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Aspects of Submanifolds on (α, β)-Type Almost Contact Manifolds with Quasi-Hemi-Slant Factor

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Abstract: In this study, the authors focus on quasi-hemi-slant submanifolds (qhs-submanifolds) of (α, β)-type almost contact manifolds, also known as trans-Sasakian manifolds. Essentially, we give sufficient and necessary conditions for the integrability of distributions using the concept of quasi-hemi-slant submanifolds of trans-Sasakian manifolds. We also consider the geometry of foliations dictated by the distribution and the requirements for submanifolds of trans-Sasakian manifolds with quasi-hemi-slant factors to be totally geodesic. Lastly, we give an illustration of a submanifold with a quasi-hemi-slant factor and discuss its application to number theory.

Keywords: trans-Sasakian manifold; quasi-hemi-slant submanifold; totally geodesic; integrability; Pontryagin number

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1. Introduction

The most exquisite and significant Riemannian manifolds are symmetric spaces. Grassmannians, compact Lie groups and bounded symmetric domains are a few of the most notable examples of this class of spaces, which are extremely important for many different areas of mathematics. Euclidean, elliptic, and hyperbolic geometry are a few examples of the unique geometry that exists in every symmetric space. However, these regions share many features in common, and there is a robust explanation for this.

There are numerous perspectives from which symmetric spaces might be seen. They can be seen as Lie triple systems, Riemannian manifolds with point reflections, parallel curvature tensors, special holonomy, homogeneous spaces with special isotropies, special Killing vector fields, or Lie groups with a specific involution.

Symmetric spaces are recognized to have certain properties in differential geometry (Riemannian geometry). The so-called symmetric submanifolds are their analogues in submanifold geometry.

A submanifold M of a Riemannian manifold $\bar{M}$ is said to be a symmetric submanifold if, for each point p in M, there exists an involutive isometry t_p of $\bar{M}$ that fixes p and leaves M invariant and whose differential at p fixes the normal vectors of M at p and reflects the tangent vectors. Any such isometry t_p is referred to as an M symmetry at p.

Ferus [1–3] examined and categorized the symmetric submanifolds in Euclidean spaces in a number of works. Interestingly, the symmetric submanifolds in Euclidean spaces are mostly the symmetric spaces amid the orbits of isotropy representations of semi-simple symmetric spaces. These orbits are referred to as symmetric real flag manifolds or symmetric R-spaces. Their classification as simply connected symmetric spaces of a compact type by Naitoh [4–6] follows further attempts by other mathematicians and classifications in compact symmetric spaces of rank one.

In irreducible symmetric spaces of a non-compact type and higher rank, i.e., a rank greater than one, the only examples of symmetric submanifolds that are symmetric are completely geodesic. In actuality, the reflecting submanifolds are fully geodesic symmetric submanifolds. If the geodesic reflection of a Riemannian manifold $\bar{M}$ in a submanifold M has a well-defined global isometry, then the submanifold M is said to be reflective. A reflecting submanifold must be completely geodesic because it is a linked part of the isometry's fixed point set. A totally geodesic submanifold of a symmetric space $\bar{M}$ is reflective precisely if it has a totally geodesic submanifold of M that is tangent to each of its normal spaces. Any such typical, completely geodesic submanifold likewise has reflection. Leung [7,8] categorized the reflective submanifolds as irreducible, simply connected symmetric spaces. Moreover, every irreducible, totally geodesic submanifold of a Hermitian symmetric space is a slant submanifold [9,10].

On the other hand, Oubina [11] popularized the concept of (α, β)-type almost contact manifolds, or trans-Sasakian manifolds, which are connected to locally conformal Kahler manifolds and include both the subclasses of Sasakian and cosymplectic structures. The types $(\alpha, 0)$, $(\beta, 0)$, and $(0, 0)$ of trans-Sasakian manifolds are α-Sasakian, β-Kenmotsu, and cosymplectic manifolds, respectively. The theory of submanifolds began with the fact that the surface extrinsic geometry evolves with ambient space. This submanifold approach plays a significant role in image processing, mathematical physics, mechanics, computer design, and economics. Because of the widespread relevance of such a topic, it is a lively and intriguing research area for all mathematical experts.

Since Chen [12] defined and introduced the geometry of slant submanifolds as a logical extension of both totally real and holomorphic immersions, various mathematicians have studied this over the past twenty years [13–17]. In 1996, Lotta [18] studied the characteristics of the immersion of a Riemannian manifold with a slant factor into an almost contact metric manifold. The concept of semi-slant submanifolds of Kaehlerian manifolds was studied in [19]. On different types of differentiable manifolds, the slant submanifolds were further extended as pseudo-slant submanifolds, semi-slant submanifolds, bi-slant submanifolds, and quasi-slant submanifolds [19–24]. Prasad et al. [25,26] recently researched the quasi-hemi-slant submanifolds of cosymplectic manifolds and Sasakian manifolds, as well as the features of integrability of distribution and completely geodesic manifolds.

We will investigate *qhs*-submanifolds of trans-Sasakian manifolds, which comprise hemi-slant and semi-slant submanifolds, as a result of the previous research.

The following is a breakdown of the structure of this article. In Section 2, we cover the fundamental concept of an almost contact metric manifold, as well as some of its features. In Section 3, we define *qhs*-submanifolds of trans-Sasakian manifolds and review some fundamental findings. Section 4 discusses the criteria for the integrability of *qsh* submanifolds. In Section 4, we also demonstrate several conditions that must be met in order for the *qhs*-submanifold of trans-Sasakian manifolds to be totally geodesic.

2. (α, β)-Type Almost Contact Manifolds

Let $\bar{M}$ be a real $(2n+1)$-dimensional manifold $\bar{M}$ endowed with an almost contact metric structure [27] if it admits a (1,1) tensor field φ, a contravariant vector field ζ, a 1-form η, and a Riemannian metric g on $\bar{M}$, which yields

$$\varphi^2 E = -E + \eta(E)\zeta, \quad \eta o \varphi = 0, \quad \varphi(\zeta) = 0, \tag{1}$$

$$g(\varphi E, F) = -g(E, \varphi F), \quad g(E, \zeta) = \eta(E), \quad \eta(\zeta) = 1, \tag{2}$$

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F) \tag{3}$$

for any vector fields E, F tangent to $\bar{M}$.

An almost contact metric manifold $\bar{M}$ $(\varphi, \zeta, \eta, g)$ is stated to be a trans-Sasakian manifold [11] if $(\bar{M} \times \mathbb{R}, J, \mathbb{G})$ belongs to the W_4 class of Hermitian manifolds, where J is the almost complex structure on $(\bar{M} \times \mathbb{R})$ defined by

$$J(E, f\frac{d}{dt}) = (\varphi E - f\zeta, \eta(E)\frac{d}{dt}),$$

for any vector field E on $\bar{M}$ and f is a smooth function on $\bar{M} \times \mathbb{R}$ with a product metric $\mathbb{G}$ on $\bar{M} \times \mathbb{R}$. This condition can be expressed as follows [27]:

$$(\bar{\nabla}_E \varphi)F = \alpha(g(E,F)\zeta - \eta(F)E) + \beta(g(\varphi E, F)\zeta - \eta(F)\varphi E), \tag{4}$$

where α, β denote smooth functions on $\bar{M}$, $\bar{\nabla}$ denotes the Riemannian connection of d on $\bar{M}$, and we can say that such structures are trans-Sasakian structures of type (α, β) or (α, β)-type almost contact manifolds.

If $\beta = 0$, then the $\bar{M}$ manifold is known as α-Sasakian.

If $\alpha = 0$, the $\bar{M}$ is known as β-Kenmotsu [28].

If $\alpha = \beta = 0$, then $\bar{M}$ is a cosymplectic manifold [27,29].

The trans-Sasakian structure or (α, β)-type almost contact manifolds, as we know, fulfills

$$\bar{\nabla}_E \zeta = -\alpha \varphi E + \beta(E - \eta(E)\zeta), \tag{5}$$

$$(\bar{\nabla}_E \eta)F = \alpha g(E, \varphi F) + \beta g(\varphi E, \varphi F). \tag{6}$$

A $(1,0)$ type of trans-Sasakian manifold is clearly a Sasakian manifold [30], whereas a $(0,1)$ type of trans-Sasakian manifold is obviously a Kenmotsu manifold [31]. A $(0,0)$ type of trans-Sasakian manifold is a cosymplectic manifold [25].

Now, let M be a Riemannian manifold immersed in $\bar{M}$, and, throughout this article, the induced Riemannian metric on M is indicated by d. The equations of Gauss and Weingarten are provided by [32]

$$\bar{\nabla}_E F = \nabla_E F + \sigma(E,F), \tag{7}$$

$$\bar{\nabla}_E U = -\mathcal{A}_U E + \nabla^{\perp}_E U \tag{8}$$

for all $E, F \in \Gamma(\mathcal{T}M)$, $U \in \Gamma(\mathcal{T}^{\perp}M)$, wherein ∇ and $\nabla^{\perp}$ are the induced connections on M and on $\mathcal{T}^{\perp}M$ of M, respectively. In addition, $\mathcal{A}_U$ is the shape operator on M with normal vector $U \in \Gamma(\mathcal{T}^{\perp}M)$ and σ is the second fundamental form of $\mathcal{A}_U$, defined as

$$g(\sigma(E,F), U) = g(\mathcal{A}_U E, F). \tag{9}$$

The mean curvature tensor H of M is defined as follows:

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^{n}\sigma(v_i, v_i) = \frac{1}{n} trace(\sigma). \tag{10}$$

wherein $\{v_1, v_2, \cdots, v_n\}$ is a local orthogonal frame of M since the $dim(M) = n$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is totally umbilical if

$$g(E,F)\mathcal{H} = \sigma(E,F), \tag{11}$$

where $\mathcal{H}$ is the mean curvature. If $\sigma(E,F) = 0$, a submanifold M is said to be totally geodesic for each $E, F \in \Gamma(\mathcal{T}M)$, and if $\mathcal{H} = 0$, then M is said to be minimal.

For any $E \in \Gamma(\mathcal{T}M)$, we have

$$\varphi E = \mathcal{T}E + \mathcal{N}E, \tag{12}$$

where $\mathcal{T}E$ and $\mathcal{N}E$ are tangential and normal components of ψE on M, respectively.

In the same way, for any $U \in \Gamma(\mathcal{T}^{\perp}M)$, we have

$$\varphi U = tU + nU, \tag{13}$$

where nU and tU indicate the normal and tangential parts of ψU on M, respectively.

In light of (2) and (12), we have

$$g(\mathcal{T}E, F) = -g(E, \mathcal{T}F) \tag{14}$$

for any $E, F \in \Gamma(\mathcal{T}M)$.

In (12) and (13), the covariant derivative of projection morphisms is defined as

$$(\bar{\nabla}_E \mathcal{T})F = \nabla_E \mathcal{T}F - \mathcal{T}\nabla_E F,$$

$$(\bar{\nabla}_E N)F = \nabla^{\perp}_E NF - N\nabla_E F,$$

$$(\bar{\nabla}_E t)U = \nabla_E tU - t\nabla_E U,$$

$$(\bar{\nabla}_X n)U = \nabla^{\perp}_X nU - n\nabla_X U,$$

for all $E, F \in \Gamma(\mathcal{T}M)$ and $U \in \Gamma(\mathcal{T}^{\perp}M)$.

Now, we have the following definitions.

Definition 1 ([33]). *Let M be a Riemannian manifold isometrically immersed in $\bar{M}$, which is almost contact metric manifold. If $\varphi(\mathcal{T}_x M) \subseteq \mathcal{T}_x M$ for every point $x \in M$, a submanifold M of an almost contact metric manifold $\bar{M}$ is said to be invariant.*

Definition 2 ([34]). *A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be anti-invariant if $\varphi(T_x M) \subseteq T_x^{\perp} M$, for every point $x \in M$.*

Definition 3 ([26]). *A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant if the angle $\theta(K)$ between ψE and $\mathcal{T}_x M$ is constant for each non-zero vector E tangent to M at $x \in M$, linearly independent on ζ for each non-zero vector E tangent to M at $x \in M$. The angle θ is referred to as the slant angle of the submanifold in this context. If neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$, a slant submanifold M is considered a valid slant submanifold.*

In addition, we can also observe the following conditions.

(i) If $\theta = 0$, a slant submanifold M is an invariant submanifold.
(ii) If $\theta = \frac{\pi}{2}$, it is an anti-invariant submanifold.

Moreover, the slant submanifold is hence an extension of invariant and anti-invariant submanifolds [35].

Definition 4 ([31]). *A semi-invariant submanifold is a submanifold M of an almost contact metric manifold $\bar{M}$ if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that*

$$\mathcal{T}M = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus <\zeta>,$$

where $\mathcal{D}$ is invariant and $\mathcal{D}^{\perp}$ is an anti-invariant distribution.

Definition 5 ([19]). *A semi-slant submanifold is a submanifold M of an almost contact metric manifold $\bar{M}$, if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\theta}$ on M such that*

$$\mathcal{T}M = \mathcal{D} \oplus \mathcal{D}^{\theta} \oplus <\zeta>,$$

where $\mathcal{D}^{\theta}$ is the slant with slant angle θ and $\mathcal{D}$ is invariant. The angle θ is known as a semi-slant angle in this scenario [36].

Definition 6 ([21])**.** *A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a hemi-slant submanifold of $\bar{M}$, if there exist two orthogonal complementary distributions $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ on M such that*

$$\mathcal{T}M = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus < \zeta >,$$

where $\mathcal{D}^\theta$ is the slant with slant angle θ and $\mathcal{D}^\perp$ is anti-invariant. The angle θ is known as a hemi-slant angle in this case.

3. Submanifolds of Trans-Sasakian Manifolds with Quasi-Hemi-Slant Factor

The quasi-hemi-slant submanifold (in short, qhs-submanifold) of trans-Sasakian manifolds is discussed in this section of the work.

Note that in the presented results, Θ denotes the "submanifold of a trans-Sasakian manifold $\bar{M}$".

Definition 7 ([26])**.** *A Θ is said to be a qhs-submanifold if there exist distributions $\mathcal{D}$, $\mathcal{D}^\theta$, and $\mathcal{D}^\perp$ on M such that*

(i) *$\mathcal{T}M$ may be broken down into its constituent parts, such as*

$$\mathcal{T}M = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus < \zeta > .$$

(ii) *The distribution $\mathcal{D}$ is φ-invariant, i.e., $\varphi\mathcal{D} = \mathcal{D}$.*

(iii) *For any non-zero vector field $E \in (\mathcal{D}^\theta)_p$, $p \in M$, the angle θ, between the distribution φE and $\mathcal{D}^\theta$, is constant and independent of the choice of p, and $E \in (\mathcal{D}^\theta)_p$.*

(iv) *The the distribution $\mathcal{D}^\perp$ is φ anti-invariant, i.e., $\varphi\mathcal{D}^\perp \subseteq \mathcal{T}^\perp$.*

In addition, the qhs-angle of M is called θ in this situation. Assume that the dimensions of $\mathcal{D}, \mathcal{D}^\theta$, and $\mathcal{D}^\perp$ are n_1, n_2, and n_3, respectively. Then, we may clearly observe the situations below.

(i) *If $n_1 = 0$, then M is a hemi-slant submanifold.*

(ii) *If $n_2 = 0$, then M is a semi-invariant submanifold.*

(iii) *If $n_3 = 0$, then M is a semi-slant submanifold.*

If $\mathcal{D} \neq \{0\}, \mathcal{D}^\perp \neq \{0\}$, and $\theta \neq 0, \frac{\pi}{2}$, we claim that a qhs-submanifold M is proper.

This entails that a *qhs*-submanifold is an extension of invariant, anti-invariant, semi-invariant, slant, hemi-slant, and semi-slant submanifolds and instances of quasi-hemi-slant submanifolds.

Remark 1. *The above definition can be extended by taking* [26]

$$\mathcal{T}M = \mathcal{D} \oplus \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \cdots \oplus \mathcal{D}^{\theta_k} \oplus \mathcal{D}^\perp \oplus < \zeta > .$$

Thus, multi-slant submanifolds, quasi-multi-slant submanifolds, quasi-hemi-multi-slant submanifolds, and so on can be defined.

Let M be a *qhs*-Θ. We indicate the projection of $E \in \Gamma(\mathcal{T}M)$ on the distribution $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ respectively. Then, we can write for $E \in \Gamma(\mathcal{T}M)$

$$E = \mathcal{P}E + \mathcal{Q}E + \mathcal{R}E + \eta(E)\zeta. \tag{15}$$

Now, we write

$$\varphi E = \mathcal{T}E + \mathcal{N}E, \tag{16}$$

where $\mathcal{T}E$ and $\mathcal{N}E$ are tangential and normal components of ψE on M.

In light of (15) and (16), we have

$$\varphi E = \mathcal{T}\mathcal{P}E + \mathcal{N}\mathcal{P}E + \mathcal{T}\mathcal{Q}E + \mathcal{N}\mathcal{Q}E + \mathcal{T}\mathcal{R}E + \mathcal{N}\mathcal{R}E.$$

Since $\varphi\mathcal{D} = \mathcal{D}$ and $\varphi\mathcal{D}^{\perp} \subseteq \mathcal{T}^{\perp}M$, we obtain $\mathcal{N}\mathcal{P}E = 0$ and $\mathcal{T}\mathcal{R}E = 0$. Thus, we find

$$\varphi E = \mathcal{T}\mathcal{P}E + \mathcal{T}\mathcal{Q}E + \mathcal{N}\mathcal{Q}E + \mathcal{N}\mathcal{R}E. \tag{17}$$

Then, for any $E \in \Gamma(\mathcal{T}M)$, we can simply express

$$\mathcal{T}E = \mathcal{T}\mathcal{P}E + \mathcal{T}\mathcal{Q}E,$$

and

$$\mathcal{N}E = \mathcal{N}\mathcal{Q}E + \mathcal{N}\mathcal{R}E.$$

As a result of (17), we may obtain the following decomposition:

$$\varphi(\mathcal{T}M) = \mathcal{D} \oplus \mathcal{T}\mathcal{D}^{\theta} \oplus \mathcal{N}\mathcal{D}^{\theta} \oplus \mathcal{N}\mathcal{D}^{\perp}$$

where $\oplus$ denotes the orthogonal direct sum.

Since $\mathcal{N}\mathcal{D}^{\theta} \subset \mathcal{T}^{\perp}M$ and $\mathcal{N}\mathcal{D}^{\perp} \subset \mathcal{T}^{\perp}M$, we obtain

$$\mathcal{T}^{\perp}M = \mathcal{N}\mathcal{D}^{\theta} \oplus \mathcal{N}\mathcal{D}^{\perp} \oplus \mu,$$

where μ is an orthogonal complement of $\mathcal{N}\mathcal{D}^{\theta} \oplus \mathcal{N}\mathcal{D}^{\perp}$ in $\Gamma(\mathcal{T}^{\perp}M)$ and it is also an invariant in terms of φ.

For every vector field with a non-zero value $U \in \Gamma(\mathcal{T}^{\perp}M)$, we write

$$\varphi U = tU + nU \tag{18}$$

for $tU \in \Gamma(\mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp})$ and $nU \in \Gamma(\mu)$.

Theorem 1. *Let M be a qhs-Θ of type (α, β). Then, we have*

$$\nabla_E \mathcal{T}F - \mathcal{A}_{\mathcal{N}F}E - \mathcal{T}\nabla_E F - t\sigma(E,F) = \alpha(g(E,F)\zeta - \eta(F)E) + \beta(g(\mathcal{T}E,F)\zeta - \eta(F)\mathcal{T}E),$$

$$\sigma(E,\mathcal{T}F) + \nabla^{\perp}_E \mathcal{N}F - \mathcal{N}\nabla_E F - n\sigma(E,F) = -\beta\eta(F)\mathcal{N}E \tag{19}$$

for any $E, F \in \Gamma(\mathcal{T}M)$.

Proof. Adopting Equations (4)–(13) and then equating the tangential and normal components, we obtain (19). □

Next, in view of Theorem 1, we present the following corollaries.

Corollary 1. *Let M be a qhs-submanifold of an α-Sasakian manifold. Then, we have*

$$\nabla_E \mathcal{T}F - \mathcal{A}_{\mathcal{N}F}E - \mathcal{T}\nabla_E F - t\sigma(E,F) = \alpha(g(E,F)\zeta - \eta(F)E),$$

$$\sigma(E,\mathcal{T}F) + \nabla^{\perp}_E \mathcal{N}F = \mathcal{N}\nabla_E F + n\sigma(E,F)$$

for any $E, F \in \Gamma(\mathcal{T}M)$.

Corollary 2. *Let M be a qhs-submanifold of a β-Kenmotsu manifold. Then, we have*

$$\nabla_E \mathcal{T}F - \mathcal{A}_{\mathcal{N}F}E - \mathcal{T}\nabla_E F - t\sigma(E,F) = \beta(g(\mathcal{T}E,F)\zeta - \eta(F)\mathcal{T}E),$$

$$\sigma(E,\mathcal{T}F) + \nabla^{\perp}_E \mathcal{N}F = \mathcal{N}\nabla_E F + n\sigma(E,F) - \beta\eta(F)\mathcal{N}E$$

for any $E, F \in \Gamma(\mathcal{T}M)$

Corollary 3. *Let M be a qhs-submanifold of a cosymplectic manifold. Then we have*

$$\nabla_E \mathcal{T}F = \mathcal{A}_{\mathcal{N}F}E + \mathcal{T}\nabla_E F + t\sigma(E,F),$$

$$\sigma(E,\mathcal{T}F) + \nabla_E^{\perp}\mathcal{N}F = \mathcal{N}\nabla_E F + n\sigma(E,F)$$

for any $E, F \in \Gamma(\mathcal{T}M)$

Theorem 2. *Let M be a qhs-Θ of type* (α,β)*. Then, we have (see page 7 in [26])*
(*i*) $\mathcal{T}\mathcal{D} = \mathcal{D}$,
(*ii*) $\mathcal{T}\mathcal{D}^{\theta} = \mathcal{D}^{\theta}$,
(*iii*) $\mathcal{T}\mathcal{D}^{\perp} = \{0\}$,
(*iv*) $t\mathcal{N}\mathcal{D}^{\theta} = \mathcal{D}^{\theta}$,
(*v*) $t\mathcal{N}\mathcal{D}^{\perp} = \mathcal{D}^{\perp}$.

Theorem 3. *Let M be a qhs-Θ of type* (α,β)*. Then,* $\mathcal{T}$ *and* $\mathcal{N}$*, t, and n in the tangent bundle of M fulfill the following relations.*
(*i*) $\mathcal{T}^2 + t\mathcal{N} = -I + \eta \otimes \zeta$ *on* $\mathcal{T}M$.
(*ii*) $\mathcal{N}\mathcal{T} + n\mathcal{N} = 0$ *on* $\mathcal{T}M$.
(*iii*) $\mathcal{N}t + n^2 = -I$ *on* $\mathcal{T}^{\perp}M$.
(*iv*) $\mathcal{T}t + tn = 0$ *on* $\mathcal{T}^{\perp}M$*, where I is the identity.*

Proof. In view of (16) and (18) and adopting (1), on equating the tangential and normal parts, we obtain the desired results. □

Now, we have a very useful lemma.

Lemma 1. *Let M be a qhs-Θ of type* (α,β)*. Then, we have the following:*
(*i*) $\mathcal{T}^2 E = -(cos^2\theta)E$;
(*ii*) $g(\mathcal{T}E,\mathcal{T}F) = (cos^2\theta)g(E,F)$;
(*iii*) $g(\mathcal{N}E,\mathcal{N}F) = (sin^2\theta)g(E,F)$
for any $E, F \in \mathcal{D}^{\theta}$.

Proof. The proof is straightforward as in [37]. Thus, we will omit it. □

Theorem 4. *Let M be a qhs-Θ of type* (α,β)*. Then, we have*

$$(\bar{\nabla}_E\mathcal{T})F = \mathcal{A}_{\mathcal{N}F}E + t\sigma(E,F) + \alpha(g(E,F)\zeta - \eta(F)E) + \beta(g(\mathcal{T}E,F)\zeta - \eta(F)\mathcal{T}E),$$

$$(\bar{\nabla}_E\mathcal{N})F = n\sigma(E,F) - \sigma(E,\mathcal{T}F) - \beta\eta(F)\mathcal{N}E,$$

$$(\bar{\nabla}_E t)U = \mathcal{A}_{nU}E - \mathcal{T}\mathcal{A}_U E,$$

$$(\bar{\nabla}_E n)U = \beta d(\mathcal{N}E,U)\zeta - \sigma(E,tU) - \mathcal{N}\mathcal{A}_U E$$

for any $E, F \in \Gamma(\mathcal{T}M)$ *and* $U \in \Gamma(\mathcal{T}^{\perp}M)$.

Proof. In light of Equations (4)–(14), and equating the tangent and normal components, we obtain the desired results. □

Now, from Theorem 4, we can articulate the following corollaries.

Corollary 4. *Let M be a qhs-submanifold of an α-Sasakian manifold* $\bar{M}$*. Then, we have*

$$(\bar{\nabla}_E\mathcal{T})F = \mathcal{A}_{\mathcal{N}F}E + t\sigma(E,F) + \alpha(g(E,F)\zeta - \eta(F)E) - \eta(F)\mathcal{T}E),$$

$$(\bar{\nabla}_E\mathcal{N})F = n\sigma(E,F) - \sigma(E,\mathcal{T}F),$$

$$(\bar{\nabla}_E t)U = \mathcal{A}_{nU}E - \mathcal{T}\mathcal{A}_U E,$$
$$(\bar{\nabla}_E n)U = -\sigma(E, tU) - \mathcal{N}\mathcal{A}_U E$$

for any $E, F \in \Gamma(\mathcal{T}M)$ *and* $U \in \Gamma(\mathcal{T}^{\perp}M)$*.*

Corollary 5. *Let* M *be a qhs-submanifold of a* β*-Kenmotsu manifold* $\bar{M}$*. Then, we have*

$$(\bar{\nabla}_E \mathcal{T})F = \mathcal{A}_{\mathcal{N}F}E + t\sigma(E, F) + \beta(g(\mathcal{T}E, F)\zeta - \eta(F)\mathcal{T}E),$$
$$(\bar{\nabla}_E \mathcal{N})F = n\sigma(E, F) - \sigma(E, \mathcal{T}F) - \beta\eta(F)\mathcal{N}E,$$
$$(\bar{\nabla}_E t)U = \mathcal{A}_{nU}E - \mathcal{T}\mathcal{A}_U E,$$
$$(\bar{\nabla}_E n)U = \beta d(\mathcal{N}E, U)\zeta - \sigma(E, tU) - \mathcal{N}\mathcal{A}_U E$$

for any $E, F \in \Gamma(\mathcal{T}M)$ *and* $U \in \Gamma(\mathcal{T}^{\perp}M)$*.*

Corollary 6. *Let* M *be a qhs-submanifold of a cosymplectic manifold* $\bar{M}$*. Then, we have*

$$(\bar{\nabla}_E \mathcal{T})F = \mathcal{A}_{\mathcal{N}F}E + t\sigma(E, F) - \gamma(F)\mathcal{T}E),$$
$$(\bar{\nabla}_E \mathcal{N})F = n\sigma(E, F) - \sigma(E, \mathcal{T}F),$$
$$(\bar{\nabla}_E t)U = \mathcal{A}_{nU}E - \mathcal{T}\mathcal{A}_U E,$$
$$(\bar{\nabla}_E n)U = -\sigma(E, tU) - \mathcal{N}\mathcal{A}_U E$$

for any $E, F \in \Gamma(\mathcal{T}M)$ *and* $U \in \Gamma(\mathcal{T}^{\perp}M)$*.*

Theorem 5. *Let* M *be a qhs-*Θ *of type* (α, β)*. Then, we have*

$$\nabla_E \zeta = -\alpha\mathcal{T}E - \beta\varphi^2 E, \qquad \sigma(E, \zeta) = -\alpha\mathcal{N}E \tag{20}$$

for any $E \in \Gamma(\mathcal{T}M)$*.*

Proof. Using (5), (7), and (12) and equating the tangent and normal components, we obtain (20). □

Hence, Theorem 5 entails the following.

Corollary 7. *A qhs-submanifold* M *of an* α*-Sasakian manifold satisfies* $\nabla_E \zeta = -\alpha\mathcal{T}E$ *and* $\sigma(E, \zeta) = -\alpha\mathcal{N}E$*.*

Corollary 8. *A qhs-submanifold* M *of a* β*-Kenmotsu manifold satisfies* $\nabla_E \zeta = -\beta\varphi^2 E$ *and* $\sigma(E, \zeta) = 0$*.*

Corollary 9. *A qhs-submanifold* M *of a cosymplectic manifold satisfies* $\nabla_E \zeta = 0$ *and* $\sigma(E, \zeta) = 0$*.*

Next, we have the following interesting result.

Theorem 6. *Let* M *be a qhs-*Θ *of type* (α, β)*. Then, we have*

$$\mathcal{A}_{\varphi F}E = \mathcal{A}_{\varphi E}F \tag{21}$$

for any $E, F \in \mathcal{D}^{\perp}$*.*

Proof. Let $E, F, G \in \mathcal{D}^{\perp}$. Adopting (2) in (9), we obtain

$$g(\mathcal{A}_{\varphi F}E, G) = -g(\varphi\sigma(E, G), F). \tag{22}$$

By virtue of (7) and (22), we have

$$g(\mathcal{A}_{\varphi F}E, G) = -(\varphi\bar{\nabla}_G E, F) + g(\varphi\nabla_G E, F). \tag{23}$$

Since $\psi\nabla_G E \in \Gamma(\mathcal{T}^\perp M)$, from (23), we obtain

$$g(\mathcal{A}_{\varphi F}E, G) = g((\bar{\nabla}_G\varphi)E, F) - g((\nabla_G\varphi)E, F). \tag{24}$$

Now, for $E \in \mathcal{D}^\perp$ and $\psi E \in \Gamma(\mathcal{T}^\perp M)$, utilizing (8) in (24), one obtains

$$g(\mathcal{A}_{\varphi F}E, G) = g((\bar{\nabla}_G\varphi)E, F) - g(\mathcal{A}_{\psi E}G, F). \tag{25}$$

Now, interchanging E and F in (22), we obtain

$$g(\mathcal{A}_{\varphi E}F, G) = g(\sigma(F, G), \varphi E). \tag{26}$$

Due to the symmetry of σ such that $\sigma(F, G) = \sigma(G, F)$, by (26), we have

$$g(\mathcal{A}_{\varphi E}G, F) = g(\mathcal{A}_{\varphi E}F, G). \tag{27}$$

Employing (27) in (25), we have

$$g(\mathcal{A}_{\varphi F}E, G) - g(\mathcal{A}_{\varphi E}F, G) = g((\bar{\nabla}_G\psi)E, F). \tag{28}$$

Adopting (2) and (4) in (28), we obtain

$$\begin{aligned} &g(\mathcal{A}_{\varphi F}E - \mathcal{A}_{\varphi E}F, G) \\ &\quad= \alpha\eta(F)g(E, G) - \alpha\eta(E)g(F, G) - \beta\eta(F)g(\varphi E, G) + \beta\eta(E)(\varphi F, G). \end{aligned} \tag{29}$$

Thus, (29) yields

$$\mathcal{A}_{\varphi F}E - \mathcal{A}_{\psi E}F = \alpha\eta(F)E - \alpha\eta(E)F - \beta\eta(F)\varphi E + \beta\eta(E)\varphi F. \tag{30}$$

Since $E, F \in \mathcal{D}^\perp$ is a distribution orthogonal to the distribution $< \zeta >$, it follows that

$$\eta(E) = \eta(F) = 0.$$

As a result of the restrictions in (30), we obtain

$$\mathcal{A}_{\varphi E}F = \mathcal{A}_{\varphi F}E.$$

This completes the proof. □

Lemma 2. *Let M be a qhs-Θ of type (α, β). Then, we have*

$$\mathcal{T}([G, H]) = 0,$$

$$\mathcal{N}([G, H]) = \nabla^\perp_G\varphi H - \nabla^\perp_H\varphi G$$

for any $G, H \in \mathcal{D}^\perp$.

Proof. Let $G, H \in \mathcal{D}^\perp$. Adopting covariant differentiation as in (4), we obtain

$$\bar{\nabla}_G\varphi H - \varphi(\bar{\nabla}_G H) = \alpha g(G, H)\zeta. \tag{31}$$

Employing (7), (8), (12), and (13) in (31), we obtain

$$-\mathcal{A}_{\varphi H}G + \nabla^\perp_G\varphi H - \mathcal{T}(\nabla_G H) - \mathcal{N}(\nabla_G H) - t\sigma(G, H) - n\sigma(G, H) = \alpha g(G, H)\zeta. \tag{32}$$

Now, equating the tangent and normal parts of (32), we obtain

$$\mathcal{A}_{\varphi H}G + \mathcal{T}(\nabla_G H) + t\sigma(G,H) = -\alpha g(G,H)\zeta, \tag{33}$$

$$\nabla^{\perp}_G \varphi H = \mathcal{N}(\nabla_G H) + n\sigma(G,H). \tag{34}$$

Interchanging G and H in (33) and (34) and using (21), we can easily obtain the desired results. □

Lemma 3. *Let M be a qhs-Θ of type (α,β). Then, we have*

$$g([E,F],\zeta) = 2\alpha g(\mathcal{T}E,F),$$

$$g(\bar{\nabla}_E F,\zeta) = \alpha g(\mathcal{T}E,F) - \beta g(E,F)$$

for any $E,F \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp})$.

Proof. If $F \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp})$, then

$$g(\bar{\nabla}_E F,\zeta) + g(F,\bar{\nabla}_E\zeta) = 0. \tag{35}$$

Adopting Equation (2) in (35), we obtain

$$g(\bar{\nabla}_E F,\zeta) - \alpha g(\varphi E,F) + \beta g(E,F) - \beta\eta(E)\eta(F) = 0. \tag{36}$$

Since $Y \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp})$, and it is a distribution orthogonal to the $< \zeta >$-distribution, it entails that

$$\eta(E) = \eta(F) = 0.$$

Thus, in view of (36), we have

$$g(\bar{\nabla}_E F,\zeta) = \alpha g(\varphi E,F) - \beta g(E,F). \tag{37}$$

By swapping E and F in the equation above, we obtain

$$g(\bar{\nabla}_F E,\zeta) = \alpha g(\varphi F,E) - \beta g(F,E) \tag{38}$$

By (2) and (38), we obtain

$$g(\bar{\nabla}_F E,\zeta) = -\alpha g(\varphi E,F) - \beta g(E,F). \tag{39}$$

Subtracting (39) from (37), we have

$$g([E,F],\zeta) = 2\alpha(\varphi E,F). \tag{40}$$

Using (12), (37), and (40), we obtain the desired results. □

Corollary 10. *For a qhs-submanifold M of an α-Sasakian manifold, we have*

$$g([E,F],\zeta) = 2\alpha g(\mathcal{T}E,F), \quad g(\bar{\nabla}_E F,\zeta) = \alpha(\mathcal{T}E,F).$$

Corollary 11. *For a qhs-submanifold M of a β-Kenmotsu manifold, we have*

$$g([E,F],\zeta) = 0, \qquad (\bar{\nabla}_E F,\zeta) = -\beta g(E,F).$$

Corollary 12. *For a qhs-submanifold M of a cosymplectic manifold, we have*

$$g([E,F],\zeta) = 0, \qquad g(\bar{\nabla}_E F,\zeta) = 0.$$

4. Integrability of Distributions

The integrability criteria of the distributions involved in the formulation of *qhs*-Θ are examined in this section.

Theorem 7. *Let M be a proper qhs-Θ of type (α, β). Then, the distribution $\mathcal{D} \oplus < \zeta >$ is integrable if and only if*

$$g(\nabla_E \mathcal{T}F - \nabla_F \mathcal{T}E, \mathcal{T}\mathcal{Q}G) = g(\sigma(F, \mathcal{T}E) - \sigma(E, \mathcal{T}F), \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G)$$

for all $E, F \in \Gamma(\mathcal{D} \oplus < \zeta >)$ and $G \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$.

Proof. For all $E, F \in \Gamma(D \oplus < \zeta >)$ and $G \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$, adopting (3), we have

$$g([E, F], G) = g(\varphi \bar{\nabla}_E F, \psi G) - g(\varphi \bar{\nabla}_F E, \varphi G). \tag{41}$$

Using (4) and the concept of covariant differentiation, we have

$$g(\varphi(\bar{\nabla}_E F), \varphi G) = g(\bar{\nabla}_E \varphi F, \varphi G).$$

Therefore, from (41), we obtain

$$g([E, F], G) = g(\bar{\nabla}_E \varphi F, \varphi G) - g(\bar{\nabla}_F \varphi E, \varphi G). \tag{42}$$

Using (7) and (12) in (42), we obtain

$$g([E, F], G) = g(\nabla_E \mathcal{T}F + \sigma(E, \mathcal{T}F), \varphi G) - g(\nabla_F \mathcal{T}E + \sigma(F, \mathcal{T}E), \psi G). \tag{43}$$

Setting $G = \mathcal{Q}G + \mathcal{R}G$ and using Equation (16) in (43), we have

$$\begin{aligned} g([E, F], G) &= g(\nabla_E \mathcal{T}F + \sigma(E, \mathcal{T}F), \mathcal{T}\mathcal{Q}G + \mathcal{T}\mathcal{R}G + \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G) \\ &\quad - g(\nabla_F \mathcal{T}E + \sigma(F, \mathcal{T}E), \mathcal{T}\mathcal{Q}G + \mathcal{T}\mathcal{R}G + \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G). \end{aligned}$$

Since $\varphi\mathcal{D}^\perp \in \mathcal{T}^\perp M$, which implies that $\mathcal{T}\mathcal{R}G = 0$, as a result of the above expressions, we obtain

$$g([E, F], G) = g(\nabla_E \mathcal{T}F - \nabla_F \mathcal{T}E, \mathcal{T}\mathcal{Q}G) + g(\sigma(E, \mathcal{T}F) - \sigma(F, \mathcal{T}E), \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G).$$

Since the distribution $\mathcal{D} \oplus < \zeta >$ is integrable, we obtain the required result. $\square$

Theorem 8. *Let M be a proper qhs-Θ of type (α, β). Then, the slant distribution $\mathcal{D}^\theta \oplus < \zeta >$ is integrable if and only if*

$$\begin{aligned} g(\mathcal{A}_{\mathcal{N}\mathcal{T}F} G - \mathcal{A}_{\mathcal{N}\mathcal{T}G} F, H) &= g(\mathcal{A}_{\mathcal{N}F} G - \mathcal{A}_{\mathcal{N}G} F, \mathcal{T}\mathcal{P}H) \\ &\quad + g(\nabla^\perp_F \mathcal{N}G - \nabla^\perp_G \mathcal{N}F, \mathcal{N}\mathcal{R}H) \end{aligned}$$

for all $Y, Z \in \Gamma(\mathcal{D}^\theta \oplus < \zeta >)$ and $H \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$.

Proof. For all $F, G \in \Gamma(\mathcal{D}^\theta \oplus < \zeta >)$, using $H = \mathcal{P}H + \mathcal{R}H \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and employing (3) and (4), we obtain

$$g([F, G], H) = g(\bar{\nabla}_F \psi G, \psi H) - g(\bar{\nabla}_G \varphi F, \varphi H). \tag{44}$$

Adopting (8) and (12) in (44), we obtain

$$\begin{aligned} g([F, G], H) &= g(\mathcal{A}_{\mathcal{N}F} G - \mathcal{A}_{\mathcal{N}G} F, \varphi H) + g(\nabla^\perp_F \mathcal{N}G - \nabla^\perp_G \mathcal{N}F, \varphi H) \\ &\quad - g(\bar{\nabla}_F \mathcal{T}^2 G - \bar{\nabla}_G \mathcal{T}^2 F, H) + g(\mathcal{A}_{\mathcal{N}\mathcal{T}G} F - \mathcal{A}_{\mathcal{N}\mathcal{T}F} G, H) \end{aligned} \tag{45}$$

Now, using (16) and Lemma 1 in (45), we obtain

$$sin^2\theta g([F,G],H) = g(\mathcal{A}_{\mathcal{N}F}G - \mathcal{A}_{\mathcal{N}G}F, \mathcal{T}\mathcal{P}H) + g(\nabla^{\perp}_{F}\mathcal{N}G - \nabla^{\perp}_{G}NY, \mathcal{N}\mathcal{R}H)$$
$$-g(\mathcal{A}_{\mathcal{N}\mathcal{T}F}G - \mathcal{A}_{\mathcal{N}\mathcal{T}G}F, H).$$

Since the distribution $\mathcal{D}^{\theta} \oplus < \zeta >$ is integrable, we obtain the required result. □

Theorem 9. *If a proper qhs-*Θ *of type* (α,β) *satisfies the conditions that, for all* $F,G \in \Gamma, (\mathcal{D}^{\theta} \oplus < \zeta >)$

$$\nabla^{\perp}_{F}\mathcal{N}G - \nabla^{\perp}_{G}\mathcal{N}F \in \mathcal{N}\mathcal{D}^{\theta} \oplus \mu,$$
$$\mathcal{A}_{\mathcal{N}\mathcal{T}F}G - \mathcal{A}_{\mathcal{N}\mathcal{T}G}F \in \mathcal{D}^{\theta},$$
$$\mathcal{A}_{\mathcal{N}F}G - \mathcal{A}_{\mathcal{N}G}F \in \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta},$$

then the slant distribution $\mathcal{D}^{\theta} \oplus < \zeta >$ *is integrable.*

Theorem 10. *Let M be a proper qhs-*Θ *of type* (α,β)*. Then, the anti-invariant* $\mathcal{D}^{\perp}$ *is integrable if and only if*

$$\nabla^{\perp}_{G}\varphi H - \nabla^{\perp}_{H}\varphi G \in \mathcal{N}\mathcal{D}^{\theta} \oplus \mu \tag{46}$$

for all $G,H \in \Gamma(\mathcal{D}^{\perp})$ *and* $F = \mathcal{P}F + \mathcal{Q}F \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta})$*.*

Proof. For all $G,H \in \Gamma(\mathcal{D}^{\perp})$ and $F = \mathcal{P}F + \mathcal{Q}F \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta})$, adopting (3) and (4), we have

$$g([G,H],F) = g(\bar{\nabla}_{G}\varphi H, \varphi F) - g(\bar{\nabla}_{H}\varphi G, \varphi F). \tag{47}$$

Using (8) and (16) in (47), we have

$$g([G,H],F) = g(\mathcal{A}_{\varphi G}H - \mathcal{A}_{\varphi iH}G, \mathcal{T}\mathcal{P}F + \mathcal{T}\mathcal{Q}F) + g(\nabla^{\perp}_{G}\varphi H - \nabla^{\perp}_{H}\varphi G, \mathcal{N}\mathcal{Q}F). \tag{48}$$

By Lemma 3 and (48), we find

$$g([G,H],F) = g(\nabla^{\perp}_{G}\varphi H - \nabla^{\perp}_{H}\varphi G, \mathcal{N}\mathcal{Q}F).$$

Since the anti-invariant distribution $\mathcal{D}^{\perp}$ is integrable, we obtain the desired result. □

5. Totally Geodesic Foliations

Geodesicness and foliations are important geometric qualities that are associated with submanifolds. We examine the geometry of the foliations of *qhs*-Θ of type (α,β) in this section, as well as some requirements for total geodesicness.

Theorem 11. *Let M be a proper qhs-*Θ *of type* (α,β)*. Then, M is totally geodesic if and only if*

$$g(\sigma(E,\mathcal{P}F) + cos^2\theta\sigma(E,\mathcal{Q}F),U) = g(\nabla^{\perp}_{E}\mathcal{N}\mathcal{T}\mathcal{Q}F,U) - g(\nabla^{\perp}_{E}\mathcal{N}F, nU)$$
$$+g(\mathcal{A}_{\mathcal{N}\mathcal{Q}F}E + \mathcal{A}_{\mathcal{N}\mathcal{R}F}E, tU)$$

for all $E,F \in \Gamma(\mathcal{T}M)$ *and* $U \in \Gamma(\mathcal{T}^{\perp}M)$*.*

Proof. For all $E,F \in \Gamma(\mathcal{T}M), U \in \Gamma(\mathcal{T}^{\perp}M)$, using (15), we obtain

$$g(\bar{\nabla}_{E}F,U) = g(\bar{\nabla}_{E}\mathcal{P}F,U) + g(\bar{\nabla}_{E}\mathcal{Q}F,U) + g(\bar{\nabla}_{E}\mathcal{R}F,U). \tag{49}$$

In view of (3), we obtain

$$g(\varphi\bar{\nabla}_{E}F, \varphi U) = g(\bar{\nabla}_{E}F,U). \tag{50}$$

Using (4), we obtain

$$g(\varphi\bar{\nabla}_E F, \varphi U) = g(\bar{\nabla}_E \varphi F, \varphi U). \tag{51}$$

Employing (2), (7), (12), (50), and (51) in (49), we obtain

$$\begin{aligned} g(\bar{\nabla}_E F, U) = g(\sigma(E, \mathcal{P}F), U) - g(\bar{\nabla}_E \mathcal{T}^2 \mathcal{Q}F, U) - g(\bar{\nabla}_E \mathcal{N}\mathcal{T}\mathcal{Q}F, U) \\ + g(\bar{\nabla}_E \mathcal{N}\mathcal{Q}F, \varphi U) + g(\bar{\nabla}_E \mathcal{N}\mathcal{R}F, \psi U). \end{aligned}$$

Adopting (7), (12), and Lemma 1 in the above equation, we have

$$g(\bar{\nabla}_E F, U) = g(\sigma(E, \mathcal{P}F) + cos^2\theta\sigma(E, \mathcal{Q}F), U) - g(\bar{\nabla}^{\perp}_E \mathcal{N}\mathcal{T}\mathcal{Q}F, U) \tag{52}$$

$$-g(\mathcal{A}_{\mathcal{N}\mathcal{Q}F}E + \mathcal{A}_{\mathcal{N}\mathcal{R}F}E, tU + nU) + g(\nabla^{\perp}_E \mathcal{N}\mathcal{Q}F + \nabla^{\perp}_E \mathcal{N}\mathcal{R}F, tU + nU).$$

Since $\mathcal{N}F = \mathcal{N}\mathcal{P}F + \mathcal{N}\mathcal{Q}F + \mathcal{N}\mathcal{R}F$ and $\mathcal{N}\mathcal{P}F = 0$, we obtain

$$g(\bar{\nabla}_E F, U) = g(\sigma(E, \mathcal{P}F) + cos^2\theta\sigma(F, \mathcal{Q}F), U) - g(\bar{\nabla}^{\perp}_E \mathcal{N}\mathcal{T}\mathcal{Q}F, U)$$

$$-g(\mathcal{A}_{\mathcal{N}\mathcal{Q}F}E + \mathcal{A}_{\mathcal{N}\mathcal{R}F}E, tU) + g(\nabla^{\perp}_E \mathcal{N}F, nU).$$

Since M is totally geodesic, we obtain the desired result. □

Theorem 12. *Let M be a proper qhs-Θ of type (α, β). Then, anti-invariant distribution $\mathcal{D}^{\perp}$ is a totally geodesic foliation on M if and only if*

$$g(\mathcal{A}_{\psi G}F, \mathcal{T}\mathcal{P}H + \mathcal{T}\mathcal{Q}H) = g(\nabla^{\perp}_F \varphi G, \mathcal{N}\mathcal{Q}H),$$

$$g(\mathcal{A}_{\varphi G}F, tU) = g(\nabla^{\perp}_F \varphi G, nU)$$

for all $F, G \in \Gamma(\mathcal{D}^{\perp}), H \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta})$ and $U \in \Gamma(\mathcal{T}^{\perp}M)$.

Proof. For any $F, G \in \Gamma(\mathcal{D}^{\perp})$, $H = \mathcal{P}H + \mathcal{Q}H \in \Gamma(\mathcal{D} \oplus \mathcal{D}^{\theta})$. Now using (3) and (4), we have

$$g(\bar{\nabla}_F G, H) = g(\bar{\nabla}_F \varphi G, \varphi H).$$

Adopting (8) and (16), we obtain

$$g(\bar{\nabla}_F G, H) = -g(\mathcal{A}_{\varphi G}F, \mathcal{T}\mathcal{P}H + \mathcal{T}\mathcal{Q}H) + g(\nabla^{\perp}_F \varphi G, \mathcal{N}\mathcal{Q}\mathcal{H}). \tag{53}$$

Now, for any $F, G \in \Gamma(\mathcal{D}^{\perp})$, $U \in \Gamma(\mathcal{T}^{\perp}M)$, employing (3) and (4), we have

$$g(\bar{\nabla}_F G, U) = g(\bar{\nabla}_F \varphi G, \varphi U).$$

Using (8) and (13), we obtain

$$\begin{aligned} g(\bar{\nabla}_F G, U) &= -g(\mathcal{A}_{\varphi G}F + \nabla^{\perp}_F \varphi G, tU + nU) \\ &= -g(\mathcal{A}_{\varphi G}F, tU) + g(\nabla^{\perp}_F \mathcal{N}G, nU). \end{aligned} \tag{54}$$

Since anti-distribution $(\mathcal{D}^{\perp})$ defines a totally geodesic foliation on M, (53) and (54) give the desired result. □

Theorem 13. *Let M be a proper qhs-Θ of type (α, β). Then, the distribution $\mathcal{D}^{\theta} \oplus <\zeta>$ defines a totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_G \mathcal{N}H, \mathcal{N}\mathcal{R}E) = g(\mathcal{A}_{\mathcal{N}H}G, \mathcal{T}\mathcal{P}E) - g(\mathcal{A}_{\mathcal{N}\mathcal{T}H}G, E),$$

$$g(\mathcal{A}_{\mathcal{N}H}G, tU) = g(\nabla^{\perp}_G \mathcal{N}H, nU) - g(\nabla^{\perp}_G \mathcal{N}\mathcal{T}H, U)$$

for all $G, H \in \Gamma(\mathcal{D}^\theta \oplus <\zeta>), E \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ *and* $U \in \Gamma(\mathcal{T}^\perp M)$.

Proof. Let $E = \mathcal{P}E + \mathcal{R}E \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $G, H \in \Gamma(\mathcal{D}^\theta \oplus <\zeta>)$. Then, using (3), (4), (8), (12), (13), and Lemma 1, we have

$$sin^2\theta g(\bar{\nabla}_G H, E) = g(\mathcal{A}_{\mathcal{N}\mathcal{T}H} G, E) - g(\mathcal{A}_{\mathcal{N}H} G, \varphi E) + g(\nabla^\perp_G \mathcal{N}H, \varphi E). \tag{55}$$

Now, employing (16) with the fact that $\mathcal{N}\mathcal{P}E = 0$ in (55), one obtains

$$sin^2\theta g(\bar{\nabla}_G H, E) = g(\mathcal{A}_{\mathcal{N}\mathcal{T}H} G, E) - g(\mathcal{A}_{\mathcal{N}H} G, \mathcal{T}\mathcal{P}E) + g(\nabla^\perp_G \mathcal{N}H, \mathcal{N}\mathcal{R}E). \tag{56}$$

Now, for any $G, H \in \Gamma(\mathcal{D}^\theta \oplus <\zeta>), U \in \Gamma(\mathcal{T}^\perp M)$, adopting (3), (4), (8), and (12), we have

$$g(\bar{\nabla}_G H, U) = -g(\bar{\nabla}_G \mathcal{T}^2 H, U) - g(\bar{\nabla}_G \mathcal{N}\mathcal{T}H, U) + g(-\mathcal{A}_{\mathcal{N}H} G + \nabla^\perp_G \mathcal{N}H, \varphi U).$$

Now, in light of (13) and Lemma 1, and from the above equation, we obtain

$$sin^2\theta g(\bar{\nabla}_G H, U) = -g(\nabla^\perp_G \mathcal{N}\mathcal{T}H, U) - g(\mathcal{A}_{\mathcal{N}H} G, tU) + g(\nabla^\perp_G \mathcal{N}H, nU). \tag{57}$$

Since the distribution $\mathcal{D}^\theta \oplus <\zeta>$ defines a totally geodesic foliation on M, from (56) and (57), we obtain the desired result. □

Theorem 14. *Let M be a proper qhs-Θ of type* (α, β)*. Then, the distribution* $\mathcal{D} \oplus <\zeta>$ *defines a totally geodesic foliation on M if and only if*

$$g(\nabla_E \mathcal{T}F, \mathcal{T}\mathcal{Q}G) = -g(\sigma(E, \mathcal{T}F), \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G),$$

$$g(\nabla_E \mathcal{T}F, tU) = -g(\sigma(E, \mathcal{T}F), nU)$$

for all $E, F \in \Gamma(\mathcal{D} \oplus <\zeta>), G = \mathcal{Q}G + \mathcal{R}G \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$ *and* $U \in \Gamma(\mathcal{T}^\perp M)$.

Proof. Let $G = \mathcal{Q}G + \mathcal{R}G \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$ and $E, F \in \Gamma(\mathcal{D} \oplus <\zeta>)$. Using (3) and (4), we gain

$$g(\bar{\nabla}_E F, G) = g(\bar{\nabla}_E \varphi F, \psi G).$$

Now, employing (7), (12), (16), and $\mathcal{N}F = 0$ in the above equation, we find

$$g(\bar{\nabla}_E F, G) = g(\nabla_E \mathcal{T}F, \mathcal{T}\mathcal{Q}G) + g(\sigma(E, \mathcal{T}F), \mathcal{N}\mathcal{Q}G + \mathcal{N}\mathcal{R}G). \tag{58}$$

For any $E, F \in \Gamma(\mathcal{D} \oplus <\zeta>)$ and $U \in \Gamma(\mathcal{T}^\perp M)$, adopting (3), (4), (16), and $\mathcal{N}F = 0$, we obtain

$$g(\bar{\nabla}_E F, U) = g(\bar{\nabla}_E \mathcal{T}F, \varphi U).$$

In view of (7) and (13), we have

$$g(\bar{\nabla}_E F, U) = g(\nabla_E \mathcal{T}F, tU) + g(\sigma(E, \mathcal{T}F), nU). \tag{59}$$

Since distribution $(\mathcal{D} \oplus <\zeta>)$ describes a totally geodesic foliation on M, from (58) and (59), we obtain the desired result. □

6. Related Example

Example 1. *Let* $\bar{M} = \{(x_1, \cdots, x_{10}, z) \in \mathbb{R}^{11} : z \neq 0\}$*, where* $(x_1, x_2, x_3, \cdots, x_9, x_{10}, z)$ *is the standard coordinate in* $\mathbb{R}^{11}$*. We choose the vector fields*

$E_i = e^{-z}\frac{\partial}{\partial x_i}$, where $i = 1, 2, \cdots, 10$, $E_{11} = e^{-z}\frac{\partial}{\partial z}$.
The metric g is defined as

$$g = e^{2z}\mathbb{G},$$

where $\mathbb{G}$ is the Euclidean metric on $\mathbb{R}^{11}$. Then, $\{E_i\}_{i=1,2,\cdots,11}$ is an orthonormal frame basis of $\bar{M}$.

Define a 1-form η by

$$\eta = e^z dz, \qquad \eta(U) = g(U, E_{11}), \qquad \forall \quad U \in \mathcal{T}\bar{M}.$$

Next, we define a tensor field φ of type $(1,1)$ by

$$\varphi\left\{\sum_{i=1}^{5}\left(x_i\frac{\partial}{\partial x_i} + x_{i+5}\frac{\partial}{\partial x_{i+5}} + z\frac{\partial}{\partial z}\right)\right\} = \sum_{i=1}^{5}\left(x_i\frac{\partial}{\partial x_{i+5}} - x_{i+5}\frac{\partial}{\partial x_i}\right).$$

Then, we have

$$\varphi(E_i) = E_{i+5}, \qquad \varphi(E_{i+5}) = -E_i, \quad \varphi(E_{11}) = 0, 1 \leq i \leq 5.$$

The linearity of d and ψ yields that

$$\eta(E_{11}) = 1, \;\; \varphi^2(U) = -U + \eta(U)E_{11},$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V)$$

for any vector fields U, V on $\bar{M}$. As a result, $\bar{M}$ $(\varphi, \zeta, \eta, g)$ defines an almost contact metric manifold with $\zeta = E_{11}$. In addition, let $\bar{\nabla}$ be the Levi–Civita connection with respect to metric g. Using basic calculations, the following expressions are obtained:

$$[E_i, \zeta] = e^{-z}E_i, \quad [E_i, E_j] = 0, \quad 1 \leq i \neq j \leq 10.$$

The Riemannian connection $\bar{\nabla}$ of the metric d is given by

$$\begin{aligned} 2g(\bar{\nabla}_U V, W) = {} & Ug(V, W) + Vg(W, U) - Wg(U, V) \\ & - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]). \end{aligned}$$

By Koszul's formula, we obtain the following relations:

$$\bar{\nabla}_{E_i}E_i = -e^{-z}\zeta, \bar{\nabla}_{\zeta}\zeta = 0, \; \bar{\nabla}_{\zeta}E_i = 0, \; \bar{\nabla}_{E_i}\zeta = e^{-z}E_i, \; 1 \leq i \leq 10.$$

Hence, considering Equations (4)–(6), we observe that M is a trans-Sasakian manifold of type $(0, e^{-z})$, where $\alpha = 0$ and $\beta = e^{-z}$.

Next, we define a submanifold M of $\bar{M}$ by the immersion f as follows:

$$\begin{aligned} f(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = e^{-z}(u_1, u_3, 0, \frac{1}{\sqrt{2}}u_5, \frac{1}{\sqrt{2}}u_6, u_2, u_4\cos\theta, \\ u_4\sin\theta, \frac{1}{\sqrt{2}}u_5, \frac{1}{\sqrt{2}}u_6, u_7), \end{aligned}$$

where $0 < \theta < \frac{\pi}{2}$.

Now, it is easy to observe that tangent bundle $\mathcal{T}M = Span\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$, where

$$X_1 = e^{-z}\frac{\partial}{\partial x_1}, X_2 = e^{-z}\frac{\partial}{\partial x_6}, X_3 = e^{-z}\frac{\partial}{\partial x_2}, X_4 = e^{-z}\{\cos\theta\frac{\partial}{\partial x_7} + \sin\theta\frac{\partial}{\partial x_8}\},$$

$$X_5 = e^{-z}\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_9}\}, X_6 = e^{-z}\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_{10}}\}, X_7 = e^{-z}\frac{\partial}{\partial z}.$$

Using the almost contact structure φ, we obtain

$$\varphi X_1 = e^{-z}\frac{\partial}{\partial x_6}, \varphi X_2 = -e^{-z}\frac{\partial}{\partial x_1}, \varphi X_3 = e^{-z}\frac{\partial}{\partial x_7},$$

$$\varphi X_4 = -e^{-z}\{\cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial x_3}\}, \varphi X_5 = e^{-z}\frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_9}\},$$

$$\varphi X_6 = e^{-z}\frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_{10}}\}, \qquad \varphi X_7 = 0.$$

If have the following distributions

$$\mathcal{D} = Span\{X_1, X_2\}, \mathcal{D}^\theta = Span\{X_3, X_4\}, \mathcal{D}^\perp = Span\{X_5, X_6\}$$

then the distributions $\mathcal{D}$, $\mathcal{D}^\theta$, and $\mathcal{D}^\perp$ will be invariant, slant with slant angle θ, and anti-invariant distributions, respectively. Taking into account the above and Definition 7, we state that M is a qhs-submanifold of $\bar{M}$.

7. Some Applications of Pontryagin Numbers in Number Theory to Submanifolds

According to the Hirzebruch signature theorem [38], the signature of a smooth manifold can be expressed by the linear combination of *Pontryagin numbers*. These numbers are certain characteristic classes or Pontryagin classes of real vector bundles. The Pontryagin classes lie in cohomology groups with degrees of multiples of four.

Moreover, for a real vector bundle $\mathcal{B}$ over a manifold M, its i-th Pontryagin class $p_i(\mathcal{B})$ is defined as

$$p_i(\mathcal{B}) = p_i(\mathcal{B}, \mathbb{Z}) \in H^{4i}(M, \mathbb{Z}), \tag{60}$$

where $H^{4i}(M, \mathbb{Z})$ is a $4i$-cohomology group of manifold M with integer coefficients. Similarly, the total Pontryagin class

$$p(\mathcal{B}) = 1 + p_1(\mathcal{B}) + p_2(\mathcal{B}) + \cdots \in H^*(M, \mathbb{Z}),$$

for two vector bundles $\mathcal{B}_1$ and $\mathcal{B}_2$ over M. In terms of the individual Pontryagin classes p_i,

$$2p_1(\mathcal{B}_1 \oplus \mathcal{B}_2) = 2p_1(\mathcal{B}_1) + 2p_1(\mathcal{B}_2). \tag{61}$$

It should be noted that the Pontryagin classes of a smooth manifold are defined to be the Pontryagin classes of its tangent bundle.

Now, in light of (60), (61), and Definition 7, we have

$$p(\mathcal{T}M) = 1 + p_1(\mathcal{D}) + p_2(\mathcal{D}^\theta) + p_3(\mathcal{D}^\perp) + p_4(<\zeta>) \in H^*(M, \mathbb{R}), \tag{62}$$

where p_1, p_2, p_3, p_4 are the Pontryagin numbers.

$$2p_1(\mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus <\zeta>) = 2p_1(\mathcal{D}) + 2p_1(\mathcal{D}^\theta) + 2p_1(\mathcal{D}^\perp) + 2p_1(<\zeta>). \tag{63}$$

Thus, we articulate the following.

Theorem 15. *Let M be a proper qhs-Θ of type (α, β). Then, the Pontryagin classes of tangent bundle TM are given by (63).*

Corollary 13. *Let M be a proper qhs-Θ of type (α, β) and the Pontryagin classes of tangent bundle TM be given by (63); then, $H^{4i}(M, \mathbb{R})$ is a cohomology group of trans-Sasakian manifolds $\bar{M}$ of type (α, β).*

8. Conclusions

In this paper, we have quantified the submanifolds (qhs-submanifolds) of trans-Sasakian manifolds or (α, β)-type almost contact metric manifolds with with quasi-hemi-slant factors. Essentially, we present some sufficient and necessary criteria for the integrability of distributions using the notion of quasi-hemi-slant submanifolds in trans-Sasakian manifolds. We have also analyzed the distribution and specifications of quasi-hemi-slant submanifolds of trans-Sasakian manifolds, which determine the geometry of foliations as a totally geodesic geometry. Finally, we illustrate an example of a quasi-hemi-slant submanifold of a trans-Sasakian manifold and describe a link between number theory in terms of Pontryagin classes and cohomology groups.

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