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Certain Class of Bi-Univalent Functions Defined by Sălăgean q-Difference Operator Related with Involution Numbers

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Abstract: We introduce and examine two new subclass of bi-univalent function Σ, defined in the open unit disk, based on Sălăgean-type q-difference operators which are subordinate to the involution numbers. We find initial estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass introduced here. We also obtain a Fekete–Szegö inequality for the new function class. Several new consequences of our results are pointed out, which are new and not yet discussed in association with involution numbers.

Keywords: univalent functions; starlike and convex functions; bi-univalent functions; Sălăgean operator; q-difference operator; coefficient bounds; Fekete–Szegö inequality

MSC: 30C45; 30C80; 30C50

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1. Introduction and Preliminaries

Let $\mathcal{H}$ represent the class of holomorphic functions expressed as

$$\zeta(\varepsilon) = \varepsilon + \sum_{n=2}^{\infty} a_n \varepsilon^n \tag{1}$$

normalized as $\zeta(0) = 0 = \zeta'(0) - 1$ defined in the open unit disk

$$\triangle = \{\varepsilon \in \mathbb{C} : |\varepsilon| < 1\}.$$

Let $\mathcal{S} \subset \mathcal{H}$ consist of functions given in (1) and which are also univalent in $\triangle$. Let the class of starlike and convex functions of order α, $(0 \leq \alpha < 1)$, be given by the following:

$$\mathcal{ST}(\alpha) = \left\{\zeta \in \mathcal{H} : \Re\left(\frac{\varepsilon\, \zeta'(\varepsilon)}{\zeta(\varepsilon)}\right) > \alpha\right\}$$

and

$$\mathcal{CV}(\alpha) = \left\{\zeta \in \mathcal{H} : \Re\left(\frac{\varepsilon(\, \zeta'(\varepsilon))'}{\zeta'(\varepsilon)}\right) > \alpha\right\}$$

respectively.

A function $\zeta \in \mathcal{H}$ is called a strongly starlike function $\mathcal{SST}(\alpha)$ of order α $(0 < \alpha \leq 1)$ if

$$\left|arg\left(\frac{\varepsilon\zeta'(\varepsilon)}{\zeta(\varepsilon)}\right)\right| < \frac{\alpha\pi}{2}, \quad \varepsilon \in \Delta$$

holds. Analytic functions $\zeta, \xi \in \mathcal{H}$ and ζ are subordinate to ξ, written $\zeta(\varepsilon) \prec \xi(\varepsilon)$, provided there exist $\varpi \in \mathcal{H}$ defined on $\triangle$ with $\varpi(0) = 0$ and $|\varpi(\zeta)| < 1$ satisfying $\zeta(\varepsilon) = \xi(\varpi(\varepsilon))$. In [1], Ma and Minda assumed more general superordinate functions expressed as

$$\phi(\varepsilon) = 1 + B_1\varepsilon + B_2\varepsilon^2 + B_3\varepsilon^3 + \cdots, \quad (B_1 > 0).$$

with positive real parts in $\triangle$ with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Further, they unified various subclasses of starlike and convex functions for which either of the quantities

$$\frac{\varepsilon\,\zeta'(\varepsilon)}{\zeta(\varepsilon)} \quad \text{or} \quad 1 + \frac{\varepsilon\,\zeta''(\varepsilon)}{\zeta'(\varepsilon)}$$

is subordinate to a more general superordinate function given in (1).

1.1. Quantum Calculus

The application of *q-calculus* was initiated by Jackson in the paper [2]. A comprehensive study on applications of *q-calculus* in operator theory may be found in the paper [3]. Research work in connection with function theory and q-theory together was first introduced by Ismail et al. [4].

We recall some basic definitions and concept details of *q-calculus* (see [5] and references cited therein) which are used in this paper.

For $0 < q < 1$ the Jackson's *q-derivative* [2] of a function $\zeta \in \mathcal{H}$ is given by the following definition:

$$\mathcal{Q}_q\zeta(\varepsilon) = \begin{cases} \frac{\zeta(\varepsilon)-\zeta(q\varepsilon)}{(1-q)\varepsilon} & for \quad \varepsilon \neq 0, \\ \zeta'(0) & for \quad \varepsilon = 0, \end{cases} \tag{2}$$

and $\mathcal{Q}_q^2\zeta(\varepsilon) = \mathcal{Q}_q(\mathcal{Q}_q\zeta(\varepsilon))$. From (2), we have

$$\mathcal{Q}_q\zeta(\varepsilon) = 1 + \sum_{n=2}^{\infty} a_n[n]_q\varepsilon^{n-1} \tag{3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \tag{4}$$

is sometimes called *the basic number* n. If $q \to 1^-, [n]_q \to n$. For a function $h(\varepsilon) = \varepsilon^n$, we obtain $\mathcal{Q}_q\varepsilon^n = \mathcal{Q}_q h(\varepsilon) == [n]_q\varepsilon^{n-1} = \frac{1-q^n}{1-q} = \varepsilon^{n-1}$ and $\lim_{q\to 1^-} \mathcal{Q}_q h(\varepsilon) = \lim_{q\to 1^-}([n]_q\varepsilon^{n-1}) = nz^{n-1} = h'(\varepsilon)$, where h' is the ordinary derivative. For $\zeta \in \mathcal{H}$, the Sălăgean *q-differential operator* is defined and discussed by Govindaraj and Sivasubramanian [6] as given below:

$$\begin{aligned} \mathcal{Q}_q^0\zeta(\varepsilon) &= \zeta(\varepsilon) \\ \mathcal{Q}_q^1\zeta(\varepsilon) &= \varepsilon\mathcal{Q}_q\zeta(\varepsilon) \\ \mathcal{Q}_q^\kappa\zeta(\varepsilon) &= \varepsilon\mathcal{Q}_q(\mathcal{Q}_q^{\kappa-1}\zeta(\varepsilon)) \\ \mathcal{Q}_q^\kappa\zeta(\varepsilon) &= \varepsilon + \sum_{n=2}^{\infty}[n]_q^\kappa a_n\varepsilon^n \quad (\kappa \in \mathbb{N}_0, \varepsilon \in \Delta) \end{aligned} \tag{5}$$

We note that $\lim_q \to 1^-$

$$\mathcal{Q}^\kappa\zeta(\varepsilon) = \varepsilon + \sum_{n=2}^{\infty} n^\kappa a_n\varepsilon^n \quad (\kappa \in \mathbb{N}_0, \varepsilon \in \Delta) \tag{6}$$

is the familiar Sălăgean derivative [7].

1.2. Generalized Telephone Numbers (GTNs)

The classical telephone numbers (TN), prominent as involution numbers, are specified by the recurrence relation

$$\mathrm{Y}(n) = \mathrm{Y}(n-1) + (n-1)\mathrm{Y}(n-2) \quad for \quad n \geq 2$$

with

$$\mathrm{Y}(0) = \mathrm{Y}(1) = 1.$$

Associates of these numbers with symmetric groups were perceived for the first time in 1800 by Heinrich August Rothe, who pointed out that $\mathrm{Y}(n)$ is the number of involutions (self-inverse permutations) in the symmetric group (see, for example [8,9]). Since involutions resemble the standard Young tableaux, it is noticeable that the nth involution number is consistently the number of Young tableaux on the set $1, 2, \ldots, n$ (for details, see [10]). It's worth citing that, according to John Riordan [11], recurrence relation, in fact, yields the number of construction patterns in a telephone system with n subscribers. In 2017, Wlochand Wolowiec-Musial [12] identified **GTNs** with the following recursion:

$$\mathrm{Y}(x,n) = \tau\mathrm{Y}(\tau, n-1) + (n-1)\mathrm{Y}(\tau, n-2) \quad n \geq 0 \quad \text{and} \quad \tau \geq 1$$

with

$$\mathrm{Y}(\tau, 0) = 1, \mathrm{Y}(\tau, 1) = \tau,$$

and studied some properties. In 2019, Bednarz and Wolowiec-Musial [13] presented a new generalization of TN by

$$\mathrm{Y}_\tau(n) = \mathrm{Y}_\tau(n-1) + \tau(n-1)\mathrm{Y}_\tau(n-2), \quad n \geq 2 \quad \text{and} \quad \tau \geq 1$$

with

$$\mathrm{Y}_\tau(0) = \mathrm{Y}_\tau(1) = 1.$$

Recently, they found the exponential generating function and the summation formula **GTNs** represented by $\mathrm{Y}_\tau(n)$, given by:

$$e^{x+\tau\frac{x^2}{2}} = \sum_{n=0}^{\infty} \mathrm{Y}_\tau(n)\frac{x^n}{n!} \quad (\tau \geq 1).$$

As we can observe, if $\tau = 1$, then we obtain classical telephone numbers $\mathrm{Y}(n)$. Clearly, $\mathrm{Y}_\tau(n)$ is for some values of n given as

1. $\mathrm{Y}_\tau(0) = \mathrm{Y}_\tau = 1,$
2. $\mathrm{Y}_\tau(2) = 1 + \tau,$
3. $\mathrm{Y}_\tau(3) = 1 + 3\tau$
4. $\mathrm{Y}_\tau(4) = 1 + 6\tau + 3\tau^2$
5. $\mathrm{Y}_\tau(5) = 1 + 10\tau + 15\tau^2$
6. $\mathrm{Y}_\tau(6) = 1 + 15\tau + 45\tau^2 + 15\tau^3.$

We now consider the function

$$\digamma(\varepsilon) := e^{(\varepsilon+\tau\frac{\varepsilon^2}{2})} = 1 + \varepsilon + \frac{1+\tau}{2}\varepsilon^2 + \frac{1+3\tau}{6}\varepsilon^3 + \frac{3\tau^2+6\tau+1}{24}\varepsilon^4 + \frac{1+10\tau+15\tau^2}{120}\varepsilon^5 + \cdots.$$

for $\varepsilon \in \mathbb{D}$ and study $\zeta \in \mathcal{H}$ (see [14,15]).

1.3. Bi-Univalent Functions

The Koebe One-quarter Theorem [16] ensures that the image of $\triangle$ under every univalent function $\zeta \in \mathcal{H}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function ζ has an inverse ζ^{-1} satisfying $\zeta^{-1}(\zeta(\varepsilon)) = \varepsilon,\ (\varepsilon \in \triangle)$ and $\zeta(\varepsilon^{-1}(\varsigma)) = \varsigma\ (|\varsigma| < r_0(\zeta),\ r_0(\zeta) \geq \frac{1}{4}).$

A function $\zeta \in \mathcal{H}$ is said to be bi-univalent in $\triangle$ if both ζ and ζ^{-1} are univalent in $\triangle$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\triangle$. The functions $\frac{\varepsilon}{1-\varepsilon} \quad -\log(1-\varepsilon), \quad \frac{1}{2}\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ are in the class Σ so it is not empty(see details in [17]). Since $\zeta \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $\xi = \zeta^{-1}$ has the expansion

$$\xi(\varsigma) = \zeta^{-1}(\varsigma) = \varsigma - a_2\varsigma^2 + (2a_2^2 - a_3)\varsigma^3 + \cdots . \tag{7}$$

Various classes of bi-univalent functions were introduced and studied in recent times. The study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al. [17]. Motivated by this, many researchers [18–33] (also the references cited therein) recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor–Maclaurin coefficients. Motivated by recent study on telephone numbers [34] and using the Sălăgean q-differential operator defined by (5), for functions ξ of the form (7) as given in [33], we have

$$\mathcal{Q}_q^{\kappa}\xi(\varsigma) = \varsigma - a_2[2]_q^{\kappa}\varsigma^2 + (2a_2^2 - a_3)[3]_q^{\kappa}\varsigma^3 + \cdots \tag{8}$$

using that in this article first time we introduce a new subclass $\mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$ of Σ in association with involution numbers and find estimates on the coefficients $|a_2|$ and $|a_3|$ for $\zeta \in \mathcal{P}\Sigma_q^{\kappa}(\lambda, F)$ by Ma–Minda subordination. We also obtain the Fekete–Szegö problem by using the initial coefficient values of a_2 and a_3.

Definition 1. *Let $0 \leq \vartheta \leq 1$. We say that $\zeta \in \Sigma$ belongs to the class $\mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$ if*

$$\frac{(1-\vartheta)\mathcal{Q}_q^{k+1}\zeta(\varepsilon) + \vartheta\mathcal{Q}_q^{k+2}\zeta(\varepsilon)}{(1-\vartheta)\mathcal{Q}_q^{\kappa}\zeta(\varepsilon) + \vartheta\mathcal{Q}_q^{k+1}\zeta(\varepsilon)} \prec F(\varepsilon) \tag{9}$$

and

$$\frac{(1-\vartheta)\mathcal{Q}_q^{\kappa+1}\xi(\varsigma) + \vartheta\mathcal{Q}_q^{k+2}\xi(\varsigma)}{(1-\vartheta)\mathcal{Q}_q^{\kappa}\xi(\varsigma) + \vartheta\mathcal{Q}_q^{k+1}\xi(\varsigma)} \prec F(\varsigma), \tag{10}$$

where $\mathcal{Q}_q^{\kappa}\xi$ is given by (8).

Example 1. *Taking $\vartheta = 0$ we have $\mathcal{P}\Sigma_q^{\kappa}(0, F) \equiv \mathcal{S}\Sigma_q^{\kappa}(F)$ and $\zeta \in \Sigma$ is in $\zeta \in \mathcal{S}\Sigma_q^{\kappa}(F)$ if the following subordination holds:*

$$\frac{\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)}{\mathcal{Q}_q^{\kappa}\zeta(\varepsilon)} \prec F(\zeta) \quad \text{and} \quad \frac{\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)}{\mathcal{Q}_q^{\kappa}\xi(\varsigma)} \prec F(\varsigma),$$

where $\mathcal{Q}_q^{\kappa}\xi$ is given by (8).

Example 2. *Taking $\vartheta = 1$ we have $\mathcal{P}\Sigma_q^{\kappa}(1, F) \equiv \mathcal{K}\Sigma_q^{\kappa}(F)$ and $\zeta \in \Sigma$ is in $\zeta \in \mathcal{K}\Sigma_q^{\kappa}(F)$ if the following subordination holds:*

$$\frac{\mathcal{Q}_q^{\kappa+2}\zeta(\varepsilon)}{\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)} \prec F(\varepsilon) \quad \text{and} \quad \frac{\mathcal{Q}_q^{\kappa+2}\xi(\varsigma)}{\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)} \prec F(\varsigma)$$

where $\mathcal{Q}_q^{\kappa}\xi$ is given by (8).

We need the following lemmas for our investigation.

Lemma 1. *(see [16], p. 41) Let $\mathcal{P}$ be the class of all analytic functions $p(\varepsilon)$ of the form*

$$p(\varepsilon) = 1 + \sum_{n=1}^{\infty} p_n \varepsilon^n \tag{11}$$

satisfying $\Re(p(\varepsilon)) > 0$ $(\varepsilon \in \Delta)$ and $p(0) = 1$. Then

$$|p_n| \leq 2 \ \ (n = 1, 2, 3, \ldots).$$

This inequality is sharp for each n. In particular, equality holds for all n for the function

$$p(\varepsilon) = \frac{1+\varepsilon}{1-\varepsilon} = 1 + \sum_{n=1}^{\infty} 2\varepsilon^n.$$

2. Coefficient Bounds for $\zeta \in \mathcal{P}\Sigma_q^{\kappa}(\vartheta, \digamma)$

Theorem 1. *Let ζ given by (1) be in the class $\mathcal{P}\Sigma_q^{\kappa}(\vartheta, \digamma)$. Then*

$$|a_2| \leq \frac{2}{\sqrt{|2[q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa} - q(1+\vartheta q)^2[2]_q^{2k}] + q^2(1+\vartheta q)^2[2]_q^{2k}(1-\tau)|}} \tag{12}$$

and

$$|a_3| \leq \frac{1}{q}\left(\frac{1}{q(1+\vartheta q)^2[2]_q^{2k}} + \frac{1}{\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}\right). \tag{13}$$

Proof. We can write $s(\varepsilon)$ and $t(\varepsilon)$ as

$$s(\varepsilon) := \frac{1+u(\varepsilon)}{1-u(\varepsilon)} = 1 + s_1\varepsilon + s_2\varepsilon^2 + \cdots$$

and

$$t(\varepsilon) := \frac{1+v(\varepsilon)}{1-v(\varepsilon)} = 1 + t_1\varepsilon + t_2\varepsilon^2 + \cdots$$

or, equivalently,

$$u(\varepsilon) := \frac{s(\varepsilon)-1}{s(\varepsilon)+1} = \frac{1}{2}\left[s_1\varepsilon + \left(s_2 - \frac{s_1^2}{2}\right)\varepsilon^2 + \cdots\right] \tag{14}$$

and

$$v(\varepsilon) := \frac{t(\varepsilon)-1}{t(\varepsilon)+1} = \frac{1}{2}\left[t_1\varepsilon + \left(t_2 - \frac{t_1^2}{2}\right)\varepsilon^2 + \cdots\right]. \tag{15}$$

Then $s(\varepsilon)$ and $t(\varepsilon)$ are analytic in $\triangle$ where $s(0) = 1 = t(0)$. Since

$$u, v : \triangle \to \triangle,$$

we say that $s(\varepsilon)$ and $t(\varepsilon)$ have a positive real part in $\triangle$, and

$$|s_i| \leq 2 \quad \text{and} \quad |t_i| \leq 2, \quad (i = 1, 2, 3, \ldots).$$

Further we have

$$\begin{aligned} \digamma(u(\varepsilon)) = e^{u(\varepsilon) + \tau\frac{[u(\varepsilon)]^2}{2}} \quad &= \quad e^{\left(\frac{s(\varepsilon)-1}{s(\varepsilon)+1} + \tau\frac{[\frac{s(\varepsilon)-1}{s(\varepsilon)+1}]^2}{2}\right)} \\ &= \quad 1 + \frac{s_1}{2}\varepsilon + \left(\frac{s_2}{2} + \frac{(\tau-1)s_1^2}{8}\right)\varepsilon^2 \\ &+ \quad \left(\frac{s_3}{2} + (\tau-1)\frac{s_1 s_2}{4} + \frac{(1-3\tau)}{48}s_1^3\right)\varepsilon^3 + \cdots. \end{aligned} \tag{16}$$

$$\begin{aligned} F(v(\varsigma)) = e^{v(\varsigma)+\tau\frac{[v(\varsigma)]^2}{2})} \quad &= \quad e^{(\frac{t(\varsigma)-1}{t(\varsigma)+1}+\tau\frac{[\frac{t(\varsigma)-1}{t(\varsigma)+1}]^2}{2})} \\ &= \quad 1+\frac{t_1}{2}\varsigma+\Big(\frac{t_2}{2}+\frac{(\tau-1)t_1^2}{8}\Big)\varsigma^2 \\ &+ \quad \Big(\frac{t_3}{2}+(\tau-1)\frac{t_1t_2}{4}+\frac{(1-3\tau)}{48}t_1^3\Big)\varsigma^3+\cdots. \end{aligned}$$

Using (14) and (15) in (9) and (10) respectively, we have

$$\frac{(1-\vartheta)\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)+\vartheta\mathcal{Q}_q^{\kappa+2}\zeta(\varepsilon)}{(1-\vartheta)\mathcal{Q}_q^{\kappa}\zeta(\varepsilon)+\vartheta\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)} = F(u(\varepsilon)) = 1+\frac{s_1}{2}\varepsilon+\Big(\frac{s_2}{2}+\frac{(\tau-1)s_1^2}{8}\Big)\varepsilon^2+\cdots \tag{17}$$

and

$$\frac{(1-\vartheta)\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)+\vartheta\mathcal{Q}_q^{\kappa+2}\xi(\varsigma)}{(1-\vartheta)\mathcal{Q}_q^{\kappa}\xi(\varsigma)+\vartheta\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)} = F(v(\varsigma)) = 1+\frac{t_1}{2}\varsigma+\Big(\frac{t_2}{2}+\frac{(\tau-1)t_1^2}{8}\Big)\varsigma^2+. \tag{18}$$

We obtain the following relations

$$q(1+\vartheta q)[2]_q^{\kappa}a_2 = \frac{1}{2}s_1, \tag{19}$$

$$q\Big\{1+\vartheta(q+q^2)\Big\}[2]_q[3]_q^{\kappa}a_3-q(1+\vartheta q)^2[2]_q^{2\kappa}a_2^2 = \frac{1}{2}(s_2-\frac{s_1^2}{2})+\frac{1+\tau}{8}s_1^2, \tag{20}$$

$$-q(1+\vartheta q)[2]_q^{\kappa}a_2 = \frac{1}{2}t_1 \tag{21}$$

and

$$q\Big\{1+\vartheta(q+q^2)\Big\}[2]_q[3]_q^{\kappa}(2a_2^2-a_3)-q(1+\vartheta q)^2[2]_q^{2\kappa}a_2^2 = \frac{1}{2}(t_2-\frac{t_1^2}{2})+\frac{1+\tau}{8}t_1^2. \tag{22}$$

From (19) and (21) it follows that

$$s_1 = -t_1 \tag{23}$$

and

$$8q^2(1+\vartheta q)^2[2]_q^{2\kappa}a_2^2 = (s_1^2+t_1^2). \tag{24}$$

From (20), (22) and (24), we obtain

$$a_2^2 = \frac{(s_2+t_2)}{2\{2[q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}-q(1+\vartheta q)^2[2]_q^{2\kappa}]+q^2(1+\vartheta q)^2[2]_q^{2\kappa}(1-\tau)\}} \tag{25}$$

Applying Lemma 1 for the coefficients s_2 and t_2, we immediately obtain the desired estimate on $|a_2|$ as asserted in (12).

By subtracting (22) from (20) and using (23) and, we have

$$a_3 = a_2^2+\frac{s_2-t_2}{4q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}. \tag{26}$$

If we use (24) in the relation (26), we will obtain

$$a_3 = \frac{s_1^2+t_1^2}{8q^2(1+\vartheta q)^2[2]_q^{2\kappa}}+\frac{s_2-t_2}{4q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}. \tag{27}$$

If we apply Lemma 1 once again for s_1, s_2, t_1 and t_2, we obtain the desired estimate on $|a_3|$ as asserted in (13). □

By taking $\vartheta = 1$ and $\vartheta = 0$ in Theorem 1 we can state the estimates for f, in the function classes $\mathcal{S}\Sigma_q^{\kappa}(F)$ and $\mathcal{K}\Sigma_q^{\kappa}(F)$ respectively given in Example 1 and 2 which are new and not yet discussed in association with involution numbers.

3. The Fekete–Szegö Problem for $\zeta \in \mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$

The Fekete–Szegö inequality is one of the well-known problems with the coefficients of univalent analytic functions. It was first given by [35], as

$$|a_3 - va_2^2| \leq \begin{cases} 3 - 4v, & if \quad v \leq 0, \\ 1 + 2e^{\frac{-2v}{1-v}}, & if \quad 0 \leq v \leq 1, \\ 4v - 3, & if \quad v \geq 1. \end{cases}$$

Lemma 2 ([36]). *Let $k, l \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$. If $|\varepsilon_1| < R$ and $|\varepsilon_2| < R$, then*

$$|(k+l)\varepsilon_1 + (k-l)\varepsilon_2| \leq \begin{cases} 2|k|R, & |k| \geq |l|, \\ 2|l|R, & |k| \leq |l|. \end{cases}$$

Now, $\zeta \in \mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$ we obtain the Fekete–Szegö inequality $|a_3 - \aleph a_2^2|$.

Theorem 2. *Let $\zeta \in \mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$ be given by (1). Then for $\aleph \in \mathbb{R}$*

$$|a_3 - \aleph a_2^2| \leq \begin{cases} \frac{1}{q\left\{1+\vartheta(q+q^2)\right\}[2]_q[3]_q^{\kappa}}, \\ \quad for \quad |\aleph - 1| \leq \left|1 - \frac{(1+\vartheta q)^2[2]_q^{2k-1}}{\left\{1+\vartheta(q+q^2)\right\}[3]_q^{\kappa}} + \frac{q(1+\vartheta q)^2[2]_q^{2\kappa-1}}{\left\{1+\vartheta(q+q^2)\right\}[3]_q^{\kappa}}\frac{(1-\tau)}{2}\right| \\ \frac{2|\aleph-1|}{\left|2[q\left\{1+\vartheta(q+q^2)\right\}[2]_q[3]_q^{\kappa} - 2q(1+\vartheta q)^2[2]_q^{2\kappa}]B_1^2 + q^2(1+\vartheta q)^2[2]_q^{2\kappa}(1-\tau)\right|}, \\ \quad for \; |\aleph - 1| \geq \left|1 - \frac{(1+\vartheta q)^2[2]_q^{2\kappa-1}}{\left\{1+\vartheta(q+q^2)\right\}[3]_q^{\kappa}} + \frac{q(1+\vartheta q)^2[2]_q^{2\kappa-1}}{\left\{1+\vartheta(q+q^2)\right\}[3]_q^{\kappa}}\frac{(1-\tau)}{2}\right| \end{cases}. \tag{28}$$

Proof. From (25) and(26) it follows that

$$a_3 - \aleph a_2^2 = \left(\varphi(\aleph) + \frac{1}{4q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}\right)s_2 + \left(\varphi(\aleph) - \frac{1}{4q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}\right)t_2,$$

where

$$\varphi(\aleph) = \frac{(1-\aleph)}{2\left\{2[q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa} - q(1+\vartheta q)^2[2]_q^{2\kappa}] + q^2(1+\vartheta q)^2[2]_q^{2\kappa}(1-\tau)\right\}}.$$

Then, applying the above Lemma 1 and Lemma 2, we get

$$\left|a_3 - \aleph a_2^2\right| \leq \begin{cases} \frac{1}{q\left\{1+\vartheta(q+q^2)\right\}[2]_q[3]_q^{\kappa}}, & for \;\; 0 \leq |\varphi(\aleph)| \leq \frac{1}{4\left\{1+\vartheta(q+q^2)\right\}[2]_q[3]_q^{\kappa}} \\ 4|\varphi(\aleph)|, & for \quad |\varphi(\aleph)| \geq \frac{1}{4\left\{1+\vartheta(q+q^2)\right\}[2]_q[3]_q^{\kappa}} \end{cases}$$

which yields the desired inequality. □

Specifically by fixing $\aleph = 1$ we obtain

$$\left|a_3 - \aleph a_2^2\right| \leq \frac{1}{q\{1+\vartheta(q+q^2)\}[2]_q[3]_q^{\kappa}}.$$

Further by fixing $\vartheta = 0$ and $\vartheta = 1$ in the Theorem 3, respectively we arrive at the Fekete–Szegö inequality for $\zeta \in \mathcal{S}\Sigma_q^{\kappa}(\digamma)$ and $\zeta \in \mathcal{K}\Sigma_q^{\kappa}(\digamma)$.

4. Bi-Univalent Function Class $\mathcal{F}\Sigma_q^{\kappa}(\wp, \beta)$

In the section, motivated by Frasin et al. [20], we will give the following new subclass involving the Sălăgean type q-difference operator linked with **GTNs** and also its related classes its worthy to note that these classes have not been discussed so far.

Definition 2. *A function $\zeta \in \Sigma$ given by (1) is said to be in the class*

$$\mathcal{F}\Sigma_q^{\kappa}(\wp, \digamma) \quad (0 \leq \wp \leq 1, \varepsilon, \varsigma \in \Delta)$$

if the following conditions hold:

$$\left((1-\wp)\frac{\mathcal{Q}_q^{\kappa}\zeta(\varepsilon)}{\varepsilon} + \wp\frac{\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)}{\varepsilon}\right) \prec \digamma(\varepsilon) \tag{29}$$

and

$$\left((1-\wp)\frac{\mathcal{Q}_q^{\kappa}\xi(\varsigma)}{\varsigma} + \wp\frac{\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)}{\varsigma}\right) \prec \digamma(\varepsilon). \tag{30}$$

Example 3. *A function $\zeta \in \Sigma$, members of which are given by (1) and*

1. *for $\wp = 0$, let $\mathcal{F}\Sigma_q^{\kappa}(0, \digamma) =: \mathcal{R}\Sigma_q^{\kappa}(\digamma)$, denotes the subclass of Σ, and the conditions*

$$\left(\frac{\mathcal{Q}_q^{\kappa}\zeta(\varepsilon)}{\varepsilon}\right) \prec \digamma(\varepsilon) \quad \textit{and} \quad \Re\left(\frac{\mathcal{Q}_q^{\kappa}\xi(\varsigma)}{\varsigma}\right) \prec \digamma(\varsigma)$$

hold.

2. *For $\wp = 1$, let $\mathcal{F}\Sigma_q^{\kappa}(1, \digamma) =: \mathcal{H}\Sigma_q^{\kappa}(\digamma)$ denote the subclass of Σ and satisfy the conditions*

$$\left(\frac{\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)}{\varepsilon}\right) \prec \digamma(\varepsilon) \quad \textit{and} \quad \left(\frac{\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)}{\varsigma}\right) \prec \digamma(\varsigma).$$

Theorem 3. *Let $\zeta \in \mathcal{F}\Sigma_q^{\kappa}(\wp, \digamma)$. Then*

$$|a_2| \leq \frac{2}{\sqrt{|2[1+(q+q^2)\wp][3]_q^{\kappa}+q^2(1+q\wp)^2[2]_q^{2\kappa}(1-\tau)|}} \tag{31}$$

$$|a_3| \leq \frac{1}{(1+q\wp)^2[2]_q^{2\kappa}} + \frac{1}{(1+(q+q^2)\wp)[3]_q^{\kappa}}. \tag{32}$$

and

$$\left|a_3 - \hbar a_2^2\right| \leq \begin{cases} \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}}, & for\ \ 0 \leq |\psi(\hbar)| \leq \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}} \\ 4|\psi(\hbar)|, & for\ \ |\psi(\hbar)| \geq \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}} \end{cases}$$

where

$$\psi(\hbar) = \frac{1-\hbar}{2\left\{2[1+(q+q^2)\wp][3]_q^{\kappa} + (1+q\wp)^2[2]_q^{2\kappa}(1-\tau)\right\}}.$$

Proof. Suppose that $\zeta \in \mathcal{F}\Sigma_q^\kappa(\wp, F,)$ satisfies the conditions given in Definition 2 and, following the steps as in Theorem 1,

$$(1-\wp)\frac{\mathcal{Q}_q^\kappa \zeta(\varepsilon)}{\varepsilon} + \wp\frac{\mathcal{Q}_q^{\kappa+1}\zeta(\varepsilon)}{\varepsilon} = 1 + \frac{s_1}{2}\varepsilon + \left(\frac{s_2}{2} + \frac{(\tau-1)s_1^2}{8}\right)\varepsilon^2 +, \tag{33}$$

$$(1-\wp)\frac{\mathcal{Q}_q^\kappa \xi(\varsigma)}{\varsigma} + \wp\frac{\mathcal{Q}_q^{\kappa+1}\xi(\varsigma)}{\varsigma} = 1 + \frac{t_1}{2}\varsigma + \left(\frac{t_2}{2} + \frac{(\tau-1)t_1^2}{8}\right)\varsigma^2 +, \tag{34}$$

Now, by comparing the corresponding coefficients in (33) and (34), we obtain,

$$(1+q\wp)[2]_q^\kappa a_2 = \frac{1}{2}s_1, \tag{35}$$

$$(1+(q+q^2)\wp)[3]_q^\kappa a_3 = \frac{1}{2}(s_2 - \frac{s_1^2}{2}) + \frac{1+\tau}{8}s_1^2, \tag{36}$$

$$-(1+q\wp)[2]_q^\kappa a_2 = \frac{1}{2}t_1, \tag{37}$$

$$(1+(q+q^2)\wp)[3]_q^\kappa (2a_2^2 - a_3) = \frac{1}{2}(t_2 - \frac{t_1^2}{2}) + \frac{1+\tau}{8}t_1^2, \tag{38}$$

From (35) and (37), we obtain

$$a_2 = \frac{1}{2(1+q\wp)[2]_q^\kappa}s_1 = -\frac{1}{2(1+q\wp)[2]_q^\kappa}t_1, \tag{39}$$

which implies

$$s_1 = -t_1 \tag{40}$$

and

$$8(1+q\wp)^2[2]_q^{2\kappa}a_2^2 = s_1^2 + t_1^2. \tag{41}$$

Adding (36) and (38), then using (41), we obtain

$$a_2^2 = \frac{s_2 + t_2}{2\left\{2[1+(q+q^2)\wp][3]_q^\kappa + (1+q\wp)^2[2]_q^{2\kappa}(1-\tau)\right\}} \tag{42}$$

Applying Lemma 1 for the coefficients s_2 and t_2, we immediately have the desired estimate on $|a_2|$ as asserted in (31). By subtracting (38) from (36) and using (40) and, we obtain

$$a_3 = a_2^2 + \frac{s_2 - t_2}{4\{1+(q+q^2)\wp\}[3]_q^\kappa}. \tag{43}$$

Next using (41) in (43), we finally obtain

$$a_3 = \frac{s_1^2 + t_1^2}{8(1+q\wp)^2[2]_q^{2\kappa}} + \frac{s_2 - t_2}{4(1+(q+q^2)\wp)[3]_q^\kappa}. \tag{44}$$

Applying Lemma 1 once again for the coefficients s_1, s_2, t_1 and t_2, we obtain the desired estimate on $|a_3|$ as asserted in (32). From (43) and (42) it follows that

$$a_3 - \hbar a_2^2 = \left(\psi(\hbar) + \frac{1}{4(1+(q+q^2)\wp)[3]_q^\kappa}\right)s_2 + \left(\psi(\hbar) - \frac{1}{4(1+(q+q^2)\wp)[3]_q^\kappa}\right)t_2,$$

where

$$\psi(\hbar) = \frac{1-\hbar}{2\Big\{2[1+(q+q^2)\wp][3]_q^{\kappa} + (1+q\wp)^2[2]_q^{2\kappa}(1-\tau)\Big\}}.$$

Then, applying Lemma 1, we have

$$\left|a_3 - \hbar a_2^2\right| \leq \begin{cases} \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}}, & for \;\; 0 \leq |\psi(\hbar)| \leq \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}} \\ 4|\psi(\hbar)|, & for \;\; |\psi(\hbar)| \geq \frac{1}{4(1+(q+q^2)\wp)[3]_q^{\kappa}} \end{cases}$$

which yields the desired inequality. □

By allowing fixing $\wp = 0$ and $\wp = 1$ in Theorem 3 we can state the estimates for f, in the function classes $\mathcal{R}\Sigma_q^{\kappa}(F)$ and $\mathcal{H}\Sigma_q^{\kappa}(F)$ respectively given in Example 3, further by taking $q \to 1^-$ we state various subclasses of Σ and above results, which are new and not yet discussed in association with involution numbers.

5. Conclusions

The results presented in this paper followed by the work of Srivastava et al. [17] related with Generalized telephone phone number (GTN). This work presented the initial Taylor coefficient and the Fekete–Szegö problem results for this newly defined function class $\mathcal{P}\Sigma_q^{\kappa}(\vartheta, F)$ and $\mathcal{F}\Sigma_q^{\kappa}(\wp, F)$. By specializing the parameters in Theorem 1 and 3, given in Examples 1–3, we can investigate problems not yet examined for GTN. Also by taking $q \to 1^-$ we state various subclasses of Σ and state results analogues to Theorem 1 and 3. This paper can motivate many researchers to extend this idea to another classes of bi-univalent functions [37], Sakaguchi-type functions [38] (other classes of functions cited in this article) and further second Hankel determinant results for function class Σ, as discussed in [39].

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