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Results for a Control Problem for a SIS Epidemic Reaction–Diffusion Model

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Abstract: This article is focused on investigating the mathematical model calibration of a reaction–diffusion system arising in the mathematical model of the spread of an epidemic in a society. We consider that the total population is divided into two classes of individuals, called susceptible and infectious, where a susceptible individual can become infectious, and that upon recovery, an infected individual can become susceptible again. We consider that the population lives in a spatially heterogeneous environment, and that the spread of the dynamics is governed by a reaction–diffusion system consisting of two equations, where the variables of the model are the densities of susceptible and infected individuals. In the reaction term, the coefficients are the rates of disease transmission and the rate of infective recovery. The main contribution of this study is the identification of the reaction coefficients by assuming that the infective and susceptible densities at the end time of the process and on overall spatial domain are observed. We apply the optimal control methodology to prove the main findings: the existence of positive solutions for the state system, the existence of at least one solution for the identification problem, the introduction of first-order necessary conditions, and the local uniqueness of optimal solutions.

Keywords: inverse problem; optimal control; SIS model

MSC: 35R30; 49N45; 49K20; 92D30



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1. Introduction

The mathematical approach to biological problems has become an important area of research in recent decades, see for instance [1–6] and references therein. In this context, mathematical models are important tools that are used to simulate a given biological phenomenon, analyze the causes or consequences without laboratory experiments, estimate some parameters which are relevant to their development, and control the dynamics. Particularly, in this paper, we are interested in the problem of parameter identification arising from the phenomenon of spreading epidemics. More precisely, we are concerned with the following optimization problem:

$$\inf_{(\beta, \gamma)} J(\mathbf{u}, \beta, \gamma) := \|\mathbf{u}(\cdot, T) - \mathbf{u}^{obs}\|_{L^2(\Omega)}^2 + \delta \|\nabla(\beta, \gamma)\|_{L^2(\Omega)}^2, \quad (1)$$

subject to $\mathbf{u} = (u_1, u_2)$ solution of the following IBVP:

$$\partial_t u_1 - \Delta u_1 = -\beta(x)\phi(u_1, u_2) + \gamma(x)u_2, \quad \text{in } Q_T := \Omega \times]0, T], \quad (2)$$

$$\partial_t u_2 - \Delta u_2 = \beta(x)\phi(u_1, u_2) - \gamma(x)u_2, \quad \text{in } Q_T, \quad (3)$$

$$\nabla u_1 \cdot \mathbf{v} = \nabla u_2 \cdot \mathbf{v} = 0, \quad \text{in } \Gamma_T := \partial\Omega \times]0, T], \quad (4)$$

$$(u_1, u_2)(x, 0) = (u_{10}, u_{20})(x) := \mathbf{u}_0(x), \quad \text{on } \Omega, \quad (5)$$

where $\phi(u_1, u_2) := u_1 u_2 (u_1 + u_2)^{-1}$, $T > 0$, $\delta > 0$, $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, $\mathbf{u}^{obs}$ is a measurement of the state variable $\mathbf{u}$ at $t = T$, and $\mathbf{u}_0$ is a given function modeling the initial conditions. The set $L^2(\Omega)$ is the Banach space of all functions from Ω to $\mathbb{R}$ that are square integrable on Ω in the sense of Lebesgue.

The state system (2)–(5) is a mathematical model introduced to study the dynamics of disease transmission in a population distributed over the spatial domain Ω and partitioned into two classes of individuals: susceptible (with density u_1) and infective (with density u_2). Basically, the following assumptions on the behavior of populations are considered: the susceptible individuals in the population can contract the disease from cross contacts with infected individuals, which is modeled by the term $\beta\phi(u_1, u_2)$ with β the transmission rate coefficient; the infected individuals who are recovered are not immune and can contract the disease, which is described by the term γu_2 , with γ the recovery rate coefficient; the space domain Ω is closed during the infection disease process, in the sense that there is neither emigration nor immigration through $\partial\Omega$, which is described by the boundary condition (4); and the spreading of the disease is also occasioned by the movement of individuals on the domain, which is modeled by diffusion terms with identity diffusion [7,8]. We remark that, in the recent literature, there have been other proposals for the dynamics of disease transmission that use advanced concepts of mathematical analysis; see, for instance, refs. [9–16] for fractional calculus approaches.

The current analysis of reaction–diffusion equations can be developed by using several mathematical methods, including, for instance, semigroup theory, fixed point theory, the variational formulation approach, and symmetry-based methods. Particularly, related with the symmetry analysis of reaction–diffusion, we refer to [17–21], where some reaction–diffusion systems with nonlinear structures similar to (2)–(5) are studied.

The optimization problem (1)–(5) is the reformulation, as an optimal control problem, of the inverse problem arising in the problem of coefficients identification from measurements of the state variables at the end time of the process. Notice that the cost function, given in (1), is more appropriate for determination of rates β and γ , since the first term of J is a comparison of the state solution and the observation in the L^2 norm and the second term is a regularization term, where δ is a parameter to be appropriately selected in order to obtain a unique solution of the optimization problem.

Recently, several works related with the control problem and parameter identification problems in epidemiological compartmental models (SIS and SIR) were developed [8,22–26]. In [8], the authors obtained results for the one-dimensional case, which were extended to the multidimensional case in [22,23]. In [24], a generalized SIR model for indirectly transmitted diseases was considered, and the authors proved results for the existence of optimal solutions, a first-order optimal condition, and the local uniqueness of the coefficient identification problem. In [25,26], the authors study the optimal control problems in SIR models related with vaccination such that the control variable force immunity. In a broad sense, we should emphasize that the results in [8,22–24] are deduced by assuming that the coefficients and the initial condition belong to appropriate holder spaces.

The main objective of this result is the extension of the results given on [8,22–24] to the case when the coefficients belong $C^2(\Omega)^2$ and the initial condition belongs to the Sobolev space $H^2(\Omega)^2$.

In this paper, our analysis of the control problem (1)–(5) is given by the application of Dubovitskii and Milyutin’s methodology [27]. In order to formulate the optimal control problem in the context of Dubovitskii and Milyutin formalism, we begin by recalling the standard notation for the function spaces which are used to analyze parabolic Equations [28–31], then we rewrite (1)–(5) in terms of some appropriate operators. Concerning the Banach spaces that are considered in the analysis of the well-posedness of (2)–(5), we consider the following notation:

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} : \|f\|_{L^p(\Omega)} < C \right\}, \quad \|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_{\Omega} |f(x)|^p \right)^{1/p} & p \in [1, \infty], \\ \text{ess sup}_{x \in \Omega} |f(x)| & p = \infty, \end{cases} \quad (6a)$$

$$W^{m,p}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} : D^{\alpha} f \in L^p(\Omega), \forall |\alpha| = 0, \dots, m \right\}, \quad \alpha \in (\mathbb{N} \cup \{0\})^d, \quad |\alpha| = |\alpha_1| + \dots + |\alpha_d|, \quad (6b)$$

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H = L^2(\Omega)^3, \quad \text{where } H' \text{ is the topological dual of } H, \quad (6c)$$

$$W^{1,2}(0, T; H) = \left\{ f \in L^2(0, T; H) : f_t \in L^2(0, T; H') \right\}, \quad \|f\|_{W^{1,2}(0, T; H)} = \|f\|_{L^2(H)} + \|f_t\|_{L^2(H)}, \quad (6d)$$

$$\mathbf{L}(T, \Omega) = L^2(0, T; H^2(\Omega))^2 \cap L^{\infty}(0, T; H^1(\Omega))^2. \quad (6e)$$

In order to introduce the terminology and notation of Dubovitskii and Milyutin's theory, we define the set of functions E and $\tilde{E}$ by the following relations:

$$E = W^{1,2}(0, T; H)^2 \times L^2(\Omega)^2, \quad \tilde{E} = \mathbf{L}(T, \Omega) \times H^2(\Omega)^2, \quad (7)$$

Additionally, we define the operator $M : E \rightarrow \tilde{E}$ such that:

$$M(\mathbf{u}, \beta, \gamma) = (\psi_1, \psi_2, \psi_3, \psi_4) \quad (8)$$

if and only if:

$$\partial_t u_1 - \Delta u_1 + \beta(x)\phi(u_1, u_2) - \gamma(x)u_2 = \psi_1, \quad \text{in } Q_T, \quad (9)$$

$$\partial_t u_2 - \Delta u_2 - \beta(x)\phi(u_1, u_2) + \gamma(x)u_2 = \psi_2, \quad \text{in } Q_T, \quad (10)$$

$$\nabla u_1 \cdot \nu = \nabla u_2 \cdot \nu = 0, \quad \text{in } \Gamma_T, \quad (11)$$

$$(u_1, u_2)(x, 0) - \mathbf{u}_0(x) = (\psi_3, \psi_4), \quad \text{on } \Omega. \quad (12)$$

Hence, we have that $J : E \rightarrow \mathbb{R}$, and we can rewrite (1)–(5) as the following optimization problem:

$$\inf_{(\mathbf{u}, \beta, \gamma) \in \mathcal{D}} J(\mathbf{u}, \beta, \gamma), \quad \mathcal{D} = \left\{ (\mathbf{u}, \beta, \gamma) \in E : M(\mathbf{u}, \beta, \gamma) = 0 \right\}. \quad (13)$$

Notice that the system (9)–(12) is reduced to the system (2)–(5) when $\psi_i = 0$ for $i = 1, \dots, 4$. Recently, in [32] and [33], an application of Dubovitskii–Milyutin formalism to control problems for a reaction–diffusion modeling of the solidification phenomena and the pest–predator–plants ecosystem under the assumptions of Lotka–Volterra competence, respectively.

The main contributions of this paper are the following four results: (a) the existence and uniqueness of a positive solution of system (2)–(5); (b) existence of optimal solutions; (c) the explicit calculus of descent and dual cones of J and other differentiability properties of the operators M ; and (d) the introduction of a first-order optimality condition, which is characterized in terms of the optimal solution, the state system, and their corresponding adjoint system. The results are deduced by considering the following assumptions on the coefficients, the spatial domain, and the data:

$$\Omega \subset \mathbb{R}^d \text{ is an open and bounded set of } C^{\infty} \text{ class,} \quad (14)$$

$$\mathbf{u}_0 \in \mathcal{U}_0 = \left\{ (u, v) \in H^2(\Omega)^2 : \nabla u \cdot \nu = \nabla v \cdot \nu = 0 \text{ on } \partial\Omega \right\}, \quad \mathbf{u}_0(x) \geq 0 \text{ on } \Omega, \quad (15)$$

$$(\beta, \gamma) \in \mathcal{U}_{ad} = \left\{ (\beta, \gamma) \in C(\Omega)^2 : (\beta, \gamma)(x) \in [\beta_*, \beta^*] \times [\gamma_*, \gamma^*] \subset]0, 1[^2 \text{ on } \Omega \right\}, \quad (16)$$

$$\mathbf{u}^{obs} \in L^2(\Omega)^2. \quad (17)$$

In (15) the notation $\mathbf{u}_0(x) \geq 0$ is used for $u_{10}(x) \geq 0$ and $u_{20}(x) \geq 0$. Moreover, we comment that the local uniqueness can be deduced by following the recent results given in [22,23].

2. Well-Posedness of State System (2)–(5)

The proof of the well-posedness will be developed following the ideas given in [26,33,34], where the authors applied the theory of evolution equations in Banach spaces [35–37]. More specifically, we apply the following theorem.

Theorem 1 ([35], Proposition 1.2, p. 175). *Let X a Banach, $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup of contractions on X , $f : [0, T] \times X \rightarrow X$ be a function which is measurable in t and is Lipschitz in $x \in X$ uniformly with respect to $t \in [0, T]$. If $y_0 \in X$, then the initial value problem:*

$$y'(t) = Ay(t) + f(t, y(t)), \quad t \in [0, T], \quad (18a)$$

$$y(0) = y_0, \quad (18b)$$

has a unique mild solution $y \in C([0, T]; X)$. Moreover, if the assumptions X is a Hilbert space, A is self-adjoint and dissipative on X , and $y_0 \in D(A)$ are satisfied, then the mild solution is a strong solution such that $y \in W^{1,2}(0, T; X)$.

Theorem 2. *Consider the notation (6). If the assumptions (14)–(16) are satisfied, the system (2)–(5) admits a unique positive global strong solution $\mathbf{u} \in W^{1,2}(0, T; H)^2$ such that $\mathbf{u} \in \mathbf{L}(T, \Omega)$. Furthermore, the estimate:*

$$\|\partial_t \mathbf{u}\|_{L^2(Q_T)^2} + \|\mathbf{u}\|_{L^2(0, T; H^2(\Omega))^2} + \|\mathbf{u}(\cdot, t)\|_{H^1(\Omega)} + \|\mathbf{u}\|_{L^\infty(Q_T)^2} \leq C, \quad (19)$$

is satisfied.

Proof. We rewrite the system (2)–(5) in the context of Banach spaces. Let us consider the following notation:

$$y = (u_1, u_2), \quad y^0 = (u_{10}, u_{20}), \quad H = L^2(\Omega)^3, \quad A : D(A) \subseteq H \rightarrow H, \quad (20)$$

$$Ay = (\Delta y_1, \Delta y_2), \quad \forall y \in D(A) = \left\{ y \in H^2(\Omega)^2 : \nabla y_1 \cdot \nu = \nabla y_2 \cdot \nu = 0 \text{ on } \partial\Omega \right\}, \quad (21)$$

$$f(t, y(t)) = \left(-\beta(x)\phi(y_1(t), y_2(t)) + \gamma(x)y_2(t), \beta(x)\phi(y_1(t), y_2(t)) - \gamma(x)y_2(t) \right). \quad (22)$$

We observe that the initial boundary value problem (2)–(5) can be rewritten as the abstract Cauchy problem (18), but we can not apply the Theorem 1, since f is not Lipschitz in the second variable. Then, we consider the truncated Cauchy problem:

$$\frac{d}{dt} y^N(t) = Ay^N(t) + f^N(t, y^N(t)), \quad t \in [0, T], \quad (23a)$$

$$y^N(0) = y^0, \quad (23b)$$

where f^N is defined in terms of the function f and $N > 0$, as follows:

$$f^N(t, y(t)) = \begin{cases} f(t, y(t)), & y(t) \in [-N, N]^2, \\ f(t, \hat{y}(t)), & y(t) \in \mathbb{R}^2 - [-N, N]^2, \end{cases} \quad (23c)$$

where

$$\hat{y}(t) = \left(\min\{|y_1(t)|, N\} \operatorname{sgn}(y_1(t)), \min\{|y_2(t)|, N\} \operatorname{sgn}(y_2(t)) \right).$$

We notice that the function f^N satisfies the hypothesis of Theorem 1, and consequently, (23) admits a unique strong solution $y^N \in W^{1,2}(0, T; H)$ with $y^N \in L^2(0, T; H^2(\Omega))^2 \cap L^\infty(0, T; H^1(\Omega))^2$.

Let us prove that that y^N is bounded on Q_T . Let us consider the following notation:

$$M = \max \left\{ \|f^N\|_{L^\infty(Q_T)}, \|y^0\|_{L^\infty(\Omega)} \right\}, \quad \mathbb{A} : D(\mathbb{A}) \subseteq L^2(\Omega) \rightarrow L^2(\Omega), \quad (24)$$

$$\mathbb{A}y = \Delta y \quad \forall y \in D(\mathbb{A}) = \left\{ y \in H^2(\Omega) : \nabla y \cdot \nu = 0 \text{ on } \partial\Omega \right\}, \quad (25)$$

$$\left\{ S(t), t \geq 0 \right\} \quad \text{the } C_0\text{-semi-group generated by the operator } \mathbb{A}, \quad (26)$$

then, for $i = 1, 2$, we can deduce that the functions $Y_i^N(x, t) = y_i^N(x, t) \pm Mt \pm \|y^0\|_{L^\infty(\Omega)}$ are solutions of the following initial value problems:

$$\frac{d}{dt} Y_i^N(t) = \mathbb{A}Y_i^N(t) + f_i^N(t, y^N(t)) \pm M, \quad t \in [0, T], \quad (27a)$$

$$Y_i^N(0) = y_i^0 \pm \|y_i^0\|_{L^\infty(\Omega)}, \quad (27b)$$

for $i = 1, 2$. The mild solutions of (27) are given by:

$$Y_i^N(t) = S(t) \left(y_i^0 \pm \|y_i^0\|_{L^\infty(\Omega)} \right) + \int_0^t S(t-s) \left(f_i^N(s, y^N(s)) \pm M \right) ds, \quad (28)$$

for $i = 1, 2$. The relation (28) implies that $y_i^N(t, x) \in \left[-Mt - \|y^0\|_{L^\infty(\Omega)}, Mt + \|y^0\|_{L^\infty(\Omega)} \right]$ for $(x, t) \in Q_T$, since, by the definition of M , we can deduce that $y_i^0 - \|y_i^0\|_{L^\infty(\Omega)} \leq 0$, $f_i^N(s, y^N(s)) - M \leq 0$, $y_i^0 + \|y_i^0\|_{L^\infty(\Omega)} \geq 0$, and $f_i^N(s, y^N(s)) - M \geq 0$. Thus, we have deduced that $y^N \in L^\infty(Q_T)$.

Let us prove that y^N is positive on Q_T . From (23), we have that:

$$\partial_t y_1^N - \Delta y_1^N = -\beta(x)\phi(y_1^N, y_2^N) + \gamma(x)y_2^N, \quad \text{in } Q_T, \quad (29)$$

$$\partial_t y_2^N - \Delta y_2^N = \beta(x)\phi(y_1^N, y_2^N) - \gamma(x)y_2^N, \quad \text{in } Q_T, \quad (30)$$

$$\nabla y_1^N \cdot \nu = \nabla y_2^N \cdot \nu = 0, \quad \text{in } \Gamma_T, \quad (31)$$

$$(y_1^N, y_2^N)(x, 0) = (u_{10}, u_{20})(x) := \mathbf{u}_0(x), \quad \text{on } \Omega. \quad (32)$$

Let us introduce the following notation:

$$y_i^N = (y_i^N)^+ - (y_i^N)^-, \quad (y_i^N)^+(t, x) = \sup \left\{ y_i^N(t, x), 0 \right\}, \quad (y_i^N)^-(t, x) = -\inf \left\{ y_i^N(t, x), 0 \right\},$$

for $(x, t) \in Q_T$ and $i = 1, 2$. Multiplying (30) by $(y_2^N)^-$ and integrating on Ω , we obtain:

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} \left((y_2^N)^- \right)^2 dx \right) \\ &= \int_{\Omega} \left| \nabla (y_2^N)^- \right|^2 dx + \int_{\Omega} \beta(x)\phi(y_1^N, y_2^N)(y_2^N)^- dx - \int_{\Omega} \gamma(x)y_2^N(y_2^N)^- dx \\ &= \int_{\Omega} \left| \nabla (y_2^N)^- \right|^2 dx + \int_{\Omega} \beta(x)\phi(y_1^N, y_2^N)(y_2^N)^- dx + \int_{\Omega} \gamma(x) \left((y_2^N)^- \right)^2 dx \\ &\geq - \int_{\Omega} \beta(x) \frac{y_1^N}{y_1^N + y_2^N} \left((y_2^N)^- \right)^2 dx \geq -\beta^* \int_{\Omega} \left((y_2^N)^- \right)^2 dx. \end{aligned}$$

By application of Gronwall inequality, we obtain that $(y_2^N)^-(x, t) = 0$ on Q_T , which implies that $y_2^N(x, t) \geq 0$ on Q_T . Similarly, multiplying (29) by $(y_1^N)^-$ and integrating on Ω , we deduce the following inequality:

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} \left((y_1^N)^- \right)^2 dx \right) \\ &= \int_{\Omega} \left| \nabla (y_1^N)^- \right|^2 dx - \int_{\Omega} \beta(x) \phi(y_1^N, y_2^N) (y_1^N)^- dx + \int_{\Omega} \gamma(x) y_2^N (y_1^N)^- dx \\ &\geq -\beta^* \int_{\Omega} \left((y_1^N)^- \right)^2 dx, \end{aligned}$$

which implies that $(y_1^N)^-(x, t) = 0$ on Q_T , and consequently, $y_1^N(x, t) \geq 0$ on Q_T . Thus, we deduce that y^N is positive on Q_T .

From (2) and integration by parts, we obtain:

$$\begin{aligned} & \int_0^t \int_{\Omega} \left| \partial_s u_1 \right|^2 ds dx + \int_0^t \int_{\Omega} \left| \Delta u_1 \right|^2 ds dx \\ &= 2 \int_0^t \int_{\Omega} \partial_s u_1 \Delta u_1 ds dx + \int_0^t \int_{\Omega} \left(-\beta(x) \phi(u_1, u_2) + \gamma(x) u_2 \right)^2 ds dx \\ &= -2 \int_{\Omega} \left| \Delta u_1 \right|^2 dx + 2 \int_{\Omega} \left| \Delta u_{10} \right|^2 dx + \int_0^t \int_{\Omega} \left(-\beta(x) \phi(u_1, u_2) + \gamma(x) u_2 \right)^2 ds dx. \end{aligned} \quad (33)$$

Similarly, Equation (3) implies that:

$$\begin{aligned} & \int_0^t \int_{\Omega} \left| \partial_s u_2 \right|^2 ds dx + \int_0^t \int_{\Omega} \left| \Delta u_2 \right|^2 ds dx + 2 \int_{\Omega} \left| \Delta u_2 \right|^2 dx \\ &= 2 \int_{\Omega} \left| \Delta u_{20} \right|^2 dx + \int_0^t \int_{\Omega} \left(\beta(x) \phi(u_1, u_2) - \gamma(x) u_2 \right)^2 ds dx. \end{aligned} \quad (34)$$

Then, using the facts that $\mathbf{u}_0 \in H^2(\Omega)^2$, $(\beta, \gamma) \in \mathcal{U}_{ad}$ and $\mathbf{u}$ is bounded on Q_T , we deduce that $\mathbf{u} \in \mathbf{L}(T, \Omega)$ and clearly (19) is satisfied. $\square$

3. Existence of an Optimal Solution

Theorem 3. *If the assumptions (14)–(17) are satisfied, the optimal control problem (13) admits at least one solution.*

Proof. Let us consider the notation $\bar{J} = \inf\{J(\mathbf{u}, \beta, \alpha) : (\mathbf{u}, \beta, \alpha) \in \mathcal{D}\}$, where $\mathbf{u}$ is the corresponding solution of (2)–(5) for $(\beta, \alpha) \in \mathcal{U}_{ad}$. We observe that $\bar{J}$ is finite, and consequently, there exists a sequence $(\mathbf{u}^n, \beta^n, \alpha^n) \in E$ such that:

$$\bar{J} \leq J(\mathbf{u}^n, \beta^n, \alpha^n) \leq \bar{J} + \frac{1}{n} \quad (35)$$

and satisfying the identity $M(\mathbf{u}^n, \beta^n, \alpha^n) = 0$.

Let us study the compactness of the sequence $\mathbf{u}^n$. We observe that the relation $M(\mathbf{u}^n, \beta^n, \alpha^n) = 0$ is equivalently to:

$$\partial_t u_1^n - \Delta u_1^n = -\beta^n(x) \phi(u_1^n, u_2^n) + \gamma^n(x) u_2^n, \quad \text{in } Q_T, \quad (36)$$

$$\partial_t u_2^n - \Delta u_2^n = \beta^n(x) \phi(u_1^n, u_2^n) - \gamma^n(x) u_2^n, \quad \text{in } Q_T, \quad (37)$$

$$\nabla u_1^n \cdot \nu = \nabla u_2^n \cdot \nu = 0, \quad \text{in } \Gamma_T, \quad (38)$$

$$(u_1^n, u_2^n)(x, 0) = (u_{10}, u_{20})(x) := \mathbf{u}_0(x), \quad \text{on } \Omega. \quad (39)$$

Then, by application of Theorem 2 to system (36)–(39), we have that $\mathbf{u}^n \in \mathbf{L}(T, \Omega)$ and that it satisfies the following estimates:

$$\|\partial_t \mathbf{u}^n\|_{L^2(Q_T)^2} \leq C, \quad \|\mathbf{u}^n\|_{L^2(0,T;H^2(\Omega))^2} \leq C, \quad \|\mathbf{u}^n(\cdot, t)\|_{H^1(\Omega)} \leq C, \quad \|\mathbf{u}^n\|_{L^\infty(Q_T)^2} \leq C. \quad (40)$$

Moreover, we note that:

$$\begin{aligned} \int_{\Omega} (u_1^n)^2(t, x) dx &= \int_{\Omega} (u_{10})^2(x) dx \\ &\quad + 2 \int_0^t \int_{\Omega} \left[(\Delta u_1^n) u_1^n - \beta^n(x) \phi(u_1^n, u_2^n) u_1^n + \gamma^n(x) u_2^n u_1^n \right] dx d\zeta, \end{aligned} \quad (41)$$

$$\begin{aligned} \int_{\Omega} (u_2^n)^2(t, x) dx &= \int_{\Omega} (u_{20})^2(x) dx \\ &\quad + 2 \int_0^t \int_{\Omega} \left[(\Delta u_2^n) u_2^n + \beta^n(x) \phi(u_1^n, u_2^n) u_1^n - \gamma^n(x) u_2^n u_1^n \right] dx d\zeta, \end{aligned} \quad (42)$$

for all $t \in [0, T]$, by testing (36) and (37) by u_1^n and u_2^n , respectively. Then, from (40), the relations (41) and (42), we obtain:

$$\left| \int_{\Omega} (u_i^n)^2(t, x) dx - \int_{\Omega} (u_i^n)^2(s, x) dx \right| \leq C|s - t|, \quad i = 1, 2. \quad (43)$$

Thus, we deduce the following two compactness properties related to the sequence $\mathbf{u}^n$: (i) $\mathbf{u}^n(\cdot, t)$ is compact in $L^2(\Omega)^2$ and (ii) $\mathbf{u}^n$ is compact in $C([0, T]; L^2(\Omega))^2$. The first result is deduced from (40) and the compact embedded of $H^1(\Omega)$ in $L^2(\Omega)$ and the second ones is proved by (43) and application of the Ascoli–Arzela Theorem.

From the compactness results, the estimates (40), and the definition of $\mathcal{U}_{ad}$, we determine that there is $(\bar{\mathbf{u}}, \bar{\beta}, \bar{\alpha}) \in E$ and a subsequence of the minimizing sequence, denoted again by $(\mathbf{u}^n, \beta^n, \alpha^n)$, such that:

$$\mathbf{u}^n \rightharpoonup \bar{\mathbf{u}} \text{ in } L^2(\Omega)^2 \text{ uniformly with respect to } t, \quad \Delta \mathbf{u}^n \rightharpoonup \Delta \bar{\mathbf{u}} \text{ in } L^2(Q_T)^2, \quad (44)$$

$$\partial_t \mathbf{u}^n \rightharpoonup \partial_t \bar{\mathbf{u}} \text{ in } L^2(Q_T)^2, \quad \mathbf{u}^n \rightharpoonup \bar{\mathbf{u}} \text{ in } L^2(0, T; H^2(\Omega))^2, \quad (45)$$

$$\mathbf{u}^n \rightharpoonup \bar{\mathbf{u}} \text{ in } L^2(0, T; H^2(\Omega))^2, \quad (\beta^n, \alpha^n) \rightharpoonup (\bar{\beta}, \bar{\alpha}) \text{ in } L^\infty(\Omega)^2. \quad (46)$$

Hence, passing to the limit in (36)–(39), we obtain that $(\bar{\mathbf{u}}, \bar{\beta}, \bar{\alpha}) \in \mathcal{D}$, and from (35), we deduce that $\bar{J} = (\bar{\mathbf{u}}, \bar{\beta}, \bar{\alpha})$. $\square$

4. Some Fundamental Results for Dubovitskii–Milyutin Formalism Application

Theorem 4. Consider the application M given on (8)–(12) and defining the constraint of the problem (13). Then, the following assertions hold:

(a) The descent and dual cones of J are defined by the following sets:

$$\begin{aligned} DC(J, (\mathbf{u}, \beta, \gamma)) &= \left\{ (\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) \in E : \left\langle \mathbf{u} - \mathbf{u}^{obs}, \tilde{\mathbf{u}} \right\rangle + \delta \left\langle \nabla(\beta, \gamma), \nabla(\tilde{\beta}, \tilde{\gamma}) \right\rangle < 0 \right\}, \\ \left[DC(J, (\mathbf{u}, \beta, \gamma)) \right]^* &= \{ -\lambda J' : \lambda \geq 0 \}. \end{aligned}$$

Hereinafter $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$ are the canonical inner products of $L^2(\Omega)$ and $L^2(\Omega)^2$, respectively.

- (b) M is Gâteaux differentiable with Gâteaux derivative M'_G , satisfying $M'_G(\mathbf{u}, \beta, \gamma)(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}) = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4)$ for any $(\mathbf{u}, \beta, \gamma) \in E$ if and only if:

$$\partial_t \hat{u}_1 - \Delta \hat{u}_1 + \beta(x) \nabla \phi(\mathbf{u}) \cdot \hat{\mathbf{u}} + \hat{\beta}(x) \phi(\mathbf{u}) - \gamma(x) \hat{u}_2 - \hat{\gamma}(x) u_2 = \hat{\psi}_1, \quad \text{in } Q_T, \quad (47)$$

$$\partial_t \hat{u}_2 - \Delta \hat{u}_2 - \beta(x) \nabla \phi(\mathbf{u}) \cdot \hat{\mathbf{u}} - \hat{\beta}(x) \phi(\mathbf{u}) + \gamma(x) \hat{u}_2 + \hat{\gamma}(x) u_2 = \hat{\psi}_2, \quad \text{in } Q_T, \quad (48)$$

$$\nabla \hat{u}_1 \cdot \boldsymbol{\nu} = \nabla \hat{u}_2 \cdot \boldsymbol{\nu} = 0, \quad \text{in } \Gamma_T, \quad (49)$$

$$(\hat{u}_1, \hat{u}_2)(x, 0) = (\hat{\psi}_3, \hat{\psi}_4), \quad \text{on } \Omega. \quad (50)$$

- (c) M is strictly differentiable and $M'(\mathbf{u}, \beta, \gamma) = M'_G(\mathbf{u}, \beta, \gamma)$ is a surjective operator.

- (d) The tangent and dual cones to the set $\mathcal{D}$ at $(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma})$ are given by the following sets:

$$TC(\mathcal{D}, (\mathbf{u}, \beta, \gamma)) = \{(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}) \in E : M'(\mathbf{u}, \beta, \gamma)(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}) = 0\}, \quad (51)$$

$$[TC(\mathcal{D}, (\mathbf{u}, \beta, \gamma))]^* = \left\{ g \in E' : g(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}) = 0, \forall (\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}) \in TC(\mathcal{D}, (\mathbf{u}, \beta, \gamma)) \right\}, \quad (52)$$

respectively. Moreover, $TC(\mathcal{D}(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma}))$ is a vector space.

Proof. [(a)] From the definition of J given on (1), we deduced that J is Gâteaux differentiable and continuous, and therefore, it is Frechet differentiable and J' is defined by:

$$J'(\mathbf{u}, \beta, \gamma)(\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) = 2 \left\langle \mathbf{u} - \mathbf{u}^{obs}, \tilde{\mathbf{u}} \right\rangle + 2\delta \left\langle \nabla(\beta, \gamma), \nabla(\tilde{\beta}, \tilde{\gamma}) \right\rangle. \quad (53)$$

Moreover, J' is a convex and continuous operator, then from Theorem 7.3 of [27], we deduce that J is regularly decreasing for all $(\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) \in E$ and

$$\begin{aligned} DC(J, (\mathbf{u}, \beta, \gamma)) &= \left\{ (\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) \in E : J'(\mathbf{u}, \beta, \gamma)(\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) < 0 \right\} \\ &= \left\{ (\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) \in E : 2 \left\langle \mathbf{u} - \mathbf{u}^{obs}, \tilde{\mathbf{u}} \right\rangle + 2\delta \left\langle \nabla(\beta, \gamma), \nabla(\tilde{\beta}, \tilde{\gamma}) \right\rangle < 0 \right\}, \end{aligned}$$

since $DC(J, (\mathbf{u}, \beta, \gamma))$ is given the continuous linear functional $J'(\mathbf{u}, \beta, \gamma)$, its dual cone is defined by

$$[DC(J, (\mathbf{u}, \beta, \gamma))]^* = \left\{ -\lambda J'(\mathbf{u}, \beta, \gamma) : \lambda \geq 0 \right\},$$

for details, we refer to ([27], p. 69).

[(b)] Considering the notation $\delta \mathbf{M} := M(\mathbf{u} + k\hat{\mathbf{u}}, \beta + k\hat{\beta}, \gamma + k\hat{\gamma}) - M(\mathbf{u}, \beta, \gamma)$ and using the definition of M given on (9)–(12), we obtain that the components of $\delta \mathbf{M}$ are given by:

$$\begin{aligned} \delta \mathbf{M}_1 &= k\partial_t \hat{u}_1 - k\Delta \hat{u}_1 + [(\beta + k\hat{\beta})(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \beta(x)\phi(\mathbf{u})] \\ &\quad - [(\gamma + k\hat{\gamma})(x)(u_2 + k\hat{u}_2) - \gamma(x)u_2], \\ \delta \mathbf{M}_2 &= k\partial_t \hat{u}_2 - k\Delta \hat{u}_2 - [(\beta + k\hat{\beta})(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \beta(x)\phi(\mathbf{u})] \\ &\quad + [(\gamma + k\hat{\gamma})(x)(u_2 + k\hat{u}_2) - \gamma(x)u_2], \\ \delta \mathbf{M}_3 &= k\hat{u}_1(x, 0), \quad \delta \mathbf{M}_4 = k\hat{u}_2(x, 0). \end{aligned}$$

Then, using the relations for $\hat{\psi}_i$ considered on system (47)–(50), we deduce that:

$$\frac{\delta \mathbf{M}_1}{k} - \hat{\psi}_1 = \frac{1}{k} \left[(\beta + k\hat{\beta})(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \beta(x)\phi(\mathbf{u}) \right] - \frac{1}{k} \left[(\gamma + k\hat{\gamma})(x)(u_2 + k\hat{u}_2) - \gamma(x)u_2 \right] - \beta(x)\nabla\phi(\mathbf{u}) \cdot \hat{\mathbf{u}} - \hat{\beta}(x)\phi(\mathbf{u}) + \gamma(x)\hat{u}_2 + \hat{\gamma}(x)u_2, \quad (54)$$

$$\frac{\delta \mathbf{M}_2}{k} - \hat{\psi}_2 = -\frac{1}{k} \left[(\beta + k\hat{\beta})(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \beta(x)\phi(\mathbf{u}) \right] + \frac{1}{k} \left[(\gamma + k\hat{\gamma})(x)(u_2 + k\hat{u}_2) - \gamma(x)u_2 \right] + \beta(x)\nabla\phi(\mathbf{u}) \cdot \hat{\mathbf{u}} + \hat{\beta}(x)\phi(\mathbf{u}) - \gamma(x)\hat{u}_2 - \hat{\gamma}(x)u_2, \quad (55)$$

$$\frac{\delta \mathbf{M}_3}{k} - \hat{\psi}_3 = \frac{\delta \mathbf{M}_4}{k} - \hat{\psi}_4 = 0. \quad (56)$$

We observe that the following identity:

$$\begin{aligned} & \frac{1}{k} \left[(\zeta + k\hat{\zeta})(x)H(a + k\hat{a}, b + k\hat{b}) - \zeta(x)H(a, b) \right] \\ &= \zeta(x) \left[\frac{H(a + k\hat{a}, b + k\hat{b}) - H(a, b + k\hat{b})}{a + k\hat{a} - a} \right] \hat{a} + \zeta(x) \left[\frac{H(a, b + k\hat{b}) - H(a, b)}{b + k\hat{b} - b} \right] \hat{b} \\ &+ \hat{\zeta}(x) H(a + k\hat{a}, b + k\hat{b}), \quad \forall (a, b, \hat{a}, \hat{b}) \in \mathbb{R}^4, \forall k > 0, \end{aligned}$$

is valid for any functions $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\zeta, \hat{\zeta} : \mathbb{R} \rightarrow \mathbb{R}$. In particular, for the case of $(H, \zeta, \hat{\zeta}) = (\phi(\mathbf{u}), \beta, \hat{\beta})$ and $(H, \zeta, \hat{\zeta}) = (u_2, \gamma, \hat{\gamma})$, this implies that

$$\begin{aligned} & \frac{1}{k} \left[(\beta + k\hat{\beta})(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \beta(x)\phi(\mathbf{u}) \right] \\ &= \beta(x) \left(\frac{\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \phi(u_1, u_2 + k\hat{u}_2)}{u_1 + k\hat{u}_1 - u_1}, \frac{\phi(u_1, u_2 + k\hat{u}_2) - \phi(\mathbf{u})}{u_2 + k\hat{u}_2 - u_2} \right) \cdot \hat{\mathbf{u}} + \hat{\beta}(x)\phi(\mathbf{u} + k\hat{\mathbf{u}}), \end{aligned} \quad (57)$$

$$\frac{1}{k} \left[(\gamma + k\hat{\gamma})(x)(u_2 + k\hat{u}_2) - \gamma(x)u_2 \right] = \gamma(x)\hat{u}_2 + \hat{\gamma}(x)(u_2 + k\hat{u}_2), \quad (58)$$

respectively. Now, replacing (57) and (58) in (54) and (55), and using the fact that:

$$\lim_{k \rightarrow 0} \left\| \left(\frac{\phi(\mathbf{u} + k\hat{\mathbf{u}}) - \phi(u_1, u_2 + k\hat{u}_2)}{u_1 + k\hat{u}_1 - u_1}, \frac{\phi(u_1, u_2 + k\hat{u}_2) - \phi(\mathbf{u})}{u_2 + k\hat{u}_2 - u_2} \right) - \nabla\phi(\mathbf{u}) \right\|_{\tilde{E}} = 0,$$

we deduce that $\|\delta \mathbf{M}/k - (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4)\|_{\tilde{E}} \rightarrow 0$, when $k \rightarrow 0$. Hence, by the Gâteaux derivative definition, we conclude the proof of item [(b)].

[(c)] We prove that M' is a surjective operator, by arbitrarily selecting $(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) \in \tilde{E}$ and then applying similar ideas to Theorem 2 to deduce that the solution of (47)–(50) belong to $(\hat{u}_1, \hat{u}_2) \in W^{1,2}(0, T; H)^2$. Furthermore, using the facts that M' is continuous, to obtain that M is strictly differentiable is enough to prove that the application $(\mathbf{u}, \beta, \gamma) \mapsto M'(\mathbf{u}, \beta, \gamma)$ is continuous or equivalently that $M'(\mathbf{u}, \beta, \gamma)$ is bounded in $\tilde{E}$, which can be deduced by the definition of the norm in $\tilde{E}$ and the continuous Sobolev inclusion $H^2(\Omega) \subset L^\infty(\Omega)$.

[(d)] From the items (b) and (c) of Theorem 4, we have that M is Gâteaux differentiable and in a neighborhood of $(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma})$, and $M'(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma})$ is continuous in a neighborhood of $(\hat{\mathbf{u}}, \hat{\beta}, \hat{\gamma})$ and surjective. By applying the Theorem of Lyusternik (see for instance [38]) we obtain that $\mathcal{D}$ at $(\mathbf{u}, \beta, \gamma)$ is the kernel of the operator $M'(\mathbf{u}, \beta, \gamma)$ and is given by the set defined on (51). In particular, $TC(\mathcal{D}, \mathbf{u}, \beta, \gamma)$ is a vector space, since is the kernel of a linear operator. Meanwhile, the proof of (52) is a consequence of the definition of dual cone. $\square$

5. First-Order Necessary Optimality Condition and a Uniqueness Result

Theorem 5. Let us consider the hypothesis of Theorem 3. If $(\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma})$ is a solution of the optimal control problem (13), then there exists a $\mathbf{p} = (p_1, p_2) \in L^\infty(Q_T)^2$ solution of the following system:

$$\partial_t p_1 + \Delta p_1 = -\bar{\beta}(x) \partial_1 \phi(\bar{\mathbf{u}})(p_1 - p_2), \quad \text{in } Q_T, \quad (59)$$

$$\partial_t p_2 + \Delta p_2 = [\bar{\beta}(x) \partial_2 \phi(\bar{\mathbf{u}}) - \bar{\gamma}(x)](p_1 - p_2), \quad \text{in } Q_T, \quad (60)$$

$$\nabla p_1 \cdot \nu = \nabla p_2 \cdot \nu = 0, \quad \text{in } \Gamma_T, \quad (61)$$

$$\mathbf{p}(x, T) = \bar{\mathbf{u}}(x, T) - \mathbf{u}^{obs}(x), \quad \text{on } \Omega, \quad (62)$$

such that the following inequality:

$$\begin{aligned} \int_0^T \int_\Omega \left\{ ((\tilde{\gamma} - \bar{\gamma})(x) \bar{u}_2 - (\tilde{\beta} - \bar{\beta})(x) \phi(\bar{\mathbf{u}}))(p_1 - p_2) \right\} dx dt \\ + \delta \left\langle \nabla(\beta, \gamma), \nabla(\tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) \right\rangle \geq 0, \end{aligned} \quad (63)$$

is satisfied for any $(\tilde{\beta}, \tilde{\gamma}) \in \mathcal{U}_{ad}$.

Proof. To prove the existence of solutions for (59)–(62), we adapt appropriately the corresponding proof given in ([8], Theorem 3.3), which, in a broad sense, consists of the following three steps: (i) transform the system (59)–(62) in an initial boundary value problem by introducing the change of variable $s = T - t$ and for $(x, s) \in Q_T$, such that $\mathbf{w}(x, s) = \mathbf{p}(x, T - s)$ and $\bar{\mathbf{u}}(x, s) = \mathbf{u}(x, T - s)$; (ii) the existence of a solution for the transformed system (consequently, for (59)–(62)) is deduced by applying similar ideas to those used in the proof of Theorem 2; and (iii) the fact that $\mathbf{p} \in L^\infty(Q_T)^2$ is a consequence of H^2 energy estimates and the continuous embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$.

If $(\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma})$ is an optimal solution of problem (13), we have that:

$$DC(J, (\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma})) \cap TC(\mathcal{D}, (\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma})) = \emptyset.$$

Then, by the Dubovitskii and Milyutin Theorem [27], we have that there exist two continuous functionals $g_1 \in [DC(J, (\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma}))]^*$ and $g_2 \in [TC(\mathcal{D}, (\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma}))]^*$, not both identically zero, that satisfy the Euler–Lagrange equation:

$$g_1 + g_2 = 0. \quad (64)$$

Furthermore, let us consider $(\tilde{\mathbf{u}}, \tilde{\beta}, \tilde{\gamma}) \in E$ satisfying the following system:

$$\partial_t \tilde{u}_1 - \Delta \tilde{u}_1 + \tilde{\beta}(x) \nabla \phi(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}} + \tilde{\beta}(x) \phi(\tilde{\mathbf{u}}) - \tilde{\gamma}(x) \tilde{u}_2 - \tilde{\gamma}(x) \bar{u}_2 = 0, \quad \text{in } Q_T, \quad (65)$$

$$\partial_t \tilde{u}_2 - \Delta \tilde{u}_2 - \tilde{\beta}(x) \nabla \phi(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}} - \tilde{\beta}(x) \phi(\tilde{\mathbf{u}}) + \tilde{\gamma}(x) \tilde{u}_2 + \tilde{\gamma}(x) \bar{u}_2 = 0, \quad \text{in } Q_T, \quad (66)$$

$$\nabla \tilde{u}_1 \cdot \nu = \nabla \tilde{u}_2 \cdot \nu = 0, \quad \text{in } \Gamma_T, \quad (67)$$

$$\tilde{\mathbf{u}}(x, 0) = 0, \quad \text{on } \Omega. \quad (68)$$

We notice that the application of Theorem 4 implies that $(\tilde{\mathbf{u}}, \tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) \in TC(\mathcal{D}, (\bar{\mathbf{u}}, \bar{\beta}, \bar{\gamma}))$, and consequently, $g_2(\tilde{\mathbf{u}}, \tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) = 0$. From the characterization of decent and dual cones given on Theorem 4 it follows that:

$$g_1(\tilde{\mathbf{u}}, \tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) = \lambda \left\langle \bar{\mathbf{u}} - \mathbf{u}^{obs}, \tilde{\mathbf{u}} \right\rangle + \lambda \delta \left\langle \nabla(\bar{\beta}, \bar{\gamma}), \nabla(\tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) \right\rangle \geq 0$$

for some $\lambda \geq 0$. We note that $\lambda \neq 0$, since if we assume that $\lambda = 0$, we have that $g_1 = 0$, and from Equation (64), we deduce that $g_2 = 0$, which is a contradiction with the Dubovitskii and Milyutin Theorem. Moreover, we have that:

$$\left\langle \bar{\mathbf{u}} - \mathbf{u}^{obs}, \tilde{\mathbf{u}} \right\rangle + \delta \left\langle \nabla(\bar{\beta}, \bar{\gamma}), \nabla(\tilde{\beta} - \bar{\beta}, \tilde{\gamma} - \bar{\gamma}) \right\rangle \geq 0 \quad (69)$$

by dividing (64) by λ and we have fixed $\lambda = 1$, for presentation convenience.

On the other hand, by multiplying the first and second equations of system (59)–(62) by $\tilde{u}_1$ and $\tilde{u}_2$, respectively, and integrating by parts over Q_T , we obtain:

$$\begin{aligned} & \int_{\Omega} (\bar{u}_1(x, T) - u_1^{obs}(x)) \tilde{u}_1(x, T) dx \\ &= \int_{Q_T} \left\{ [\partial_t \tilde{u}_1 - \Delta \tilde{u}_1 - \bar{\beta}(x) \partial_1 \phi(\bar{\mathbf{u}}) \tilde{u}_1] p_1 + \bar{\beta}(x) \partial_1 \phi(\bar{\mathbf{u}}) \tilde{u}_1 p_2 \right\} dx dt \end{aligned} \quad (70)$$

$$\begin{aligned} & \int_{\Omega} (\bar{u}_2(x, T) - u_2^{obs}(x)) \tilde{u}_2(x, T) dx \\ &= \int_{Q_T} \left\{ [\partial_t \tilde{u}_2 - \Delta \tilde{u}_2 + (\bar{\beta} \partial_2 \phi(\bar{\mathbf{u}}) - \bar{\gamma}) \tilde{u}_2] p_2 - (\bar{\beta} \partial_2 \phi(\bar{\mathbf{u}}) - \bar{\gamma}) \tilde{u}_2 p_1 \right\} dx dt. \end{aligned} \quad (71)$$

Summing (70) and (71) and using Equations (65) and (66), we obtain:

$$\begin{aligned} \langle \bar{\mathbf{u}} - \mathbf{u}^{obs}, \bar{\mathbf{u}} \rangle &= \int_0^T \int_{\Omega} \left\{ [\partial_t \tilde{u}_1 - \Delta \tilde{u}_1 - \bar{\beta} \nabla \phi(\bar{\mathbf{u}}) \cdot \bar{\mathbf{u}} + \bar{\gamma} \tilde{u}_2] p_1 + [\partial_t \tilde{u}_2 - \Delta \tilde{u}_2 + \bar{\beta} \nabla \phi(\bar{\mathbf{u}}) \cdot \bar{\mathbf{u}} - \bar{\gamma} \tilde{u}_2] p_2 \right\} dx dt \\ &= \int_0^T \int_{\Omega} \left\{ [-(\tilde{\beta} - \bar{\beta})(x) \phi(\bar{\mathbf{u}}) + (\tilde{\gamma} - \bar{\gamma})(x) \bar{u}_2] p_1 + [(\tilde{\beta} - \bar{\beta})(x) \phi(\bar{\mathbf{u}}) - (\tilde{\gamma} - \bar{\gamma})(x) \bar{u}_2] p_2 \right\} dx dt \\ &= \int_0^T \int_{\Omega} \left\{ ((\tilde{\gamma} - \bar{\gamma})(x) \bar{u}_2 - (\tilde{\beta} - \bar{\beta})(x) \phi(\bar{\mathbf{u}})) (p_1 - p_2) \right\} dx dt. \end{aligned} \quad (72)$$

Thus, replacing (72) in (69), we deduce (63). $\square$

Using the properties of $\mathbf{u}$ and $\mathbf{p}$ and the first-order optimality condition, we can prove the continuous dependence of state variables and adjoint system solutions with respect to the coefficients and observation functions, and we can also deduce the local uniqueness of the optimal control problem. More precisely, we obtain the theorem.

Theorem 6. *Let us consider the hypothesis of Theorem 3 and the notation of Theorem 5. Then, the following mappings:*

$$\begin{aligned} (\beta, \gamma) &\mapsto \mathbf{u} \text{ from } U_{ad} \subset [L^2(\Omega)]^2 \text{ to } L^\infty(0, t; L^2(\Omega))^2, \\ (\beta, \gamma, \mathbf{u}^{obs}) &\mapsto \mathbf{p} \text{ from } U_{ad}(\Omega) \times L^2(\Omega) \times L^2(\Omega) \subset [L^2(\Omega)]^4 \text{ to } L^\infty(0, t; L^2(\Omega)), \end{aligned}$$

are continuous for almost all time t in $[0, T]$. Moreover, given $\mathbf{c} = (c_1, c_2) \in \mathbb{R}_+^2$ (fixed), there exists $\bar{\gamma} \in \mathbb{R}^+$ such that the solution of (13) is uniquely defined, up an additive constant, on the set $\mathcal{U}_{\mathbf{c}}(\Omega) = \left\{ (\beta, \gamma) \in U_{ad}(\Omega) : \int_{\Omega} (\beta, \gamma) d\mathbf{x} = \mathbf{c} \right\}$ in the $L^2(\Omega)$ sense for any regularization parameter $\gamma > \bar{\gamma}$.

Proof. We omit the details of the proof, since it can be deduced by a straightforward application of the proof of ([22], Theorem 2.2, items (iv)–(vi)). $\square$

6. Conclusions

In this paper, we present the application of the Dubovitskii and Milyutin Theorem for a control problem arising in the parameter calibration of an SIS reaction–diffusion model. The observed data consist of the profiles for state variables (susceptible and infective densities) on the spatial domain at the end of the disease process. The parameter calibration or the identification problem is formulated as an optimal control problem where the cost functional consists of two terms: a comparison of the model solution with the observed data in the L^2 sense and a regularization term. We have assumed the following behavior of domain and data: the behavior of the transmission dynamics is developed under the requirements of an SIS model; the spatial domain where the epidemics occur is an open and bounded set; neither emigration nor immigration through the boundary spatial domain take place during the disease process; the initial conditions are positive and belong to

H^2 ; the admissible set is a subset of the continuous functions; and the observed data are L^2 integrable functions. Thus, we have deduced the following results: the existence and uniqueness of a positive solution of the state system; the existence of optimal solutions; the explicit calculus of descent and dual cones of cost functional and some differentiability properties; and the introduction of a first-order optimality condition, which is characterized in terms of the optimal solution, the state system and their corresponding adjoint system. Moreover, we have observed that a local uniqueness result can be proved.

This paper generalizes the results of [8,22,23] in the sense that the assumption “the initial condition and coefficients belong a Hölder space” is replaced by “the initial condition belongs H^2 and coefficients are continuous functions”. However, the paper also considers the case of strong solutions. Thus, it will be interesting to study the case of the parameter identification problem for an SIS reaction–diffusion model in the context of weak solutions in a future work.

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