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An Analysis of the Factors Influencing the Strong Chromatic Index of Graphs Derived by Inflating a Few Common Classes of Graphs

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Abstract: The problem of *strong edge coloring* discusses assigning colors to the edges of a graph such that distinct colors are assigned to any two edges which are either adjacent to each other or are adjacent to a common edge. The least number of colors required to define a strong edge coloring of a graph is called its *strong chromatic index*. This problem is equivalent to the problem of assigning collision-free frequencies to the links between the elements of a wireless sensor network. In this article, we discuss a novel way of generating new graphs from existing graphs. This graph construction is known as *inflating* a graph. We discuss the strong chromatic index of graphs generated by *inflating* some common classes of graphs and graphs derived from them. In particular, we consider the cycle graph, which is symmetric in nature, and graphs such as the path graph and the star graph, which are not symmetric. Further, we analyze the factors which influence the strong chromatic index of these inflated graphs.

Keywords: strong chromatic index; strong edge coloring; inflated graphs; wireless sensor networks; induced matchings

MSC: 05C15; 05C38

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1. Introduction

A proper *edge coloring* of a graph assigns distinct colors to any two edges which are adjacent to each other such that the color classes thus obtained partition the edge set of the graph into matchings. We consider a special kind of edge coloring called strong edge coloring in this article. Two edges e_1 and e_2 are said to be *visible* to each other if they are adjacent to each other or are adjacent to a common edge. The *strong edge coloring* of a graph assigns distinct colors to any two edges which are *visible* to each other. If the subgraph induced by a subset of $E(G)$ is a matching, the subgraph is called an *induced matching*. The color classes generated as result of the strong edge coloring of the graph G partition the edge set of G into *induced matchings*. The minimum number of color classes required to define a strong edge coloring of a graph G is called the *strong chromatic index* of G and is denoted by $\chi'_s(G)$.

1.1. Factors Influencing $\chi'_s(G)$

The initial spark which kindled the interest of researchers was the article authored by Hale [1] which discussed assigning frequencies to transceivers in a wireless sensor network. This was followed by a significant contribution published by Erdős and Nešetřil [2], who formally defined the problem. Over the past years, analysis of the parameter $\chi'_s(G)$ has focused on various attributes of graph G. Contributions by Lv et al. [3] and Huang et al. [4] determined the bounds for $\chi'_s(G)$ based on the maximum degree Δ_G of the graph G. In particular, the article by Huang et al. established an upper bound for $\chi'_s(G)$ for every planar graph with a maximum degree 4. The article by Lv et al. obtained results for the upper bound of $\chi'_s(G)$ based on a new attribute called the *maximum average degree* of graph G. This article established the proof using a partition of the vertex set of graph G. The influence of the planar nature of graphs on $\chi'_s(G)$ was explored in the article [5] by Chang and Duh, and in the article [6] by Wang et al. The contributions made by Chang and Duh classified graphs based on their girth and the maximum degree Δ of the graph. The results derived by Wang et al. proved that any graph with a maximum degree of 4 can be colored using a maximum of 19 colors by classifying certain vertices and faces as *interior* vertices and *interior* faces. Conjectures on upper bounds of $\chi'_s(G)$ for bipartite graphs were explored by Bensmail and Huang. The results by Bensmail et al. [7] showed that any $(3, \Delta)-$ bipartite graph can be colored using 4Δ colors. This bound was improved to 3Δ in an article [8] published by Huang et al.

1.2. Graph Coloring and Communication Networks

Communication networks can be efficiently modeled using structures called graphs. Problems of coloring the vertices and edges of a graph are known to have various applications in the design of networks. The survey of ref. [9] lists the most recent of such contributions. The problem of coloring the vertices of a graph can be applied to assigning frequencies in a Wi-Fi network. This was explored by Orden et al. in [10]. Dey et al. studied the problem of coloring vertices and edges of vague graphs in [11] and included a discussion on applying the results to solve problems related to traffic flow management.

Recently, there have been contributions which have applied the study performed on strong edge coloring in the field of wireless sensor networks. A wireless sensor network can be represented by an undirected simple graph with a vertex for every transceiver and an edge between two vertices whenever two transceivers are in the transmission range of each other.

The performance of a wireless sensor network can be affected if there is a primary or secondary type of interference between two pairs of transceivers. A primary type of interference is observed between edges (e_1, e_2) and (e_3, e_4) if the transceivers denoted by e_1, e_2, e_3 and e_4 are not distinct. A secondary type of interference occurs when the edge (e_1, e_4) or (e_2, e_3) is also part of the network. Barette et al. in their article [12] proved that the problem of assigning interference-free frequencies to a wireless network is equivalent to the problem of defining a strong edge coloring for the edges of the graph which models this network.

1.3. Inflation Graphs—A New Method to Construct Graphs for Modeling Wireless Sensor Networks

Generating new graphs from a given graph has contributed more toward understanding the properties of graph structures. One such graph structure is a bipartite spanning graph generated from a given graph G. Edge deletion is employed to generate spanning bipartite graphs from a given graph. The articles [13,14] discussed the minimum number of colors to be deleted to generate bipartite spanning subgraphs from cactus chains, carbon nanotubes and boron nanotubes. Carbon nanotubes (CNT) and boron nanotubes (BNT) are employed in the field of electronics to build sensors and transistors. In this paper, we consider the inflation of a graph as a procedure to generate new graphs from existing graphs. The construction of the inflation graph G' of a graph G was introduced by Casselgren and Pedersen, who carried out an investigation of Hadwiger's conjecture on inflations of 3-chromatic graphs in their article [15]. Constructing larger graphs from a given graph have contributed more toward understanding the properties of a given graph G. One such method was discussed by Hayat et al. in [16]. This method generates new graphs by replacing a vertex of a given graph by clique of size s, for a positive integer s. However, the proposed method is different from the method of s-clique extension since the method of inflating a graph G uses a more specific value of s, namely, the degree of the vertices of the graph G. In the method of s-clique extension, every vertex is replaced by a clique of order s. On the other hand, while inflating a graph, a pendant vertex is not replaced by a clique of a larger order. Further, any graph G can have only one graph G' representing its inflated version but the s-clique extension of a graph can generate many graphs from a given graph based on various values of s.

Definition 1 ([17]). *Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ denote the vertex set of a graph G without isolated vertices. Its inflation graph G' is constructed by replacing each vertex u_i of the graph G by a clique A_i on $d(u_i)$ vertices, where $d(u_i)$ is the degree of the vertex u_i, and replacing each edge u_iu_j by an edge v_iv_j where $v_i \in A_i$ and $v_j \in A_j$. Any two different edges of G are replaced by non-adjacent edges of G'.*

Recent contributions on inflation graphs include a study on the square difference labeling, cube difference labeling and square multiplicative labeling of inflated triangle snake graphs [18] and inflated ladder graphs [19] by Thirusangu et al. and the neighborhood-prime labeling of inflations of some graphs [17] by Palani et al.

1.4. Novelty of the Article

In this paper, we explore the value of the parameter χ'_s for graphs obtained by inflating graphs, such as paths, cycles and star graphs and their derivatives. Similar studies on analyzing derivatives of path graphs and cycle graphs have been discussed in the literature before. The article [20] by Hayat et al. discusses a zero forcing number and the propagation of oriented versions of graphs derived from path graphs and cycle graphs. We aimed at identifying what factors other than Δ influence the value of χ'_s for inflating graphs which are derivatives of path graphs, cycle graphs and star graphs. We believe that this novel approach will contribute more toward modeling wireless sensor networks using inflated graphs and assigning frequencies to the networks by using the results on the strong chromatic index of such graphs.

We introduce the following classification of edges in an inflated graph G' to establish the various results on the strong chromatic index of inflated graphs. Further, we define the lower bound for the strong chromatic index of any given inflated graph.

Definition 2. *Inflating a vertex of degree $d(u_i)$ will yield a clique A_i on $d(u_i)$ vertices. This clique will have $\binom{d(u_i)}{2}$ edges. The edges in such cliques are called clique edges. If the vertex is a pendant vertex, inflating the vertex will yield a clique on only one vertex without any new edges.*

Definition 3. *Let u_i and u_j denote two adjacent vertices connected by an edge e in the graph G. In the inflated graph G', the edge e will connect a vertex in the clique A_i and a vertex in the clique A_j. Such edges are called link edges. If G has $|E|$ edges, the inflated graph G' has $|E|$ link edges.*

Based on the above classifications, we propose the following results.

Theorem 1. *Let u_i denote a non-pendant vertex of degree, $d(u_i)$. Each of the $\binom{d(u_i)}{2}$ clique edges in this clique A_i is visible to every other clique edge. Further, each clique edge is visible to exactly $d(u_i)$ link edges which connect every vertex of A_i to a corresponding vertex in the cliques generated by inflating the neighbors of u_i.*

Proof. Since there is an edge between any two vertices in a clique, any two clique edges in the clique A_i are either adjacent or are adjacent to a common edge. Hence, any two clique edges are *visible* to each other. Further, we know that a link edge e, incident with the vertex w in the inflated graph G', connects w to exactly 1 vertex in every clique generated by inflating the neighbors of u_i in G. Thus, all clique edges incident to w are *visible* to e since they are adjacent to e. Every other clique edge in A_i is incident with some neighbor of w. Thus, they are adjacent to the clique edges incident to w, which is incident to e. Thus, every clique edge is *visible* to each of the $d(u_i)$ link edges of clique A_i. □

Corollary 1. *Let v denote a vertex of graph G with maximum degree Δ. The strong chromatic index of graph G' obtained by inflating graph G is bounded below by $\binom{\Delta}{2} + \Delta$, where Δ is the maximum degree of graph G.*

Proof. Let $v \in E(G)$ be a vertex with $d(v) = \Delta$. We consider the $\binom{\Delta}{2} + \Delta$ edges in the clique generated by inflating the vertex v. From Theorem 1, each of these edges is *visible* to every other edge. Thus, every edge must be assigned a different color. Hence $\binom{\Delta}{2} + \Delta$ colors are necessary. We have the inequality, $\chi'_s(G') \geq \binom{\Delta}{2} + \Delta$. □

Lemma 1. *Let A_i and A_j denote two cliques obtained by inflating two adjacent vertices u_i and u_j of graph G. Let e_1 and e_2 denote two clique edges in inflated graph G'. These edges are visible to each other if and only if the following hold:*

- *e_1 and e_2 belong to the same clique A_i of order $d(u_i)$ generated by inflating a vertex u_i.*
- *Alternatively, e_1 and e_2 belong to two different cliques A_i and A_j and are adjacent to a link edge e.*

Proof. Suppose e_1 and e_2 belong to the same clique A_i. The edges will be *visible* to each other since they will be either adjacent to each other or will be adjacent to a common edge in the same clique A_i.

Suppose e_1 and e_2 belong to different cliques A_i and A_j generated by inflating two adjacent vertices u_i and u_j. If they are adjacent to a common link edge e between a vertex in clique A_i and a vertex in clique A_j, they will be *visible* to each other, by definition.

Conversely, if two clique edges e_1 and e_2 are *visible* to each other, they are adjacent to each other or are adjacent to a common edge. If e_1 and e_2 are adjacent to each other, they certainly belong to the same clique A_i since only link edges connect vertices belonging to cliques generated by inflating adjacent vertices. If they are not adjacent to each other, e_1 and e_2 can be *visible* to each other only if they are adjacent to a common clique edge in clique A_i or if they are adjacent to link edge e between a vertex in clique A_i and a vertex in clique A_j. □

Lemma 2. *Let G' denote the inflated graph of G. Let A_i and A_j denote two cliques generated by inflating two adjacent vertices u_i and u_j. Clique edge e_1 in A_i and link edge e are visible to each other if and only if e is a link edge between one vertex in the clique A_i and another vertex in the clique A_j.*

Proof. Suppose u_i and u_j are adjacent vertices in graph G. Let A_i and A_j denote the cliques generated by inflating u_i and u_j. Suppose e is a link edge between vertex w_1 in clique A_i and vertex w_2 in clique A_j, and e_1 is a clique edge incident with w_1 in clique A_i. Then e_1 and e are adjacent. Thus, they are *visible* to each other. If e_1 is not incident to w_1, e_1 is adjacent to some clique edge e', between w_1 and one of the end vertices of e_1. Thus, e is *visible* to e_1 since both e and e_1 are adjacent to e'.

Conversely, if clique edge e_1 and link edge e are *visible* to each other, they are adjacent to each other or are adjacent to a common edge by definition. Let w denote the end vertices of clique edge e_1. If they are adjacent to each other, w is incident to the link edge e connecting w with a vertex in another clique. If they are adjacent to a common edge, this common edge must be a clique edge between w and another vertex in A_i, which is incident to link edge e. □

Lemma 3. *Two link edges e and e' are visible to each other in inflated graph G' if and only if they are two adjacent edges in graph G.*

Proof. If e and e' are adjacent to each other in graph G, they are incident to the same vertex u_i. Thus, they are the link edges each connecting a vertex of clique A_i with vertices in other cliques obtained by inflating neighbors of u_i. The two edges are *visible* to each other since they are adjacent to a common clique edge e_1 connecting one end vertex of e in clique A_i and another end vertex of e' in the same clique.

Conversely, if e and e' are *visible* to each other in G', they can only be adjacent to a common clique edge e_1 of some clique A_i since no two link edges can be adjacent in the inflated graph G'. Since they are adjacent to a common clique edge in A_i, they represent two edges incident to vertex u_i in graph G. □

Lemma 4. *Let u_1 and u_2 denote two non-adjacent vertices in graph G. The clique edges in cliques A_1 and A_2 are not visible to each other.*

Proof. Case (i): Let G be a connected graph. Then, there is a path from u_1 to u_2. Without loss of generality, let us assume that Let $P = u_1 u_k u_2$ denotes one such path. Inflating u_1, u_k and u_2 will result in cliques A_1, A_k and A_2 in the inflated graph G'. If e_1 and e_2 are two clique edges in cliques A_1 and A_2, they are not adjacent to a common link edge e and hence they cannot be *visible* to each other according to Lemma 1.

Case (ii): Let G be a disconnected graph. If u_1 and u_2 belong to different components, the two clique edges A_1 and A_2 cannot be *visible* to each other. Supposing that they belong to the same component of G, an argument similar to the one in Case (i) can be repeated to establish the correctness of the result. □

In the rest of the paper, we denote the vertices of graph G by $u_1, u_2, \ldots, u_n$ and the vertices of inflated graph G' by $v^k_{j,i}$ representing the i^{th} vertex of the k^{th} clique on j vertices.

2. Strong Chromatic Index of Inflations of Path Graphs and Its Derivatives

In this section, we discuss the strong chromatic index of path graphs and two graphs which are derived from path graphs. Path graphs are one of the simplest forms of graphs which are both connected and acyclic. A graph $G(V, E)$ exhibits the property of symmetry if a non-trivial automorphism can be defined on its vertex set $V(G)$. Since the vertices of a path graph follow a linear order, any automorphism which preserves this order of vertices can only be trivial. Hence, path graphs are not symmetric in nature.

2.1. Inflation of the Path Graph

Theorem 2. *$\chi'_s(G') = 3$ when G' is obtained by inflating a path on $n \geq 3$ vertices is 3.*

Proof. Let G denote a path $u_1 u_2 u_3 \ldots u_n$ on n vertices. If $n = 2$, G' is a path on 2 vertices too. Hence, G' can be colored using only one color. For $n \geq 3$, G contains two pendant vertices and $n-2$ vertices of degree 2. Since $\Delta = 2$, by Corollary 1, at least $\binom{2}{2} + 2 = 3$ colors are necessary. The inflated graph G' is constructed by considering $n-2$ cliques of order 2 and two cliques of order 1. The two pendant vertices u_1 and u_n are replaced by two cliques on 1 vertices $v^1_{1,1}$ and $v^2_{1,1}$. The remaining $n-2$ vertices are replaced by $n-2$ cliques of order 2 denoted by $v^k_{2,1} v^k_{2,2}, 1 \leq k \leq n-2$. Thus, inflating the path $u_1 u_2 u_3 \ldots u_{n-1} u_n$ results in a path on $2n-2$ vertices with $2n-3$ edges denoted by

$$v^1_{1,1} - v^1_{2,1} - v^1_{2,2} - v^2_{2,1} - v^2_{2,2} - \cdots - v^{n-2}_{2,1} - v^{n-2}_{2,2} - v^2_{1,1}.$$

For $n \geq 3$, $2n-3 \geq 3$, we know three colors are sufficient to define a strong edge coloring of a path containing at least three edges. Hence, we have $\chi'_s(G') = 3$. □

2.2. Inflation of the Comb Graph

Definition 4 ([21]). *A comb graph on $2n$ vertices is constructed by joining a single pendant edge to every vertex of a path on n vertices. It has $2n$ vertices and $2n-1$ edges.*

(Figure 1) shows a comb graph (Figure 1a) and its inflated graph (Figure 1b). Recently, the labeling of the square graph of comb graphs [21] and k-graceful labeling of comb graphs [22] were discussed by the research community working on graph theory.

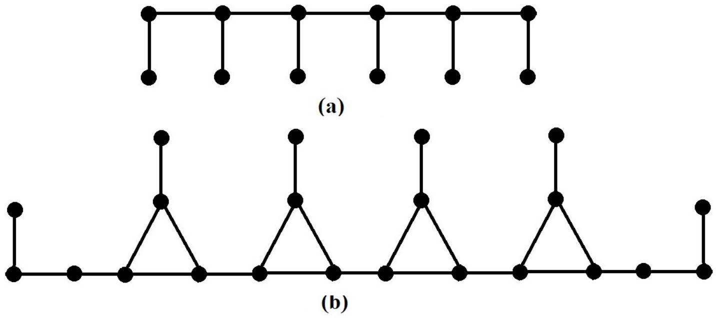

Figure 1. A comb graph (**a**) and its inflated graph (**b**).

Theorem 3. *$\chi'_s(G') = 6$ when G' is obtained by inflating a comb graph on $2n, \geq 3$ vertices.*

Proof. For $n = 2$, the comb graph on 4 vertices is a path on 4 vertices. Hence, edges of G' can be colored using 3 colors as per Theorem 2. For $n \geq 3$, there are n pendant vertices, 2 vertices of degree 2 and $n-2$ vertices of degree 3 in the graph G. Thus, the inflated comb graph has n cliques on 1 vertex, 2 cliques on 2 vertices, and $n-2$ cliques on 3 vertices. The existence of a vertex of maximum degree 3 in G can be used to conclude that $\chi'_s(G')$ is bounded below by 6 as per Corollary 1. Next, we show that these 6 colors are sufficient to color the edges of the inflated comb graph.

The $n-2$ vertices of degree 3 denoted by $u_2, u_3, \ldots, u_{n-1}$ are replaced by $n-2$ cliques on 3 vertices. The vertices on these cliques on 3 vertices are denoted by $v^k_{3,1}, v^k_{3,2}$ and $v^k_{3,3}, 1 \leq k \leq n-2$. The n pendant vertices of the graph G are replaced by n one-vertex cliques. These are denoted by $v^k_{1,1}, 1, 2, \ldots, n$. The 2 vertices of degree 2 of the graph G are replaced by 2 cliques on 2 vertices each. We denote these vertices by $v^k_{2,1}, v^k_{2,2}, k = 1, 2$. We claim that the coloring scheme given in Table 1 colors all edges of the inflated comb graph G' using only 6 colors.

Table 1. Coloring scheme to color the edges of an inflated comb graph.

Color	**Edge**	**Range of k, m**
c_1	$v^k_{1,1}v^m_{3,1}$	$k = 2, 3, \ldots, n-1$; $m = 1, 2, \ldots, n-2$
c_2	$v^k_{3,1}v^k_{3,3}$	$k = 1, 2, \ldots, n-2$
c_3	$v^k_{3,1}v^k_{3,2}$	$k = 1, 2, \ldots, n-2$

The three other colors can be used to color the edges of the path defined by the following sequence of vertices,

$$v^1_{1,1} - v^1_{2,1} - v^1_{2,2} - v^1_{3,3} - v^1_{3,2} - v^2_{3,3} - v^2_{3,2} \ldots v^{n-2}_{3,3} - v^{n-2}_{3,2} - v^2_{2,1} - v^2_{2,2} - v^n_{1,1}$$

Thus, it is established that the coloring of the edges in the inflated comb graph can be performed using just six colors. □

2.3. Inflation of the Y-Tree Graph

Definition 5. *The $Y-tree$ graph is constructed by choosing a vertex v adjacent to a pendant vertex of a path on n vertices and introducing a new vertex v' adjacent to v through a new edge $e = vv'$ [23].*

(Figure 2) shows a Y-tree graph (Figure 2a) and its inflated graph (Figure 2b).

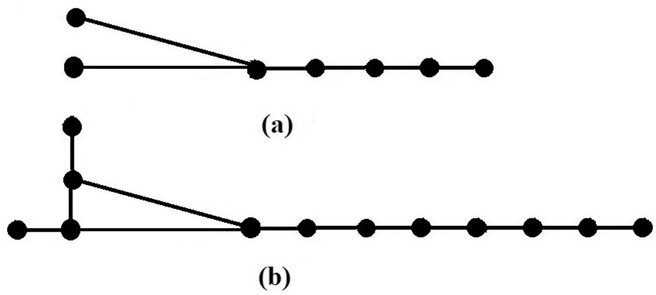

Figure 2. A Y-tree graph (**a**) and its inflated graph (**b**).

Theorem 4. $\chi'_s(G') = 6$ *when* G' *is obtained by inflating Y-tree graph* G *on* $n+1$ *vertices.*

Proof. Let the $n+1$ vertices of the Y-tree graph be denoted by $u_1, u_2, \ldots u_{n-1}, u_n, u_{n+1}$, with u_{n+1} denoting the newly introduced vertex adjacent to vertex u_{n-1}. Hence, there are $n-3$ vertices, $u_2, u_3, \ldots u_{n-2}$ of degree 2 , 3 vertices u_1, u_n, u_{n+1} of degree 1 and one vertex, u_{n-1} of degree $\Delta = 3$. By Corollary 1, 6 colors are necessary to color the edges generated by inflating the vertex u_{n-1}. The remaining edges of the inflated $Y - tree$ graph form a path as described below:

$$v^1_{1,1} - v^1_{2,1} - v^1_{2,2} - v^2_{2,1} - v^2_{2,2} - \cdots - v^{n-2}_{2,1} - v^{n-2}_{2,2}.$$

From Theorem 2, three colors are sufficient to color the edges of an inflated path graph. These three colors can be chosen optimally from the colors used so far to complete the coloring of the edges in G'. □

3. Strong Chromatic Index of Inflations of the Cycle Graph and Its Derivatives

In this section, we discuss the strong chromatic index of cycle graphs and two graphs which are derived from cycle graphs. cycle graphs are one of the simplest examples of graphs which are regular in nature. Since all vertices of a cycle graph are of degree 2, we can define a non-trivial automorphism mapping any vertex of the cycle graph to any other vertex. Hence, cycle graphs are symmetric in nature.

3.1. Inflation of the Cycle Graph

Theorem 5. $\chi'_s(G') = \chi'_s(C_{2n})$ *when* G *is a cycle on* n *vertices.*

Proof. Since every vertex of a cycle on n vertices is of degree 2, inflated graph G' will have n cliques of order 2 and size 1 joined by n edges. Hence, the inflated graph G' will be a cycle on $2n$ vertices. Thus, the edges of G' can be colored using $\chi'_s(C_{2n})$ colors. □

3.2. Inflation of the Wheel Graph

Definition 6. *The wheel graph* W_n *is obtained by connecting each vertex of the cycle* $C_n = u_1u_2 \ldots u_nu_1$ *to a common vertex* u_0 *called the apex. It has* $2n+1$ *vertices and* $2n$ *edges.*

(Figure 3) shows a wheel graph (Figure 3a) and its inflated graph (Figure 3b). Recently, the distance matrices of wheel graphs with odd vertices were discussed by Balaji et al. [24]. The article by Kapuhennayake and Perera explored the anti-magic labeling of wheel graphs [25].

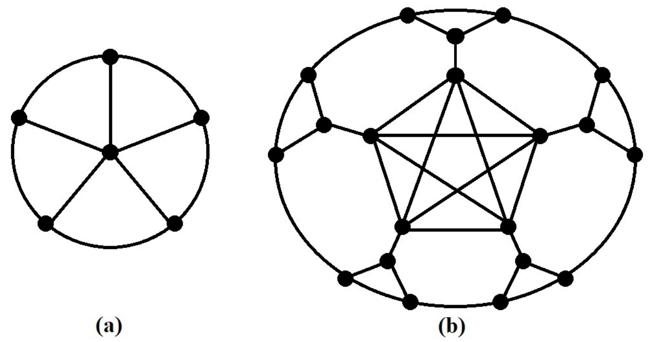

Figure 3. A wheel graph (**a**) and its inflated graph (**b**).

Theorem 6. $\chi'_s(G') = \binom{n}{2} + n$ *when G is a wheel graph* W_n *on* $n+1$ *vertices.*

Proof. The inflated wheel graph G' will have 1 clique on n vertices denoted by $v^0_{n,i}$ where $1 \le i \le n$ and n cliques on 3 vertices, denoted by $v^k_{3,i}, 1 \le k \le n, 1 \le i \le 3$. These vertices $u_1, u_2, \ldots, u_n$ are inflated into cliques containing three edges connecting vertices denoted by $v^k_{3,1}, v^k_{3,2}$ and $v^k_{3,3}, 1 \le k \le n$. The apex vertex u_0 is inflated into a clique A_0 containing n vertices denoted by $v^0_{n,i}, 1 \le i \le n$ and $\binom{n}{2}$ edges.

Further, the edges connecting vertices on the cycle $u_1 u_2 \ldots u_n u_1$ are replaced by link edges. Edge $u_i u_{i+1}$ is replaced by $v^i_{3,3} v^{i+1}_{3,2}$ for all $i = 1, 2 \ldots n-1$. Edge $u_n u_1$ is replaced by $v^n_{3,3} v^1_{3,2}$.

n edges connecting the apex vertex to n vertices on the cycle are replaced by n edges connecting the vertices of the clique on n vertices to vertices on the cliques on three vertices. We denote these edges as follows. Edges $u_0 u_i, 1 \le i \le n$ are replaced by $v^0_{n,i} v^i_{3,1}, 1 \le i \le n$.

We establish the correctness of the result stated by considering three different values of n.

Case (i): $n = 3$

By Corollary 1, a minimum of $\binom{3}{2} + 3$ colors is essential to color the three clique edges in the clique $v^0_{3,1}, v^0_{3,2}\ v^0_{3,3} v^0_{3,1}$ and the three link edges $v^0_{3,1} v^k_{3,1}, k = 1, 2, 3$ incident to the vertices of this clique. We claim that these six colors are sufficient to color the remaining edges based on the following coloring scheme defined in Table 2.

Table 2. Coloring scheme to color the edges of inflated wheel graph on 4 vertices.

Color	Edge
c_1	$v^0_{3,1} v^0_{3,3}, v^3_{3,3} v^1_{3,2}, v^2_{3,2} v^3_{3,3}$
c_2	$v^0_{3,1} v^2_{3,2}, v^3_{3,2} v^3_{3,3}, v^1_{3,3} v^2_{3,2}$
c_3	$v^0_{3,2} v^0_{3,3}, v^1_{3,2} v^1_{3,3}, v^2_{3,3} v^3_{3,2}$
c_4	$v^0_{3,1} v^3_{3,1}, v^1_{3,1} v^1_{3,3}, v^2_{3,1} v^2_{3,3}$
c_5	$v^3_{3,1} v^3_{3,3}, v^1_{3,1} v^1_{3,2}, v^0_{3,3} v^2_{3,1}$
c_6	$v^2_{3,1} v^2_{3,2}, v^3_{3,1} v^3_{3,2}, v^0_{3,2} v^1_{3,1}$

Case (ii): $n = 4$

By Corollary 1, a minimum of $\binom{4}{2} + 4$ colors are essential to color the edges since there is one vertex of maximum degree 4. Table 3 defines a coloring scheme demonstrating that these 10 colors are sufficient to color the remaining edges of G'.

Table 3. Coloring scheme to color the edges of inflated wheel graph on 5 vertices.

Color	Edge
c_1	$v^0_{4,1}v^0_{4,2}, v^3_{3,3}v^2_{3,2}, v^4_{3,1}v^4_{3,3}, v^3_{3,1}v^3_{3,3}$
c_2	$v^0_{4,2}v^0_{4,3}, v^3_{3,3}v^2_{3,2}, v^4_{3,1}v^4_{3,2}, v^1_{3,1}v^1_{3,3}$
c_3	$v^0_{4,4}v^0_{4,1}, v^4_{3,3}v^1_{3,2}, v^2_{3,1}v^2_{3,3}, v^3_{3,1}v^3_{3,2}$
c_4	$v^0_{4,3}v^0_{4,4}, v^3_{3,3}v^4_{3,2}, v^2_{3,1}v^2_{3,2}, v^1_{3,1}v^1_{3,2}$
c_5	$v^4_{3,2}v^4_{3,3}, v^0_{4,4}v^0_{4,2}, v^2_{3,2}v^2_{3,3}$
c_6	$v^1_{3,2}v^1_{3,3}, v^0_{4,1}v^0_{4,3}, v^3_{3,2}v^3_{3,3}$
c_7	$v^4_{3,1}v^0_{4,4}$
c_8	$v^1_{3,1}v^0_{4,1}$
c_9	$v^0_{4,2}v^2_{3,1}$
c_{10}	$v^0_{4,3}v^3_{3,1}$

Case (iii): $n \geq 5$
The apex vertex of the wheel graph has degree n. Therefore, a minimum of $\binom{n}{2} + n$ colors are necessary by Corollary 1. We claim that the remaining edges of G which are inflated can be colored using no additional colors. Following the convention established earlier, we observe that the cycle $u_1u_2 \dots u_nu_1$ is inflated to form a cycle C on $2n$ vertices.

$$v^1_{3,2} - v^1_{3,3} - v^2_{3,2} - v^2_{3,3} \dots v^n_{3,2} - v^n_{3,3} - v^1_{3,2}$$

The edges of this cycle are either clique edges or are link edges connecting two clique edges. These edges are *not visible* to the edges in the cycle by Lemmas 1 and 2. According to the result established in [26], $\chi'_s(C_{2n}) = 3$ or 4. Hence the cycle edges can be colored using any of the four colors from the $\binom{n}{2}$ colors used to color the edges in the clique.

Consider vertex u_i of degree 3 in G. Inflating this vertex generates clique A_i on three vertices and three edges. One of these edges is part of cycle C and its coloring is well defined above. The other two edges are *visible* to exactly three link edges as per Theorem 1. Two of these edges are the edges in cycle C. The color assigned to the other link edge cannot be assigned to these two edges since they are *visible* to each other. However, the remaining $n - 1$ link edges incident to the vertices of clique A_0 are not *visible* to these two clique edges. Hence, two of those colors can be optimally chosen to color these edges. The same argument can be repeated to the $n - 1$ vertices of degree 3. Thus, no new colors are essential to complete the strong edge coloring of the edges in the inflated wheel graph. This establishes the result that exactly $\binom{n}{2}$ colors are sufficient to color all the edges in inflated graph G'. □

3.3. Inflation of the Kite Graph

Definition 7. *An $(n,t)-$kite graph is constructed joining a $t-$edge path to a vertex of a cycle $C_n = u_1u_2 \dots u_nu_1$. It has order $n + t$ and size $n + t$.*

(Figure 4) shows a kite graph (Figure 4a) and its inflated graph (Figure 4b). Recently, the total edge irregularity of the $(n,t)-$kite graph was discussed by Winarsih and Indriati [27].

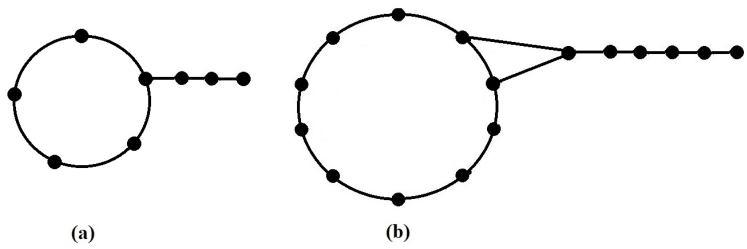

Figure 4. A kite graph (**a**) and its inflated graph (**b**).

Theorem 7. $\chi'_s(G') = 6$ *when G is an* $(n,t)-$*Kite graph* $n+t$ *vertices.*

Proof. Let us denote the cycle in the kite graph by $u_1u_2\ldots u_nu_1$. There are $n+t-3$ vertices of degree 2, $n-1$ of them lie on the cycle, 1 of them is the terminal vertex of the $t-edge$ path and the remaining $t-2$ of them are on the path which has just a solitary pendant vertex. Without loss of generality, let us assume that the $t-$edge path is attached to vertex u_2 of the cycle on n vertices. Vertex u_2 is of degree 3. From Corollary 1, a different color is necessary to color the six edges generated by inflating u_2. Three of these six colors can be chosen optimally to color the edges of the path defined below:

$$v^2_{3,2} - v^3_{2,1} - v^3_{2,2} \cdots - v^{n-1}_{2,1} - v^{n-1}_{2,2} - v^n_{2,1} - v^n_{2,2} - v^1_{2,1}$$

Further, the edges obtained by inflating the edges of the $t-$edge path are *not visible* to the edges of the path listed above. Hence, no additional colors are necessary. This establishes the proof that no more than six colors are essential to complete the strong edge coloring of the edges of the inflated kite graph. □

4. Strong Chromatic Index of Inflations of the Star Graph and Its Derivatives

In this section, we examine the strong chromatic index of graphs obtained by inflating the star graph and two graphs derived from the star graph. Like path graphs, star graphs are acyclic and connected graphs. They do not exhibit either vertex symmetry or edge symmetry since only the trivial automorphism can be defined on its vertex set.

4.1. Inflation of the Star Graph

Theorem 8. $\chi'_s(G') = \binom{n}{2} + n$ *when G is the Star graph* $K_{1,n}$ *on* $n+1$ *vertices.*

Proof. Inflating the graph $K_{1,n}$ results in one clique A_0 on n vertices and n cliques on 1 vertex. This results in graph G' having just $\binom{n}{2} + n$ edges. Since there is only one vertex of maximum degree n in G, we observe that the $\binom{n}{2} + n$ colors identified as a lower bound in the Corollary 1 are just sufficient to color the $\binom{n}{2} + n$ edges in inflated graph G'. □

4.2. Inflation of the Coconut Tree Graph

Definition 8. *The coconut tree graph* $T(m,k)$ *with parameters m and k can be constructed by identifying one vertex of degree m in the graph* $K_{1,m}$ *and a pendant vertex of path graph* P_k.

(Figure 5) shows a coconut tree graph (Figure 5a) and its inflated graph (Figure 5b). Extensive research has been carried out on the family of coconut tree graphs by exploring the aspects of properly even harmonious labeling [28] and equitable irregular coloring [29].

Theorem 9. $\chi'_s(G') = \binom{m+1}{2} + m + 1$ *when G is coconut tree graph* $T(m,k)$ *on* $m+k$ *vertices,* $m \geq 2$.

Proof. Inflating graph $T(m,k)$ results in one clique A_0 on $m+1$ vertices, $k-1$ cliques on two vertices and $m+1$ pendant vertices. We establish the correctness of the result by considering the following cases of m and k:

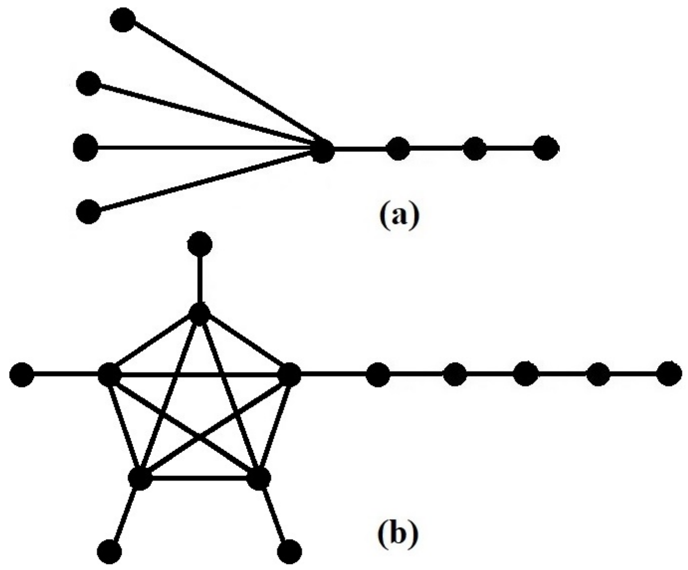

Figure 5. A coconut tree graph (**a**) and its inflated graph (**b**).

Case (i) $m \geq 2, k = 2$:
When $k = 2$, coconut tree graph $T(m, 2)$ generated is the star graph $K_{1,m+1}$. The result follows from Theorem 8.
Case (ii) $m \geq 2, k \geq 3$:
By Corollary 1, $\binom{m+1}{2} + m + 1$ colors are essential to color the edges of the inflated graph G'. The other edges of the graph form a path. The three colors required to color the edges of this path can be chosen optimally from the $\binom{m+1}{2} + m + 1$ colors used to color the edges of the clique on $m + 1$ vertices. This proves the result.
□

4.3. Inflation of the Banana Tree Graph

Definition 9. *The banana tree graph $B_{n,m}$ is generated by linking one pendant vertex from n copies of the star graph $K_{1,m-1}$ with a single root vertex. There are $nm + 1$ vertices and nm edges in banana tree graph $B_{n,m}$. Figure 6 shows a banana tree graph $B_{3,5}$, and Figure 7 shows its inflated graph.*

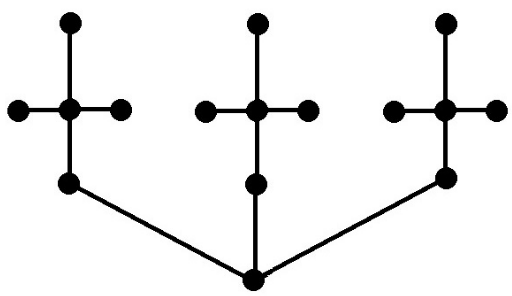

Figure 6. Banana tree graph $B(3, 5)$.

The family of banana tree graphs has been studied in depth for the properties exhibited by them in the area of topological indices. Sardar et al. computed topological indices of the line graphs of banana tree graphs [30] in 2017. Ajmal et al. analyzed the forgotten polynomial and the forgotten index for the line graphs of banana tree graphs [31] in 2017. Recently, this family of graphs was explored in the aspect of Peg solitaire games by JH de Wiljes [32] in 2020.

Theorem 10. *$\chi'_s(G') = \binom{N}{2} + N$, where $N = max.\{n, m-1\}$, when G is the banana tree graph, $B_{n,m-1}$.*

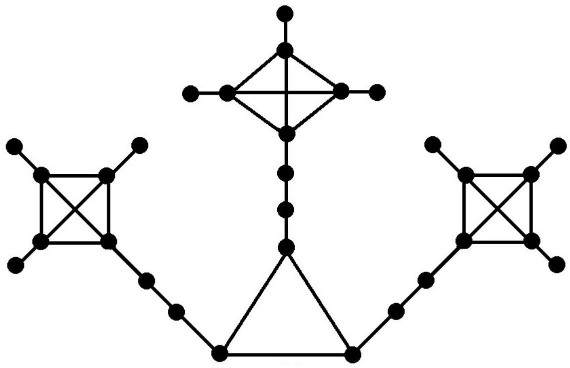

Figure 7. The inflated graph of a banana tree

Proof. Banana tree graph G has one vertex of degree n, n vertices of degree $m-1$, n vertices of degree 2 and $nm-2n$ pendant vertices. We denote the solitary vertex of degree n by u_0 and each of the n vertices of degree $m-1$ by $u_i, 1 \le i \le n$.

Case (i): $n \ge m-1$

Let A_0 denote the clique obtained by inflating the vertex u_0. By Corollary 1, $\binom{n}{2} + n$ colors are necessary to color the clique edges in A_0 and the n link edges connecting vertices of A_0 to cliques of adjacent vertices. Let $A_1, A_2, \ldots, A_n$ denote the cliques generated by inflating the n vertices $u_1, u_2, \ldots, u_n$, each of degree $m-1$. Considering one such vertex u_1, we observe that it is not adjacent to u_0. Hence, by Lemma 4, the $\binom{n}{2}$ colors used so far are sufficient to color the $\binom{m-1}{2}$ edges in clique A_1.

Next, we consider the $m-1$ link edges incident to the vertices of clique A_i. Since $n \ge m-1$, the n colors assigned to the link edges incident to the vertices of clique A_0 can be optimally chosen to color these $m-1$ edges.

The same argument can be repeated for the remaining $n-1$ vertices of degree $m-1$ since these n vertices are pairwise non-adjacent and none of them is adjacent to u_0. Thus, the $\binom{n}{2}$ colors are sufficient to color the edges in the n cliques of order $m-1$ and the link edges incident to the vertices of these cliques. This coloring scheme covers the inflations of the $nm-2n$ pendant vertices. Each of the remaining n vertices of degree 2, which are pairwise non-adjacent, is inflated to n copies of 1 edge in inflated graph G'. Thus by Lemma 4, a single color can be assigned to each of these n edges. This color can be optimally chosen from the $\binom{n}{2}$ colors used thus far. Since $n \ge m-1$, the availability of this color is guaranteed since $\binom{n}{2} \ge \binom{m-1}{2}$.

Thus, we establish that $\binom{n}{2} + n$ colors are sufficient to define a proper strong edge coloring of the inflated banana tree graph.

Case (ii): $n < m-1$

There are n vertices, each having a degree $m-1$, the maximum degree amongst the vertices of the graph. We choose one such vertex u_1. By Corollary 1, $\binom{m-1}{2} + m - 1$ colors are necessary to color the clique edges in clique A_1 and $m-1$ link edges between the vertices in clique A_1 and the vertices in the cliques generated by inflating the vertices adjacent to u_1. Since u_1 is not adjacent to the remaining $n-1$ vertices of degree $m-1$, the same set of $\binom{m-1}{2} + m - 1$ colors can be used to define a strong edge coloring of the edges obtained by inflating these $n-1$ vertices. The only vertex of degree n, namely u_0, is another vertex not adjacent with u_1. Since $n < m-1$, we have $\binom{n}{2} + n < \binom{m-1}{2} + m - 1$; thus, $\binom{n}{2}$ clique edges in the clique A_0 can be colored with the $\binom{m-1}{2} + m - 1$ colors introduced thus far. n link edges incident to n vertices of clique A_0 can be colored by optimally choosing a different color from the colors introduced thus far. Hence, no additional colors are necessary.

n edges obtained by inflating n vertices of degree 2 can be colored using a similar procedure described in the previous case.

Thus, we conclude that no additional colors are required. Summing up, we observe that $\chi'_s(G') = \binom{N}{2} + N$, where N is the greatest of the two values n and $m-1$ when G is banana tree graph $B(n, m-1)$. □

5. Strong Chromatic Index of Inflation of Complete Bipartite Graphs and Complete Multipartite Graphs

Theorem 8 deals with a particular case of complete bipartite graphs in which $m = 1$ or $n = 1$. In this section, we discuss the graph obtained by inflating the complete bipartite graph $K_{m,n}$, where $m, n \geq 2$, and derive its strong chromatic index.The generalized form of complete bipartite graphs is multipartite graphs, which are used to model probabilistic neural networks (PNNs), cellular neural networks (CNNs) and Tickysym spiking neural networks (TSNNs) by Khan et al. in their article [33].

Inflation of a Complete Bipartite Graph

Let G denote a complete bipartite graph $K_{m,n}, m, n \geq 2$ with partitions X and Y such that $|X| = m$ and $|Y| = n$. Figure 8 shows a complete bipartite graph $K_{2,3}$ (Figure 8a) and its inflated graph (Figure 8b).

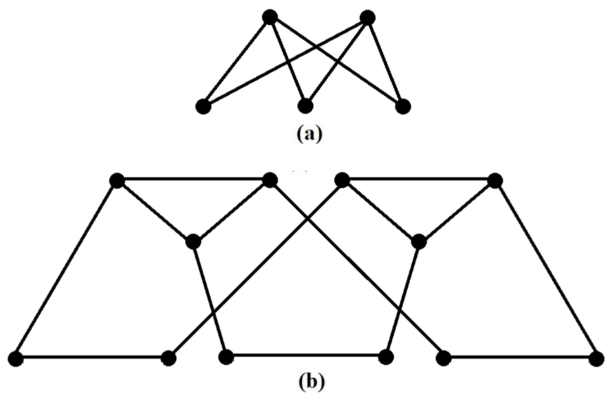

Figure 8. Complete bipartite graph $K_{2,3}$ (**a**) and its inflated graph (**b**).

Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots v_n$ denote the vertices in the partitions X and Y. Each $u_i \in X$ has a degree n and each $v_i \in Y$ has a degree m. The edge set of G has mn edges. The inflated graph G' obtained from G has m cliques of order n and size $\binom{n}{2}$ and n cliques of order m and size $\binom{m}{2}$. Let $A_1, A_2, \ldots A_m$ and $B_1, B_2, \ldots, B_n$ denote the cliques replacing the vertices of sets X and Y in inflated graph G'. Following the nomenclature established in Section 1, we observe that there are $n\binom{m}{2} + m\binom{n}{2}$ clique edges and mn link edges in the inflated graph. When $m, n = 2$, graph G is a cycle on four vertices. Its strong chromatic index is discussed in Theorem 5. The following theorem determines the strong chromatic index of bipartite graphs for values of m and n, where at least one of them is greater than 2.

Theorem 11. *Let G denote a complete bipartite graph $K_{m,n}$, where at least one of m or n is greater than 2 and $m \neq n$. Then $\chi'_s(G') = \binom{N}{2} + N$, where $N = Max.\{n, m\}$*

Proof. We build this proof based on the following two cases.

Case (i): $n > m$

We consider a vertex $u_i \in X, 1 \leq i \leq m$. In G', this vertex is replaced by a clique A_i of order n and size $\binom{n}{2}$. By Corollary 1, we observe that $\binom{n}{2}$ colors are necessary to color the edges in clique A_i. Since X is one of the partitions of bipartite graph G, no vertex $u_i \in X$ is adjacent to any of the remaining $m-1$ vertices in X. Hence, by Lemma 4, the clique edges in A_i are *not visible* to the clique edges in any of the remaining $m-1$ cliques $A_j, j \neq i$.

Hence, the same set of $\binom{n}{2}$ colors are sufficient to color the edges in the other cliques: $A_j, j \neq i$.

Next we consider a vertex $v_j \in Y, 1 \leq j \leq n$. In inflated graph G', this vertex is replaced by clique B_j of order m and size $\binom{m}{2}$. By Corollary 1, $\binom{m}{2}$ colors are necessary. Since u_i and v_j are adjacent in G, there exists link edge e connecting vertex w in clique A_i to vertex w' in clique B_j. It can be noted that a clique edge in clique B_j is *visible* to exactly $n-1$ clique edges in clique A_i. In particular, $m-1$ clique edges which are incident to w' are *visible* to $n-1$ clique edges incident to w. Hence, $n-1$ colors of the $\binom{n}{2}$ assigned to clique edges in A_i are not available for coloring $m-1$ clique edges in B_j. We observe that

$$\binom{n}{2} - (n-1) \geq m-1.$$

As long as $n > m$, the largest value of m is $n-1$. Therefore $m-1 = n-2$, which implies

$$\binom{n}{2} - (n-1) \geq n-2.$$

For all $n \geq 3$, we see that exactly one color is available for each of the $m-1$ edges. Hence, whenever $n > m$, the colors for these $m-1$ edges can be chosen from the remaining $\binom{n}{2} - (n-1)$ colors. Hence, the same set of $\binom{n}{2}$ colors is sufficient to color the edges of clique B_j. Since set Y is one of the partitions of the bipartite graph, vertices v_i and v_j are no adjacent. Hence, by Lemma 4, the clique edges in B_j are *not visible* to the clique edges in other cliques $B_k, k \neq j$. Therefore, the same set of $\binom{n}{2}$ colors is sufficient to color the edges in every clique, which replaces the vertices in partition Y without including any additional colors.

Next, there are n link edges incident to the vertices of clique A_i, and all these n edges are incident to the $\binom{n}{2}$ clique edges in clique A_i and are *visible* to each other. Therefore, n new colors are necessary. In addition, these n link edges incident to the vertices of A_i are *not visible* to the n link edges incident to the vertices of clique A_j since u_i and u_j are not adjacent. Therefore, by Lemma 3, the n colors used to color the link edges incident to the vertices of the clique A_i can be used to color the remaining $(m-1)n$ link edges.

Summing up, we observe that $\binom{n}{2} + n$ colors are necessary and sufficient to color the edges of the graph obtained by inflating the complete bipartite graph $K_{m,n}$.

Case (ii): $m > n$

The argument for the case when $m > n$ can be extended in a manner similar to Case (i). The correctness of the result is thus established. □

Theorem 12. *Let G denote a complete bipartite graph $K_{m,n}$, where $m, n \geq 2$ and $m = n$. Then $\chi'_s(G') = n^2$.*

Proof. Repeating an argument from Theorem 11, $\binom{n}{2} + n$ colors are sufficient to color the edges in each of the n cliques, replacing the vertices of partition X. Considering vertex $v_j \in Y$, we note that it is adjacent to $u_i \in X$ and hence a clique incident to vertex w' in clique B_j is *visible* to $n-1$ edges incident to vertex w in clique A_i.

Case (i): $n = 3$

For $n = m = 3$, we observe that only one color used to color the edges in A_i is available for coloring the two edges in B_j. Hence one extra color is necessary. Repeating this observation for each of the remaining $n-1$ cliques, we note that three additional colors are necessary. To complete the coloring of these edges, a coloring scheme can be defined by choosing a second set of $\binom{3}{2}$ colors for coloring the edges in clique $B_j, 1 \leq j \leq 3$. Since v_j is not adjacent to any of the other vertices in the partitions Y, the same set of $\binom{3}{2}$ colors can be used for the cliques in the remaining cliques. Summing up, we see that two sets of $\binom{3}{2}$ colors are sufficient to color all the clique edges, and one set of three colors is sufficient for all of the link edges. Therefore, if $G = K_{3,3}$, we have $\chi'_s(G') = 9$.

Case (ii): $n \geq 4$

For values of n greater than 3, we observe that $\binom{n}{2} - (n-1) \geq n-1$. Therefore no extra colors are necessary to color the $n-1$ edges incident at w'. However, since the same set of $\binom{n}{2}$ colors is used for coloring all n cliques $A_1, A_2, \ldots, A_n$, at least one new color is necessary for each of the cliques $B_1, B_2, \ldots, B_n$ based on the values of n. Hence, we suggest an alternate coloring scheme. Under this scheme, we use $\binom{n}{2}$ additional colors to color the edges in each of the n cliques $B_1, B_2, \ldots, B_n$. Since no two vertices of partition Y are adjacent to each other, this second set of $\binom{n}{2}$ colors is sufficient. Repeating an argument similar to the one in Theorem 11, we see that n colors are necessary and sufficient to color all the link edges in inflated graph G'.

Summing up, we see that $2\binom{n}{2} + n = n^2$ colors are necessary and sufficient to complete a strong edge coloring of the complete bipartite graph $K_{n,n}$. Hence, $\chi'_s(G') = n^2$. □

We observe that the value established by the theorem is the upper bound of the result conjectured by Brualdi and Massey [34]. A similar argument can be extended for the strong chromatic index of the inflation of a complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$.

6. Comparison of Factors Influencing the Strong Chromatic Index of Inflated Graphs

The construction of inflated graph G' of given graph G is based on replacing every vertex v_i by a clique on $d(v_i)$ vertices where $d(v_i)$ denotes the degree of vertex v_i. However, we observe that the lower bound of $\chi'_s(G')$ is $\binom{\Delta}{2} + \Delta$ when the graph has a vertex with maximum degree Δ. For graphs such as paths, cycles, Y-tree graphs and comb graphs, we observe that the strong chromatic index does not vary as a function of n, the number of vertices. On the other hand, the values of $\chi'_s(G')$ for the wheel graph, the star graph, the coconut tree graph and the banana tree graph vary as a quadratic function of n. We tabluate the results in Table 4 obtained to demonstrate this in detail. We attribute this to the presence of the star graph $K_{1,n}$ as a subgraph of graph G. Inflating a vertex of degree n in the the star graph $K_{1,n}$ results in a clique containing $\binom{n}{2}$ clique edges and n link edges. Thus, a minimum of $\binom{n}{2} + n$ colors is necessary to complete the coloring of the edges in G' whenever G has $K_{1,n}$ as its subgraph. Thus, the value of $\chi'_s(G')$ varies as a quadratic function of the number of vertices in the graph.

Table 4. Comparison of the strong chromatic index of various inflated graphs.

Graph G	$\chi'_s(G')$ of the Inflated Graph	Variation of $\chi'_s(G)$
Path on $n \geq 3$ vertices	3	Independent of n
Comb graph on $2n$ vertices, $n \geq 3$	6 the Inflated graph	Independent of n
Cycle on n vertices	$\chi'_s(C_{2n})$	Independent of n
Wheel on $n+1$ vertices	$\binom{n}{2} + n$	Quadratic function of n
Star graph on $n+1$ vertices	$\binom{n}{2} + n$	Quadratic function of n
Coconut tree graph on $m+k$ vertices	$\binom{m+1}{2} + m + 1$	Quadratic function of $m+1$
Banana tree graph on $nm+1$ vertices	$\binom{N}{2} + N + 1$ where $N = max\{N, m-1\}$	Quadratic function of N

7. Conclusions

In this paper, we explored the strong chromatic index of graphs generated from existing classes of graphs using a novel operation called inflation of the graph. We determined the exact values of $\chi'_s(G)$ for graphs obtained by *inflating* popular classes of graphs, such as paths, cycles, star graphs and their derivatives. The results obtained indicate how the strong chromatic index of the inflated graph G', namely $\chi'_s(G')$, varies as a quadratic function of n

whenever $K_{1,n}$ is a subgraph of the parent graph G. We believe that the knowledge of the exact values of parameter $\chi'_s(G)$ for various graphs discussed in this paper will contribute toward the design of interference-free wireless networks, which will accommodate more transceivers and channels. Future research can be extended by identifying other factors that influence χ'_s for graphs, such as circulant networks, which do not have star graph $K_{1,n}$ as its subgraph.

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