat{\theta}$ = 1.854428	0.3528995	409.5022	825.0044	833.5605	0.015023	0.092074	0.029928	0.999843
	$\hat{a}$ = 1.303778	0.6083500							
	$\hat{b}$ = 55.746420	20.1269541							
WLx	$\hat{\alpha}$ = 0.072862	0.137502	410.0418	828.0837	839.4918	0.029264	0.194151	0.041024	0.982411
	$\hat{\beta}$ = 7.312808	6.808461							
	$\hat{a}$ = 75.243002	247.138134							
	$\hat{b}$ = 1.525593	0.284570							
CGTPLx	$\hat{\lambda}$ = −0.838684	0.144250	409.6202	827.2403	838.6484	0.018824	0.120356	0.035205	0.997360
	$\hat{\theta}$ = 23.777878	106.066481							
	$\hat{a}$ = 0.227014	1.364232							
	$\hat{b}$ = 25.997754	44.284429							
LxW	$\hat{\alpha}$ = 2.070040	0.968163	409.7399	827.4798	838.8879	0.019457	0.130771	0.035056	0.997519
	$\hat{\beta}$ = 12.323896	433.967972							
	$\hat{a}$ = 0.482693	11.905508							
	$\hat{b}$ = 1.427611	0.1779201							
TWLx	$\hat{\alpha}$ =0.132298	0.326495	409.8028	829.6057	843.8658	0.023896	0.157834	0.038419	0.991586
	$\hat{\beta}$ = 12.150835	15.252054							
	$\hat{\lambda}$ = 0.669019	0.452082							
	$\hat{a}$ = 28.320694	130.023884							
	$\hat{b}$ =1.462765	0.258484							
ELx	$\hat{\alpha}$ = 1.586198	0.279744	410.0718	826.1436	834.6997	0.0283230	0.190211	0.040512	0.984604
	$\hat{a}$ = 4.585705	2.226673							
	$\hat{b}$ = 24.741450	16.685800							
KwLx	$\hat{\alpha}$ = 0.397583	2.539788	409.9403	827.8806	839.2887	0.025854	0.172632	0.038924	0.990172
	$\hat{\beta}$ = 12.340590	18.279746							
	$\hat{a}$ = 1.516220	0.266744							
	$\hat{b}$ = 11.801720	89.335600							
MKLx	$\hat{\alpha}$ = 1.612191	0.390874	410.8951	827.7901	836.3462	0.050520	0.325224	0.049121	0.916932
	$\hat{\lambda}$ = 0.392557	0.152109							
	$\hat{\beta}$ = 1.848593	1.565668							
BXLx	$\hat{\alpha}$ = 0.298273	0.051134	411.1459	828.2918	836.8479	0.054390	0.354999	0.047712	0.932685
	$\hat{\beta}$ = 1.020066	0.664080							
	$\hat{\theta}$ = 0.933832	0.249918							
BELx	$\hat{\beta}$ =2.021947	2.967803	409.9135	829.8269	844.0871	0.0254417	0.169137	0.039074	0.989719
	$\hat{\theta}$ = 0.657188	2.449987							
	$\hat{\lambda}$ = 0.093312	0.123866							
	$\hat{a}$ = 0.744411	1.078977							
	$\hat{b}$ = 6.620361	29.576475							
McLx	$\hat{\alpha}$ = 0.808465	3.545156	409.9125	829.8249	844.0851	0.025352	0.168538	0.039115	0.989595
	$\hat{\beta}$ = 11.292883	18.295065							
	$\hat{a}$ = 1.506001	0.283714							
	$\hat{b}$ = 4.188648	26.365111							
	$\hat{c}$ = 2.104531	3.111132							
OEHLLx	$\hat{\alpha}$ =1.426955	0.261373	409.728	827.456	838.8642	0.021328	0.140743	0.036817	0.995086
	$\hat{\beta}$ = 26.076262	93.761718							
	$\hat{a}$ = 0.106983	0.316311							
	$\hat{b}$ = 9.722434	8.822504							
TLx	$\hat{\theta}$ = 4.020827	1.492284	409.6258	825.2516	833.8077	0.019549	0.125509	0.035842	0.996591
	$\hat{\beta}$ = 0.052398	0.025906							
	$\hat{\lambda}$ = −0.844517	0.139368							
Lx	$\hat{\theta}$ = 13.93842	15.38628	413.8329	831.6658	837.3698	0.080679	0.487362	0.096670	0.182695
	$\hat{\beta}$ = 121.02227	142.71830							
TPMOLx	$\hat{\theta}$ = 2.810465	0.227464	418.1563	840.3125	846.0166	1.308544	7.103445	0.199602	0.000074
	$\hat{b}$ = 179.408493	35.185927							

The relative histogram of cancer data with the fitted functions of the OLxLx model, including the fitted PDF, CDF, SF, and (probability-probability) P-P plots, are given in Figure 3. The plots in this figure support and confirm the results in Table 10. Hence, the OLxLx distribution can be used more effectively to model bladder cancer data compared to other competing Lx extensions.

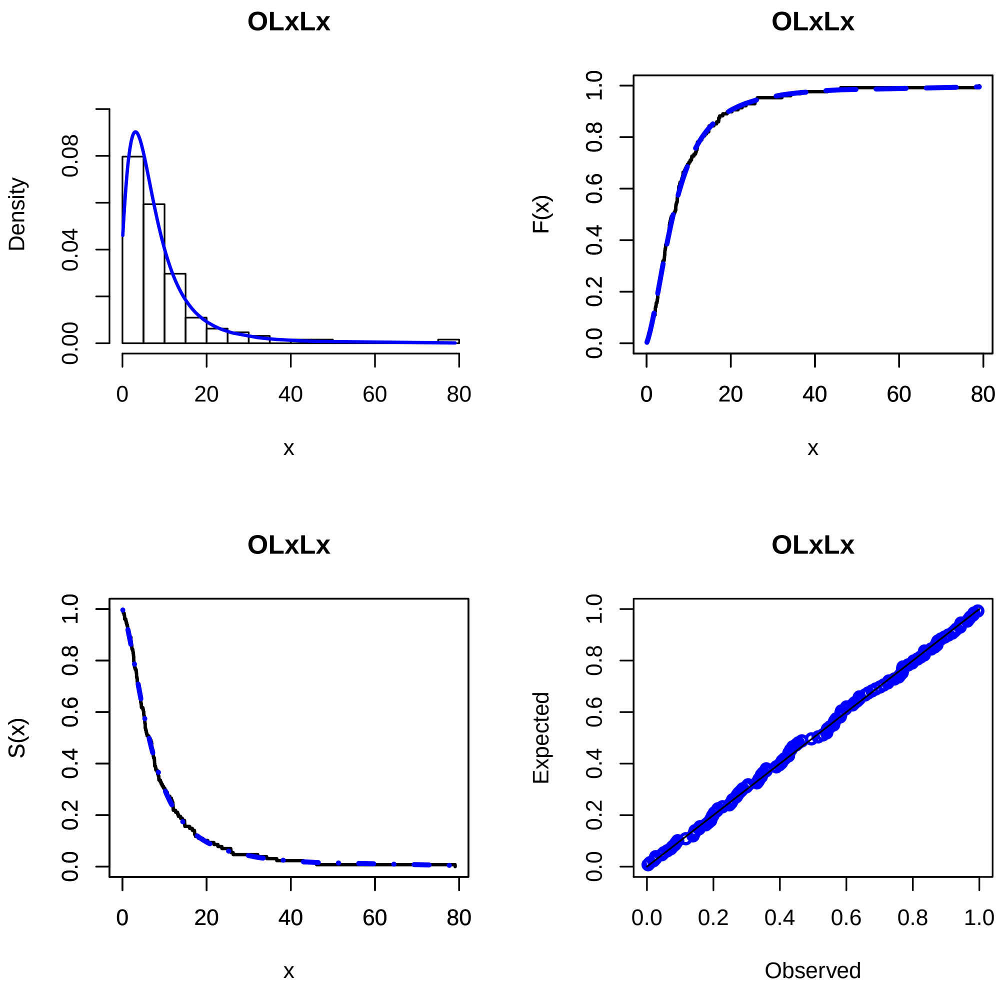

Figure 3. Plots of different fitted functions of the OLxLx distribution for bladder cancer data.

Table 11 shows the parameter estimates of the OLXLX model, KS, and KS p-values under several estimation methods for cancer data. From Table 11, and based on the KS p-values, all estimation methods, except the PCEs, are recommended for estimating the OLxLx parameters for cancer data. Additionally, Figures 4 and 5 display the probability–probability (PP) plots and histograms along with fitted OLxLx density under various estimation methods for bladder cancer data. The plots support the results in Table 11 visually.

Table 11. The OLxLx parameter estimates using several estimation methods along with KS and KS p-value for bladder cancer data.

Method	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	KS	KS p-Value
WLSEs	1.817852	1.259341	51.54454	0.032765	0.999396
LSEs	1.854045	1.193350	51.61323	0.033634	0.999052
MLEs	1.854428	1.303778	55.746420	0.029928	0.999843
MPSEs	1.752837	1.470533	56.83993	0.033684	0.999028
CVMEs	1.876271	1.200982	54.52999	0.0337999	0.998971
ADEs	1.794702	1.400382	56.83796	0.030627	0.999832
RADEs	1.813133	1.326467	54.75095	0.032264	0.999542
PCEs	1.642911	1.273914	28.02273	0.099726	0.173070

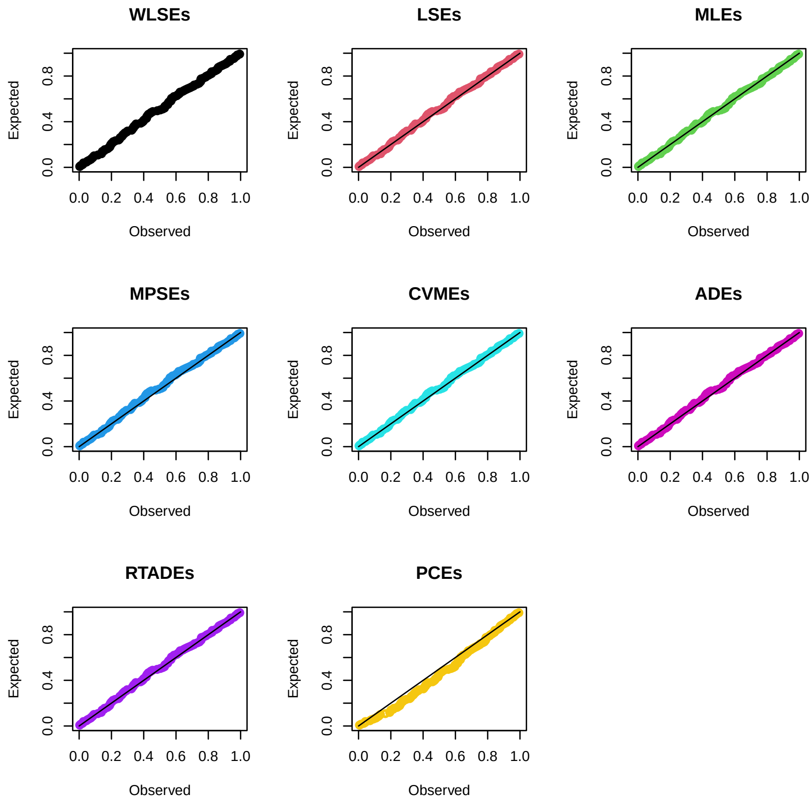

Figure 4. The PP plots of the OLxLx distribution under various methods for bladder cancer data.

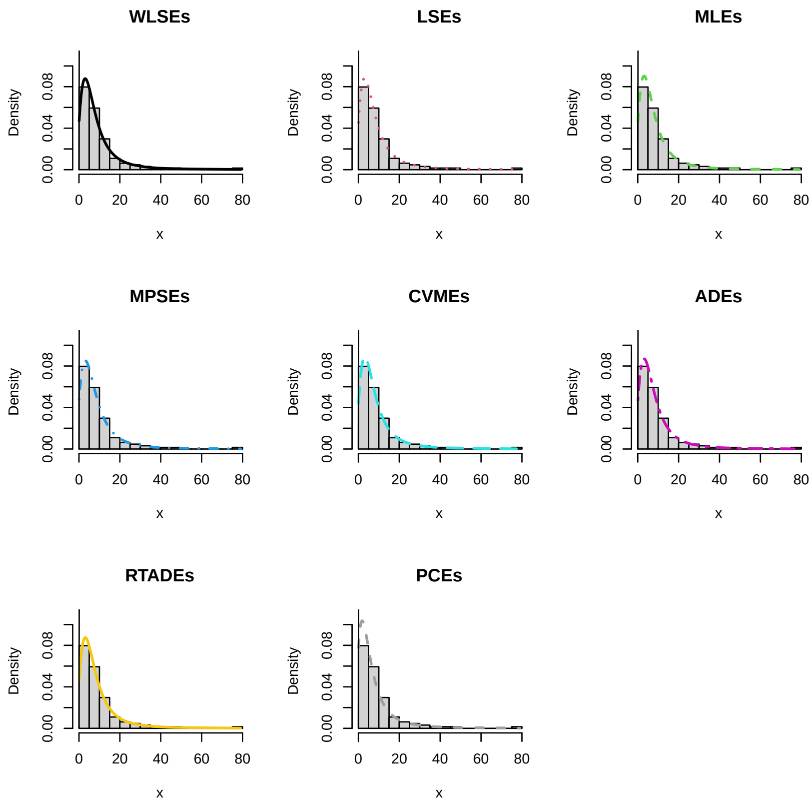

Figure 5. The histogram for bladder cancer data of the OLxLx distribution under various methods.

7. Conclusions

In this article, a new three-parameter extension of the Lomax distribution, called the odd Lomax–Lomax (OLxLx) distribution, is studied. The hazard function of the OLxLx distribution provides constant, increasing, decreasing, and unimodal shapes. Its density provides symmetrical, right-skewed, left-skewed, and J shaped densities. The parameters of the OLxLx distribution are estimated by eight estimation methods. A detailed simulation study is presented showing that maximum product of spacings has the best performance in estimating the OLxLx parameters. A real-life data from the medical field is fitted using the OLxLx distribution and other fourteen competing models. The real-life application shows the flexibility and superiority of the OLxLx distribution in fitting the analyzed data.

The work of this paper can be extended in light of skew-elliptical distributions following the works of Azzalini and Valle [28], Branco and Dey [29], Loperfido [30], and Adcock et al. [31]. Furthermore, the estimation of the model parameters can be addressed using the Bayesian approach under complete and censored samples. Additionally, quantile regression model by exploiting the flexibility of the OLxLx distribution can be constructed.

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Data Availability Statement: This work is mainly a methodological development and has been applied on secondary data; however, if required, data will be provided.

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