

Article

New Applications of Gaussian Hypergeometric Function for Developments on Third-Order Differential Subordinations

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Abstract: The main objective of this paper is to present classical second-order differential subordination knowledge extended in this study to include new results regarding third-order differential subordinations. The focus of this study is on the main problems examined by differential subordination theory. Hence, the new results obtained here reveal techniques for identifying dominants and the best dominant of certain third-order differential subordinations. Another aspect of novelty is the new application of the Gaussian hypergeometric function. Novel third-order differential subordination results are obtained using the best dominant provided by the theorems and the operator previously defined as Gaussian hypergeometric function's fractional integral. The research investigation is concluded by giving an example of how the results can be implemented.

Keywords: differential subordination; best dominant; analytic function; convex function; Gaussian hypergeometric function; fractional integral

MSC: 30C80; 33C15; 30C45



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1. Introduction

The concept of differential subordination was proposed by S.S. Miller and P.T. Mocanu in two publications that appeared in 1978 [1] and 1981 [2], as an effort to expand the idea of inequality from the real numbers to the complex plane. A completely new theory known as the theory of differential subordination or admissible functions theory was created in response to this idea. The theory of differential subordination was established by S.S. Miller and P.T. Mocanu for the case of second-order differential subordinations. This theory was expanded by J.A. Antonino and S.S. Miller [3] for the case of third-order differential subordinations, opening the way for a fresh line of investigation into differential subordination theory.

This line of study is mainly concerned with the identification of already known results from the theory of second-order differential subordinations that remain valid for the theory of third-order differential subordinations through the appropriate extension. The first such extensions of the results established for second-order differential subordinations could be realized having as starting point one of the fundamental concepts of the theory of differential subordinations, namely the class of admissible functions. New results in the theory of the third-order differential subordinations were presented by defining specific classes of admissible functions, obtained considering the extension of the classes of admissible functions defined for second-order differential subordinations so that they are usable in the theory of third-order differential subordinations. Notable results have been obtained in recent years using this approach.

A new approach is implemented for the study described in this paper which follows another key problem regarding differential subordination theory, namely finding dominants for the differential subordinations under investigation and, furthermore, finding

the best dominant when admitted by the differential subordination. New information regarding techniques for identifying dominants and the best dominant of certain third-order differential subordinations are made available in the theorems and corollaries obtained in this research, hence contributing to the development of the branch of differential subordination theory concerning the third-order differential subordinations. The new results are derived by applying established knowledge familiar to the theory of third-order differential subordination given in the form of a lemma.

Gaussian hypergeometric function, a function particularly renowned for its significant contribution to geometric function theory of univalent functions, is involved in providing an application of obtaining the best dominant for a certain third-order differential subordination. The operator previously defined as Gaussian hypergeometric function's fractional integral is applied as best dominant of a certain third-order differential subordination, revealing an interesting result. The soundness of the results obtained in this manner is validated by a numerical example.

The theory of differential subordination is considered in the known context of geometric function theory involving specific classes of functions defined in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with its connected notations $\bar{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial U = \{z \in \mathbb{C} : |z| = 1\}$. The unit disc presents a wide variety of symmetries, such as rotational, reflection and inversion symmetries, which offer effective tools for examining the geometric characteristics of functions in this symmetry domain such as starlikeness and convexity.

Designation $H(U)$ is associated to the class of analytic functions and the significant subclasses listed below are invoked when a is a complex number and n is a positive integer:

$$H[a, n] = \left\{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U \right\},$$

writing $H_0 = H[0, 1]$ and $H_1 = H[1, 1]$,

$$A_n = \left\{ f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots, z \in U \right\},$$

writing $A_1 = A$.

A noteworthy subclass of A describing the convex functions is

$$K = \left\{ f \in A : \operatorname{Re} \left(\frac{z f''(z)}{f'(z)} + 1 \right) > 0, z \in U \right\},$$

where $f(0) = 0, f'(0) \neq 0$. Due to the high correlation between convexity and symmetry properties, involving convex functions into the studies offers a wide perspective since the results obtained while working on one of the concepts can be applied to the other as well.

Definition 1 ([4,5]). We let f and F be members of $H(U)$. The function f is said to be subordinate to F , written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U , with $w(0) = 0$ and $|w(z)| < 1, z \in U$ and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

In their article [3], Antonino and Miller considered functions $\psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ and evaluated the requirements that an analytic function p must comply with in order to have the following implication hold true:

$$\left\{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z)) : z \in U \right\} \subset \Omega \implies p(U) \subset \Delta. \quad (1)$$

With $h \in H(U)$, Implication (1) is written in [3] in the form of the corresponding third-order differential subordination

$$\left\{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z)) : z \in U \right\} \prec h(z). \quad (2)$$

In the same paper [3], function $p \in H(U)$ complying with (2) is referred to as a solution of the third-order differential subordination and a univalent function q achieving $p \prec q$ for all solutions p of (2) is known as a dominant of the solutions of the differential subordination, or simply a dominant. The best dominant of (2) is a dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2).

Two fundamental classes are needed for the study on third-order differential subordination. They are defined in [3] and are described in the next two definitions:

Definition 2 ([3]). We let Q denote the set of functions q that are analytic and univalent on $\overline{U} \setminus E(q)$ where

$$E(q) = \left\{ \zeta \in \partial U; \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that $\min |q'(\zeta)| = \varrho > 0$, for $q \in \partial U \setminus E(q)$. The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$.

Definition 3 ([3]). We let Ω be a set in $\mathbb{C}$, $q \in Q$, and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those $\psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t, u; z) \notin \Omega, z \in U, r = q(w), s = n w q'(w), \quad (3)$$

$$\operatorname{Re} \left(\frac{t}{s} + 1 \right) \geq n \left[\operatorname{Re} \frac{w q''(w)}{q'(w)} + 1 \right],$$

$$\operatorname{Re} \frac{u}{s} \geq n^2 \operatorname{Re} \frac{w^2 q''(w)}{q'(w)},$$

$$w \in \partial U \setminus E(q).$$

Applications of the results presented in [3] followed soon by considering suitable classes of admissible functions involving normalized analytic functions [6], meromorphic functions [7], generalized Bessel functions [8]. The research on the topic of third-order differential subordination was continued in the years that followed, with the addition of the dual concept of third-order differential superordination [9], with studies that include both third-order differential subordination and superordination [10], with the addition of different operators to the study [11–14], or with studies pertaining to special functions associated to third-order differential subordination and superordination [15,16]. The topic continues to spark fresh ideas and recent papers present interesting outcome. Certain classes of admissible functions are described and particular uses of third-order differential subordination for p -valent functions associated with generalized fractional differintegral operator are examined in [17] using the third-order differential subordination fundamental results. Using the same idea of defining suitable classes of admissible functions generates interesting results involving a generalized operator in [18] and concerning special functions in [19,20].

Another approach on third-order differential subordination is investigating one of the primary concerns in the theory of differential subordinations, namely finding the dominants for differential subordinations and, when possible, the best dominant. This approach is likely to obtain results that are applicable to the geometric theory of analytical functions by using some remarkable functions in light of their geometric characteristics as the best dominants. This approach has already been exploited in [21] and key conclusions drawn from research on the well-known, classical second-order differential subordinations have been adjusted to apply to third-order differential subordinations. The main aim of the investigation presented by this paper is to develop the idea of extending outcomes from

the theory of second-order differential subordinations that have not yet been taken into account by other authors by using this second type of approach.

The renowned Gaussian hypergeometric function combined with the fractional integral given in [22,23] has generated a new operator investigated in [24]. The same operator is used for constructing some applications of the theoretical findings reported in the theorems found in the main results of this study. Gaussian hypergeometric function has been recently investigated by means of geometric function theory [25,26] but also in relation to third-order differential subordination theory applied in source-sink dynamics theory [27] proving its significance and also the potential for future studies. The symmetry properties that this function possesses refer to permutation symmetry and mirror symmetry and they contribute to obtaining univalence characteristics.

2. Materials and Methods

The definitions of the notable tools of this investigation are listed next.

Definition 4 ([22,23]). The fractional integral of order λ ($\lambda > 0$) is defined for a function f by the following expression:

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \quad (4)$$

where f is an analytic function in a simply connected region of the z -plane containing the origin and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Definition 5 ([4]). We let $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$

The function

$$\begin{aligned} F(a, b, c; z) &= 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!}, \quad z \in U, \end{aligned} \quad (5)$$

is called Gaussian hypergeometric function, where $(d)_k$ is Pochhammer symbol defined by

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2) \dots (d+k-1) \text{ with } (d)_0 = 1,$$

and

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \text{ with } \Gamma(z+1) = z \cdot \Gamma(z), \Gamma(1) = 1, \Gamma(n+1) = n!.$$

Definition 6 ([24]). We let a, b and c be complex numbers with $c \neq 0, -1, -2, \dots$ and $\lambda > 0$. We define the fractional integral of Gaussian hypergeometric function:

$$\begin{aligned} D_z^{-\lambda} F(a, b, c; z) &= \frac{1}{\Gamma(\lambda)} \int_0^z \frac{F(a, b, c; t)}{(z-t)^{1-\lambda}} \cdot dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^z \frac{\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \cdot \frac{t^k}{k!}}{(z-t)^{1-\lambda}} dt \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(\lambda+k+1)} \cdot z^{k+\lambda}, \quad z \in U. \end{aligned} \quad (6)$$

The next lemma is an indispensable tool for proving the theorems in the following section.

Lemma 1 ([3,10]). We let $q \in Q(a)$ and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ be analytic in U , with $p(z) \neq a$, and $n \geq 2$. If p is not subordinate to q , then there exist points $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$ and $\xi_0 \in \partial U \setminus E(q)$ for which $p(U_{r_0}) \subset q(U)$ and $p(z_0) = q(\xi_0)$ such that the following conditions are satisfied:

- (i) $\operatorname{Re} \frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} \geq 0$ and $\left| \frac{z p'(z)}{q'(\xi_0)} \right| \leq n$;
- (ii) $z_0 p'(z_0) = n \xi_0 q'(\xi_0)$;
- (iii) $\operatorname{Re} \left(\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq n \operatorname{Re} \left(\frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} + 1 \right)$;
- (iv) $\operatorname{Re} \frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 \operatorname{Re} \frac{\xi_0 q'''(\xi_0)}{q'(\xi_0)}$.

The main findings of this research provide extensions to certain classical research results authored by Miller and Mocanu, comprehensively presented in [4], concerning the second-order differential subordination theory. The new outcome is provided in the theorems established in this study and presented in the following section of the paper. A dominant for the third-order differential subordination related to the function $p \in H[a, n]$, when $n \geq 2$ can be obtained by using the method in Theorem 1. It is demonstrated that the best dominant of a third-order differential subordination associated with a specific convex function can be obtained using the technique given by Theorem 2. The basic idea behind this technique resides in determining the univalent solutions for the differential equations that correlate with the differential subordinations taken into consideration in the analysis. As a potential use for the conclusions drawn in Theorem 2, Corollary 1 employs the Gaussian hypergeometric function's fractional integral given by (6) to examine a specific third-order differential subordination. Based on this specific outcome, a numerical example is developed as well, as a final result to this study.

3. Results

The first outcome in the study extends a significant theorem that is well-known for second-order differential subordinations. The following theorem offers a dominant in the context of third-order differential subordination. In tandem with the condition of admissibility seen in (3), the proof applies Lemma 1 presented in the Introduction.

Theorem 1. We let $h \in K$ and consider $p \in H[a, n]$, $n \geq 2$, with $p(0) = h(0)$. Also, we take an analytic function $\phi : D \rightarrow \mathbb{C}$ with $\phi(w) \neq 0$ and $p(U) \subset D$.

We assume that the following conditions are satisfied:

$$\operatorname{Re} \frac{\xi h''(\xi)}{h'(\xi)} \geq 0 \text{ and } \left| \frac{z p'(z)}{h'(\xi)} \right| \leq m, z \in U, \xi \in \partial U, m \geq n \geq 2.$$

If function p satisfies the third-order differential subordination

$$p(z) + z p'(z) \cdot \phi(p(z)) + z^2 p''(z) + z^3 p'''(z) \prec h(z), \quad (7)$$

then

$$p(z) \prec h(z), z \in U.$$

Proof. We must assume that the hypothesis of Lemma 1 is in compliance with the functions p and h in $\bar{U}$. Otherwise, functions p and h could be replaced by $p_r(z) = p(rz)$ and $h_r(z) = h(rz)$, $0 < r < 1$ and the hypothesis of Lemma 1 would be satisfied.

Knowing that $h \in K$ and that $p(0) = h(0)$, the third-order differential subordination (7) can be seen as

$$\left\{ p(z) : p(z) + z p'(z) \cdot \phi(p(z)) + z^2 p''(z) + z^3 p'''(z); z \in U \right\} \subset h(U).$$

By employing a particular value $z = z_0 \in U$, the above inclusion can be interpreted as

$$p(z_0) + zp'(z_0) \cdot \phi(p(z_0)) + z^2 p''(z_0) + z^3 p'''(z_0) \in h(U). \quad (8)$$

For the proof of the assertions of this theorem, Lemma 1 is applied in conjunction with the admissibility condition found in Definition 3.

The function $\psi : \mathbb{C}^4 \times \overline{U} \rightarrow \mathbb{C}$ is introduced as

$$\psi(r, s, t, u; z) = r + s \cdot \phi(r) + t + u. \quad (9)$$

We assume that $\psi \in \psi_n[h(U), h], n \geq 2$.

Considering particular values $r = p(z_0), s = z_0 \cdot p'(z_0), t = z_0^2 p''(z_0), u = z_0^3 p'''(z_0)$, Relation (9) passes into

$$\psi(p(z_0), z_0 \cdot p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) = p(z_0) + z_0 \cdot p'(z_0) \cdot \phi(p(z_0)) + z_0^2 p''(z_0) + z_0^3 p'''(z_0). \quad (10)$$

Applying (10) in (8), we obtain

$$\psi(p(z_0), z_0 \cdot p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \in h(U). \quad (11)$$

We assume now that $p \not\prec h$. In this case, Lemma 1 states that certain $z_0 = r_0 e^{i\theta_0} \in U$ and $\xi_0 \in \partial U$ satisfy the following inequalities:

$$p(z_0) = h(\xi_0), z_0 p'(z_0) = m \xi_0 h'(\xi_0), t = z_0^2 p''(z_0), u = z_0^3 p'''(z_0), \quad (12)$$

when

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq m \operatorname{Re}\left(\frac{\xi_0 h''(\xi_0)}{h'(\xi_0)} + 1\right),$$

$$\operatorname{Re}\frac{u}{s} \geq m^2 \operatorname{Re}\frac{\xi_0 h'''(\xi_0)}{h'(\xi_0)}.$$

Taking $r = h(\xi_0), s = m \xi_0 h'(\xi_0), t = z_0^2 p''(z_0), u = z_0^3 p'''(z_0)$ in Definition 3, we conclude that

$$\psi(h(\xi_0), m \xi_0 h'(\xi_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \notin h(U). \quad (13)$$

We now use in (13) Equations (12) and obtain

$$\psi(p(z_0), z_0 \cdot p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \notin h(U),$$

which contradicts (11). Hence, the assumption that $p \not\prec h$ is false, and so the theorem is proved. $\square$

In the next theorem, the best dominant for a specific third-order differential subordination is identified highlighting the necessary and sufficient conditions that must be met by a function in order for it to serve as the best dominant for the third-order differential subordination. The method used to achieve those conditions is finding the univalent solutions to specific differential equations that correspond to the differential subordinations.

Theorem 2. We let $h \in K$ and take $\phi : D \rightarrow \mathbb{C}$, an analytic function in D with $\phi(w) \neq 0$. We consider also a function $p \in H[a, n], n \geq 2$, with $p(0) = h(0)$ and $p(U) \subset D$.

We assume that the differential equation

$$q(z) + zq'(z) \cdot \phi(q(z)) + z^2 q''(z) + z^3 q'''(z) = h(z) \quad (14)$$

has a univalent solution q that satisfies the conditions

$$\operatorname{Re} \frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \text{ and } \left| \frac{z p'(z)}{q'(\xi)} \right| \leq m, z \in U, \xi \in \partial U \setminus E(q), m \geq n \geq 2. \quad (15)$$

If the following third-order differential subordination is satisfied,

$$p(z) + z p'(z) \cdot \phi(p(z)) + z^2 p''(z) + z^3 p'''(z) \prec h(z), \quad (16)$$

then

$$p(z) \prec q(z), z \in U,$$

q being the best dominant of for the third-order differential Subordination (16).

Proof. Just as we have assumed for the proof of Theorem 1, functions p, q, h comply with the conditions of Lemma 1 in $\overline{U}$. Using the function given by (9) in the proof of Theorem 1 and substituting $r = p(z), s = z \cdot p'(z), t = z^2 p''(z), u = z^3 p'''(z)$, we obtain

$$\psi(p(z), z \cdot p'(z), z^2 p''(z), z^3 p'''(z); z) = p(z) + z \cdot p'(z) \cdot \phi(p(z)) + z^2 p''(z) + z^3 p'''(z). \quad (17)$$

Using (17), Subordination (16) transforms into

$$\psi(p(z), z \cdot p'(z), z^2 p''(z), z^3 p'''(z); z) \prec h(z), z \in U. \quad (18)$$

Since $h \in S$, Subordination (18) can be interpreted as

$$\left\{ \psi(p(z), z \cdot p'(z), z^2 p''(z), z^3 p'''(z); z) \right\} \subset h(U). \quad (19)$$

By employing a particular value $z = z_0 \in U$, (19) provides

$$\psi(p(z_0), z_0 \cdot p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \in h(U). \quad (20)$$

The next step in the proof is to assume that $p \not\prec h$. Then, in Lemma 1, it is provided that there are certain points $z_0 \in U$ and $\xi_0 \in \partial U \setminus E(q)$ such that:

$$p(z_0) = q(\xi_0), z_0 p'(z_0) = m \xi_0 q'(\xi_0), t = z_0^2 p''(z_0), u = z_0^3 p'''(z_0), \quad (21)$$

satisfying the conditions stated in Lemma 1,

$$\operatorname{Re} \left(\frac{t}{s} + 1 \right) \geq m \operatorname{Re} \left(\frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} + 1 \right), \operatorname{Re} \frac{u}{s} \geq m^2 \operatorname{Re} \frac{\xi_0 q'''(\xi_0)}{q'(\xi_0)}.$$

Substituting in Definition 3 $r = q(\xi_0), s = m \xi_0 q'(\xi_0), t = z_0^2 p''(z_0), u = z_0^3 p'''(z_0)$, we write

$$\psi(q(\xi_0), m \xi_0 q'(\xi_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \notin h(U). \quad (22)$$

Using Equations (21) in (22), we deduce

$$\psi(p(z_0), z_0 \cdot p'(z_0), z_0^2 p''(z_0), z_0^3 p'''(z_0)) \notin h(U). \quad (23)$$

But (23) contradicts (20), hence the assumption that $p \not\prec h$ is false and we can conclude that $p(z) \prec q(z), z \in U$.

Since q is a univalent solution of the differential Equation (14), it is the best dominant for the third-order differential Subordination (16). $\square$

As an application for the results proved in Theorem 2, we use $p(z) = D_z^{-\lambda} F(a, b, c; z)$, $a, b, c \in \mathbb{C}, c \neq 0, -1, -2, \dots$, the fractional integral of Gaussian hypergeometric function

given by (6), and the convex function $h(z) = z - \frac{3}{2}z^2 + (z - z^2) \cdot \phi\left(z - \frac{1}{2}z^2\right)$, and the following corollary emerges:

Corollary 1. We let $h \in K$, $h(z) = z - \frac{3}{2}z^2 + (z - z^2) \cdot \phi\left(z - \frac{1}{2}z^2\right)$ and take $\phi : D \rightarrow \mathbb{C}$, an analytic function in D with $\phi(w) \neq 0$. We consider function $p \in H[a, n]$, $n \geq 2$, with $p(0) = h(0)$ and $p(U) \subset D$. We let $q(z) = z - \frac{1}{2}z^2$ be the univalent solution of the differential equation

$$z - \frac{3}{2}z^2 + (z - z^2) \cdot \phi\left(z - \frac{1}{2}z^2\right) = h(z), z \in U.$$

We let $p(z) = D_z^{-\lambda}F(a, b, c; z)$, $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, given by (6) with $p(0) = q(0)$ satisfy

$$\begin{aligned} \operatorname{Re} \frac{\xi q''(\xi)}{q'(\xi)} = \operatorname{Re} \frac{\xi}{\xi - 1} \geq \frac{1}{2} \text{ and} \\ \left| \frac{z[D_z^{-\lambda}F(a, b, c; z)]'}{1 - \xi} \right| \leq m, z \in U, \xi \in \partial U \setminus E(q), m \geq n \geq 2. \end{aligned} \quad (24)$$

Then, the third-order differential subordination

$$\begin{aligned} D_z^{-\lambda}F(a, b, c; z) + z[D_z^{-\lambda}F(a, b, c; z)]' \cdot \phi(D_z^{-\lambda}F(a, b, c; z)) \\ + z^2[D_z^{-\lambda}F(a, b, c; z)]'' + z^3[D_z^{-\lambda}F(a, b, c; z)]''' \\ \prec z - \frac{3}{2}z^2 + (z - z^2) \cdot \phi\left(z - \frac{1}{2}z^2\right), \end{aligned} \quad (25)$$

implies

$$D_z^{-\lambda}F(a, b, c; z) \prec z - \frac{1}{2}z^2,$$

$q(z) = z - \frac{1}{2}z^2$ being the best dominant of for the third-order differential Subordination (25).

Proof. We show that functions p, q and h used in the statement of the corollary satisfy the conditions of Theorem 2.

Conditions (15) of Theorem 2 are replaced by (24) in Corollary 1. All that is left to be demonstrated is that the function $q(z) = z - \frac{1}{2}z^2$ is convex in U . For the proof of this statement, we evaluate

$$\operatorname{Re} \left(\frac{z q''(z)}{q'(z)} + 1 \right) = \operatorname{Re} \frac{1 - 2z}{1 - z} = \frac{2\rho^2 + 1 - 3\rho \cos \theta}{1 - 2\rho \cos \theta + \rho^2},$$

where $z = \rho(\cos \theta + i \sin \theta)$, $0 < \rho < 1$.

Letting $\rho \rightarrow 1$,

$$\lim_{\rho \rightarrow 1} \operatorname{Re} \left(\frac{z q''(z)}{q'(z)} + 1 \right) = \lim_{\rho \rightarrow 1} \frac{2\rho^2 + 1 - 3\rho \cos \theta}{1 - 2\rho \cos \theta + \rho^2} = \frac{3(1 - \cos \theta)}{2(1 - \cos \theta)} = \frac{3}{2} \geq 0,$$

hence, $q \in K$.

All the conditions required by Theorem 2 being satisfied, for $p(z) = D_z^{-\lambda}F(a, b, c; z)$, $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, given by (6) and $q(z) = z - \frac{1}{2}z^2$ applied in Theorem 2, we obtain

$$D_z^{-\lambda}F(a, b, c; z) \prec z - \frac{1}{2}z^2,$$

with $q(z) = z - \frac{1}{2}z^2$ being the best dominant of this differential subordination. $\square$

Example. For $a = -2, b = 2 - i, c = 2 + i, F(-1, -i, i; z) = 1 + z, \lambda = 1$, we obtain

$$D_z^{-1}F(-2, 2 - i, 2 + i; z) = z + \frac{-3 + 4i}{5}z^2 - \frac{z^3}{3}.$$

Applying Corollary 1, we write the following:

The third-order differential subordination

$$z + \frac{-3 + 4i}{5}z^2 - \frac{z^3}{3} + z \left[1 + \frac{2(-3 + 4i)}{5}z - z^2 \right] \cdot \phi \left(z + \frac{-3 + 4i}{5}z^2 - \frac{z^3}{3} \right) + z^2 \left[\frac{2(-3 + 4i)}{5} - 2z \right] + z^3(-2) \\ \prec z - \frac{1}{2}z^2, z \in U$$

implies

$$z + \frac{-3 + 4i}{5}z^2 - \frac{z^3}{3} \prec z - \frac{1}{2}z^2, z \in U,$$

and $z - \frac{1}{2}z^2$ is the best dominant.

4. Conclusions

The research described in this paper offers entirely novel results regarding the extension of a series of findings related to the familiar second-order differential subordination established by Miller and Mocanu in [4] to third-order differential subordination theory. The two theorems contained in the paper demonstrate the third-order differential subordination results that were attained involving second-order differential subordination findings which were not previously taken into consideration for an extension to third-order. Theorem 1 provides a means for determining a third-order differential subordination's dominant. The techniques used to determine the best dominants of third-order differential subordinations are highlighted in the following theorem. The results presented in Theorem 2 demonstrate that the challenge of identifying the best dominant of a third-order differential subordination is essentially resolved when the univalent solution of the related differential equation is discovered. A nice corollary develops as an application for the results obtained in Theorem 2 in the case of investigating Gaussian hypergeometric function's fractional integral and a particular convex function to obtain a specific third-order differential subordination, and its best dominant is also indicated. As a conclusion to this study, an illustration of how the theoretical findings from Corollary 1 may be applied is also included.

Since the findings of this study are fundamental in analyzing the theory of third-order differential subordinations, they could eventually be applied in future research to establish new third-order differential subordinations.

- The Gaussian hypergeometric function's fractional integral employed as the application here could be replaced by other fractional operators inspired by the study contained in this paper.
- The analysis of this pattern for third-order differential subordinations may be related to other differential-integral operators. Recent such results can be found in [28–30].
- Additionally, corresponding third-order differential superordinations can be explored employing the dual theory of third-order differential superordination, potentially connecting the findings of such a study with current findings through sandwich-type theorems as it can be seen in recent investigations like [31,32].
- Applications of third-order differential subordination theory in source-sink dynamics theory have already been mentioned citing the work seen in [27]; hence, the study presented here could be adapted to fit this theory. Furthermore, applications for fluid mechanics can also be derived by building on ideas from [33].
- Symmetry properties which result from the involvement of the Gauss hypergeometric function and the fractional integral could be further investigated using the techniques of third-order differential subordination and its dual, third-order differential superordination.

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