symmetry

Article

Bipolar b-Metric Spaces in Graph Setting and Related Fixed Points

Haroon Ahmad [1], **Mudasir Younis** [2,*] and **Afrah Ahmad Noman Abdou** [3,*]

1 Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan; haroon_ahmad_22@sms.edu.pk or haroonrao3@gmail.com
2 Department of Mathematics and Statistics, Indian Institute of Technology Kanpur-(IIT Kanpur), Kanpur 208016, India
3 Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia
* Correspondence: myounis@iitk.ac.in or mudasiryouniscuk@gmail.com (M.Y.); aabdou@uj.edu.sa (A.A.N.A.); Tel.: +91-7006354561 (M.Y.)

Abstract: In this article, we develop a new notion that combines fixed-point theory and graph theory: graphical bipolar b-metric spaces. We demonstrate fixed-point solutions in the framework of graphical bipolar b-metric spaces, employing covariant and contravariant mapping contractions, which is a new addition to this end. This article also features illustrative examples drawn from various contexts to further demonstrate our findings. This is a significant study since it melds ideas from graph theory with those from generalized bipolar metric spaces, and considers that the symmetry of the edges of the underlying graphs connected with the enunciated metric spaces is essential in the graphical metric spaces.

Keywords: directed graph; edge symmetry; fixed point; graphic contraction; graphical bipolar b-metric space

Citation: Ahmad, H.; Younis, M.; Abdou, A.A.N. Bipolar b-Metric Spaces in Graph Setting and Related Fixed Points. *Symmetry* **2023**, *15*, 1227. https://doi.org/10.3390/sym15061227

Academic Editor: Alice Miller

Received: 22 May 2023
Revised: 3 June 2023
Accepted: 7 June 2023
Published: 8 June 2023

1. Introduction

Fixed-point theory (FPT), which serves as a bridge between topology and analysis, is a fundamental and commonly used tool in modern mathematics. It investigates the circumstances in which a self-map on a non-empty set admits one or more fixed points. FPT is divided into three categories: metric FPT, topological FPT, and discrete FPT. This field's researchers operate in a variety of directions and generalize their findings. Poincare and Brouwer made substantial contributions to this field, particularly Brouwer's (topological) fixed-point theorem. The metric fixed-point theorem rose to popularity due to its applications in both pure and applied mathematics, particularly in determining the presence of solutions to nonlinear systems. Banach's contraction principle, established in 1922, is a noteworthy achievement in FPT and approximation theory, and it is well respected for its adaptability.

In FPT, the concept of metric space (MS) and the Banach contraction principle (BCP) are fundamental. Many scholars have been drawn to metric spaces (MSs) because of their axiomatic clarity. Several writers have generalized the BCP by using various forms of contraction mappings in different MSs (for example, [1,2]). Mutlu and Gürdal [3] introduced the concept of bipolar MS in 2016, which is a type of partial distance that connects MSs and bipolar MSs. After that, Mutlu and Ozkan generalized coupled fixed-point theorems in bipolar MSs in 2017 [4]. Furthermore, Gürdal et al. extended fixed-point results to multivalued mappings in these spaces in 2020 [5]. Researchers interested in an in-depth analysis may refer to [6,7].

Many researchers are concentrating on the field of graphical FPT, as evidenced by pioneering articles such as [8–11]. Graph theory has become increasingly significant in the study of FPT. For solving DEs with infinite delay, Hammad and Zayed [12] used fixed-point analysis on b-MS in conjunction with graph symmetry, while Shukla et al. defined graphical MSs [13], and Chuensupantharat et al. [8] defined graphical b-MSs with accompanying graphs. In 2019, Younis et al. expanded on previous work by extending these notions to graphical rectangular b-MSs and controlled-graphical MSs [11,14]. Hooda et al. [15] provided a good summary of the literature on b-MSs and their generalized forms. The authors emphasized specific ordered vector spaces derived from the b-MSs and discussed their relevance in economic research. Then, using a graph endowed with edge symmetry, they built shared fixed-point outcomes on the vector b-MSs. Popescu [16], on the other hand, used the graphic contraction principle to deal with a new type of operator and produced fixed-point theorems that complement prior discoveries in nonlinear operator theory. A connection is shown between iterative techniques and Ulam–Hyers stability issues in particular. The illuminating article given in [17] provides a comprehensive explanation of graphical structures.

Considering the aforementioned discussion, in this article, we establish the notion of graphical structure in bipolar b-MSs, which is a generalization of bipolar MSs, prompted by the existing literature and following the work of Younis et al. and Mutlu and Gürdal [3,11]. We demonstrate the correctness of fixed-point theorems by using covariant and contravariant mapping contractions, respectively, and give examples with different cases.

2. Preliminaries

In this part of the article, we will discuss some primary definitions and certain primary facts necessary for the continued examination of the subject.

Consider $\Theta = (V(\Theta), E(\Theta))$, a digraph having no parallel edges carrying Δ as a diagonal of $Ł \times Ł$, $\phi \neq Ł$. It is assumed that the vertex set (node-set) $V(\Theta)$ coincides with the set Ł, whereas $E(\Theta)$ is equipped with all the loops of the graph.

We refer to the graph with symmetric edges as $\breve{\Theta}$ (graph symmetry). With this concept, we have

$$E(\Theta^{-1}) \cup E(\Theta) = E(\breve{\Theta}),$$

where Θ^{-1} is a symmetric graph obtained by inverting the edges of θ.

Let us say that the nodes of the digraph Θ are κ and w. A sequence $\{\kappa_j\}_{j=0}^{m}$ with $(1+m)$ nodes is a path in Θ if the following contends:

$$\kappa_0 = \kappa, \kappa_m = w \text{ such that } (\kappa_{j-1}, \kappa_j) \in E(\Theta); j = 1, 2, \ldots, m.$$

If a path connects a node of Θ to every other node, we say Θ is connected. In the case of a graph, Θ is undirected and possesses a path between its very pair of nodes; we call Θ to be a weakly connected graph. A graph $H = (V(H), E(H))$ is considered to be a sub-graph of Θ if $V(H) \subseteq V(\Theta)$ and $E(H) \subseteq E(\Theta)$.

On the other side, a novel development in graph-metric theory was introduced by Shukla [13], where the authors used the following notions:

- $[\kappa]^l_\Theta$ consists of the set of all nodes $w \in Ł$ making a directed path from κ in Θ with length l.
- The symbol P represents a relation on Ł with $(\kappa P w)_\Theta$ representing that there is a path from κ to w in the graph Θ.
- If $w \in (\kappa P w)_\Theta$, then w is somewhere on the path $(\kappa P w)_\Theta$.

Additionally, we say a sequence $\{\kappa_m\} \subset Ł$ is $\Theta-$ term-wise-connected ($\Theta - TWC$) if $(\kappa_m P \kappa_{m+1})_\Theta$ for all $m \in \mathbb{N}$. Unless otherwise specified, all graphs will thereafter be interpreted as being directed.

Bipolar MSs are a new kind of partial distance introduced by Mutlu and Gürdal [3]. In particular, they focus on the completeness of MSs and their connection to bipolar MSs. The following is the formal definition.

Definition 1 ([3]). *A bipolar* MS *is triple* $(Ł, Ł^*, d)$ *such that* $Ł, Ł^* \neq \phi$ *and* $d : Ł \times Ł^* \to \mathbb{R}^+$ *is a function satisfying the underlying properties:*

1. *If* $d(\partial, \eth) = 0$, *then* $\partial = \eth$, $(\partial, \eth) \in Ł \times Ł^*$.
2. *If* $\partial = \eth$, *then* $d(\partial, \eth) = 0$, $(\partial, \eth) \in Ł \times Ł^*$.
3. $d(\partial, \eth) = d(\eth, \partial)$, *for all* $\partial, \eth \in Ł \cap Ł^*$.
4. $d(\partial_1, \eth_2) \leq d(\partial_1, \eth_1) + d(\partial_2, \eth_1) + d(\partial_2, \eth_2)$,
for all $\partial_1, \partial_2 \in Ł$, $\eth_1, \eth_2 \in Ł^*$ *and* $\mathbb{R}^+$ *is set of all non-negative real numbers. Then, d is called a bipolar* MS *on the pair* $(Ł, Ł^*)$.

The following characteristics are listed following the aforementioned definition:

(i) If conditions (2) and (3) are satisfied, d is referred to as a bipolar pseudo-semi metric on the pair $(Ł, Ł^*)$.
(ii) A bipolar pseudo-metric (BPSM) is one where d is a bipolar pseudo-semi metric meeting (4).
(iii) A bipolar metric is a BPSM d satisfying (2).
A triple $(Ł, Ł^*, d)$, where d is a bipolar (pseudo-(semi)) metric on $(Ł, Ł^*)$, is referred to as a bipolar (pseudo-(semi)) MS. Specifically, the space is referred to as a disjoint if $Ł \cap Ł^* = \phi$; otherwise, it is referred to as a joint. The left and right poles of $(Ł, Ł^*, d)$ are the sets L and $L*$, respectively.

A BPSM typically only considers the distance between points that are in different poles of space. It also reveals some details about these poles' internal structure.

Example 1 ([3]). *The following are some examples related to the defined terms:*

(i) *If* $(Ł, Ł^*)$ *is a (pseudo-(semi))* MS *with distance function d, then* $(Ł, Ł^*, d)$ *is a bipolar (pseudo-(semi))* MS. *Conversely, if* $(Ł, Ł^*, d)$ *is a bipolar (pseudo-(semi))* MS *with* $Ł = Ł^*$, *then* $(Ł, Ł^*)$ *is also a (pseudo-(semi))* MS.
(ii) *In a quasi-*MS $(Ł, Ł^*)$, *if we define the set* $Ł^* = Ł \times Ł$, *then* $(Ł, Ł^*, d')$ *is a disjoint bipolar* MS, *where* $d'(\partial_1, (\partial_2, Ł)) = d(\partial_1, \partial_2)$ *for every* $\partial_1, \partial_2 \in Ł$. *The axiom of regularity requires that the sets* Ł *and* $Ł^*$ *be disjoint.*
(iii) *Let* Ł *be a set, and d be a pseudo-semimetric on* Ł *with the property that* $Ł \cap P(Ł) = \emptyset$. *The point-to-set distance function* $d' : Ł \times P(Ł) \to \mathbb{R}^+$ *is a bipolar pseudo-semimetric on the pair* $(Ł, P(Ł))$.
(iv) *Consider the nonempty sets* Ł *and* $Ł^*$, *a pseudo-*MS (Z, ρ), *and a function* $f : Ł \cup Ł^* \to Z$. *The function* $d : Ł \times Ł^* \to \mathbb{R}^+$ *defined as* $d(\partial, \eth) = \rho(f(\partial), f(\eth))$ *represents a* BPSM *on* $(Ł, Ł^*)$. *If* Ł *and* $Ł^*$ *are normed spaces, then* $d(\partial, \eth) = |||\partial|| - ||\eth|||$ *denotes a* BPSM *on* $(Ł, Ł^*)$.
(v) *Let C represent the set of all functions,* $f : \mathbb{R} \to [1, 3]$, *and* $d : C \times \mathbb{R} \to \mathbb{R}^+$ *can be defined as* $d(f, \partial) = f(\partial)$. *The space* $(C, \mathbb{R}, d)$ *is a disjoint bipolar* MS.

Definition 2 ([4]). *Let* $(Ł_1, Ł_1^*)$ *and* $(Ł_2, Ł_2^*)$ *be two sets and consider a mapping* $T : Ł_1 \cup Ł_1^* \rightrightarrows Ł_2 \cup Ł_2^*$; *then, T is called a covariant* (CoVM) *if T fulfills the underlying property:*

$$T(Ł_1) \subseteq Ł_2 \text{ and } T(Ł_1^*) \subseteq Ł_2^*.$$

On the other hand, T is called a contravariant mapping (CnVM) *if the following condition is met:*

$$T(Ł_1) \subseteq Ł_2^* \text{ and } T(Ł_1^*) \subseteq Ł_2.$$

Such types of mappings are denoted by $T : (Ł_1, Ł_1^*) \rightleftarrows (Ł_2, Ł_2^*)$. *One should take care that* d_1 *and* d_2 *are bipolar* MSs *on* $(Ł_1, Ł_1^*)$ *and* $(Ł_2, Ł_2^*)$, *respectively.*

Definition 3 ([5]). *Suppose* $(Ł, Ł^*, d)$ *is a bipolar* MS. *A point* $\kappa \in Ł \cup Ł^*$ *is the left point if* $\kappa \in Ł$, *the right point if* $\kappa \in Ł^*$ *and a central point if* $\kappa \in Ł \cap Ł^*$. *A sequence* (∂_n) *on* Ł *is the left sequence and* $(\eth_n)$ *on* $Ł^*$ *is the right sequence. A left or right sequence is simply a sequence in* $(Ł, Ł^*, d)$. *A sequence* (κ_n) *is said to be convergent to a point* κ*; if* (κ_n) *is the left sequence and* κ

is the right point, then $\lim_{n\to\infty} d(\kappa_n, \kappa) = 0$*. Similarly, if* (κ_n) *is the right sequence and* κ *is the left point, then* $\lim_{n\to\infty} d(\kappa, \kappa_n) = 0$*. A sequence* $(\partial_n, \eth_n) \in Ł \times Ł^*$*, is said to be a bi-sequence* (BC) *on* $(Ł, Ł^*, d)$*. If both sequences* (∂_n) *and* $(\eth_n)$ *converge, then the BC is said to be bi-convergent, and if both sequences converge to the same point* $\kappa \in Ł \cap Ł^*$*, then BC is said to be bi-convergent. Every bi-convergent BC in* $(Ł, Ł^*, d)$ *is a Cauchy BC, and every convergent BC is bi-convergent.*

Definition 4 ([3]). *Let* $(Ł_1, Ł_1^*, d_1)$ *and* $(Ł_2, Ł_2^*, d_2)$ *be a bipolar* MS*. A mapping T is continuous if it is left continuous at each point* $\partial \in Ł_1$ *and right continuous at each point* $\eth \in Ł_1^*$*. A* CnVM $T : (Ł_1, Ł_1^*) \rightleftarrows (Ł_2, Ł_2^*)$ *is continuous if and only if it is continuous as a* CoVM $T : (Ł_1, Ł_1^*) \rightrightarrows (Ł_2, Ł_2^*)$*.*

It can be seen from Definition 3 that a covariant or a CnVM *is continuous if and only if* $(\partial_n) \to \eth$ *on* $(Ł_1, Ł_1^*, d_1)$*, implying that* $T(\partial_n) \to T(\eth)$ *on* $(Ł_2, Ł_2^*, d_2)$*.*

3. Graphical Bipolar *b*-MSs

In this section, we introduce the notion of graphical bipolar b-MS $(Ł, Ł^*, d_\Theta)$ as follows:

Definition 5. *Consider a metric* $d_\Theta : Ł \times Ł^* \to \mathbb{R}^+$*, endowed with the graph* Θ *with* $Ł, Ł^* \neq \phi$ *and* $b \geq 1$*, contending the following axioms:*

1. *If* $d_\Theta(\partial, \eth) = 0$*, then* $\partial = \eth$*.*
2. *If* $\partial = \eth$*, then* $d_\Theta(\partial, \eth) = 0$*.*
3. $d_\Theta(\partial, \eth) = d_\Theta(\eth, \partial)$*, for all* $\partial, \eth \in Ł \cap Ł^*$*.*
4. $(\partial_1 P \eth_2)_\Theta, \partial_2, \eth_1 \in (\partial_1 P \eth_2)_\Theta \Rightarrow d_\Theta(\partial_1, \eth_2) \leq b[d_\Theta(\partial_1, \eth_1) + d_\Theta(\partial_2, \eth_1) + d_\Theta(\partial_2, \eth_2)]$*, for all* $\partial_1, \partial_2 \in Ł$ *and* $\eth_1, \eth_2 \in Ł^*$ *, where* $(\partial, \eth) \in Ł \times Ł^*$*.*

   A triplet $(Ł, Ł^*, d_\Theta)$ *is called a graphical bipolar b-*MS *if* d_Θ *satisfies the postulates (1–4).*

Remark 1. *It is noteworthy that graphical bipolar b-*MS $(Ł, Ł^*, d_\Theta)$ *is the generalization of bipolar* MS $(Ł, Ł^*, d)$ *if we take* $b = 1$*.*

Example 2. *Let* $Ł = \{1, 3, 5, 7, 13\}$*,* $Ł^* = \{2, 4, 5, 7, 15\}$*, and the function* $d_\Theta : Ł \times Ł^* \to \mathbb{R}^+$ *is defined as*

$$d_\Theta(\partial, \eth) = \begin{cases} 0, & \text{if } \partial = \eth \\ \min\{\partial, \eth\}, & \text{otherwise,} \end{cases}$$

where $b = \frac{13}{4}$ *for all* $\partial \in Ł$*,* $\eth \in Ł^*$ *and equipped with graph* Θ*. The graph* $\Theta = (V(\Theta), E(\Theta))$*, having the set of vertices*

$$V(\Theta) = Ł \cup Ł^* = \{1, 2, 3, 4, 5, 7, 13, 15\},$$

and the set of edges given by the following:

$$E(\Theta) = \triangle \cup \left\{ \begin{array}{c} (1,2), (1,3), (1,4), (1,5), (1,7), (1,13), \\ (1,15), (3,2), (2,4), (5,2), (2,7), (13,2), \\ (2,15), (3,4), (3,5), (3,7), (3,13), (3,15), \\ (5,4), (4,7), (13,4), (4,15), (5,7), (5,13), \\ (5,15), (7,13), (7,15), (13,15). \end{array} \right\}$$

The corresponding edges of the graph Θ *are propounded in Figure 1.*

In order to verify that $(Ł, Ł^*, d_\Theta)$ *is graphical bipolar b-*MS*, we discuss some cases for validating the underlying inequality of Definition 5.*

$$d_\Theta(\partial_1, \eth_2) \leq b[d_\Theta(\partial_1, \eth_1) + d_\Theta(\partial_2, \eth_1) + d_\Theta(\partial_2, \eth_2)]. \tag{1}$$

Case (i): *If we take* $\partial_1 = 13, \partial_2 = 1, \eth_1 = 2$, *and* $\eth_2 = 15$, *we have*

$$\begin{aligned} d_\Theta(13,15) &\leq b\,[d_\Theta(13,2) + d_\Theta(1,2) + d_\Theta(1,15) \\ 13 &\leq b\,(4). \end{aligned}$$

Consequently, for $b = \frac{13}{4}$, (1) *holds.*
Case (ii): *If* $\partial_1 = 1, \partial_2 = 13, \eth_1 = 4$, *and* $\eth_2 = 2$, *then*

$$\begin{aligned} d_\Theta(1,2) &\leq b\,[(1,4) + d_\Theta(13,4) + d_\Theta(13,2)] \\ 7 &\leq b\,(7). \end{aligned}$$

This amounts to saying that (1) *holds for* $b = \frac{13}{4}$.
Case (iii): *When* $\partial_1 = 3, \partial_2 = 5, \eth_1 = 2$, *and* $\eth_2 = 4$, *we find*

$$\begin{aligned} d_\Theta(3,4) &\leq b\,[d_\Theta(3,2) + d_\Theta(5,2) + d_\Theta(5,4)] \\ 3 &\leq b\,(6). \end{aligned}$$

Evidently, (1) *is valid for* $b = \frac{13}{4}$.

For different cases, we adopt the same analysis to infer that $(Ł, Ł^*, d_\Theta)$ *is a graphical bipolar* b-MS *with* $b = \frac{13}{4}$.

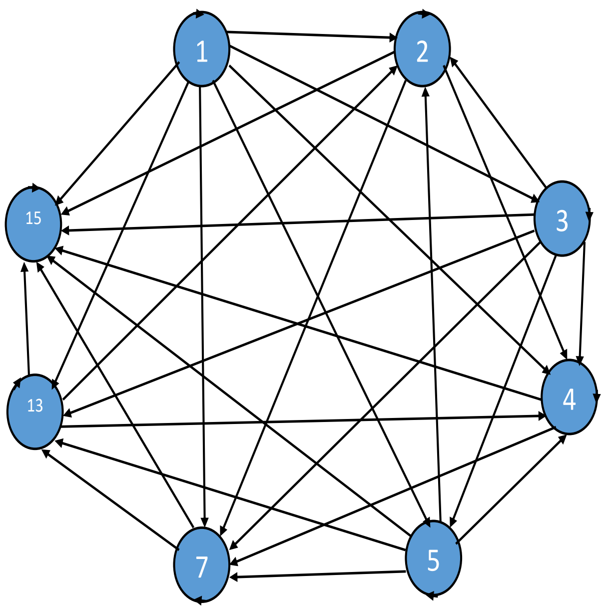

Figure 1. Underlying graph.

Definition 6. *Let* $f : Ł_1 \cup Ł_1^* \to Ł_2 \cup Ł_2^*$ *be a function on pairs of sets* $(Ł_1, Ł_1^*)$ *and* $(Ł_2, Ł_2^*)$. *If* $f(Ł_1) \subseteq Ł_2$ *and* $f(Ł_1^*) \subseteq Ł_2^*$, f *is known as* `CoVM` *from* $(Ł_1, Ł_1^*)$ *to* $(Ł_2, Ł_2^*)$ *and denoted as*

$$f : (Ł_1, Ł_1^*) \rightrightarrows (Ł_2, Ł_2^*).$$

If $f(Ł_1) \subseteq Ł_2^*$ *and* $f(Ł_1^*) \subseteq Ł_2$, f *is known as* `CnVM` *from* $(Ł_1, Ł_1^*)$ *to* $(Ł_2, Ł_2^*)$ *and denoted as*

$$f : (Ł_1, Ł_1^*) \rightleftarrows (Ł_2, Ł_2^*).$$

If d_{Θ_1} *and* d_{Θ_1} *are graphical bipolar* `MSs` *on* $(Ł_1, Ł_1^*)$ *and* $(Ł_2, Ł_2^*)$, *respectively, then, we use the notions*

$$f : (Ł_1, Ł_1^*, d_{\Theta_1}) \rightrightarrows (Ł_2, Ł_2^*, d_{\Theta_2}), \ (\texttt{CoVM})$$

and

$$f : (Ł_1, Ł_1^*, d_{\Theta_1}) \rightleftarrows (Ł_2, Ł_2^*, d_{\Theta_2}), \ (\texttt{CnVM}).$$

Definition 7. *Let* $(Ł, Ł^*, d_\Theta)$ *be a graphical bipolar* b-MS*:*

(i) *A point* $z \in Ł \cup Ł^*$ *is the left point if* $z \in Ł$*, the right point if* $z \in Ł^*$*, and a central point if* $z \in Ł \cap Ł^*$.

(ii) *A sequence* (∂_n) *on* Ł *is the left sequence and* $(\eth_n)$ *on* $Ł^*$ *is the right sequence. A left or right sequence is simply a sequence in* $(Ł, Ł^*, d_\Theta)$*. A sequence* (z_n) *is said to be convergent to a point* z *if* (z_n) *is the left sequence and* z *is a right point; then,*

$$\lim_{n\to\infty} d_\Theta(z_n, z) = 0.$$

Similarly, if (z_n) *is the right sequence and* z *is a left point then*

$$\lim_{n\to\infty} d_\Theta(z, z_n) = 0.$$

(iii) *A sequence* $(\partial_n, \eth_n) \in Ł \times Ł^*$*, is said to be* (BC) *on* $(Ł, Ł^*, d_\Theta)$*. If both sequences* (∂_n) *and* $(\eth_n)$ *converge, then the BC is said to be bi-convergent, and if both sequences converge to the same point* $z \in Ł \cap Ł^*$*, then BC is said to be bi-convergent.*

(iv) *Every bi-convergent BC in* $(Ł, Ł^*, d_\Theta)$ *is a Cauchy BC and every convergent BC is bi-convergent.*

(v) *A graphical bipolar* b-MS *is called complete if every Cauchy BC in* $(Ł, Ł^*, d_\Theta)$ *is convergent.*

Definition 8. *Let* $(Ł_1, Ł_1^*, d_{\Theta_1})$ *and* $(Ł_2, Ł_2^*, d_{\Theta_2})$ *be the graphical bipolar* b-MSs*:*

(i) *A mapping* $f : (Ł_1, Ł_1^*, d_{\Theta_1}) \rightrightarrows (Ł_2, Ł_2^*, d_{\Theta_2})$ *is said to be left continuous at a point* $\partial_0 \in Ł_1$*, if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$d_{\Theta_1}(\partial_0, \eth) < \delta \Longrightarrow d_{\Theta_2}(f(\partial_0), f(\eth)) < \epsilon \text{ for all } \eth \in Ł_1^*.$$

Similarly, it is right continuous at a point $\eth_0 \in Ł_1^*$*, if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$d_{\Theta_1}(\partial, \eth_0) < \delta \Longrightarrow d_{\Theta_2}(f(\partial), f(\eth_0)) < \epsilon \text{ for all } \partial \in Ł,$$

and a mapping f *is called continuous if it is left continuous at each point* $\partial \in Ł_1$ *and right continuous at each point* $\eth \in Ł_1^*$.

(ii) *A mapping* $f : (Ł_1, Ł_1^*, d_{\Theta_1}) \rightleftarrows (Ł_2, Ł_2^*, d_{\Theta_2})$ *is continuous if it is continuous as a covariant mapping* $f : (Ł_1, Ł_1^*, d_{\Theta_1}) \rightrightarrows (Ł_2, Ł_2^*, d_{\Theta_2})$.
Now, we can say that a CoVM *and* CnVM *are continuous from* $(Ł_1, Ł_1^*, d_{\Theta_1})$ *to* $(Ł_2, Ł_2^*, d_{\Theta_2})$ *if:*

$$(z_n) \to \kappa \text{ on } (Ł_1, Ł_1^*, d_{\Theta_1}) \text{ implies that } (f(z_n)) \to f(\kappa) \text{ on } (Ł_2, Ł_2^*, d_{\Theta_2}).$$

4. Main Results

In this section, we will give some fixed-point theorems for CoVMs and CnVMs, satisfying various contractive conditions in the environment of graphical complete bipolar b-MSs $(Ł, Ł^*, d_\Theta)$. We treated the graph Θ as a weighted graph. Let $\partial_0 \in Ł$ and $\eth_0 \in Ł^*$ be the initial values of the BC $(\partial_m, \eth_m)$; we say $(\partial_m, \eth_m)$ is $f-$Picards BC $(f - P_bS)$ if $\partial_m = f\partial_{m-1}$ and $\eth_m = f\eth_{m-1}$ for all $m \in \mathbb{N}$.

Definition 9. *Consider a relation* P *on* $Ł \times Ł^*$ *with* $(\partial_1 P \eth_2)_\Theta$ *if there exists a path from* ∂_1 *to* $\eth_2$ *in* Θ*. If* $\partial_2, \eth_1 \in (\partial_1 P \eth_2)_\Theta$*, then a BC* $(\partial_m, \eth_m) \in Ł \times Ł^*$ *is* Θ^*-*termwise connected* $(\Theta^* - TWC)$ *if* $(\partial_m P \eth_{m+1})$ *for all* $\partial \in Ł$*,* $\eth \in Ł^*$ *, and* $m \in \mathbb{N}$*. Furthermore, we say a graph* $\Theta = (V(\Theta), E(\Theta))$ *satisfies the property* (P^*) *if a* $\Theta^* - TWC$ $f-$*Picards BC* $(\partial_m, \eth_m)$ *is bi-convergent in* $Ł \cap Ł^*$*, which guarantees that there is a limit* κ *such that* $((\partial_m, \eth_m), \kappa) \in E(\Theta)$ *or* $(\kappa, (\partial_m, \eth_m)) \in E(\Theta)$ *for all* $m > m_0$.

Definition 10. *Let* Θ *be a graph containing all the loops associated with graphical bipolar* b-MS $(Ł, Ł^*, d_\Theta)$*. A* CoVM $f : (Ł, Ł^*, d_\Theta) \rightrightarrows (Ł, Ł^*, d_\Theta)$ *is said to be* Θ_b-*contraction (on* $Ł \cup Ł^*$*) on a graphical bipolar* b-MS $(Ł, Ł^*, d_\Theta)$ *if*

(i) *For $(\partial, \eth) \in E(\Theta)$, the graph is preserved,*

$$(f\partial, f\eth) \in E(\Theta), \text{ for all } \partial \in Ł,\ \eth \in Ł^*. \tag{2}$$

(ii) *There exists $0 < \gamma < b$ for all $\partial \in Ł$, and $\eth \in Ł^*$ with $(\partial, \eth) \in E(\Theta)$ implies*

$$d_\Theta(f\partial, f\eth) \leq \frac{\gamma}{b^2} d_\Theta(\partial, \eth). \tag{3}$$

Theorem 1. *Let $f : (Ł, Ł^*, d_\Theta) \rightrightarrows (Ł, Ł^*, d_\Theta)$ be a graphical Θ_b-contraction on a $\Theta-$complete graphical bipolar metric $(Ł, Ł^*, d_\Theta)$. If the underneath axioms are contended,*

(i) *The property (P^*) is asserted by the graph Θ.*
(ii) *There exists $\partial_0 \in Ł$, and $\eth_0 \in Ł^*$ with $(f\partial_0, f\eth_0) \in [(\partial_0, \eth_0)]^l_\Theta$ for some $l \in \mathbb{N}$.*

Then, there exists $\kappa \in Ł \cap Ł^*$ such that $f - P_bS$ $(\partial_m, \eth_m)$ with the initial value $\partial_0 \in Ł$, and $\eth_0 \in Ł^*$ is $\Theta^* - fWC$ and bi-converges to both κ and $f\kappa$.

Proof. Let $\partial_0 \in Ł$ and $\eth_0 \in Ł^*$ such that $(f\partial_0, f\eth_0) \in [(\partial_0, \eth_0)]^l_\Theta$ for some $l \in \mathbb{N}$. By taking $\partial_0 \in Ł$ and $\eth_0 \in Ł^*$ as initial values of $f - P_bS$ $(\partial_m, \eth_m)$, there exists a path $\{(\partial_j, \eth_j)\}^l_{j=0}$, such that $\partial_{n+1} = f\partial_n$ and $\eth_{n+1} = f\eth_n$, where $(\partial_j, \eth_j) \in E(\Theta)$ for $j = 0, 1, 2 \ldots, l$. By using (2), we have $(f\partial_{j-1}, f\eth_j) \in E(\Theta)$ for $j = 1, 2 \ldots, l$. This implies that $\{(f\partial_j, f\eth_j)\}^l_{j=0}$ is a path from $\partial_2 = f\partial_1 = f^2\partial_0$ to $\eth_2 = f\eth_1 = f^2\eth_0$ having length l such that $(\partial_2, \eth_2) \in [(\partial_1, \eth_1)]^l_\Theta$. Continuing this procedure, we conclude that $\{(f^m\partial_j, f^m\eth_j)\}^l_{j=0}$ is a path from $(f^{m-1}\partial_0, f^{m-1}\eth_0) = (\partial_m, \eth_m)$ to $(f^{m+l}\partial_0, f^{m+l}\eth_0) = (\partial_{m+l}, \eth_{m+l})$ of length l and hence $(\partial_{m+l}, \eth_{m+l}) \in [(\partial_m, \eth_m)]^l_\Theta$ for all $m \in \mathbb{N}$. This confirms that $(\partial_m, \eth_m)$ is a $\Theta^* - fWC$ BC, which shows that

$$(f^m\partial_j, f^m\eth_j) \in E(\Theta) \text{ for } j = 1, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Utilizing (3), we attain

$$\begin{aligned} d_\Theta(f^m\partial_j, f^m\eth_j) &\leq \frac{\gamma}{b^2} d_\Theta(f^{m-1}\partial_j, f^{m-1}\eth_j) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta(f^{m-2}\partial_j, f^{m-2}\eth_j). \end{aligned}$$

Continuing the same procedure, we conclude that

$$d_\Theta(f^m\partial_j, f^m\eth_j) \leq \frac{\gamma^m}{b^{2m}} d_\Theta(\partial_j, \eth_j).$$

Since the BC $(f\partial_n, f\eth_n)$ is a $\Theta^* - fWC$ BC, now, we have to show that $(\partial_n, \eth_n)$ is Cauchy BC in graphical bipolar b-MS $(Ł, Ł^*, d_\Theta)$ for each positive integer n and m. We have

$$\begin{aligned} d_\Theta(\partial_{n+1}, \eth_{n+1}) &= d_\Theta(f(\partial_n), f(\eth_n)) \\ &\leq \frac{\gamma}{b^2} d_\Theta(\partial_{n-1}, \eth_{n-1}) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta(\partial_{n-2}, \eth_{n-2}) \\ &\vdots \\ &\leq \frac{\gamma^n}{b^{2n}} d_\Theta(\partial_0, \eth_0). \end{aligned}$$

Furthermore,

$$\begin{aligned} d_\Theta(\partial_{n+1}, \eth_{n+2}) &= d_\Theta(f(\partial_n), f(\eth_{n+1})) \\ &\leq \frac{\gamma}{b^2} d_\Theta(\partial_{n-1}, \eth_n) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta(\partial_{n-2}, \eth_{n-1}) \\ &\vdots \\ &\leq \frac{\gamma^n}{b^{2n}} d_\Theta(\partial_0, \eth_1). \end{aligned}$$

Now for $m \geq n$, we assert

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_n) &\leq b[d_\Theta(f\partial_m, f\eth_{n+1}) + d_\Theta(f\partial_n, f\eth_{n+1}) + d_\Theta(f\partial_n, f\eth_n)] \\ &\leq b[d_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^n}{b^{2n}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^n}{b^{2n}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^n}{b^{2n-1}}[d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{4}$$

Furthermore, we can prove that

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_{n+1}) &\leq b[d_\Theta(f\partial_m, f\eth_{n+2}) + d_\Theta(f\partial_{n+1}, f\eth_{n+2}) + d_\Theta(f\partial_{n+1}, f\eth_{n+1})] \\ &\leq b[d_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{n+1}}{b^{2n+2}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^{n+1}}{b^{2n+2}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{n+1}}{b^{2n+1}}[d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{5}$$

Similarly, we assert

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_{m-1}) &\leq b[d_\Theta(f\partial_m, f\eth_m) + d_\Theta(f\partial_{m-1}, f\eth_m) + d_\Theta(f\partial_{m-1}, f\eth_{m-1})] \\ &\leq b[d_\Theta(f\partial_m, f\eth_m) + \frac{\gamma^{m-1}}{b^{2m-2}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^{m-1}}{b^{2m-2}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{m-1}}{b^{2m-3}}[d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{6}$$

From the observation of (4)–(6), if we set $M = d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)$, then we have

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_n) &\leq bd_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^n}{b^{2n-1}} M \\ &\leq b^2 d_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^n}{b^{2n-1}} M + \frac{\gamma^{n+1}}{b^{2n+1}} M \\ &\vdots \\ &\leq b^{m-n} d_\Theta(f\partial_m, f\eth_m) + \frac{\gamma^n}{b^{2n-1}} M + \frac{\gamma^{n+1}}{b^{2n+1}} M + \ldots + \frac{\gamma^{m-1}}{b^{2m-3}} M \\ &\leq \frac{\gamma^n}{b^{2n-1}} M + \frac{\gamma^{n+1}}{b^{2n+1}} M + \ldots + \frac{\gamma^{m-1}}{b^{2m-3}} M + \frac{\gamma^m}{b^{m+n}} M \\ &\leq (\frac{\gamma}{b})^n M + (\frac{\gamma}{b})^{n+1} M + \ldots + (\frac{\gamma}{b})^{m-1} M + (\frac{\gamma}{b})^m M \\ &\leq \left[\frac{(\frac{\gamma}{b})^n M}{1 - \frac{\gamma}{b}}\right] \to 0, \end{aligned}$$

which shows that $(\partial_n, \eth_n)$ is a Cauchy BC. Now, $(Ł, Ł^*, d_\Theta)$ being a $\Theta-$complete graphical bipolar b-MS, BC $(\partial_n, \eth_n)$ is bi-convergent to κ in $Ł \cap Ł^*$. Making use of (i), there exists some $\kappa \in Ł \cap Ł^*$, $n_0 \in \mathbb{N}$ such that $((\partial_n, \eth_n), \kappa) \in E(\Theta)$ and $(\kappa, (\partial_n, \eth_n)) \in E(\Theta)$ for all $n > n_0$ and

$$\lim_{n\to\infty} d_\Theta((\partial_n, \eth_n), \kappa) = 0,$$

which confirms that BC $(\partial_n, \eth_n)$ is bi-convergent to κ. If $((\partial_n, \eth_n), \kappa) \in E(\Theta)$, then by using (3)

$$\begin{aligned} d_\Theta((\partial_{n+1}, \eth_{n+1}), \kappa) &= d_\Theta((f\partial_n, f\eth_n), f\kappa) \\ &\leq \frac{\gamma}{b^2} d_\Theta((\partial_n, \eth_n), \kappa), \end{aligned}$$

for all $n > n_0$, which shows that

$$\lim_{n\to\infty} d_\Theta((\partial_{n+1}, \eth_{n+1}), \kappa) = 0.$$

If $(\kappa, (\partial_n, \eth_n)) \in E(\Theta)$, a similar argument to that we used above,

$$\lim_{n\to\infty} d_\Theta(\kappa, (\partial_{n+1}, \eth_{n+1})) = 0,$$

Hence, BC $(\partial_n, \eth_n)$ is bi-convergent to κ and $f\kappa$. □

To demonstrate that the mapping f has a fixed point, we define the property (Q^*).

Definition 11. *Let $f : Ł \cup Ł^* \to Ł \cup Ł^*$ be a mapping on graphical bipolar b-*`MS` *$(Ł, Ł^*, d_\Theta)$. We say that if the quadruple $(Ł, Ł^*, d_\Theta, f)$ meets the property (Q^*) corresponding to two limits $u \in Ł \cap Ł^*$ and $\kappa \in f(Ł \cap Ł^*)$ of a $\Theta^* - TWC$ $f - P_bS$ $(\partial_n, \eth_n)$, we have $u = \kappa$.*

Theorem 2. *If all the hypotheses in Theorem 1 are true and we further suppose that the quadruple $(Ł, Ł^*, d_\Theta, f)$ meets the property (Q^*), then f concedes a fixed point.*

Remark 2. *It is worth noting that in the setting of graphical bipolar b-*`MS` *$(Ł, Ł^*, d_\Theta)$, Theorem 2 is an extended form of $\mathbb{BCP}$. To clarify, we can observe that inequality (3) is more general when evaluated within the context of graphical bipolar b-*`MS` *$(Ł, Ł^*, d_\Theta)$. This is backed further by Remark 1. With these facts, it can be readily seen that the remarkable results attributed to Banach (*`BCP`*) and Mutlu and Gürdal [3] are special cases of the Theorem 2.*

Proof. Theorem 1 exhibits that $f - P_bS$ $(\partial_n, \eth_n)$ with initial values $\partial_0 \in Ł$, and $\eth_0 \in Ł^*$ bi-converges to both κ and $f\kappa$. Since $\kappa \in Ł \cap Ł^*$ and $f\kappa \in f(Ł \cap Ł^*)$, by hypothesis, we obtain $\kappa = f\kappa$ and f concedes a fixed point. □

Below, we will discuss an example related to Theorem 1 with different cases.

Example 3. *Let $Ł = \{2, 4, 6, 8\}$, $Ł^* = \{1, 6, 8, 12\}$, and the function $d_\Theta : Ł \times Ł^* \to \mathbb{R}^+$ is defined as*

$$d_\Theta(\partial, \eth) = \begin{cases} 0, \text{ if } \partial = \eth \\ \min\{\partial, \eth\}, \text{ otherwise,} \end{cases}$$

for all $\partial \in Ł$, $\eth \in Ł^*$ and equipped with graph Θ. The graph $\Theta = (V(\Theta), E(\Theta))$, having the set of vertices

$$V(\Theta) = Ł \cup Ł^* = \{1, 2, 4, 6, 8, 12\},$$

and the set of edges obtained from the vertex set $V(\Theta)$, is obtained as

$$E(\Theta) = \triangle \cup \left\{ \begin{array}{c} (2,4), (2,6), (2,8), (2,1), (2,12), \\ (4,6), (4,8), (4,1), (4,12), (6,8), \\ (6,1), (6,12), (8,1), (8,12), (1,12) \end{array} \right\}.$$

See Figure 2 for further exposition. It is simple to confirm that $(\text{Ł}, \text{Ł}^*, d_\Theta)$ *is a graphical bipolar* b-MS *with* $b = 2$.

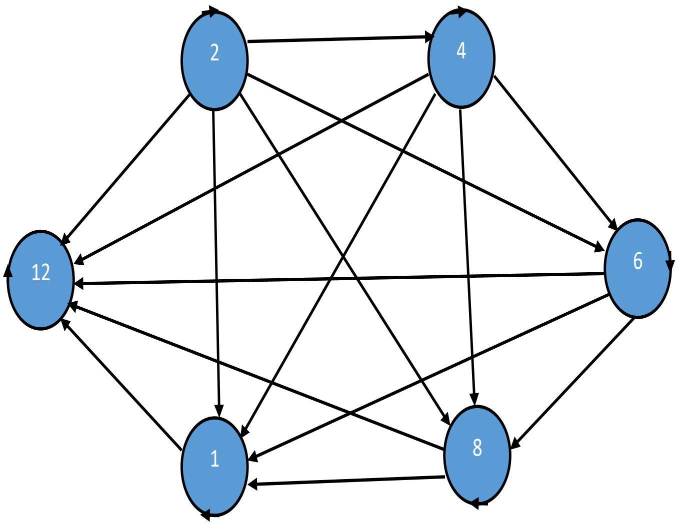

Figure 2. Graph related to Examples 3 and 5.

Now, we define a CoVM
$$f : (\text{Ł}, \text{Ł}^*, d_\Theta) \rightrightarrows (\text{Ł}, \text{Ł}^*, d_\Theta),$$
with $f(\text{Ł}) \subseteq \text{Ł}$ *and* $f(\text{Ł}^*) \subseteq \text{Ł}^*$, *where*
$$f\partial = 6, \text{ if } \partial \in \{4, 6\}, \text{ and } f\eth = 6, \text{ if } \eth \in \{1, 12\},$$
for all $\partial \in \text{Ł}$ *and* $\eth \in \text{Ł}^*$.

Furthermore, if $\partial = 4$ *and* $\eth = 12$, *we have the following*
$$\begin{aligned} d_\Theta(f4, f12) &\leq \frac{\gamma}{b^2} d_\Theta(4, 12) \\ d_\Theta(6, 6) &\leq \frac{\gamma}{b^2} d_\Theta(4, 12) \\ 0 &\leq \frac{\gamma}{b^2}(4). \end{aligned}$$

Again, if $\partial = 4$ *and* $\eth = 1$, *we infer*
$$\begin{aligned} d_\Theta(f4, f1) &\leq \frac{\gamma}{b^2} d_\Theta(4, 1) \\ d_\Theta(6, 6) &\leq \frac{\gamma}{b^2} d_\Theta(4, 1) \\ 0 &\leq \frac{\gamma}{b^2}(1), \end{aligned}$$
proving that f *is a* Θ_b-*contraction with* $b = 2$ *and* $\gamma = 1$, *where* $0 < \gamma < b$. *A simple computation leads us to obtain* $\frac{\gamma}{b^2} = \frac{1}{4}$. *This shows that all the assertions of the Theorem 1 are met, and* f *concedes a fixed point for all* $\partial \in \text{Ł}$ *and* $\eth \in \text{Ł}^*$, *which is* $6 \in \text{Ł} \cap \text{Ł}^*$.

Below, we will prove a similar result for CnVMs.

Definition 12. *Let* Θ *be a graph containing all the loops associated with graphical bipolar* b-MS $(\text{Ł}, \text{Ł}^*, d_\Theta)$. *A* CnVM $f : (\text{Ł}, \text{Ł}^*, d_\Theta) \rightleftarrows (\text{Ł}, \text{Ł}^*, d_\Theta)$ *is said to be a* Θ_b^*-*contraction* (*on* $\text{Ł} \cup \text{Ł}^*$) *on a graphical bipolar* b-MS $(\text{Ł}, \text{Ł}^*, d_\Theta)$ *if the following axioms are upheld:*

(i) *For* $(\partial, \eth) \in E(\Theta)$, *we have*

$$(f\partial, f\eth) \in E(\Theta), \text{ for all } \partial \in Ł,\ \eth \in Ł^*. \tag{7}$$

(ii) *There exists* $0 < \gamma < b$ *for all* $\partial \in Ł$, *and* $\eth \in Ł^*$ *with* $(\partial, \eth) \in E(\Theta)$ *implies*

$$d_\Theta(f\partial, f\eth) \leq \frac{\gamma}{b^2} d_\Theta(\eth, \partial). \tag{8}$$

Theorem 3. *Let* $f : (Ł, Ł^*, d_\Theta) \rightleftarrows (Ł, Ł^*, d_\Theta)$ *be a graphical* Θ_b^*-*contraction on a* $\Theta-$*complete graphical bipolar b-*`MS` $(Ł, Ł^*, d_\Theta)$ *if the following conditions hold:*

(i) Θ *exhibits the property* (P^*).

(ii) *There exists* $\partial_0 \in Ł$, *and* $\eth_0 \in Ł^*$ *with* $(f\partial_0, f\eth_0) \in [(\partial_0, \eth_0)]^l_\Theta$ *for some* $l \in \mathbb{N}$.

Then, there exists $\kappa \in Ł \cap Ł^*$ such that $f - P_bS\ (\partial_m, \eth_m)$ with the initial value $\partial_0 \in Ł$, $\eth_0 \in Ł^*$ is $\Theta^* - TWC$ and bi-converges to both κ and $f\kappa$.

Proof. Let $\partial_0 \in Ł$ such that $(f\partial_0, f\eth_0) \in [(\partial_0, \eth_0)]^l_\Theta$ for some $l \in \mathbb{N}$. By taking $\partial_0 \in Ł$ as the initial values of $f - P_bS\ (\partial_m, \eth_m)$, there exists a path $\{(\partial_j, \eth_j)\}^l_{j=0}$, such that $\eth_n = f\partial_n$ and $\partial_{n+1} = f\eth_n$ where $(\partial_j, \eth_j) \in E(\Theta)$ for $j = 0, 1, \ldots, l$. By using (7), we assert that $(f\partial_{j-1}, f\eth_j) \in E(\Theta)$ for $j = 1, \ldots, l$. This implies that $\{(f\partial_j, f\eth_j)\}^l_{j=0}$ is a path from $\partial_2 = f^2\partial_1 = f^3\partial_0$ to $\eth_2 = f^2\eth_1 = f^3\eth_0$ having length l such that $(\partial_2, \eth_2) \in [(\partial_1, \eth_1)]^l_\Theta$. Continuing this procedure, we conclude that $\{(f^m\partial_j, f^m\eth_j)\}^l_{j=0}$ is a path from $(f^m\partial_0, f^m\eth_0) = (\partial_m, \eth_m)$ to $(f^{m+l}\partial_0, f^{m+l}\eth_0) = (\partial_{m+l}, \eth_{m+l})$ of length l and hence $(\partial_{m+1}, \eth_{m+1}) \in [(\partial_m, \eth_m)]^l_\Theta$ for all $m \in \mathbb{N}$. This confirms that $(\partial_m, \eth_m)$ is a $\Theta^* - TWC\ BC$, which shows that

$$(f^m\partial_j, f^m\eth_j) \in E(\Theta) \text{ for } j = 1, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Imposing inequality (8), we infer

$$\begin{aligned} d_\Theta(f^m\partial_j, f^m\eth_j) &\leq \frac{\gamma}{b^2} d_\Theta(f^{m-1}\eth_j, f^m\partial_j) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta(f^{m-1}\eth_j, f^{m-1}\partial_j) \\ &= \frac{\gamma^2}{b^4} d_\Theta(f^{m-1}\partial_j, f^{m-1}\eth_j). \end{aligned}$$

Continuing the same approach, we arrive at the conclusion that

$$d_\Theta(f^m\partial_j, f^m\eth_j) \leq \frac{\gamma^{2m}}{b^{4m}} d_\Theta(\partial_j, \eth_j).$$

Since the BC $(\partial_n, \eth_n)$ is a $\Theta^* - TWC\ BC$, it is imperative to prove that $(\partial_n, \eth_n)$ is Cauchy BC in graphical bipolar b-`MS` $(Ł, Ł^*, d_\Theta)$. For every positive integer n and m, we proceed as follows:

$$\begin{aligned} d_\Theta(\partial_n, \eth_n) &= d_\Theta(f(\eth_{n-1}), f(\partial_n)) \\ &\leq \frac{\gamma}{b^2} d_\Theta(\partial_n, \eth_{n-1}) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta(\partial_{n-1}, \eth_{n-1}) \\ &\ \ \vdots \\ &\leq \frac{\gamma^{2n}}{b^{4n}} d_\Theta(\partial_0, \eth_0), \end{aligned}$$

Furthermore,

$$\begin{aligned} d_\Theta(\partial_n, \eth_{n+1}) &= d_\Theta(f(\partial_{n-1}), f(\eth_n)) \\ &\leq \frac{\gamma}{b^2} d_\Theta(\partial_{n-1}, \eth_n) \\ &\vdots \\ &\leq \frac{\gamma^{2n}}{b^{4n}} d_\Theta(\partial_0, \eth_1). \end{aligned}$$

Now, for $m \geq n$, we obtain

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_n) &\leq b[d_\Theta(f\partial_m, f\eth_{n+1}) + d_\Theta(f\partial_n, f\eth_{n+1}) + d_\Theta(f\partial_n, f\eth_n)] \\ &\leq b[d_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^n}{b^{4n}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^n}{b^{4n}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^n}{b^{4n-1}} [d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_{n+1}) &\leq b[d_\Theta(f\partial_m, f\eth_{n+2}) + d_\Theta(f\partial_{n+1}, f\eth_{n+2}) + d_\Theta(f\partial_{n+1}, f\eth_{n+1})] \\ &\leq b[d_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{2(n+1)}}{b^{4(n+1)}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^{2(n+1)}}{b^{4(n+1)}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{2(n+1)}}{b^{4(n+1)-1}} [d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{10}$$

Similarly, we obtain

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_{m-1}) &\leq b[d_\Theta(f\partial_m, f\eth_m) + d_\Theta(f\partial_{m-1}, f\eth_m) + d_\Theta(f\partial_{m-1}, f\eth_{m-1})] \\ &\leq b[d_\Theta(f\partial_m, f\eth_m) + \frac{\gamma^{2(m-1)}}{b^{4(m-1)}} d_\Theta(\partial_0, \eth_1) + \frac{\gamma^{2(m-1)}}{b^{4(m-1)}} d_\Theta(\partial_0, \eth_0)] \\ &\leq bd_\Theta(f\partial_m, f\eth_m) + \frac{\gamma^{2(m-1)}}{b^{4(m-1)-1}} [d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)]. \end{aligned} \tag{11}$$

From the observations (9)–(11), if we set $M = d_\Theta(\partial_0, \eth_1) + d_\Theta(\partial_0, \eth_0)$, we arrive at

$$\begin{aligned} d_\Theta(f\partial_m, f\eth_n) &\leq bd_\Theta(f\partial_m, f\eth_{n+1}) + \frac{\gamma^{2n}}{b^{4n-1}} M \\ &\leq b^2 d_\Theta(f\partial_m, f\eth_{n+2}) + \frac{\gamma^{2n}}{b^{4n-1}} M + \frac{\gamma^{2(n+1)}}{b^{4(n+1)-1}} M \\ &\vdots \\ &\leq b^{m-n} d_\Theta(f\partial_m, f\eth_m) + \frac{\gamma^{2n}}{b^{4n-1}} M + \frac{\gamma^{2(n+1)}}{b^{4(n+1)-1}} M + \ldots + \frac{\gamma^{2(m-1)}}{b^{4(m-1)-1}} M \\ &\leq b^{m-n} \frac{\gamma^{2m}}{b^{4m}} M + \frac{\gamma^{2n}}{b^{4n-1}} M + \frac{\gamma^{2(n+1)}}{b^{4(n+1)-1}} M + \ldots + \frac{\gamma^{2(m-1)}}{b^{4(m-1)-1}} M \\ &\leq \frac{\gamma^{2n}}{b^{4n-1}} M + \frac{\gamma^{2(n+1)}}{b^{4(n+1)-1}} M + \ldots + \frac{\gamma^{2(m-1)}}{b^{4(m-1)-1}} M + \frac{\gamma^{2m}}{b^{3m+n}} M \\ &\leq \frac{1}{b}[(\frac{\gamma^2}{b^4})^n M + (\frac{\gamma^2}{b^4})^{n+1} M + \ldots + (\frac{\gamma^2}{b^4})^{m-1} M + (\frac{\gamma^2}{b^4})^m M] \\ &\leq \frac{1}{b}\left[\frac{(\frac{\gamma^2}{b^4})^n M}{1 - (\frac{\gamma^2}{b^4})} \to 0\right] \to 0, \end{aligned}$$

which shows that $(\partial_n, \eth_n)$ is a Cauchy BC. Since $(Ł, Ł^*, d_\Theta)$ is a $\Theta-$complete graphical bipolar b-$\mathtt{MS}$, then BC $(\partial_n, \eth_n)$ is bi-convergent to κ in $Ł \cap Ł^*$ and from (i) there exists some

$\kappa \in \text{Ł} \cap \text{Ł}^*$, $n_0 \in \mathbb{N}$ such that $((\partial_n, \eth_n), \kappa) \in E(\Theta)$ and $(\kappa, (\partial_n, \eth_n)) \in E(\Theta)$ for all $n > n_0$ and

$$\lim_{n\to\infty} d_\Theta((\partial_n, \eth_n), \kappa) = 0,$$

which confirms that BC $(\partial_n, \eth_n)$ is bi-convergent to κ. If $((\partial_n, \eth_n), \kappa) \in E(\Theta)$, then by using (8)

$$\begin{aligned} d_\Theta((\partial_{n+1}, \eth_{n+1}), \kappa) &= d_\Theta((f\eth_n, f\partial_{n+1}), f\kappa) \\ &\leq \frac{\gamma}{b^2} d_\Theta((\partial_{n+1}, \eth_n), \kappa) \\ &\leq \frac{\gamma^2}{b^4} d_\Theta((\partial_n, \eth_n), \kappa) \end{aligned}$$

for all $n > n_0$, which shows that

$$\lim_{n\to\infty} d_\Theta((\partial_{n+1}, \eth_{n+1}), \kappa) = 0.$$

If $(\kappa, (\partial_n, \eth_n)) \in E(\Theta)$, a similar argument to that we used above,

$$\lim_{n\to\infty} d_\Theta(\kappa, (f\partial_{n+1}, f\eth_{n+1})) = 0.$$

Hence, BC $(\partial_n, \eth_n)$ is bi-convergent to κ and $f\kappa$. $\square$

To demonstrate that mapping f has a fixed point, we employ the attribute (P^*).

Theorem 4. *If all the hypotheses are retained in Theorem 3 and we further suppose that the quadruple $(\text{Ł}, \text{Ł}^*, d_\Theta, f)$ fulfills the property (P^*), then f concedes a fixed point.*

Proof. From the Theorem 3, we find that $f - P_bS$ $(\partial_n, \eth_n)$ with the initial values $\partial_0 \in \text{Ł}$ bi-converges to both κ and $f\kappa$. Since $\kappa \in \text{Ł} \cap \text{Ł}^*$ and $f\kappa \in f(\text{Ł} \cap \text{Ł}^*)$, by hypothesis, we obtain $\kappa = f\kappa$ and f concedes a fixed point. $\square$

Below, we will discuss an example related to Theorem 3 with different cases.

Example 4. *Let $\text{Ł} = \{a, b, c\}$, $\text{Ł}^* = \{b, c, d\}$ and $d_\Theta : \text{Ł} \times \text{Ł}^* \to \mathbb{R}^+$ be defined as*

d_Θ	a	b	c	d
a	0	1	2	4
b	1	0	1	2
c	2	1	0	1
d	4	2	1	0

for all $\partial \in \text{Ł}$ and $\eth \in \text{Ł}^*$. For $b = \frac{4}{3} > 1$, it is simple to observe that $(\text{Ł}, \text{Ł}^*, d_\Theta)$ is graphical bipolar `MS` *with the vertex set $V(\Theta) = \text{Ł} \cup \text{Ł}^*$ and edge set $E(\Theta) = \{(a, b), (a, c), (a, d), (c, b), (b, d), (c, d)\}$, as shown in Figure 3.*

We define a `CnVM` *using the following:*

$$f : (\text{Ł}, \text{Ł}^*, d_\Theta) \rightleftarrows (\text{Ł}, \text{Ł}^*, d_\Theta),$$

with $f(\text{Ł}) \subseteq \text{Ł}^*$, $f(\text{Ł}^*) \subseteq \text{Ł}$ such that

$$f\partial = \begin{cases} c, \ if\, \partial = a, \\ d, \ if\, \partial = b, \end{cases}$$

and

$$f\eth = \begin{cases} b, \ if\, \eth = b, \\ c, \ if\, \eth = d, \end{cases}$$

for all $\partial \in Ł$, $\eth \in Ł^*$.

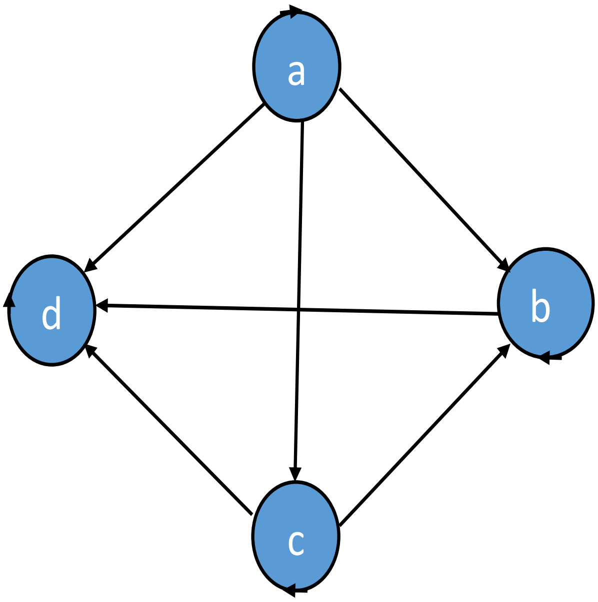

Figure 3. Underlying graph linked to Example 4.

The cases below are examined while assuming that $\gamma = \frac{11}{10}$*:*
Case (i): *When* $\partial = a$ *and* $\eth = d$*, then*

$$\begin{aligned} d_\Theta(fa, fd) &\leq \frac{\gamma}{b^2} d_\Theta(b, d) \\ d_\Theta(c, c) &\leq \frac{\gamma}{b^2} d_\Theta(b, d) \\ 0 &\leq \frac{\gamma}{b^2}(2). \end{aligned}$$

Evidently, the case holds for $\gamma = \frac{11}{10}$ *and* $b = \frac{4}{3}$.
Case (ii): *For* $\partial = b$ *and* $\eth = d$*,*

$$\begin{aligned} d_\Theta(fb, fd) &\leq \frac{\gamma}{b^2} d_\Theta(b, d) \\ d_\Theta(d, c) &\leq \frac{\gamma}{b^2} d_\Theta(b, d) \\ 1 &\leq \frac{\gamma}{b^2}(2). \end{aligned}$$

Hence, Θ_b^*-*contraction is satisfied for* $b = \frac{4}{3}$ *and* $\gamma = \frac{11}{10}$*, where* $0 < \gamma < b$*. After simple calculation, we obtain* $\frac{\gamma}{b^2} = 0.61875 < 1$*. This suggests that all the requirements of Theorem 3 are satisfied, and* f *concedes a fixed point for all* $\partial \in Ł$ *and* $\eth \in Ł^*$*, which is* $b \in Ł \cap Ł^*$.

Similarly, we can continue Example 3 related to Theorem 1 with `CnVM`.

Example 5. *If we take all the terms and conditions of Example 3 and define a* `CnVM`

$$f : (Ł, Ł^*, d_\Theta) \rightleftarrows (Ł, Ł^*, d_\Theta),$$

with $f(Ł) \subseteq Ł^*$ *and* $f(Ł^*) \subseteq Ł$ *such that*

$$f\partial = 1, \text{ if } \partial = 6, \text{ and } f\eth = 8, \text{ if } \eth \in \{8, 12\},$$

for all $\partial \in Ł$ *and* $Ł^* \in Ł^*$.

For $\partial = 6$ *and* $\eth = 12$, *then*

$$\begin{aligned} d_\Theta(f6, f12) &\leq \frac{\gamma}{b^2} d_\Theta(6,12) \\ d_\Theta(1,8) &\leq \frac{\gamma}{b^2} d_\Theta(6,12) \\ 1 &\leq \frac{\gamma}{b^2}(6), \end{aligned}$$

which is true for $b = 2$ *and* $\gamma = 1$.

Hence, Θ_b^*-*contraction is satisfied for* $b = 2$ *and* $\gamma = 1$, *where* $0 < \gamma < b$. *With routine calculations, we obtain* $\frac{\gamma}{b^2} = \frac{1}{4}$, *showing that all the requirements of Theorem 3 are met, and* f *concedes a fixed point for all* $\partial \in Ł$ *and* $\eth \in Ł^*$, *which is* $8 \in Ł \cap Ł^*$.

Remark 3. *It is important to note that all of the conclusions presented in this article apply equally to symmetric graphs, or graphs in which edge symmetry exists.*

5. Conclusions

Metric spaces and their various generalizations allow us to consider the distances between points in a set, either classically or non-classically. However, in some situations, distances might occur between components of two distinct sets compared to between points of a single set. Distances among the same type of points are either indefinite or unclear in some circumstances due to a lack of data. Illustrations of these distances abound in mathematics and science. Some examples include distance within lines and points in a Euclidean space, suitability measurement of habitats to species, an affinity across a group of students, and a set of duties, etc. We formalize such distances as the graphical bipolar b-metric but only examine them isometrically without deeply delving into their topological aspects. We begin by offering basic definitions and examples of graphical bipolar b-MS, then explain maps and bi-sequences, investigate the completeness, explore some related features, and conclude with convergence results, employing covariant and contravariant mapping contractions in the form of graphs.

With this aim, we instigate a novel notion, termed graphical bipolar b-metric spaces, which combines graphical analysis and fixed-point theory. We propound fixed-point results using covariant and contravariant mapping contractions within the context of graphical bipolar b-metric spaces, a novel study within the field of graphical metric spaces, considering the symmetry of the edges of the underlying graph. These results are the first of their kind in the current state of the art. Illustrative examples with appropriate graphs are provided to demonstrate the findings. These examples help to improve the comprehension and implementation of the concepts that have been developed.

We offer the following open questions for future developments of this study:

- Can the proposed convergence theorems in this study be generalized to multi-valued mappings?
- Is it possible to use `CnVM` and `CoVM` to obtain the fixed points for Reich-type contractions in a graph setting?

Author Contributions: Conceptualization, M.Y.; methodology, M.Y. and H.A.; software, M.Y. and A.A.N.A.; validation, M.Y., H.A. and A.A.N.A.; formal analysis, H.A. and M.Y.; investigation, M.Y.; writing—original draft preparation, M.Y. and H.A.; writing—review and editing, A.A.N.A.; visualization, H.A.; supervision, M.Y.; project administration, A.A.N.A.; funding acquisition, A.A.N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant no. UJ-22-DR-4.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the referees for their informative remarks and helpful feedback, which considerably improved our article. Their competence and meticulousness improved our research and work. We appreciate their comprehensive evaluation of our work and valuable recommendations reinforcing our findings. This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant no. UJ-22-DR-4. The authors, therefore, thank the University of Jeddah for its technical and financial support.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jabeen, S.; Rehman, S.U.; Zheng, Z.; Wei, W. Weakly compatible and quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations. *Adv. Differ. Equ.* **2020**, *2020*, 280. [CrossRef]
2. Yazici, N.E.; Ege, O.; Mlaiki, N.; Mukheimer, A. Controlled *S*-Metric-Type Spaces and Applications to Fractional Integrals. *Symmetry* **2023**, *15*, 1100. [CrossRef]
3. Mutlu, A.; Gürdal, U. Bipolar metric spaces and some fixed point theorems. *J. Nonlinear Sci. Appl.* **2016**, *9*, 5362–5373. [CrossRef]
4. Mutlu, A.; Ozkan, K.; Gürdal, U. Coupled fixed point theorems on bipolar metric spaces.*Eur. J. Pure Appl. Math.* **2017**, *10*, 655–667.
5. Gürdal, U.; Mutlu, A.; Özkan, K. Fixed point results for α-ψ contractive mappings in bipolar metric spaces. *J. Inequalities Spec. Funct.* **2020**, *11*, 1.
6. Radenović, S.; Zoto, K.; Dedovixcx, N.; Šešum-Cavic, V.; Ansari, A.H. Bhaskar-Guo-Lakshmikantam-Ćiri ć type results via new functions with applications to integral equations. *Appl. Math. Comput.* **2019**, *357*, 75–87. [CrossRef]
7. Radenović, S.; Došenovixcx, T.; Aleksixcx-Lampert, T.; Golubovixcx, Z. A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations. *Appl. Math. Comput.* **2016**, *273*, 155–164. [CrossRef]
8. Chuensupantharat, N.; Kumam, P.; Chauhan, V.; Singh, D.; Menon, R. Graphic contraction mappings via graphical *b*-metric spaces with applications. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 3149–3165. [CrossRef]
9. Hussain, N.; Arshad, M.; Shoaib, A. Common fixed point results for α-ψ-contractions on a metric space endowed with graph. *J. Inequalities Appl.* **2014**, *1*, 136. [CrossRef]
10. Hussain, N.; Ahmad, J.; Kutbi, M.A. Fixed point theorems for generalized Mizoguchi-Takahashi graphic contractions. *J. Funct. Spaces* **2016**, *2016*, 6514920. [CrossRef]
11. Younis, M.; Singh, D.; Goyal, A. A novel approach of graphical rectangular *b*-metric spaces with an application to the vibrations of a vertical heavy hanging cable. *J. FPT Appl.* **2019**, *21*, 33. [CrossRef]
12. Hammad, H.A.; Zayed, M. Solving a system of differential equations with infinite delay by using tripled fixed point techniques on graphs. *Symmetry* **2022**, *14*, 1388. [CrossRef]
13. Shukla, S.; Radenović, S.; Vetro, C. Graphical metric space: A generalized setting in FPT, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. *Matematicas* **2017**, *111*, 641–655.
14. Younis, M.; Bahuguna, D. A unique approach to graph-based metric spaces with an application to rocket ascension. *Comp. Appl. Math.* **2023**, *42*, 44. [CrossRef]
15. Hooda, R.; Kamra, M.; Malik, A. Common fixed point results for three and four mappings on vector-*b*-metric space with a graph. *Rend. Circ. Mat. Palermo II Ser.* **2023**, *72*, 2721–2743. [CrossRef]
16. Popescu, O. Fixed-point results for convex orbital operators. *Dem. Math.* **2023**, *56*, 20220184. [CrossRef]
17. de Hierro, A.F.; Shahzad, N. From graphical metric spaces to fixed point theory in binary related distance spaces. *Filomat* **2017**, *31*, 3209–3231.