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Multiscale Asymptotic Computations for the Elastic Quadratic Eigenvalue Problem in Periodically Composite Structure

Qiang Ma ^{1,*}, Hongyu Wang ¹, Lin Bi ^{2,*}, Junzhi Cui ³, Tingyan Chen ¹ and Yuting Wu ¹

¹ School of Mathematics, Sichuan University, Chengdu 610041, China; why442639843@163.com (H.W.); chenty0121@163.com (T.C.); wyt2389775652@163.com (Y.W.)

² China Aerodynamics Research and Development Center, Mianyang 621000, China

³ LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS), Beijing 100190, China; cjz@lsec.cc.ac.cn

* Correspondence: maqiang809@scu.edu.cn (Q.M.); bzbaby1010@163.com (L.B.)

Abstract: A multiscale analysis and computational method based on the Second-Order Two-Scale (SOTS) approach are proposed for the elastic quadratic eigenvalue problems in the periodic composite domain. Two typical quadratic eigenvalue problems with different damping effects are considered, and by the asymptotic expansions of both the eigenfunctions and eigenvalues, the first- and second-order cell functions, the microscale features of this heterogeneous materials are defined successively. Then, the homogenized quadratic eigenvalue problems are derived and the second-order expansions of the eigenfunctions are formed. The eigenvalues are also broadened to the second-order terms by introducing proper auxiliary elastic functions defined in the composite structure, and the nonlinear expressions of the correctors of the eigenvalues are derived. The finite element procedures are established, linearized methods are discussed for solving the quadratic eigenvalue problems and the second-order asymptotic computations are performed. Effectiveness of the asymptotic model is demonstrated by both the qualitative and quantitative comparisons between the computed SOTS approximations and the reference solutions, and the converging behavior of the eigenfunctions are numerically verified. It is also indicated that the second-order correctors are of importance to reconstruct the detailed information of the original eigenfunctions within the micro cells.

Keywords: periodic composite structure; second-order two-scale asymptotic analysis; quadratic eigenvalue problem; Rayleigh damping; linearization method; finite element algorithm



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1. Introduction

Eigenvalue problems play an important role in many areas of science and engineering. In many structural systems, while linear eigenvalue analysis is sufficient to predict the modal response, it is not an adequate description when damping are taken into account and polynomial eigenvalue problem often arise in the analysis of vibration of the system [1]. The quadratic eigenvalue problem (QEP) is a special case of polynomial eigenvalue problems. Although it is less common and less regularly solved than the generalized eigenvalue problem (GEP), a wide range of applications require solving this problem, most of which appear in the dynamic analysis of structural mechanics, harmonic systems, circuit simulation, fluid mechanics, and, more recently, in modeling microelectromechanical systems (MEMS) [2,3]. Additionally, some studies on quadratic eigenvalue problem have involved discrete ill-posed problems [4], dissipative acoustics [5], gyroscopic [6] and overdamped systems [7].

Currently, along with the rapid growth of the advanced materials and structures, the composite materials with fine microstructures have become a matter of great concern and present interesting mathematical and computational challenges. With the improvement of computing power and the demand for accurate modeling of new materials and complex systems, this field is receiving renewed attention. In fact, traditional methods cannot solve

the fine scale with reasonable calculation cost, and may be inconsistent and stable. We need a high computational cost to calculate the structure composed of heterogeneous materials. In general, a practical modeling approach is to use macroscopic or homogenized information of the material, i.e., by considering how the microstructure affects the macroscopic behavior. The asymptotic expansion homogenization (AEH) method [8–11] is introduced to simplify the control equation with fast oscillation coefficient to the medium equation with homogenization or effective coefficient in order to facilitate the analysis and economize computational memory. Hou et al. [12,13] propose a multiscale finite element method, which is systematic and self-consistent, for solving various equations with rapidly oscillating coefficients. Some studies have involved the linear static elastic equation [14], incompressible Navier–Stokes equations [15], two-phase immiscible flow simulations in heterogeneous porous media [16], the large deformation problem [17], the elastic and thermo-elastic problem in cylindrical coordinates [18,19] and the nonlinear coupled thermoelectric problem [20,21].

In order to obtain more accurate predictions of the physical and mechanical properties of composites, Cui et al. [22] develop a second-order two-scale (SOTS) method by taking into account second-order corrections. Additionally, Cao [23] gives a rigorous proof of the convergence of second-order asymptotic progression in elliptic eigenvalue problems with mixed boundaries in porous regions. Next, aiming at the dynamic thermodynamic problems of cylindrical periodic composite structures, a new second-order two-scale analysis method and corresponding numerical algorithm are proposed in [24]. Ma et al. also extend the SOTS method to the heat conduction problems of composite materials with periodic structure under coordinate transformation [25], the axisymmetric and spherically symmetric elastic problems with small periodic structure [26], the static heat conduction problems in periodic porous regions with cavity surface radiation boundary conditions [27] and the dynamic piezoelectric problem [28]. Liu et al. [29] propose that nearly all the composite materials are provided with multiscale properties, that is to say, the scale of the structure is far greater than the feature size of the materials or constituents. For the eigenvalue problems in the composite materials. Kesavan [30,31] introduces the corrector equation to obtain second-order correctors of eigenvalues and numerical experiments show that they play an important role in improving accuracy. Our previous work also involves the eigenvalue problem. The elliptic eigenvalue problem formulated in the curvilinear coordinates is proposed and the general SOTS expansions of the eigenvalues and eigenfunctions are derived [32].

Different from the asymptotic methods and finite element algorithm that are established on the linear models [26,28,32]. In this paper, as the first try, we will consider asymptotic model for the elastic quadratic eigenvalue problem in the composite structure and the high-order finite element algorithm. To the best of our knowledge, it is the new application of the SOTS method to the nonlinear complex-valued eigenvalue problem. The derivation of the second-order expansions of both the eigenfunctions and eigenvalues is also important for the numerical simulation. In the implementation of the algorithms, we will perform the numerical computations on two separate meshes, which is different from the multiscale finite element method [12,14]. For the multiscale finite element method, the expansion of the solutions is often limited to the first-order term, we will always perform the second-order expansions. Moreover, it can be seen that this numerical procedure proposed in our work is also quite effective to capture the local oscillation of the displacement and stress field in much smaller periodicity.

For this purpose, the paper is organized as follows. Two quadratic eigenvalue problems with damping terms, one of which is a Rayleigh damping that is of industrial interest, are considered and the corresponding compact forms are described in Section 2. Two-scale asymptotic analysis for the eigenfunctions and eigenvalues are derived in Section 3. Next, the SOTS finite algorithm is presented in Section 4. Numerical examples are discussed in Section 5, and concluding remarks are given in Section 6. For ease of notations, the

convention of summation with regard to repeated indices is adopted. Without loss of generality, the bold letters in the formula represent tensors or vector functions.

2. Elastic Quadratic Eigenvalue Problem in Composite Domain

The composite domain of consideration is denoted by $\Omega \subset \mathbb{R}^N$ with N being the dimension, which is shown in Figure 1. Several kinds of constituent materials are periodically scattered in the composite domain in Figure 1a and the normalized cell domain $Q = [0, 1]^N$ is plotted in Figure 1b. Let $\mathbf{x} \in \Omega$ and $\mathbf{y} \in Q$ be the macroscopic and microscopic coordinates, respectively, and the relationship between them is expressed by a small parameter ε as

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon} - \left\lfloor \frac{\mathbf{x}}{\varepsilon} \right\rfloor, \quad (1)$$

where $\lfloor \cdot \rfloor$ denotes the round down operation. It can be seen that the ε represents the periodicity of the cell Q that is distributed in the macroscopic domain Ω ($\varepsilon = 1/8$ in Figure 1). With this configuration, Ω can be denoted by

$$\Omega = \bigcup_{\mathbf{z} \in I} \varepsilon(Q + \mathbf{z}),$$

with the index set $I = \{\mathbf{z} \in \mathbb{Z}^N | \varepsilon(Q + \mathbf{z}) \subset \Omega\}$.

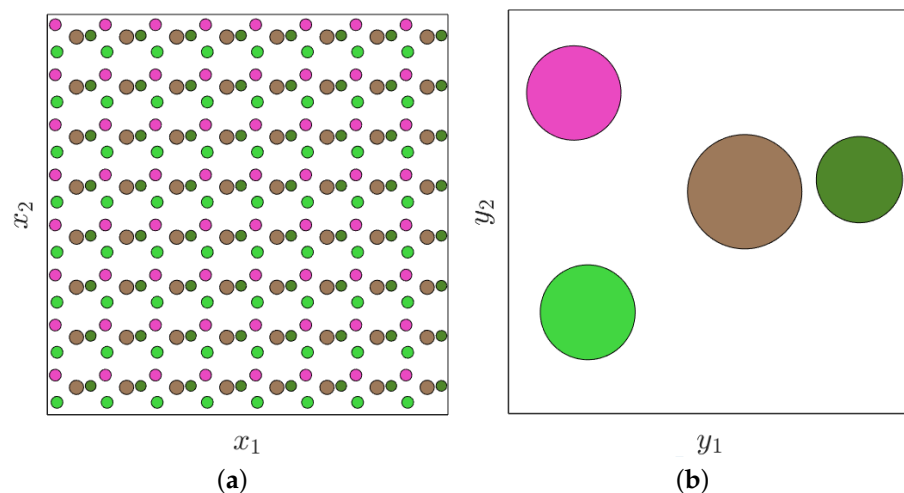


Figure 1. Illustrations of the periodic composite structure and the representative cell with $\varepsilon = 1/8$. (a) Composite domain Ω . (b) Periodic cell domain Q .

The quadratic eigenvalue problem can be typically derived from the elastodynamic equation with a damping effect formulated as follows [1,5]

$$\rho^\varepsilon \frac{\partial^2 U_i^\varepsilon}{\partial t^2} + c^\varepsilon \frac{\partial U_i^\varepsilon}{\partial t} - \frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial U_k^\varepsilon}{\partial x_l}) = f_i, \quad (2)$$

where $U_i^\varepsilon(\mathbf{x}, t)$, $i = 1, 2, \dots, N$ denotes the displacement, ρ^ε is the density, c^ε is the damping coefficient, C_{ijkl}^ε is the elasticity tensor and f_i is the component of the body force. Considering the free vibration of the structure, we have $f_i = 0$ and let $U_i^\varepsilon(\mathbf{x}, t)$ be expressed by

$$U_i^\varepsilon(\mathbf{x}, t) = e^{\lambda_\varepsilon t} u_i^\varepsilon(\mathbf{x}),$$

where λ_ε is the complex-valued eigenvalue with the real and imaginary part corresponding to the damping and oscillation, respectively. Substituting the expression of $U_i^\varepsilon(\mathbf{x}, t)$ above

into the Equation (2) and combining boundary conditions, we obtain the first case of the elastic quadratic eigenvalue problem as

$$(\text{QEP1}) : \begin{cases} \lambda_\varepsilon^2 \rho^\varepsilon u_i^\varepsilon + \lambda_\varepsilon c^\varepsilon u_i^\varepsilon - \frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l}) = 0 & \text{in } \Omega, \\ u_i^\varepsilon = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l} n_j = 0 & \text{on } \Gamma_2, \end{cases} \quad (3)$$

with $u_i^\varepsilon(\mathbf{x})$ standing for the elastic eigenfunction. By the periodic setting in Ω , the coefficients can be defined as

$$\rho^\varepsilon(\mathbf{x}) = \rho(\frac{\mathbf{x}}{\varepsilon}) = \rho(\mathbf{y}), c^\varepsilon(\mathbf{x}) = c(\frac{\mathbf{x}}{\varepsilon}) = c(\mathbf{y}), C_{ijkl}^\varepsilon(\mathbf{x}) = C_{ijkl}(\frac{\mathbf{x}}{\varepsilon}) = C_{ijkl}(\mathbf{y}),$$

in which, ρ , c , and C_{ijkl} are the $\mathbf{y}$ -periodic functions and have the regularity

$$\rho, c, C_{ijkl} \in L^\infty(Q), \rho \geq \rho_0 > 0, c \geq 0, \\ C_{ijkl} = C_{jikl} = C_{klij}, \alpha |e(\boldsymbol{\phi})|^2 \leq Ce(\boldsymbol{\phi})e(\boldsymbol{\phi}), |Ce(\boldsymbol{\phi})| \leq \beta |e(\boldsymbol{\phi})|,$$

where

$$\alpha, \beta > 0, e(\boldsymbol{\phi}) = (e_{ij})_{1 \leq i, j \leq N}, e_{ij}(\boldsymbol{\phi}) = \frac{1}{2} (\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}), \\ (Ce(\boldsymbol{\phi}))_{ij} = C_{ijkl} e_{kl}, Ce(\boldsymbol{\phi})e(\boldsymbol{\phi}) = C_{ijkl} e_{ij} e_{kl}, |e(\boldsymbol{\phi})| = (\sum_{i,j=1}^N e_{ij}^2)^{\frac{1}{2}}.$$

The homogeneous mixed boundary conditions are applied on the boundary $\partial\Omega$ as

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset,$$

where the zero displacement is prescribed on the boundary Γ_1 and the structure is free of motion on Γ_2 . $\mathbf{n} = [n_1, \dots, n_N]^T$ is the outward normal unit on Γ_2 . To some extent, the setting of damping coefficients c^ε is general, and it is not necessarily proportional to the mass ρ^ε or the elasticity C_{ijkl}^ε . Nevertheless, in the engineering, the Rayleigh damping is also of interest, and the associated quadratic eigenproblem, which forms the second case of our consideration, can be modeled by

$$(\text{QEP2}) : \begin{cases} \lambda_\varepsilon^2 \rho^\varepsilon u_i^\varepsilon + \lambda_\varepsilon \left[k_1 \rho^\varepsilon u_i^\varepsilon - k_2 \frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l}) \right] - \frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l}) = 0 & \text{in } \Omega, \\ u_i^\varepsilon = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l} n_j = 0 & \text{on } \Gamma_2, \end{cases} \quad (4)$$

where the constant Rayleigh coefficients $k_1 > 0, k_2 > 0$ are determined by experiments. When each of the constituent materials is homogeneous and isotropic, the elastic tensor can be defined as $C_{ijkl}^\varepsilon = \Lambda^\varepsilon \delta_{ij} \delta_{kl} + 2\mu^\varepsilon \delta_{ik} \delta_{jl}$, where δ_{ij} is the Kronecker delta symbol and Λ^ε and μ^ε are the Lamé coefficients of the materials, expressed by the Young's modules E^ε and the Poisson's ratio ν^ε as

$$\Lambda^\varepsilon = \frac{E^\varepsilon \nu^\varepsilon}{(1 - 2\nu^\varepsilon)(1 + \nu^\varepsilon)}, \mu^\varepsilon = \frac{E^\varepsilon}{2(1 + \nu^\varepsilon)}. \quad (5)$$

The material coefficients $\Lambda^\varepsilon, \mu^\varepsilon, E^\varepsilon, \nu^\varepsilon$ are assumed to be periodic and expressed as

$$\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\frac{\mathbf{x}}{\varepsilon}) = \Lambda(\mathbf{y}), \mu^\varepsilon(\mathbf{x}) = \mu(\frac{\mathbf{x}}{\varepsilon}) = \mu(\mathbf{y}), \\ E^\varepsilon(\mathbf{x}) = E(\frac{\mathbf{x}}{\varepsilon}) = E(\mathbf{y}), \nu^\varepsilon(\mathbf{x}) = \nu(\frac{\mathbf{x}}{\varepsilon}) = \nu(\mathbf{y}).$$

3. Second-Order Two-Scale Asymptotic Analysis for the Eigenpair

In this section, we will use the SOTS asymptotic analysis method to derive the eigenfunctions and eigenvalue expansions of these two quadratic eigenvalue problems with different damping effects.

3.1. Case 1

For the QEP1, different from static problems [14,26], we assume both the eigenfunction u_i^ε and eigenvalue λ_ε are expanded to the second order terms as [10,30]

$$\begin{aligned} u_i^\varepsilon(\mathbf{x}, \mathbf{y}) &= u_i^0(\mathbf{x}) + \varepsilon u_i^1(\mathbf{x}, \mathbf{y}) + \varepsilon^2 u_i^2(\mathbf{x}, \mathbf{y}) + O(\varepsilon^3), \\ \lambda_\varepsilon &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + O(\varepsilon^3), \end{aligned} \quad (6)$$

where the (u_i^0, λ_0) is referred to as the homogenized eigenpair and u_i^1, u_i^2, λ_1 and λ_2 are the correctors of the eigensolutions. Respecting $\mathbf{x}, \mathbf{y}$ as independent variables, the partial differential becomes

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \varepsilon^{-1} \frac{\partial}{\partial y_i}. \quad (7)$$

By substituting (6) and (7) into the governing Equation (3), considering the independence of $\mathbf{y}$ for u_k^0 , and equalizing the factor of the same power of ε on both sides of the equation, we obtain the first two equations

$$O(\varepsilon^{-1}) : -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^1}{\partial y_l}) = \frac{\partial C_{ijkl}}{\partial y_j} \frac{\partial u_k^0}{\partial x_l} \quad \text{in } \Omega \times Q, \quad (8)$$

$$\begin{aligned} O(\varepsilon^0) : -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^2}{\partial y_l}) &= \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k^1}{\partial y_l}) + \frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^1}{\partial x_l}) + \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k^0}{\partial x_l}) \\ &+ \lambda_0^2 \rho u_i^0 + \lambda_0 c u_i^0 \quad \text{in } \Omega \times Q. \end{aligned} \quad (9)$$

After that, from the Equation (8), it is suggested that the corrector $u_k^1(\mathbf{x}, \mathbf{y})$ can be represented in the form

$$u_k^1(\mathbf{x}, \mathbf{y}) = N_k^{\alpha_1 m}(\mathbf{y}) \frac{\partial u_m^0(\mathbf{x})}{\partial x_{\alpha_1}}, \alpha_1, m = 1, 2, \dots, N, \quad (10)$$

where the $N_k^{\alpha_1 m}$ is the so-called first-order cell function, which is $\mathbf{y}$ -periodic and the solution of the following cell problem

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial N_k^{\alpha_1 m}}{\partial y_l}) = \frac{\partial C_{ijm\alpha_1}}{\partial y_j} & \text{in } Q, \\ N_k^{\alpha_1 m} \in W_{per}(Q), \forall \alpha_1, m, k = 1, 2, \dots, N, \end{cases} \quad (11)$$

where

$$W_{per}(Q) = \left\{ v \in H_{per}^1(Q); \frac{1}{|Q|} \int_Q v dy = 0 \right\}.$$

Next, by the integral average of the Equation (9), we obtain the homogenized quadratic eigenvalue equation

$$\lambda_0^2 \rho^0 u_i^0 + \lambda_0 c^0 u_i^0 - C_{ijkl}^0 \frac{\partial^2 u_k^0}{\partial x_j \partial x_l} = 0,$$

with the homogenized coefficients ρ^0, c^0 , and C_{ijkl}^0 calculated by

$$\rho^0 = \frac{1}{|Q|} \int_Q \rho dy, c^0 = \frac{1}{|Q|} \int_Q c dy, C_{ijkl}^0 = \frac{1}{|Q|} \int_Q (C_{ijkl} + C_{ijm\alpha_1} \frac{\partial N_m^{lk}}{\partial y_{\alpha_1}}) dy, \quad (12)$$

respectively, where $|Q|$ represents the measure of cell. Combining with the boundary condition, we can rewritten the homogenized quadratic eigenproblem as

$$\begin{cases} \lambda_0^2 \rho^0 u_i^0 + \lambda_0 c^0 u_i^0 - C_{ijkl}^0 \frac{\partial^2 u_k^0}{\partial x_j \partial x_l} = 0 & \text{in } \Omega, \\ u_i^0 = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^0 \frac{\partial u_k^0}{\partial x_l} n_j = 0 & \text{on } \Gamma_2. \end{cases} \quad (13)$$

Since all the coefficients are constant, we can use a more coarser computational mesh with less time to obtain (u_i^0, λ_0) numerically compared with that of $(u_i^\varepsilon, \lambda_\varepsilon)$.

Now the homogenized eigenpair (u_i^0, λ_0) and the first corrector u_i^0 are well defined, but it is not enough to describe the whole oscillating behavior of the eigenfunctions. Additionally, for fixed ε , λ_1 and λ_2 should be obtained to give proper corrections of the homogenized eigenvalues.

We now turn to Equation (9) and define the expression of $u_k^2(x, y)$ as

$$u_k^2(x, y) = N_k^{\alpha_1 \alpha_2 m}(y) \frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} - \lambda_0 H_k^m(y) u_m^0(x), \alpha_1, \alpha_2, m = 1, 2, \dots, N, \quad (14)$$

where the y -periodic second-order cell functions $N_k^{\alpha_1 \alpha_2 m}$ and H_k^m satisfy the cell problems

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial N_k^{\alpha_1 \alpha_2 m}}{\partial y_l}) = C_{i\alpha_2 m \alpha_1} - \frac{\rho}{\rho^0} C_{i\alpha_2 m \alpha_1}^0 + C_{i\alpha_2 kl} \frac{\partial N_k^{\alpha_1 m}}{\partial y_l} + \frac{\partial}{\partial y_j} (C_{ijk\alpha_2} N_k^{\alpha_1 m}) & \text{in } Q, \\ N_k^{\alpha_1 \alpha_2 m} \in W_{per}(Q), \forall \alpha_1, \alpha_2, m, k = 1, 2, \dots, N, \end{cases} \quad (15)$$

$$\begin{cases} -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial H_k^m}{\partial y_l}) = (c - \frac{\rho}{\rho^0} c^0) \delta_{im} & \text{in } Q, \\ H_k^m \in W_{per}(Q), i, k, m = 1, 2, \dots, N, \end{cases} \quad (16)$$

respectively. It can be easily proved that both $N_k^{\alpha_1 \alpha_2 m}$ and H_k^m are uniquely determined in the periodic function space.

In summary, inserting the forms (10) and (14) of u_k^1 and u_k^2 into expansion (6), we obtain

$$u_k^\varepsilon(x, y) = u_k^0(x) + \varepsilon N_k^{\alpha_1 m}(y) \frac{\partial u_m^0}{\partial x_{\alpha_1}} + \varepsilon^2 \left[N_k^{\alpha_1 \alpha_2 m}(y) \frac{\partial^2 u_m^0}{\partial x_{\alpha_1} \partial x_{\alpha_2}} - \lambda_0 u_m^0 H_k^m(y) \right] + O(\varepsilon^3). \quad (17)$$

The derivation of λ_1 and λ_2 comes from the idea of “corrector equation” [30,31]. First, we introduce the auxiliary function ω_k^ε , which solves following the elasticity boundary value problem

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial \omega_k^\varepsilon}{\partial x_l}) = -\lambda_0^2 \rho u_i^0 - \lambda_0 c u_i^0 & \text{in } \Omega, \\ \omega_k^\varepsilon = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^\varepsilon \frac{\partial \omega_k^\varepsilon}{\partial x_l} n_j = 0 & \text{on } \Gamma_2, \end{cases} \quad (18)$$

and apply also the SOTS method to $\omega_k^\varepsilon(x, y)$ as mentioned above. Then, the corresponding expansion of ω_k^ε can be constructed similarly by

$$\omega_k^\varepsilon(x, y) = \omega_k^0(x) + \varepsilon N_k^{\alpha_1 m}(y) \frac{\partial \omega_m^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 \left[N_k^{\alpha_1 \alpha_2 m}(y) \frac{\partial^2 \omega_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} - \lambda_0 u_m^0(x) H_k^m(y) \right] + O(\varepsilon^3),$$

where $\omega_k^0(\mathbf{x})$ is the solution of the following homogenized problem

$$\begin{cases} -C_{ijkl}^0 \frac{\partial^2 \omega_k^0}{\partial x_j \partial x_l} = -\lambda_0^2 \rho^0 u_i^0 - \lambda_0 c^0 u_i^0 & \text{in } \Omega, \\ \omega_k^0 = 0 \text{ on } \Gamma_1, \quad C_{ijkl}^0 \frac{\partial \omega_k^0}{\partial x_l} n_j = 0 & \text{on } \Gamma_2. \end{cases} \quad (19)$$

From the uniqueness of the problem, it is seen that $\omega_k^0 = u_k^0$, and the first three expansions terms of ω_k^ε are the same as those of u_k^ε . Now, comparing the weak forms of (3) with that of (18), we have

$$\int_{\Omega} C_{ijkl}^\varepsilon \frac{\partial u_k^\varepsilon}{\partial x_l} \frac{\partial \phi_i}{\partial x_j} dx = \int_{\Omega} (-\lambda_\varepsilon^2 \rho^\varepsilon u_i^\varepsilon \phi_i - \lambda_\varepsilon c^\varepsilon u_i^\varepsilon \phi_i) dx, \quad (20)$$

$$\int_{\Omega} C_{ijkl}^\varepsilon \frac{\partial \omega_k^\varepsilon}{\partial x_l} \frac{\partial \phi_i}{\partial x_j} dx = \int_{\Omega} (-\lambda_0^2 \rho u_i^0 - \lambda_0 c u_i^0) \phi_i dx, \quad (21)$$

respectively, where ϕ_i is the arbitrary test function in Ω that vanishes on Γ_1 , i.e., $\phi_i \in V(\Omega) = \{v | v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1\}$. Letting ϕ_i be u_i^ε in (20) and ω_i^ε in (21) leads to the corrector equation:

$$\lambda_\varepsilon^2 \int_{\Omega} \rho^\varepsilon u_i^\varepsilon \omega_i^\varepsilon dx + \lambda_\varepsilon \int_{\Omega} c^\varepsilon u_i^\varepsilon \omega_i^\varepsilon dx = \lambda_0^2 \int_{\Omega} \rho^\varepsilon u_i^0 \omega_i^\varepsilon dx + \lambda_0 \int_{\Omega} c^\varepsilon u_i^0 \omega_i^\varepsilon dx. \quad (22)$$

By taking the expansions of λ_ε , u_k^ε , and ω_k^ε into the above, and comparing the power-like terms of ε , the expressions of λ_1 and λ_2 are also calculated successively by

$$\lambda_1 = -\lambda_0 \frac{\int_{\Omega} (\lambda_0 \rho^\varepsilon + c^\varepsilon) u_i^0 u_i^1 dx}{\int_{\Omega} (2\lambda_0 \rho^\varepsilon + c^\varepsilon) u_i^0 u_i^0 dx} \quad (23)$$

$$\begin{aligned} \lambda_2 = & \frac{2\lambda_1^2}{\lambda_0} - \lambda_1^2 \frac{\int_{\Omega} \rho^\varepsilon u_i^0 u_i^0 dx}{\int_{\Omega} (c^\varepsilon + 2\lambda_0 \rho^\varepsilon) u_i^0 u_i^0 dx} \\ & - 2\lambda_1 \lambda_0 \frac{\int_{\Omega} \rho^\varepsilon u_i^0 u_i^1 dx}{\int_{\Omega} (c^\varepsilon + 2\lambda_0 \rho^\varepsilon) u_i^0 u_i^0 dx} - \lambda_0 \frac{\int_{\Omega} (c^\varepsilon + \lambda_0 \rho^\varepsilon) (u_i^1 u_i^1 + u_i^0 u_i^2) dx}{\int_{\Omega} (c^\varepsilon + 2\lambda_0 \rho^\varepsilon) u_i^0 u_i^0 dx}. \end{aligned} \quad (24)$$

3.2. Case 2

For the second case, an analogous argument can be adopted as stated in the last subsection. By applying the asymptotic expansions of u_k^ε and λ_ε in Equation (6) to the problem (4) and $\lambda_0 k_2 + 1 \neq 0$, we can obtain the two equations for u_i^1 and u_i^2 as

$$-\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^1}{\partial y_l}) = \frac{\partial}{\partial y_j} (C_{ijkl}) \frac{\partial u_k^0}{\partial x_l} \quad \text{in } \Omega \times Q, \quad (25)$$

$$\begin{aligned} -\frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^2}{\partial y_l}) = & \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k^1}{\partial y_l}) + \frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^1}{\partial x_l}) + \frac{\partial}{\partial x_j} (C_{ijkl} \frac{\partial u_k^0}{\partial x_l}) \\ & + \frac{\lambda_1 k_2}{1 + \lambda_0 k_2} \left[\frac{\partial}{\partial y_j} (C_{ijkl}) \frac{\partial u_k^0}{\partial x_l} + \frac{\partial}{\partial y_j} (C_{ijkl} \frac{\partial u_k^1}{\partial y_l}) \right] \\ & - \frac{\lambda_0^2 + k_1 \lambda_0}{1 + \lambda_0 k_2} \rho u_i^0 \quad \text{in } \Omega \times Q. \end{aligned} \quad (26)$$

We similarly obtain the expansions of u_k^ε as

$$u_k^\varepsilon(\mathbf{x}, \mathbf{y}) = u_k^0(\mathbf{x}) + \varepsilon N_k^{\alpha_1 m}(\mathbf{y}) \frac{\partial u_m^0(\mathbf{x})}{\partial x_{\alpha_1}} + \varepsilon^2 N_k^{\alpha_1 \alpha_2 m}(\mathbf{y}) \frac{\partial^2 u_m^0(\mathbf{x})}{\partial x_{\alpha_1} \partial x_{\alpha_2}} + O(\varepsilon^3), \quad (27)$$

and $(u_k^0(x), \lambda_0)$ here is the eigenpair of the following homogenized eigenvalue problem:

$$\begin{cases} \lambda_0^2 \rho^0 u_i^0 + \lambda_0 [k_1 \rho^0 u_i^0 - k_2 C_{ijkl}^0 \frac{\partial^2 u_k^0}{\partial x_j \partial x_l}] - C_{ijkl}^0 \frac{\partial^2 u_k^0}{\partial x_j \partial x_l} = 0 & \text{in } \Omega, \\ u_k^0 = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^0 \frac{\partial u_k^0}{\partial x_l} n_j = 0 & \text{on } \Gamma_2. \end{cases} \quad (28)$$

Turning to the expansions of λ_1 and λ_2 , we can analogously introduce the auxiliary function $\tilde{w}_k^\varepsilon(x, y)$, which satisfies

$$\begin{cases} -\frac{\partial}{\partial x_j} (C_{ijkl}^\varepsilon \frac{\partial \tilde{w}_k^\varepsilon}{\partial x_l}) = -\frac{\lambda_0^2 + \lambda_0 k_1}{1 + \lambda_0 k_2} \rho^\varepsilon u_i^0 & \text{in } \Omega, \\ \tilde{w}_k^\varepsilon = 0 & \text{on } \Gamma_1, \quad C_{ijkl}^\varepsilon \frac{\partial \tilde{w}_k^\varepsilon}{\partial x_l} n_j = 0 & \text{on } \Gamma_2, \end{cases} \quad (29)$$

and performing second-order two-scale expansion for $\tilde{w}_k^\varepsilon$ again, one immediately has

$$\tilde{w}_k^\varepsilon(x, y) = \tilde{w}_k^0(x) + \varepsilon N_k^{\alpha_1 m}(y) \frac{\partial \tilde{w}_m^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N_k^{\alpha_1 \alpha_2 m} \frac{\partial^2 \tilde{w}_m^0(x)}{\partial x_{\alpha_1} \partial x_{\alpha_2}} + O(\varepsilon^3),$$

and the equation $\tilde{w}_k^0 = u_k^0$ still holds in this case. Then, the “corrector equation” this time leads to

$$\frac{\lambda_\varepsilon^2 + k_1 \lambda_\varepsilon}{1 + k_2 \lambda_\varepsilon} \int_\Omega \rho^\varepsilon u_i^\varepsilon \tilde{w}_i^\varepsilon dx = \frac{\lambda_0^2 + k_1 \lambda_0}{1 + k_2 \lambda_0} \int_\Omega \rho^\varepsilon u_i^0 u_i^\varepsilon dx, \quad (30)$$

and consequently the expressions of λ_1 and λ_2 are derived by

$$\lambda_1 = -\lambda_0 \frac{(1 + k_2 \lambda_0)(\lambda_0 + k_1)}{(2\lambda_0 + k_1 + k_2 \lambda_0^2)} \frac{\int_\Omega \rho^\varepsilon u_i^0 u_i^1 dx}{\int_\Omega \rho^\varepsilon u_i^0 u_i^0 dx}, \quad (31)$$

$$\begin{aligned} \lambda_2 = & -\frac{\lambda_1^2(1 + k_2 \lambda_0)}{2\lambda_0 + k_1 + k_2 \lambda_0^2} + \frac{\lambda_1^2(4\lambda_0 + 2k_1 + 3k_2 \lambda_0^2 + k_1 k_2 \lambda_0)}{\lambda_0(1 + k_2 \lambda_0)(k_1 + \lambda_0)} \\ & + \lambda_1 \frac{\int_\Omega \rho^\varepsilon (u_i^0 u_i^2 + u_i^1 u_i^1) dx}{\int_\Omega \rho^\varepsilon u_i^0 u_i^1 dx}. \end{aligned} \quad (32)$$

4. The SOTS Algorithm

Based on the expansion of eigenvalues and eigenvectors, it is natural to construct the SOTS algorithm. For simplicity of presentation, we recursively define the homogenized solution, the first-order two-scale (FOTS) and the second-order two-scale (SOTS) asymptotic estimations of the eigenfunction and eigenvalue

$$\begin{aligned} u^{\varepsilon,0} &= u^0, & u^{\varepsilon,1} &= u^{\varepsilon,0} + \varepsilon u^1, & u^{\varepsilon,2} &= u^{\varepsilon,1} + \varepsilon^2 u^2, \\ \lambda_{\varepsilon,0} &= \lambda_0, & \lambda_{\varepsilon,1} &= \lambda_{\varepsilon,0} + \varepsilon \lambda_1, & \lambda_{\varepsilon,2} &= \lambda_{\varepsilon,1} + \varepsilon^2 \lambda_2, \end{aligned} \quad (33)$$

respectively, and give out the SOTS algorithm in the two-dimensional case as follows:

1. Construct the homogenized domain and the composite cell Q and determine the coefficients ρ^ε , c^ε and C_{ijkl}^ε in each material. The above domains are discretized in space to obtain the computational meshes.
2. Compute cell problems and the homogenized coefficients.
 - (1) Obtain the first-order cell function $N_k^{\alpha_1 m}(y)$ via the following weak form:

$$\int_Q C_{ijkl} \frac{\partial N_k^{\alpha_1 m}}{\partial y_l} \frac{\partial \phi_i}{\partial y_j} dy = - \int_Q C_{ijm\alpha_1} \frac{\partial \phi_i}{\partial y_j} dy. \quad (34)$$

- (2) Computation of homogenized coefficients ρ^0 , c^0 and C_{ijkl}^0 based on the integral expressions in Equation (12).
- (3) Obtain the second-order cell functions $N_k^{\alpha_1\alpha_2m}(\mathbf{y})$ and $H_k^m(\mathbf{y})$ via the following two weak forms:

$$\int_Q C_{ijkl} \frac{\partial N_k^{\alpha_1\alpha_2m}}{\partial y_l} \frac{\partial \phi_i}{\partial y_j} dy = \int_Q (C_{i\alpha_2m\alpha_1} - \frac{\rho}{\rho^0} C_{i\alpha_2m\alpha_1}^0 + C_{i\alpha_2kl} \frac{\partial N_k^{\alpha_1m}}{\partial y_l}) \phi_i dy - \int_Q C_{ijk\alpha_2} N_k^{\alpha_1m} \frac{\partial \phi_i}{\partial y_j} dy, \quad (35)$$

$$\int_Q C_{ijkl} \frac{\partial H_k^m}{\partial y_l} \frac{\partial \phi_i}{\partial y_j} dy = \int_Q (c - c^0 \frac{\rho}{\rho^0}) \phi_m dy, \quad (36)$$

where $\phi_i \in W_{per}(Q)$.

3. Solve homogenized eigenpair (u_i^0, λ_0) .

For the ease of illustration, consider the two dimensional case, we can solve $(\mathbf{u}^0(x), \lambda_0)$ via the following weak form:

$$\lambda_0^2 \int_{\Omega} \rho^0 \boldsymbol{\phi}^T \mathbf{u}^0 dx + \lambda_0 \int_{\Omega} c^0 \boldsymbol{\phi}^T \mathbf{u}^0 dx + \int_{\Omega} (\tilde{\mathbf{e}})^T \mathbf{D}^0 \mathbf{e}^0 dx = 0, \quad (37)$$

For case 1, with the homogenized constitutive relation denoted by

$$\mathbf{D}^0 \mathbf{e}^0 = \begin{bmatrix} C_{1111}^0 & C_{1122}^0 & C_{1112}^0 \\ C_{2211}^0 & C_{2222}^0 & C_{2212}^0 \\ C_{1211}^0 & C_{1222}^0 & C_{1212}^0 \end{bmatrix} \begin{bmatrix} \frac{\partial u_1^0}{\partial x_1} \\ \frac{\partial u_2^0}{\partial x_2} \\ \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} \end{bmatrix},$$

$$\boldsymbol{\phi} = [\phi_1 \quad \phi_2]^T, \quad \tilde{\mathbf{e}} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_1} \end{bmatrix}^T.$$

$\boldsymbol{\phi}$ is the arbitrary test function in Ω and satisfies $\boldsymbol{\phi} = \mathbf{0}$ on Γ_1 . Then, after discretizing the spatial domain, let $\mathbf{u}^0(x)$ be presented by the node value $\boldsymbol{\eta}^e$ in each element, i.e.,

$$\mathbf{u}^0(x) = [\mathbf{N}(x)]_{2 \times N_v} \boldsymbol{\eta}^e,$$

where N_v is the number of node per element and $[\mathbf{N}(x)]_{2 \times N_v}$ is the shape function of $\mathbf{u}^0(x)$. Based on the weak form Equations (37), the elastic quadratic eigenvalue discretized system as

$$\sum_{e=1}^{N_e} (\lambda_0^2 \mathbf{M}^e + \lambda_0 \mathbf{K}^e + \mathbf{C}^e) \boldsymbol{\eta}^e = \mathbf{0}, \quad (38)$$

where

$$\mathbf{M}^e = \int_e \rho^0 \mathbf{N}^T \mathbf{N} dx^e, \quad \mathbf{K}^e = \int_e c^0 \mathbf{N}^T \mathbf{N} dx^e, \quad \mathbf{C}^e = \int_e \mathbf{B}^T \mathbf{D}^0 \mathbf{B} dx^e,$$

N_e is the number of elements, " $\int_e$ " represents the integral on an element and

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_{11}}{\partial x_1} & \frac{\partial N_{12}}{\partial x_1} & \dots & \frac{\partial N_{1,N_v}}{\partial x_1} \\ \frac{\partial N_{21}}{\partial x_2} & \frac{\partial N_{22}}{\partial x_2} & \dots & \frac{\partial N_{2,N_v}}{\partial x_2} \\ \frac{\partial N_{11}}{\partial x_2} + \frac{\partial N_{21}}{\partial x_1} & \frac{\partial N_{12}}{\partial x_2} + \frac{\partial N_{22}}{\partial x_1} & \dots & \frac{\partial N_{1,N_v}}{\partial x_2} + \frac{\partial N_{2,N_v}}{\partial x_1} \end{bmatrix}.$$

Then, the finite element assembly is carried out, and the discrete system of the whole space are as follows [5]

$$[\lambda_0^2 \mathbf{M} + \lambda_0 \mathbf{K} + \mathbf{C}] \boldsymbol{\eta} = \mathbf{0}, \quad (39)$$

where $\mathbf{M}, \mathbf{K}, \mathbf{C}$ are the global mass matrix, damping matrix and stiff matrix, respectively, and $\boldsymbol{\eta}$ is the global eigenvector.

There are many methods to solve the QEP, such as the Lanczos two-sided recursion [33], the SOAR method [34,35] and a Refined Second-Order Arnoldi (RSOAR) method, the Generalized Second-Order Arnoldi (GSOAR) method [36,37] and the Refined GSOAR (RGSOAR) method [38]. Tisseur [39] has solved the QEP by applying the QZ algorithm to a corresponding GEP. Following the linearization method, let $\mathbf{v} = \lambda_0 \boldsymbol{\eta}$ to obtain the generalized eigenvalue problem [33,37]

$$\begin{bmatrix} -\mathbf{C} & -\mathbf{K} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{v} \end{bmatrix} = \lambda_0 \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{v} \end{bmatrix}, \quad (40)$$

For case 2, we can follow the same procedure as case 1, and the final generalized eigenvalue problem is formed as

$$\begin{bmatrix} -\mathbf{C} & -k_1 \mathbf{M} - k_2 \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{v} \end{bmatrix} = \lambda_0 \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{v} \end{bmatrix}, \quad (41)$$

where the same $\lambda_0, \boldsymbol{\eta}$ and $\mathbf{v}$ are applied for simplicity.

4. Assembling the FOTS and SOTS approximation of the eigenfunctions and eigenvalues by the formulas (17), (27), (23), (24), and (31), (32).
5. Error computation of eigenfunctions and eigenvalues. Since we cannot construct an analytical solutions for the composite domain. We will perform the classical finite element computations in the very refined meshes and let these solutions be the reference solutions and compare them with our asymptotic solutions. The relative errors for the eigenvalues are defined as

$$e_{\lambda,i}^0 = \frac{|\lambda_\varepsilon^i - \lambda_{\varepsilon,0}^i|}{|\lambda_\varepsilon^i|}, \quad e_{\lambda,i}^1 = \frac{|\lambda_\varepsilon^i - \lambda_{\varepsilon,1}^i|}{|\lambda_\varepsilon^i|}, \quad e_{\lambda,i}^2 = \frac{|\lambda_\varepsilon^i - \lambda_{\varepsilon,2}^i|}{|\lambda_\varepsilon^i|}, \quad (42)$$

respectively. Let $(\lambda_\varepsilon^i, \mathbf{u}_i^\varepsilon)$ be the i -th eigenpair. Because the characteristic function is not unique, we cannot directly compare the characteristic function. If the eigenvalue is single, we can define the relative error of the corresponding eigenfunction as

$$e_{\mathbf{u},i}^j = \frac{\min_{\alpha_0 \in \mathbb{C}} \|\mathbf{u}_i^\varepsilon - \alpha_0 \mathbf{u}_i^{\varepsilon,j}\|_{L^2(\Omega)}}{\|\mathbf{u}_i^\varepsilon\|_{L^2(\Omega)}}, j = 0, 1, 2. \quad (43)$$

where the L^2 norm is defined by

$$\|\mathbf{u}_i^\varepsilon\|_{L^2(\Omega)} = \sqrt{\int_{\Omega} u_{k,i}^\varepsilon \bar{u}_{k,i}^\varepsilon dx},$$

where the summation form for the indices k and $u_{k,i}^\varepsilon$ is the k -th component of $\mathbf{u}_i^\varepsilon$. After minimizing the relative error in the L^2 norm based on the least square method, we can gain the scaling factors α_i as follows:

$$\alpha_j = \frac{(\mathbf{u}_i^\varepsilon, \mathbf{u}_i^{\varepsilon,j})_{L^2(\Omega)}}{\|\mathbf{u}_i^{\varepsilon,j}\|_{L^2(\Omega)}^2}, j = 0, 1, 2,$$

respectively. For the eigenfunction with the multiple eigenvalues, we assumed that the algebraic multiplicity of eigenvalue λ_i is m . $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+m-1}$. Because the

linear combination of eigenfunctions of the same eigenvalue is also an eigenfunction, the relative errors can be computed by

$$e_{u,i}^j = \frac{\min_{\alpha_k^j \in \mathbb{C}} \|u_i^\varepsilon - \alpha_k^j u_{i+k-1}^{\varepsilon,j}\|_{L^2(\Omega)}}{\|u_i^\varepsilon\|_{L^2(\Omega)}}, j = 0, 1, 2, \quad (44)$$

where we do not have summations on the indices i . $\alpha^i = [\alpha_1^i, \alpha_2^i, \dots, \alpha_m^i]^T, i = 0, 1, 2$. Similarly, we take into account the orthogonality of the eigenfunction and obtain the coefficients as

$$\alpha_k^j = \frac{(u_i^\varepsilon, u_{i+k-1}^{\varepsilon,j})_{L^2(\Omega)}}{\|u_{i+k-1}^{\varepsilon,j}\|_{L^2(\Omega)}^2}, j = 1, 2,$$

where we do not have summations on the indices i and k .

6. Compute the homogenized, FOTS and SOTS strains $e_{ij}^0, e_{ij}^{\varepsilon,1}$ and $e_{ij}^{\varepsilon,2}$ by

$$e_{ij}^0 = \frac{1}{2} \left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i} \right), e_{ij}^{\varepsilon,1} = \frac{1}{2} \left(\frac{\partial u_i^{\varepsilon,1}}{\partial x_j} + \frac{\partial u_j^{\varepsilon,1}}{\partial x_i} \right), e_{ij}^{\varepsilon,2} = \frac{1}{2} \left(\frac{\partial u_i^{\varepsilon,2}}{\partial x_j} + \frac{\partial u_j^{\varepsilon,2}}{\partial x_i} \right),$$

and the stress approximations by

$$\sigma_{ij}^0 = C_{ijkl}^0 e_{kl}^0, \quad \sigma_{ij}^{\varepsilon,1} = C_{ijkl}^\varepsilon e_{kl}^{\varepsilon,1}, \quad \sigma_{ij}^{\varepsilon,2} = C_{ijkl}^\varepsilon e_{kl}^{\varepsilon,2}.$$

5. Numerical Examples

Two typical eigenvalue problems are presented to show the availability and correctness of the SOTS model and algorithm.

5.1. Two-Dimensional Domain with Composite Cell

Let $\Omega = [0, 1]^2$ and the two-dimensional periodic composite structure shown in Figure 2a with $\varepsilon = 1/8$. The cell domain is depicted in Figure 2b. As is seen in Figure 2b that the cell is composed of two materials and the center is a square with side length 0.5. There is the material information in Table 1. Piecewise linear triangular elements are used for fine mesh and element meshes, and we employ the homogenization calculation quadratic triangular elements for more accurate estimation of the first and second derivatives of the solution. The details of computational meshes corresponding to different ε s are given in Table 2, to carry out the finite element computation and show the convergence of the SOTS asymptotic algorithm. It can be clearly known that the number of grid partitions calculated by SOTS is far greater than the number of grid partitions calculated by FE fine calculation. Because of its independence of ε , we only perform one asymptotic calculation, which deeply reduces the memory consumption and calculation time. Additionally, we give the finite element reference solution and SOTS solution calculation time for case 2 with different ε in Table 2. Through comparison, it can be seen that the SOTS algorithm has the advantage of saving time and cost.

Table 1. The elastic properties for the composite structure.

	Young's Modules E^ε	Poisson's Ratio ν^ε	Λ^ε	μ^ε	ρ^ε	c^ε
Material a_1	200	0.2	55.56	83.3	2.3	0.1
Material a_2	15	0.2	4.17	6.25	8.3	1

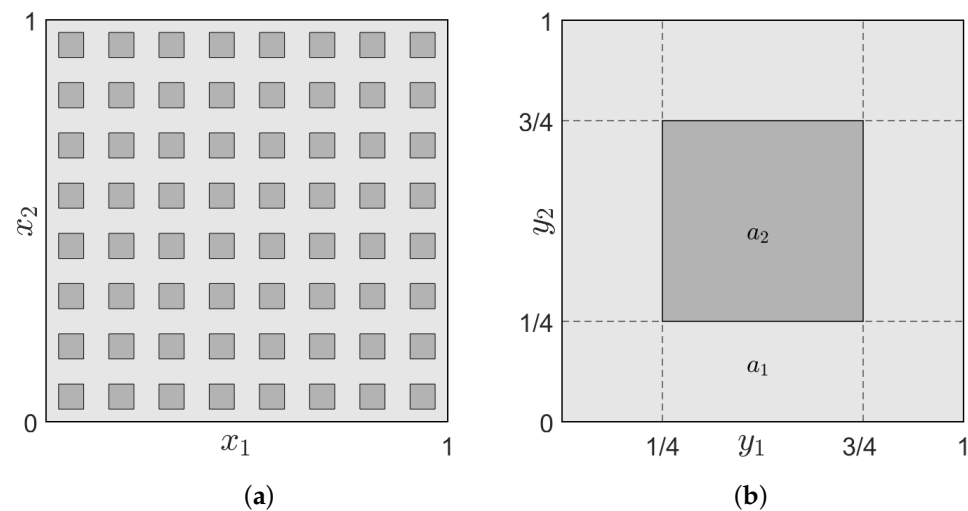


Figure 2. Composite structure Ω with $\varepsilon = 1/8$ and representative cell Q . (a) Composite domain Ω . (b) Cell domain Q .

Table 2. The information of mesh for the composite structure.

	Fine Mesh						Cell	Homo
	$\varepsilon = 1/2$	$\varepsilon = 1/4$	$\varepsilon = 1/8$	$\varepsilon = 1/16$	$\varepsilon = 1/32$	$\varepsilon = 1/64$	Mesh	Mesh
Nodes	1597	2529	4289	16,897	67,073	267,265	1193	1089
Elements	3048	4864	8320	33,280	133,120	532,480	2256	2048
Mesh size	0.0422	0.0331	0.0234	0.0117	0.00586	0.00293	0.0485	0.0442
CPU Times (s)	12	35	61	146	472	3278	68(SOTS solution)	

Based on the weak form Equation (34), we can solve the first-order cell function $N^{\alpha_1 m}(\mathbf{y})$ and the computed solutions are shown in Figure 3. By the symmetry of the domain and constitutive relation, it is seen that the Figure 3c can be obtained by rotating Figure 3a ninety degrees counterclockwise around the point (0.5, 0.5) on the xoy plane and first-order cell function $N^{12}(\mathbf{y})$ is same as $N^{21}(\mathbf{y})$, which is consistent with Figure 3b.

After the calculation of the first-order cell functions, we can naturally obtain the homogenization coefficients in Equation(12):

$$\begin{aligned}
 \rho^0 &= 3.8, c^0 = 0.328, \\
 C_{1111}^0 &= (\Lambda + 2\mu)^0 = \frac{1}{|Q|} \int_Q [\Lambda + 2\mu + \Lambda e_{mm}(N^{11}) + 2\mu e_{11}(N^{11})] dy = 132.596, \\
 C_{1122}^0 &= C_{2211}^0 = \Lambda^0 = \frac{1}{|Q|} \int_Q [\Lambda + \Lambda e_{mm}(N^{11}) + 2\mu e_{22}(N^{11})] dy = 26.896, \\
 C_{2112}^0 &= C_{1221}^0 = C_{2121}^0 = C_{1212}^0 = \mu^0 = \frac{1}{|Q|} \int_Q [\mu + 2\mu e_{12}(N^{12})] dy = 44.0353, \\
 C_{1121}^0 &= C_{1211}^0 = C_{2111}^0 = C_{1112}^0 = \frac{1}{|Q|} \int_Q [\Lambda e_{mm}(N^{12}) + 2\mu e_{11}(N^{12})] dy \approx 0, \\
 C_{1222}^0 &= C_{2122}^0 = C_{2221}^0 = C_{2212}^0 = \frac{1}{|Q|} \int_Q [\Lambda e_{mm}(N^{12}) + 2\mu e_{22}(N^{12})] dy \approx 0, \\
 C_{2222}^0 &= (\Lambda + 2\mu)^0 = \frac{1}{|Q|} \int_Q [\Lambda + 2\mu + \Lambda e_{mm}(N^{22}) + 2\mu e_{22}(N^{22})] dy = 132.596,
 \end{aligned} \tag{45}$$

where there are summations on the indices m .

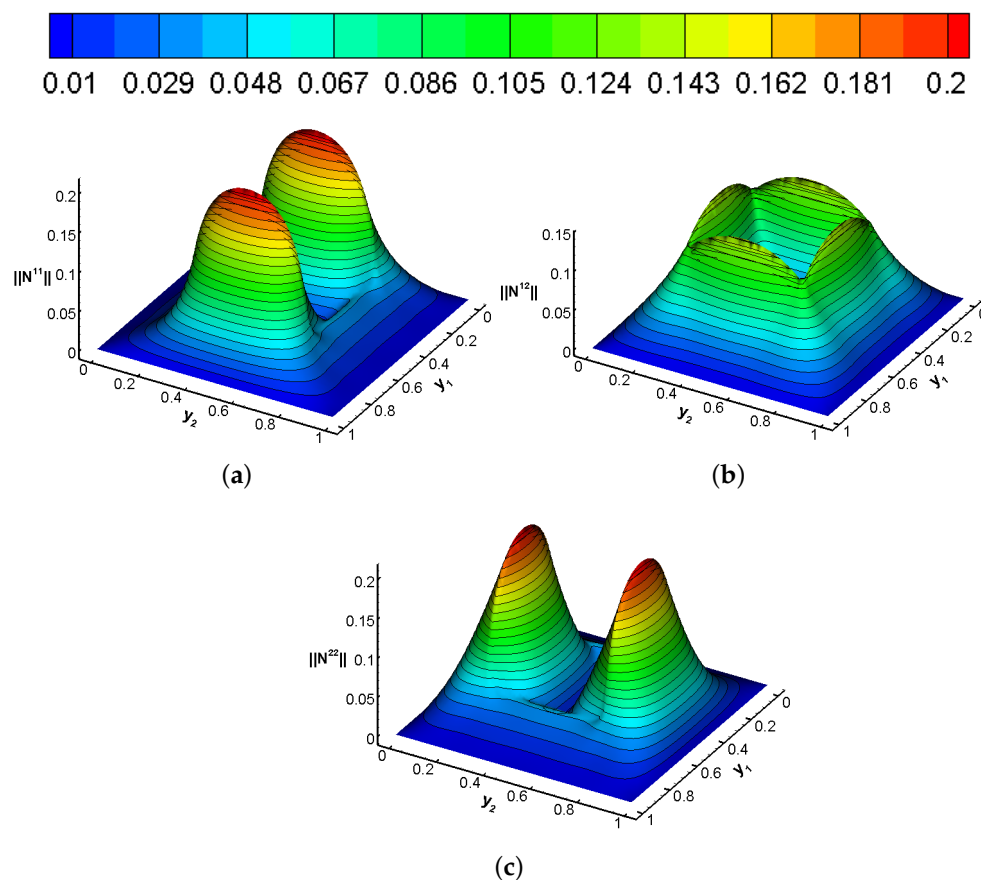


Figure 3. First-order cell functions $N^{\alpha_1 m}(\mathbf{y})$. (a) $\|N^{11}(\mathbf{y})\|_{L^2(Q)}$. (b) $\|N^{12}(\mathbf{y})\|_{L^2(Q)} = \|N^{21}(\mathbf{y})\|_{L^2(Q)}$. (c) $\|N^{22}(\mathbf{y})\|_{L^2(Q)}$.

5.2. Case 1

It is known that for the vibration analysis, only the first few smallest eigen modes are of interest, so we only compute the first twenty eigenvalues. Since the matrix of our studied QEP is real, the eigenvalues are real or come in pairs $(\lambda, \bar{\lambda})$ and if x is a right eigenvector of λ , then $\bar{x}$ is a right eigenvector of $\bar{\lambda}$. The asymptotic approximations of the imaginary part of the first component of first eigenfunction $\text{Im}(u_{1,1}^\varepsilon)$ with $\varepsilon = 1/8$ are shown in Figure 4. The imaginary part of the first component of the homogenized solution $\text{Im}(u_{1,1}^{\varepsilon,0})$ are smooth enough to characterize the macroscopic behavior. And the local oscillation can be slightly captured when we append the first-order corrector. Furthermore, by adding the second-order corrector, $\text{Im}(u_{1,1}^{\varepsilon,2})$ is more close to the fine solution significantly. We can reach a similar conclusion for the first eigenfunction with $\varepsilon = 1/16$, which are shown in Figure 5. For the simple eigenvalue, the second-order correctors also work effectively in Figure 6.

To further demonstrate the effect of asymptotic computation, we show that comparisons of the real part of the second component of the first eigenfunction $\text{Re}(u_{2,1}^\varepsilon)$ and the imaginary part of the first component of the fourth eigenfunction $\text{Im}(u_{1,4}^\varepsilon)$ on the line $x_1 = x_2$ with $\varepsilon = 1/8$ and $\varepsilon = 1/16$ in Figures 7 and 8. From the above figures, we can easily see that when $\varepsilon = 1/16$, the oscillation frequency of the broken line becomes higher. For convenience of observation, we give the details of local oscillations in Figures 7b and 8b. It is observed that the homogenized solution is smooth enough, which describes the macroscopic behavior of fine solution, and it exhibits microscopic oscillatory behavior significantly with the help of first- and second-order correctors. The two figures also show the second-order correctors work better for smaller ε .

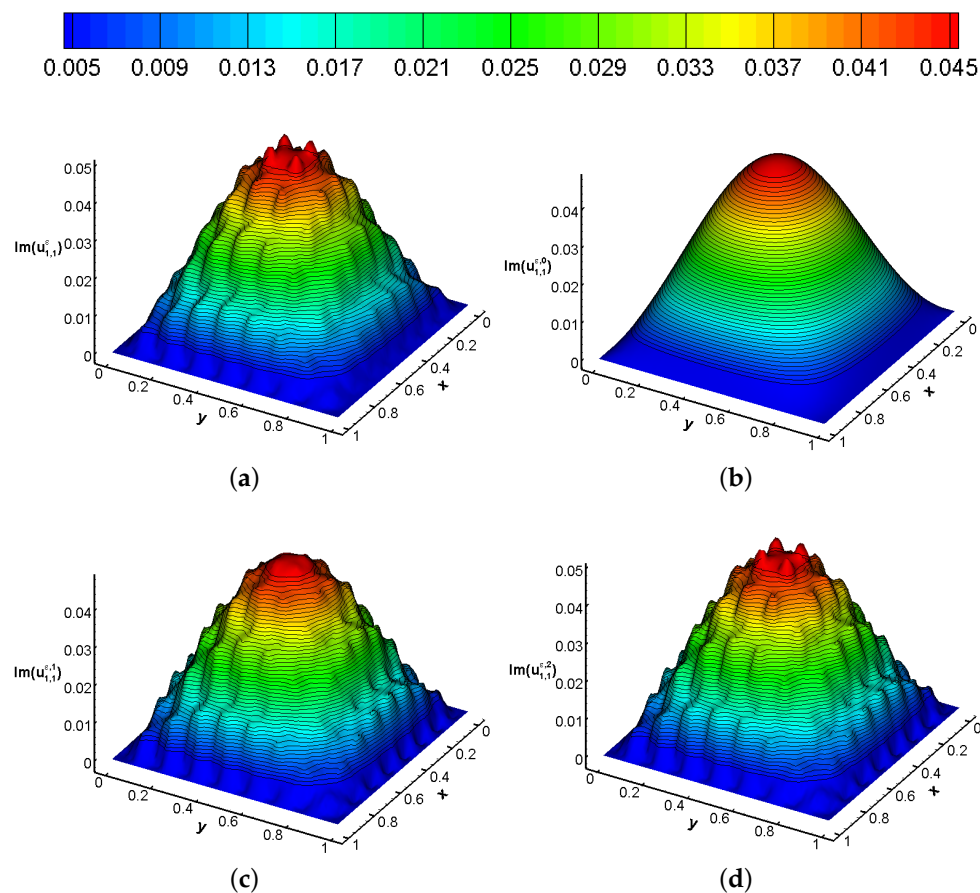


Figure 4. Approximate solutions of $\text{Im}(u_{1,1}^\epsilon)$ with $\epsilon = 1/8$. (a) Fine solution $\text{Im}(u_{1,1}^\epsilon)$. (b) Homogenized solution $\text{Im}(u_{1,1}^{\epsilon,0})$. (c) First-order approximation $\text{Im}(u_{1,1}^{\epsilon,1})$. (d) Second-order approximation $\text{Im}(u_{1,1}^{\epsilon,2})$.

Then, the asymptotic convergence of the eigenvalues λ_ϵ and eigenfunction u^ϵ are talked over in refined mesh with decreasing ϵ . Both the homogenized and cell functions are not related to ϵ , so the SOTS approximation of u^ϵ can be directly obtained for arbitrary ϵ . We can compute the correctors of eigenvalues from Equations (23) and (24). With different ϵ , the comparisons of the eigenvalue for the multiscale finite element computations are shown in Tables 3 and 4. Note that the first four eigenvalues are the two complex-valued multiple eigenvalues corresponding to the same eigenfunction. So are the 9th to 12th and the 17th to 20th eigenvalues for both Tables 3 and 4. It can be seen that the error decreases as the periodicity ϵ becomes smaller. A single table illustrates that $\lambda_{\epsilon,0}$ is enough approaching to the fine solution, and the second-order correctors make $\lambda_{\epsilon,2}$ more close to λ_ϵ .

In addition, we give the distribution of eigenvalues with different ϵ on the complex plane, which is shown in Figure 9. Comparing Figure 9a,b, we can know that correctors of eigenvalues perform better with smaller ϵ , especially for the real part and by appending first-order corrector $\lambda_{\epsilon,1}$ it is almost the same as the homogenized eigenvalue $\lambda_{\epsilon,0}$, which is in accordance with the Tables 3 and 4. By these quantitative and qualitative comparisons, it is believed that the SOTS method works more effectively for smaller eigenvalues.

By the least square approximation of the eigenfunction in Equations (43) and (44), the relative errors of the real part of the asymptotic approximation of eigenfunctions with both $\epsilon = 1/8$ and $\epsilon = 1/16$ are listed in Table 5. The relative errors decrease with the decrease in ϵ . Additionally, we can obtain better approximations with the help of the first-order and second-order correctors. Especially for the first four relative errors, it is observed that $e_{u,i}^j, j = 0, 1, 2, i = 1, 2, 3, 4$ are reduced by at least half when ϵ changes from $1/8$ to $1/16$.

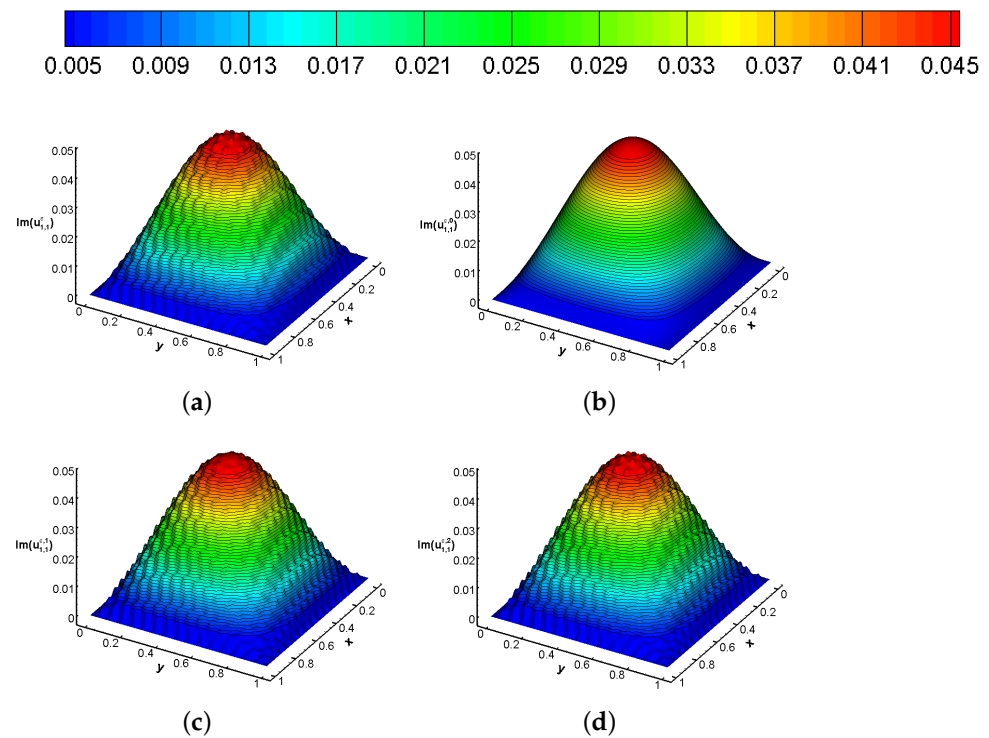


Figure 5. Approximate solutions of $\text{Im}(u_{1,1}^\epsilon)$ with $\epsilon = 1/16$. (a) Fine solution $\text{Im}(u_{1,1}^\epsilon)$. (b) Homogenized solution $\text{Im}(u_{1,1}^{\epsilon,0})$. (c) First-order approximation $\text{Im}(u_{1,1}^{\epsilon,1})$. (d) Second-order approximation $\text{Im}(u_{1,1}^{\epsilon,2})$.

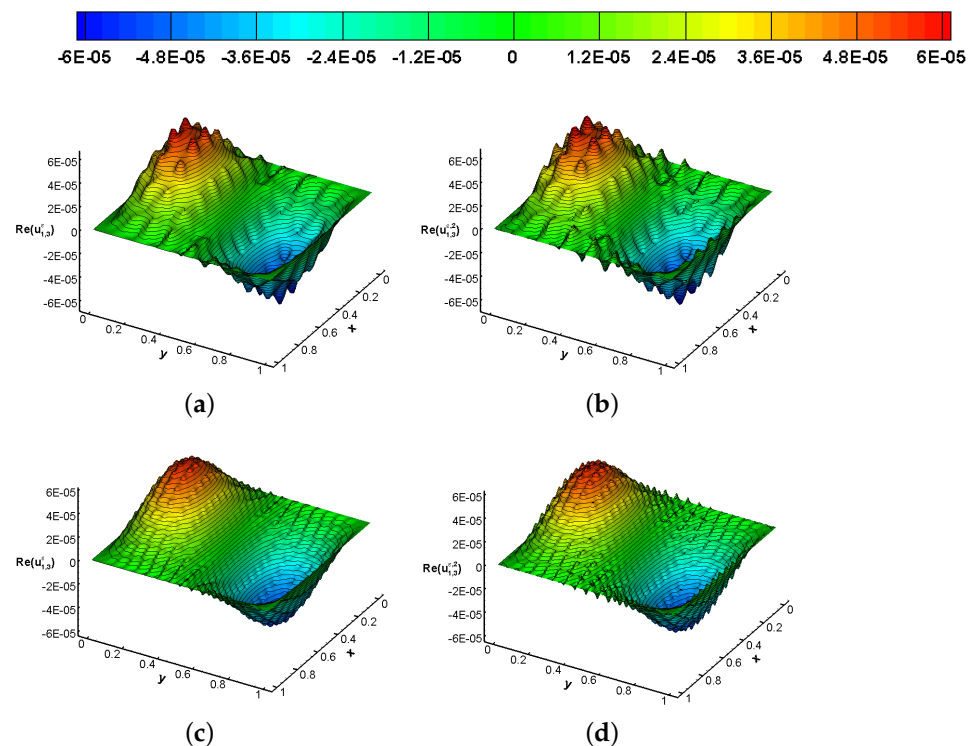


Figure 6. Approximate solution of $\text{Re}(u_{1,3}^\epsilon)$ with $\epsilon = 1/8$ and $\epsilon = 1/16$. (a) $\text{Re}(u_{1,3}^\epsilon)$ for $\epsilon = 1/8$. (b) $\text{Re}(u_{1,3}^{\epsilon,2})$ for $\epsilon = 1/8$. (c) $\text{Re}(u_{1,3}^\epsilon)$ for $\epsilon = 1/16$. (d) $\text{Re}(u_{1,3}^{\epsilon,2})$ for $\epsilon = 1/16$.

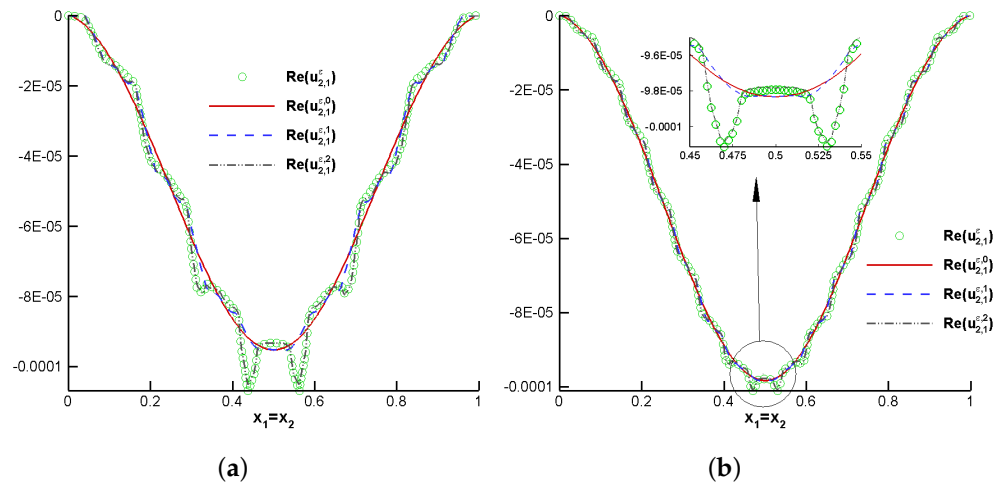


Figure 7. Comparisons of $\text{Re}(u_{2,1}^\varepsilon)$ with $\varepsilon = 1/8$ and $\varepsilon = 1/16$. (a) $\varepsilon = 1/8$. (b) $\varepsilon = 1/16$.

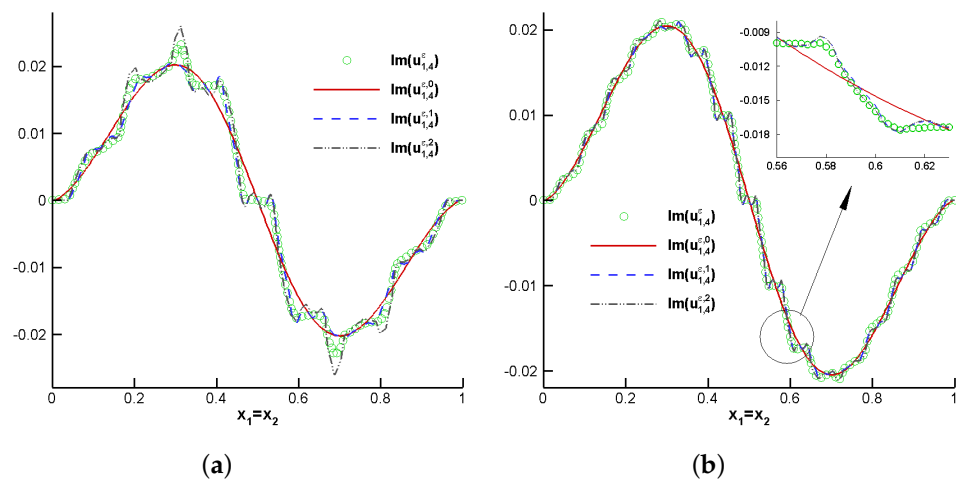


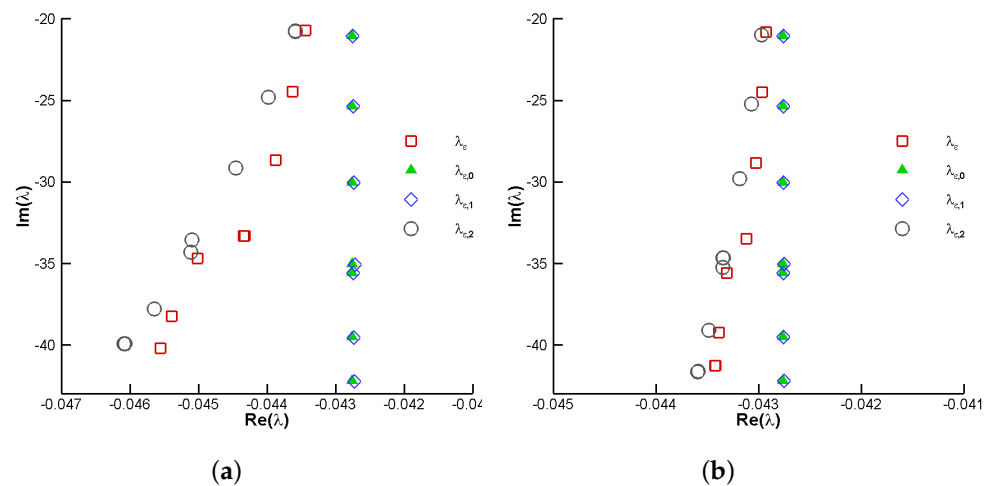
Figure 8. Comparisons of $\text{Im}(u_{1,4}^\varepsilon)$ with $\varepsilon = 1/8$ and $\varepsilon = 1/16$. (a) $\varepsilon = 1/8$. (b) $\varepsilon = 1/16$.

Table 3. Comparisons of the eigenvalues for the multiscale computations with $\varepsilon = 1/8$.

i	λ_ε^i	$\lambda_{\varepsilon,0}^i$	$\lambda_{\varepsilon,1}^i$	$\lambda_{\varepsilon,2}^i$	$e_{\lambda,i}^0$	$e_{\lambda,i}^1$	$e_{\lambda,i}^2$
1–4	-0.0434 $\pm 20.7267i$	-0.0428 $\pm 21.0413i$	-0.0428 $\pm 21.0441i$	-0.0436 $\pm 20.7604i$	0.015179	0.015313	0.0016252
5, 6	-0.0436 $\pm 24.4687i$	-0.0428 $\pm 25.3449i$	-0.0427 $\pm 25.3529i$	-0.0436 $\pm 24.7941i$	0.035812	0.036136	0.013300
7, 8	-0.0439 $\pm 28.6791i$	-0.0428 $\pm 30.033i$	-0.0427 $\pm 30.0448i$	-0.0445 $\pm 29.1393i$	0.047208	0.047621	0.016048
9–12	-0.0443 $\pm 33.3252i$	-0.0428 $\pm 35.0332i$	-0.0427 $\pm 35.0553i$	-0.0451 $\pm 33.5535i$	0.051254	0.051915	0.0068505
13, 14	-0.0450 $\pm 34.6995i$	-0.0428 $\pm 35.5699i$	-0.0427 $\pm 35.5799i$	-0.0451 $\pm 34.2895i$	0.025084	0.025374	0.011814
15, 16	-0.0454 $\pm 38.2442i$	-0.0428 $\pm 39.5220i$	-0.0427 $\pm 39.5371i$	-0.0457 $\pm 37.7639i$	0.033410	0.033805	0.012559
17–20	-0.0456 $\pm 40.1831i$	-0.0428 $\pm 42.1961i$	-0.0427 $\pm 42.2207i$	-0.0461 $\pm 39.8869i$	0.050095	0.050709	0.0073711

Table 4. Comparisons of the eigenvalues for the multiscale finite element computations with $\varepsilon = 1/16$.

i	λ_ε^i	$\lambda_{\varepsilon,0}^i$	$\lambda_{\varepsilon,1}^i$	$\lambda_{\varepsilon,2}^i$	$e_{\lambda,i}^0$	$e_{\lambda,i}^1$	$e_{\lambda,i}^2$
1–4	−0.0429 $\pm 20.8098i$	−0.0428 $\pm 21.0413i$	−0.0428 $\pm 21.0420i$	−0.0430 $\pm 20.9711i$	0.011124	0.011158	0.0077493
5, 6	−0.0430 $\pm 24.4895i$	−0.0428 $\pm 25.3449i$	−0.0428 $\pm 25.3468i$	−0.0431 $\pm 25.2071i$	0.034942	0.035010	0.029303
7, 8	−0.0430 $\pm 28.8328i$	−0.0428 $\pm 30.033i$	−0.0428 $\pm 30.0361i$	−0.0432 $\pm 29.8098i$	0.041627	0.041735	0.033887
9–12	−0.0431 $\pm 33.4947i$	−0.0428 $\pm 35.0332i$	−0.0428 $\pm 35.0386i$	−0.0433 $\pm 34.6631i$	0.045934	0.046094	0.034883
13, 14	−0.0433 $\pm 35.5708i$	−0.0428 $\pm 35.5699i$	−0.0428 $\pm 35.5722i$	−0.0434 $\pm 35.2494i$	0.000029903	0.000042627	0.0090362
15, 16	−0.0434 $\pm 39.2329i$	−0.0428 $\pm 39.5220i$	−0.0428 $\pm 39.5257i$	−0.0435 $\pm 39.0823i$	0.0073696	0.0074652	0.0038372
17–20	−0.0434 $\pm 41.2632i$	−0.0428 $\pm 42.1961i$	−0.0428 $\pm 42.2024i$	−0.0436 $\pm 41.6187i$	0.022608	0.022761	0.0086160

**Figure 9.** The distribution of eigenvalues with $\varepsilon = 1/8$ and $\varepsilon = 1/16$. (a) $\varepsilon = 1/8$. (b) $\varepsilon = 1/16$.**Table 5.** The relative errors of the real part of the eigenfunctions of the multiscale finite element computations in L^2 norm with $\varepsilon = 1/8$ and $\varepsilon = 1/16$.

i	ε	$e_{u,i}^0$	$e_{u,i}^1$	$e_{u,i}^2$
1–4	1/8	0.062853	0.049287	0.032068
	1/16	0.028503	0.018853	0.016703
5, 6	1/8	0.091823	0.073718	0.052404
	1/16	0.043631	0.029022	0.026550
7, 8	1/8	0.10173	0.066192	0.071713
	1/16	0.053611	0.034565	0.030627
9–12	1/8	0.14855	0.13038	0.089557
	1/16	0.10238	0.089901	0.087278
13, 14	1/8	0.13746	0.13078	0.057244
	1/16	0.048186	0.041428	0.030246

5.3. Case 2

In the second case, we consider the elastic quadratic eigenvalue problem with Rayleigh damping in the same composite structure and let the corresponding coefficient be $k_1 = 0.0625$, $k_2 = 0.0008$, respectively. Mesh information is listed in Table 2 and we also use a sequence of ε to discuss the convergence of both eigenvalues and eigenfunctions except the method analogy to case 1.

First, we consider approximations of the eigenvalue and eigenfunction with $\varepsilon = 1/8$ and $\varepsilon = 1/16$ as above. The approximations of the first twenty eigenvalues as well as relative errors are listed in Tables 6 and 7. Comparing the two tables, it can be seen that the first- and second-order correctors, as well as the relative errors, show the effectiveness of the correctors. Similar to case 1, the homogenized eigenvalues $\lambda_{\varepsilon,0}$ are close enough to the eigenvalues λ_{ε} , which be calculated under the fine mesh and by adding the first-order corrector, the approximation solution $\lambda_{\varepsilon,1}$ are not improved, but with the help of adding the second-order corrector to the homogenized eigenvalue $\lambda_{\varepsilon,2}$, they perform extremely well. For the first four eigenvalues, the relative error $e_{\lambda,i}^2, i = 1, 2, 3, 4$ is reduced by at least ten times compared to $e_{\lambda,i}^0$ without the corrector.

Table 6. Comparisons of the eigenvalues for the multiscale finite element computations with $\varepsilon = 1/8$.

i	λ_{ε}^i	$\lambda_{\varepsilon,0}^i$	$\lambda_{\varepsilon,1}^i$	$\lambda_{\varepsilon,2}^i$	$e_{\lambda,i}^0$	$e_{\lambda,i}^1$	$e_{\lambda,i}^2$
1–4	−0.2031 $\pm 20.7257i$	−0.2083 $\pm 21.0403i$	−0.2084 $\pm 21.0431i$	−0.2036 $\pm 20.7594i$	0.015179	0.015313	0.0016252
5, 6	−0.2707 $\pm 24.4672i$	−0.2882 $\pm 25.3433i$	−0.2884 $\pm 25.3513i$	−0.2770 $\pm 24.7926i$	0.035813	0.036137	0.013300
7, 8	−0.3602 $\pm 28.6769i$	−0.3920 $\pm 30.0305i$	−0.3923 $\pm 30.0423i$	−0.3706 $\pm 29.1370i$	0.047211	0.047624	0.016048
9–12	−0.4755 $\pm 33.3218i$	−0.5222 $\pm 35.0294i$	−0.5228 $\pm 35.0514i$	−0.4807 $\pm 33.5501i$	0.051258	0.051919	0.0068509
13, 14	−0.5129 $\pm 34.6957i$	−0.5373 $\pm 35.5658i$	−0.5376 $\pm 35.5759i$	−0.5009 $\pm 34.2859i$	0.025086	0.025376	0.011815
15, 16	−0.6163 $\pm 38.2393i$	−0.6560 $\pm 39.5166i$	−0.6565 $\pm 39.5317i$	−0.6005 $\pm 37.7592i$	0.033413	0.033809	0.012561
17–20	−0.6771 $\pm 40.1774i$	−0.7436 $\pm 42.1927i$	−0.7444 $\pm 42.2174i$	−0.6658 $\pm 39.8911i$	0.050180	0.050795	0.0071311

Table 7. Comparisons of the eigenvalues for the multiscale finite element computations with $\varepsilon = 1/16$.

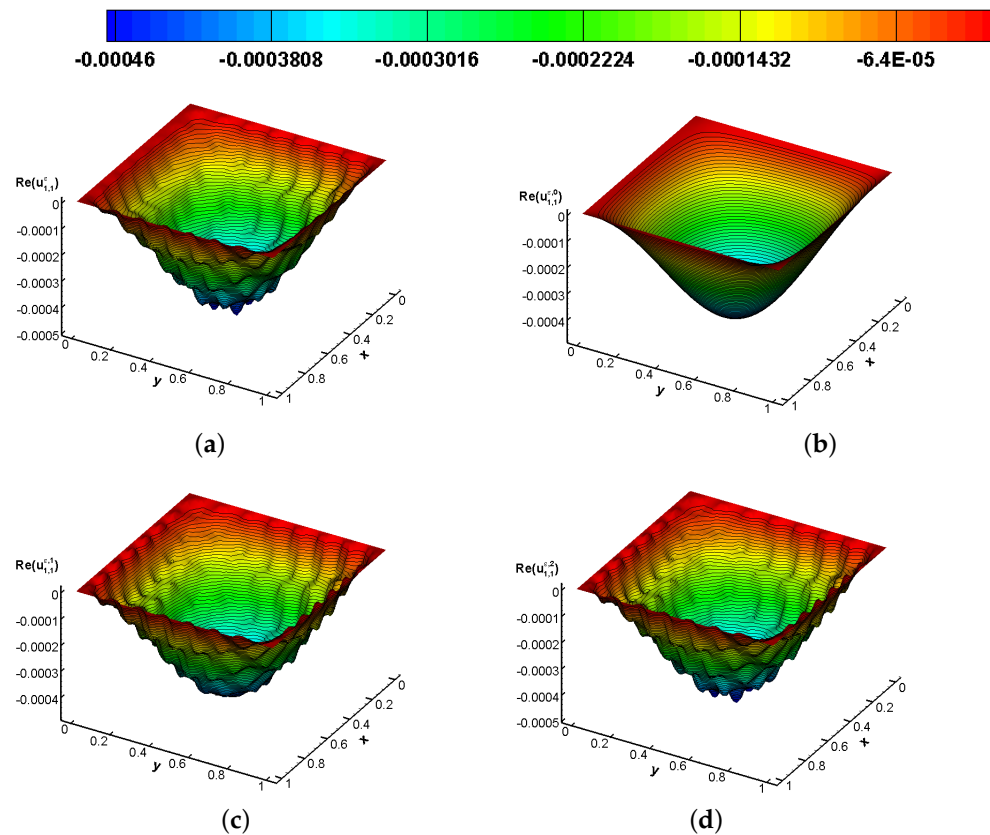
i	λ_{ε}^i	$\lambda_{\varepsilon,0}^i$	$\lambda_{\varepsilon,1}^i$	$\lambda_{\varepsilon,2}^i$	$e_{\lambda,i}^0$	$e_{\lambda,i}^1$	$e_{\lambda,i}^2$
1–4	−0.2045 $\pm 20.8088i$	−0.2083 $\pm 21.0403i$	−0.2084 $\pm 21.0410i$	−0.2072 $\pm 20.9701i$	0.011125	0.011158	0.0077494
5, 6	−0.2711 $\pm 24.4880i$	−0.2882 $\pm 25.3433i$	−0.2882 $\pm 25.3452i$	−0.2854 $\pm 25.2055i$	0.034934	0.035011	0.029304
7, 8	−0.3638 $\pm 28.8305i$	−0.3920 $\pm 30.0305i$	−0.3921 $\pm 30.0336i$	−0.3867 $\pm 29.8074i$	0.041629	0.041738	0.033889
9–12	−0.4800 $\pm 33.4913i$	−0.5222 $\pm 35.0294i$	−0.5223 $\pm 35.0347i$	−0.5118 $\pm 34.6592i$	0.045883	0.046047	0.0034823
13, 14	−0.5374 $\pm 34.5668i$	−0.5373 $\pm 35.5658i$	−0.5374 $\pm 35.5682i$	−0.5282 $\pm 34.2454i$	0.000025645	0.000039700	0.0090371
15, 16	−0.6469 $\pm 39.2275i$	−0.6560 $\pm 39.5166i$	−0.6561 $\pm 39.5203i$	−0.6421 $\pm 39.0711i$	0.0073904	0.0074660	0.0038376
17–20	−0.7123 $\pm 41.2571i$	−0.7436 $\pm 42.1927i$	−0.7438 $\pm 42.1990i$	−0.7241 $\pm 41.6174i$	0.022688	0.022841	0.0087380

Quantitative comparisons for the eigenfunction can also be computed. In Table 8, we give the relative errors of the real part of the eigenfunctions for $\varepsilon = 1/8$ and $\varepsilon = 1/16$, from which it is also indicated that the SOTS solution is not much better than the FOTS one. If we take the eigenfunction into account in the practical problem, we can obtain acceptable results through first-order approximation. This is not a good idea for eigenvalue because the error improvement is not obvious only by appending the first-order corrector. Additionally, we can know that approximations of the first four eigenfunctions are improved by half mostly as ε is smaller. So, we can say that the SOT algorithm performs better for the smallest eigenmode.

Table 8. The relative errors of the real part of the eigenfunctions for the multiscale finite element computations in L^2 norm with $\varepsilon = 1/8$ and $\varepsilon = 1/16$.

i	ε	$e_{u,i}^0$	$e_{u,i}^1$	$e_{u,i}^2$
1–4	1/8	0.053329	0.029489	0.021533
	1/16	0.026916	0.013382	0.012700
5, 6	1/8	0.080252	0.046301	0.037830
	1/16	0.041669	0.020936	0.020197
7, 8	1/8	0.10108	0.055112	0.042573
	1/16	0.051786	0.025064	0.023878
9–12	1/8	0.13398	0.093321	0.078345
	1/16	0.10544	0.090213	0.090860
13, 14	1/8	0.089517	0.073068	0.032400
	1/16	0.039818	0.029129	0.024539

The approximation for the real part of the first component of the first eigenfunction $\text{Re}(u_{1,1}^\varepsilon)$ with $\varepsilon = 1/8$ is shown in Figure 10. The real part of the first component of the homogenized solution $\text{Re}(u_{1,1}^{\varepsilon,0})$ in Figure 10b is smooth enough and it has well-modeled the macroscopic behaviors of the corresponding part of the fine solutions $\text{Re}(u_{1,1}^\varepsilon)$ shown in Figure 10a. The homogenized solution is locally modified by using the cell function and the first- and second-order solutions show good estimations. For $\text{Re}(u_{1,1}^\varepsilon)$, under the observation and comparison of some details, the second-order estimation $\text{Re}(u_{1,1}^{\varepsilon,2})$ is obviously better than the first-order one $\text{Re}(u_{1,1}^{\varepsilon,1})$ in capturing the microscopic oscillation behavior.

**Figure 10.** Approximate solutions of $\text{Re}(u_{1,1}^\varepsilon)$ with $\varepsilon = 1/8$. (a) Fine solution $\text{Re}(u_{1,1}^\varepsilon)$. (b) Homogenized solution $\text{Re}(u_{1,1}^{\varepsilon,0})$. (c) First-order approximation $\text{Re}(u_{1,1}^{\varepsilon,1})$. (d) Second-order approximation $\text{Re}(u_{1,1}^{\varepsilon,2})$.

Next, the asymptotic behavior of the first eigenvalue $\lambda_{\varepsilon,0}^1$ and corresponding eigenfunctions with different ε are more interesting. Because the homogenized solutions are not related of ε , we do not list the value of $\lambda_{\varepsilon,0}^1$ in Table 9. We show λ_{ε}^1 for different periodicity ε and the first- and second-order approximation $\lambda_{\varepsilon,1}^1, \lambda_{\varepsilon,2}^1$ as well as their relative errors in Table 9.

Table 9. Convergence of λ_{ε}^1 to the first eigenvalue $\lambda_{\varepsilon,0}^1 = -0.2083 \pm 21.0403i$ and its corrections.

ε	λ_{ε}^1	$\lambda_{\varepsilon,1}^1$	$\lambda_{\varepsilon,2}^1$	$e_{\lambda,1}^0$	$e_{\lambda,1}^1$	$e_{\lambda,1}^2$
1/2	−0.1542	−0.2084	−0.1321	0.20038	0.20067	0.058009
	$\pm 17.5282i$	$\pm 21.0453i$	$\pm 16.5116i$			
1/4	−0.1947	−0.2085	−0.1894	0.041002	0.041501	0.014665
	$\pm 20.2117i$	$\pm 21.0504i$	$\pm 19.9153i$			
1/8	−0.2031	−0.2084	−0.2036	0.015179	0.015313	0.0016252
	$\pm 20.7257i$	$\pm 21.0431i$	$\pm 20.7594i$			
1/16	−0.2045	−0.2084	−0.2072	0.011125	0.011158	0.0077494
	$\pm 20.8088i$	$\pm 21.0410i$	$\pm 20.9701i$			
1/32	−0.2049	−0.2083	−0.2081	0.0098826	0.009891	0.0090399
	$\pm 20.8344i$	$\pm 21.0405i$	$\pm 21.0228i$			
1/64	−0.2046	−0.2083	−0.2083	0.010605	0.010607	0.010394
	$\pm 20.8195i$	$\pm 20.8195i$	$\pm 21.0359i$			

The convergence of the imaginary part of first component of first eigenfunction $\text{Im}(u_{1,1}^{\varepsilon})$ is exhibited in Figures 11 and 12 for $\varepsilon = 1/2^n, n = 1, 2, \dots, 6$. Due to symmetry, we only show half of the figure. As can be seen from the Figures 11 and 12 that for bigger ε , (shown in Figures 11a,b and 12a,b) neither $\text{Im}(u_{1,1}^{\varepsilon,0})$ nor $\text{Im}(u_{1,1}^{\varepsilon,1})$ can give out good approximations of $\text{Im}(u_{1,1}^{\varepsilon})$, even $\text{Im}(u_{1,1}^{\varepsilon,2})$ is not better, while $\text{Im}(u_{1,1}^{\varepsilon,2})$ is more effective for small ε , as with $\text{Im}(u_{1,1}^{\varepsilon,0})$ and $\text{Im}(u_{1,1}^{\varepsilon,1})$ (shown in Figures 11c–f and 12c–f). So we can say that the asymptotic analysis shows better effectiveness in the case of smaller ε . As ε becomes smaller and smaller, the enlarged picture of oscillation details, see Figures 11d–f and 12d–f with $\varepsilon = 1/16, \varepsilon = 1/32$ and $\varepsilon = 1/64$, can help us to determine the subtle differences between the eigenfunctions. The above results show that when $\varepsilon < 1/8$, it is in accordance with the results obtained by the fine calculation.

Finally, the approximations of the shear stress $\sigma_{12}^{\varepsilon}$ are plotted in Figure 13 for the fifth eigenvalue, which may be of more engineering interest. Similar to the displacement, the homogenized stress σ_{12}^0 in Figure 13b shows a smooth macroscopic distribution for the fine solutions. By constructing the first and second-order approximations from the Step 6 in Section 4. The local detailing can also be reconstructed efficiently in Figure 13c,d. The approximations of the stress are not as good as the displacement field, which can be expected because the accuracy of the computed gradient (stress) is always one-order lower than the original solution (displacement) from the theory of the finite element method. Maybe more advanced finite element method such as the mixed finite element method can be introduced to improve the approximations of the stress field.

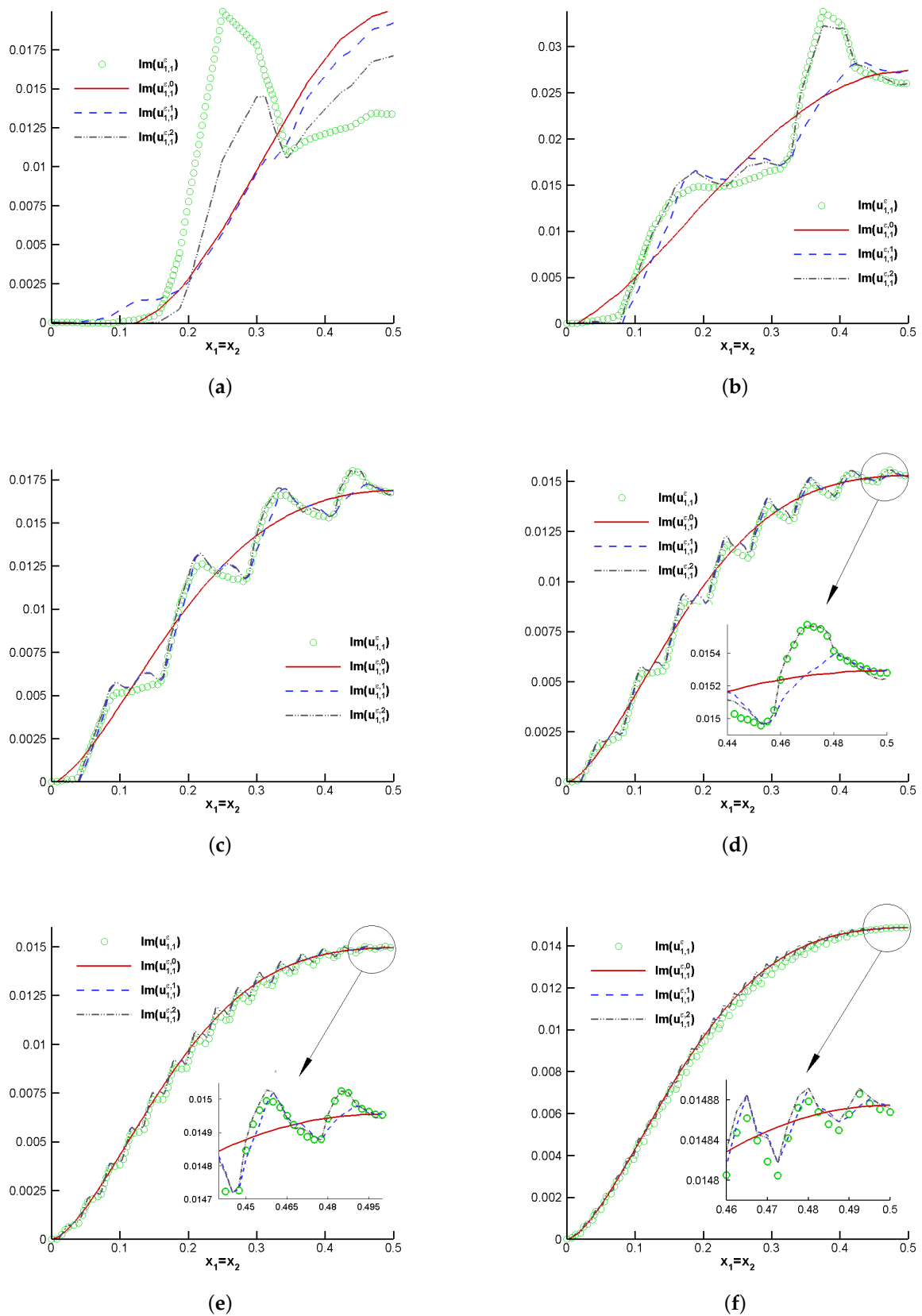


Figure 11. Convergence for $\text{Im}(u_{1,1}^\epsilon)$ with different ϵ . (a) $\epsilon = 1/2$. (b) $\epsilon = 1/4$. (c) $\epsilon = 1/8$. (d) $\epsilon = 1/16$. (e) $\epsilon = 1/32$. (f) $\epsilon = 1/64$.

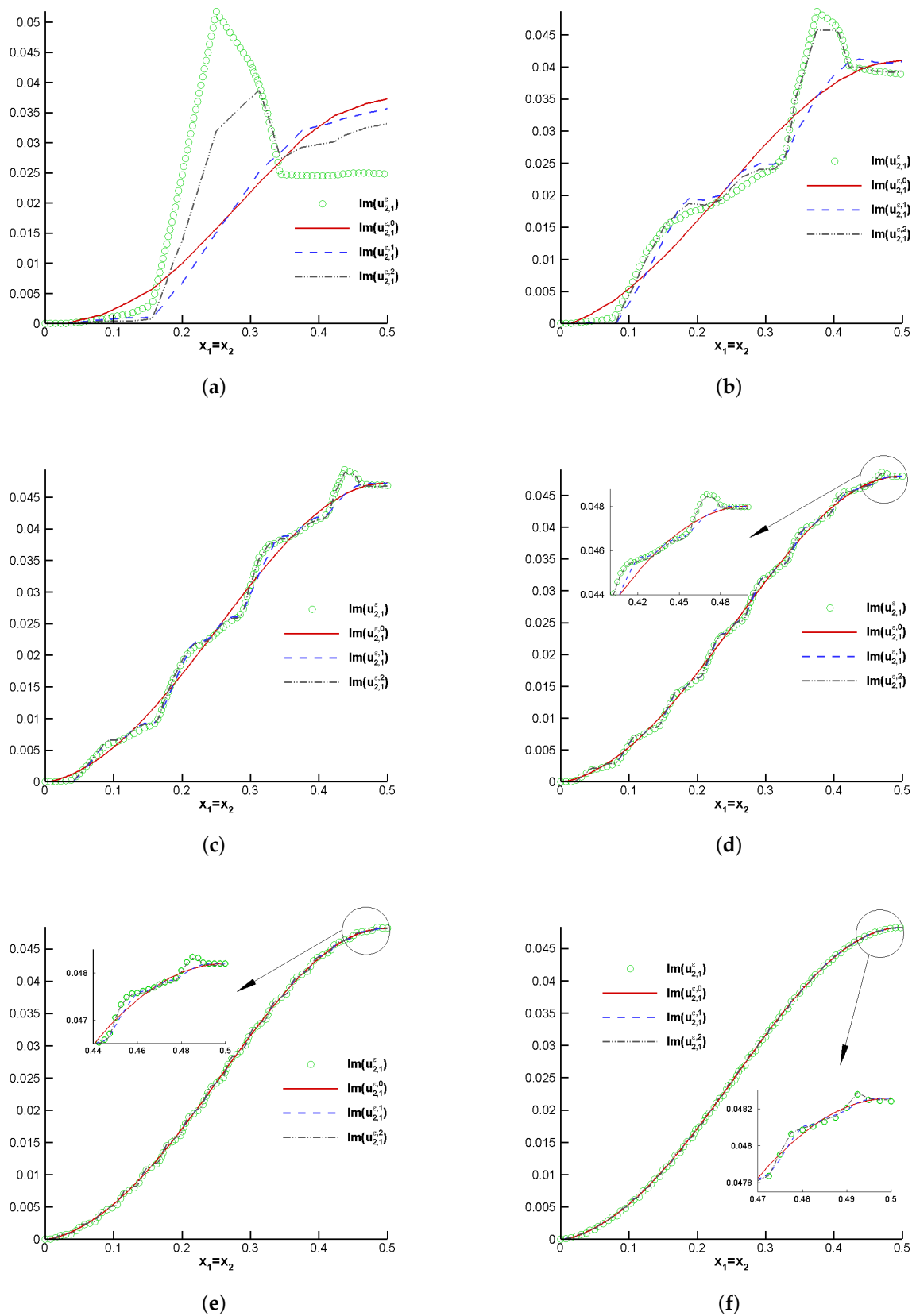


Figure 12. Convergence for $\text{Im}(u_{2,1}^\epsilon)$ with different ϵ . (a) $\epsilon = 1/2$. (b) $\epsilon = 1/4$. (c) $\epsilon = 1/8$. (d) $\epsilon = 1/16$. (e) $\epsilon = 1/32$. (f) $\epsilon = 1/64$.

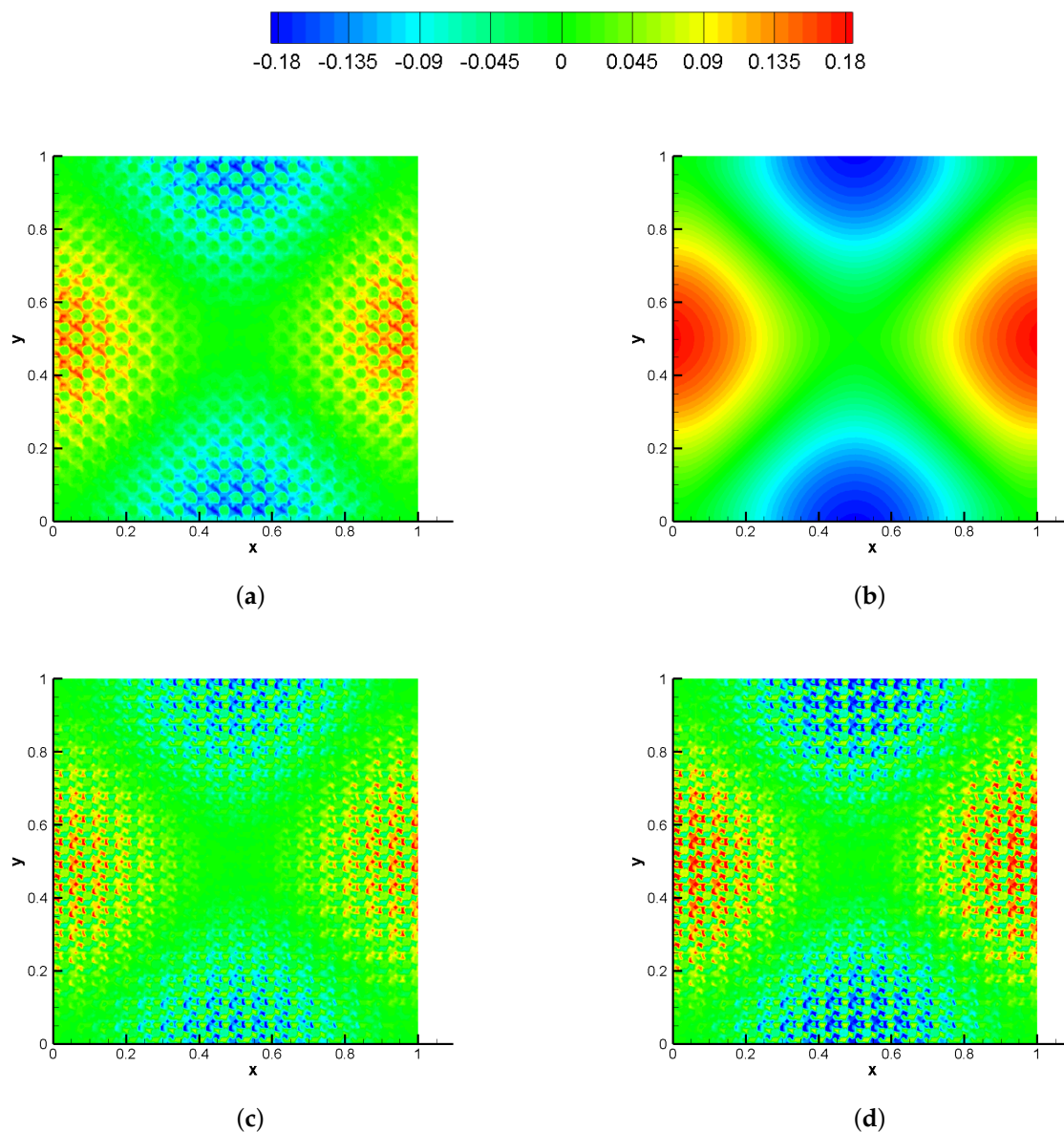


Figure 13. Approximate solution of σ_{12}^ε for the fifth eigenvalue with $\varepsilon = 1/16$. (a) Fine solution σ_{12}^ε . (b) Homogenized solution σ_{12}^0 . (c) First-order approximation $\sigma_{12}^{\varepsilon,1}$. (d) Second-order approximation $\sigma_{12}^{\varepsilon,2}$.

6. Conclusions

In this paper, the second-order two-scale asymptotic model is presented for the nonlinear quadratic eigenvalue problem in the elastic composite structure. The novel nonlinear “corrector equation” is firstly obtained for deriving the second-order expansions of the eigenvalues. The linearization method is introduced to solve the homogenized quadratic eigenvalue problem and after the assembly of the proper correctors, both the complex-valued eigenvalues and eigenfunctions are approximated quite well. The convergences of the asymptotic expansions are verified, and it is also apparent that the second-order correctors play an indispensable role for giving more accurate approximations for both the eigenvalues and eigenfunctions in the numerical simulations.

In our future work, the quadratic eigenvalue problem with more general composite domain can be considered, such as axisymmetric and spherically symmetric domain [26], the general quasi-periodic or non-periodic domain [32]. The asymptotic analysis for other nonlinear eigenvalue problems can also be considered such as the polynomial eigenvalue problem, the piezoelectric coupled problem with damping effect. It is believed that more

SOTS models can be developed and detailed consideration in constructing more efficient finite element algorithms can also be addressed.

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