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Measurable Version of Spectral Decomposition Theorem for a $\mathbb{Z}^2$-Action

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Abstract: In this article, we present a measurable version of the spectral decomposition theorem for a $\mathbb{Z}^2$-action on a compact metric space. In the process, we obtain some relationships for a $\mathbb{Z}^2$-action with shadowing property and k-type weak extending property. Then, we introduce a definition of measure expanding for a $\mathbb{Z}^2$-action by using some properties of a Borel measure. We also prove one property that occurs whenever a $\mathbb{Z}^2$-action is invariantly measure expanding. All of the supporting results are necessary to prove the spectral decomposition theorem, which is the main result of this paper. More precisely, we prove that if a $\mathbb{Z}^2$-action is invariantly measure expanding, has shadowing property and has k-type weak extending property, then it has spectral decomposition.

Keywords: spectral decomposition theorem; measure expanding; shadowing property; chain recurrence; $\mathbb{Z}^d$-action

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1. Introduction

The theorem of spectral decomposition had caught the attention of many when it was first attained by Smale in 1967 [1]. Then, it was extended by Bowen four years later [2]. The ongoing efforts to generalize Smale's spectral decomposition theorem have led to a continuously evolving theorem [3–8]. Aoki was the person who first gave the topological version of the spectral decomposition theorem for homeomorphism f on compact metric spaces [3]. He claimed that expansiveness and shadowing property are conditions for spectral decomposition to take place; that is, the set of all non-wandering points $\Omega(f)$ can be decomposed into a finite union of closed and invariant subsets $\left(\Omega(f) = \cup_{i=1}^{l} S_i\right)$ such that each restricted homeomorphism on the subset $\left(f|_{S_i}\right)$ is topologically transitive. Das et al. [4] generalized the spectral decomposition theorem for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces.

The shadowing property (or pseudo-orbit tracing property) is the main role of the study about stability [9–12]. The theory of shadowing has rapidly grown for the qualitative theory of discrete dynamical systems. Intuitively, a homeomorphism is said to have shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit is being ε-shadowed by some element. There is abundant research about shadowing property (See [13–15]). In addition, there are also other notions of shadowing that have been introduced, such as weak shadowing [16], orbital shadowing [17], eventually shadowing property [5,18], finite shadowing property [15], periodic shadowing [19,20], etc.

The concept of expansive is another main key in the spectral decomposition theorem [3,4]. The model behind expansiveness is to describe the trajectory of every nearby point separate from the initial one in the system [21]. We can observe that expansive is closely related to the concept of sensitive dependence on initial conditions. Therefore, expansivity manifests the most chaotic scenario in which predictions may have no sense at all [21,22].

Extensive literature about expansive has been developed from many different research studies. It is natural to consider other notions of expansiveness, such as N-expansive [23],

G-expansive [24] and pointwise expansive [25]. Some studies have already defined expansiveness for flows [26], group actions [27], $\mathbb{Z}^d$-actions [28], etc. Morales and Sirvent were able to develop and introduce a notion of expansivity by using the theory of measures. They considered the property of having zero measure with respect to a given Borel probability measure to define the concept of expansive measure for a homeomorphism [21]. Carrasco-Olivera and Morales studied expansive measure for flows [29].

Other than that, Cordeiro et al. [30] defined another notion of expansiveness for a Borel measure, which is called strongly expansive. Then, they introduced the notion of strongly measure expansive for a homeomorphism f by characterizing that every Borel measure is strongly expansive for f. Dong et al. [5] introduced another type of expansive measure known as measure expanding to develop a measurable version of the spectral decomposition theorem for a homeomorphism on a compact metric space. They proved that if a homeomorphism is invariantly measure expanding and eventually has shadowing property on its chain recurrent set, then the phenomenon of spectral decomposition will occur. In addition, Lee and Nguyen also defined measure expanding for flows on a compact metric space and gave a measurable version of the spectral decomposition theorem specifically for flows [7].

A $\mathbb{Z}^d$-action on a topological space is known as a multidimensional discrete dynamical system, which has been described by the two properties of its mapping. There is also abundant research about $\mathbb{Z}^d$-action in many different kinds of topics (See [28,31–35]). One of the most interesting results in the study of $\mathbb{Z}^d$-action is about the spectral decomposition theorem. Oprocha developed the spectral decomposition theorem for a $\mathbb{Z}^d$-action on a compact metric space [8]. It was said that a spectral decomposition will take place for a $\mathbb{Z}^d$-action, which has shadowing property, has k-type weak pseudo-orbit extending property and is expansive. Kim and Lee proved the spectral decomposition theorem for a $\mathbb{Z}^2$-action on a compact metric space [6]. They declared that shadowing property and expansiveness are the conditions required to admit the spectral decomposition in $\mathbb{Z}^2$-action.

The main objective of our study is to give a measurable version of the spectral decomposition theorem for a $\mathbb{Z}^2$-action on a compact metric space. This paper is organized into sections. Section 2 gives all preliminary definitions required for this article. In Section 3, we discuss some results that are related to shadowing property and k-type weak extending property of a $\mathbb{Z}^2$-action. In Section 4, we introduce some terms of measure theory for a $\mathbb{Z}^2$-action. Then, we define the concept of measure expanding of a $\mathbb{Z}^2$-action, and we prove an important lemma, which is closely related to the property of measure expanding. In Section 5, we prove the main theorem of this article, which is the measurable version of the spectral decomposition theorem for a $\mathbb{Z}^2$-action on a compact metric space.

2. Preliminary Definitions

In this section, we begin with some basic notions and properties that are necessary for this paper. Let (X, ρ) be a compact metric space. We define $\rho(a, B)$ for any point $a \in X$ and any subset $B \subset X$ as

$$\rho(a, B) = \inf_{b \in B} \rho(a, b)$$

Let $U_r(x)$ be an open ball with radius $r > 0$ centered at $x \in X$. We let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the standard canonical basis such that every $\mathbf{n} \in \mathbb{Z}^2$ can be expressed as $\mathbf{n} = (n_1, n_2) = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$ where $n_1, n_2 \in \mathbb{Z}$. We let $k \in \{1, 2, 3, 4\}$, associated to $k' = (k'_1, k'_2) \in \{0, 1\}^2$ such that

$$k = 1 + \sum_{i=1}^{2} k'_i 2^{i-1}$$

Then, we say that $\mathbf{x} >^k \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ and $k \in \{1, 2, 3, 4\}$ if $(-1)^{k'_i} x_i > (-1)^{k'_i} y_i$ for $i \in \{1, 2\}$ [6,8]. We find that the relation '$>^k$' on $\mathbb{Z}^2$ is transitive through this lemma below.

Lemma 1. *For any $k \in \{1,2,3,4\}$, the k-type inequality '$>^k$' on $\mathbb{Z}^2$ has transitive relation.*

Proof. Let $k \in \{1,2,3,4\}$ and $k' = (k'_1, k'_2)$ such that $k = 1 + \sum_{i=1}^{2} k'_i 2^{i-1}$. Assume that '$>^k$' is a relation on $\mathbb{Z}^2$ such that $\mathbf{u} >^k \mathbf{v}$ whenever $(-1)^{k'_i} u_i > (-1)^{k'_i} v_i$ for each $i \in \{1,2\}$. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and $\mathbf{z} = (z_1, z_2)$. Suppose that $\mathbf{x} >^k \mathbf{y}$ and $\mathbf{y} >^k \mathbf{z}$. Then,

$$(-1)^{k'_i} x_i > (-1)^{k'_i} y_i$$

and

$$(-1)^{k'_i} y_i > (-1)^{k'_i} z_i$$

for each $i \in \{1,2\}$. Since $(-1)^{k'_i} x_i > (-1)^{k'_i} y_i > (-1)^{k'_i} z_i$ for each $i \in \{1,2\}$, then $(-1)^{k'_i} x_i > (-1)^{k'_i} z_i$ for each $i \in \{1,2\}$. Then, $\mathbf{x} >^k \mathbf{z}$. Therefore, the relation $>^k$ on $\mathbb{Z}^2$ is transitive. □

It is obvious that the k-type inequality '$>^k$' on $\mathbb{Z}^2$ for any $k \in \{1,2,3,4\}$ does not have symmetric relation. However, for any $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$, which $\mathbf{x} >^1 \mathbf{y}$ and $\mathbf{u} >^2 \mathbf{v}$, then they imply to $\mathbf{y} >^4 \mathbf{x}$ and $\mathbf{v} >^3 \mathbf{u}$, respectively. Next, a relation between two elements of $\mathbb{Z}^2$ is described as follows.

Lemma 2. *Let $k \in \{1,2,3,4\}$. For every $\mathbf{m}, \mathbf{l} \in \mathbb{Z}^2$ with $\mathbf{m} >^k \mathbf{l}$, there exists $\mathbf{r} >^k \mathbf{0} = (0,0)$ such that $\mathbf{m} = \mathbf{l} + \mathbf{r}$.*

Proof. Let $\mathbf{m} = (m_1, m_2), \mathbf{l} = (l_1, l_2), k \in \{1,2,3,4\}$ and $\mathbf{m} >^k \mathbf{l}$. Then, let $k' = (k'_1, k'_2) \in \{0,1\}^2$ such that $k = 1 + \sum_{i=1}^{2} k'_i 2^{i-1}$. Then, $(-1)^{k'_i} m_i > (-1)^{k'_i} l_i$ for all $i \in \{1,2\}$. Then, $(-1)^{k'_i} m_i - (-1)^{k'_i} l_i > 0$ for all $i \in \{1,2\}$. Then, $(-1)^{k'_i}(m_i - l_i) > (-1)^{k'_i} 0$ for all $i \in \{1,2\}$. Let $\mathbf{r} = (r_1, r_2)$ such that $r_i = m_i - l_i$ for all $i \in \{1,2\}$. Since $(-1)^{k'_i} r_i > (-1)^{k'_i} 0_i$, which $0_i = 0$ for all $i \in \{1,2\}$, then $\mathbf{r} >^k \mathbf{0}$ and $\mathbf{l} + \mathbf{r} = \mathbf{m}$. □

Besides that, we also prove another relation as in the following.

Lemma 3. *Let $k \in \{1,2,3,4\}$. If $\mathbf{x} >^k \mathbf{y}$, then $\mathbf{x} + \mathbf{m} >^k \mathbf{y} + \mathbf{m}$ for any $\mathbf{m} \in \mathbb{Z}^2$.*

Proof. Let $k \in \{1,2,3,4\}$ and $k' = (k'_1, k'_2)$ such that $k = 1 + \sum_{i=1}^{2} k'_i 2^{i-1}$. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Suppose that $\mathbf{x} >^k \mathbf{y}$. Then, $(-1)^{k'_i} x_i > (-1)^{k'_i} y_i$ for every $i \in \{1,2\}$. Let $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$. Then, $\mathbf{x} + \mathbf{m} = (x_1 + m_1, x_2 + m_2)$ and $\mathbf{y} + \mathbf{m} = (y_1 + m_1, y_2 + m_2)$. Then,

$$\begin{aligned}
(-1)^{k'_i} x_i &> (-1)^{k'_i} y_i \\
(-1)^{k'_i} x_i + (-1)^{k'_i} m_i &> (-1)^{k'_i} y_i + (-1)^{k'_i} m_i \\
(-1)^{k'_i}(x_i + m_i) &> (-1)^{k'_i}(y_i + m_i)
\end{aligned}$$

for every $i \in \{1,2\}$. Then, $\mathbf{x} + \mathbf{m} >^k \mathbf{y} + \mathbf{m}$. □

Then, we describe a strictly monotonic increasing sequence $\{\mathbf{t}_m\}_{m \in \mathbb{N}} \subset \mathbb{Z}^2$ as follows.

Lemma 4. *Let $\{\mathbf{t}_m\}_{m \in \mathbb{N}} \subset \mathbb{Z}^2$ be a sequence with $\mathbf{t}_{m+1} >^k \mathbf{t}_m$ where $k \in \{1,2,3,4\}$. Then, there exists $M \in \mathbb{N}$ such that $\mathbf{t}_n >^k \mathbf{0}$ for all $n > M$.*

Proof. Let $\{\mathbf{t}_m\}_{m \in \mathbb{N}} = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \ldots\} \subset \mathbb{Z}^2$ be a sequence such that $\mathbf{t}_2 >^k \mathbf{t}_1$, $\mathbf{t}_3 >^k \mathbf{t}_2$ and so on. If $\mathbf{t}_1 >^k \mathbf{0}$, then we are done since the relation '$>^k$' is transitive according to Lemma 1. Suppose that $\mathbf{t}_1 <^k \mathbf{0}$. Then, by Lemma 2, there exists $\mathbf{r} >^k \mathbf{0}$ such that $\mathbf{0} = \mathbf{t}_1 + \mathbf{r}$. Since $\mathbf{t}_{i+1} >^k \mathbf{t}_i$ for each $i \in \mathbb{N}$, then clearly $\mathbf{t}_{i+1} >^k \mathbf{t}_1$ for all $i \in \mathbb{N}$ by transitivity of '$>^k$' from Lemma 1. Then, by Lemma 2, we let $\mathbf{r}_i >^k \mathbf{0}$ for each $i \in \mathbb{N}$ such that $\mathbf{t}_{i+1} = \mathbf{t}_1 + \mathbf{r}_i$. Then, we have $\mathbf{r}_{i+1} >^k \mathbf{r}_i$ for each $i \in \mathbb{N}$. Let $\{\mathbf{r}_i\}_{i \in \mathbb{N}}$ be a sequence. Suppose that the sequence $\{\mathbf{r}_i\}_{i \in \mathbb{N}}$ is bounded above. Let $S \in \mathbb{N}$ such that $\mathbf{r}_i <^k \mathbf{r}_S$ for every $i \in \mathbb{N}$ and $\mathbf{t}_{S+1} = \mathbf{t}_1 + \mathbf{r}_S$. Since $\{\mathbf{t}_m\}_{m \in \mathbb{N}}$ is strictly monotonic increasing, then $\mathbf{t}_{S+2} >^k \mathbf{t}_{S+1}$. Then,

$\mathbf{t}_1 + \mathbf{r}_{S+1} >^k \mathbf{t}_1 + \mathbf{r}_S$. Then, $\mathbf{r}_{S+1} >^k \mathbf{r}_S$, and this is a contradiction since $\mathbf{r}_i <^k \mathbf{r}_S$ for every $i \in \mathbb{N}$. Therefore, the sequence $\{\mathbf{r}_i\}_{i\in\mathbb{N}}$ is not bounded above, and thus, it is divergent. Since $\mathbf{r} >^k \mathbf{0}$ and $\mathbf{r}_{i+1} >^k \mathbf{r}_i >^k \mathbf{0}$ for each $i \in \mathbb{N}$, then we can take some $L \in \mathbb{N}$, which $\mathbf{r}_L >^k \mathbf{r}$, and $\mathbf{t}_{L+1} \in \{\mathbf{t}_m\}_{m\in\mathbb{N}}$, which corresponds to $\mathbf{r}_L$ such that $\mathbf{t}_{L+1} = \mathbf{t}_1 + \mathbf{r}_L$. Since $\mathbf{t}_1 = -\mathbf{r}$, then $\mathbf{t}_{L+1} = \mathbf{t}_1 + \mathbf{r}_L = -\mathbf{r} + \mathbf{r}_L$. Since $\mathbf{r}_L - \mathbf{r} >^k \mathbf{0}$, then clearly $\mathbf{t}_{L+1} >^k \mathbf{0}$ and $\mathbf{t}_{L+i+1} >^k \mathbf{0}$ for all $i \in \mathbb{N}$ by transitivity of '$>^k$' from Lemma 1. Therefore, we have $M = L + 1$, which $\mathbf{t}_n >^k \mathbf{0}$ for all $n > M$. $\square$

Next, we recall a $\mathbb{Z}^2$-action on X is a continuous map $T : \mathbb{Z}^2 \times X \to X$ such that

1. $T(\mathbf{0}, x) = x$ for $\mathbf{0} = (0,0) \in \mathbb{Z}^2$ and any $x \in X$,
2. $T(\mathbf{n}, T(\mathbf{m}, x)) = T(\mathbf{n} + \mathbf{m}, x)$ for any $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^2$ and $x \in X$.

The map $T^{\mathbf{n}} : X \to X$ is defined by $T^{\mathbf{n}}(x) = T(\mathbf{n}, x)$ for all $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$. Note that the map $T^{\mathbf{n}}$ is a homeomorphism on compact metric space X [33]. Moreover, we can observe that each $T^{\mathbf{n}}$ can be expressed as a finite composition of $T^{\mathbf{e}_1}$ and $T^{\mathbf{e}_2}$; that is, $T^{\mathbf{n}} = T^{\mathbf{e}_{i_1}} \circ T^{\mathbf{e}_{i_2}} \circ \cdots \circ T^{\mathbf{e}_{i_j}}$ for $i_1, i_2, \ldots, i_j \in \{1,2\}$ [6].

For $x \in X$, we write $O_T(x) = \{T^{\mathbf{n}} | \mathbf{n} \in \mathbb{Z}^2\}$ as the orbit of $x \in X$ under T. A point $p \in X$ is called a periodic point of T if its orbit $O_T(x)$ is finite. Equivalently, we say there is some $\mathbf{n} \in \mathbb{Z}^2$ such that $T^{\mathbf{n}}(x) = x$. We denote $Per(T)$ as the set of all periodic points of T. Next, a subset $A \subset X$ is said to be T-invariant if for every $y \in A$, then $T^{\mathbf{e}_i}(y) \in A$ for any $i \in \{1,2\}$. Then, a sequence known as a δ-pseudo orbit is defined as follows.

Definition 1 ([6,8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). A sequence $\xi = \{x_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ in (X, ρ) is said to be a δ-pseudo orbit of T if $\rho(T^{\mathbf{e}_i}(x_{\boldsymbol{n}}), x_{\boldsymbol{n}+\boldsymbol{e}_i}) < \delta$ for any $\boldsymbol{n} \in \mathbb{Z}^2$ and $i \in \{1,2\}$.*

The definition of shadowing property for a $\mathbb{Z}^2$-action can be seen below.

Definition 2 ([6,8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). We say that a $\mathbb{Z}^2$-action T has shadowing property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit $\{x_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2} \subset X$ is ε-shadowed by a point of X; that is, there exists $y \in X$ such that $\rho(T^{\boldsymbol{n}}(y), x_{\boldsymbol{n}}) < \varepsilon$ for every $\boldsymbol{n} \in \mathbb{Z}^2$.*

Next, we describe a k-type δ-chain by a definition as follows.

Definition 3 ([8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). Let $x \in X$ and $\delta > 0$. It is said that a k-type δ-chain for x is a δ-pseudo orbit $\gamma = \{y_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ with two conditions as follows:*

(i) $y_{\mathbf{0}} = x$,
(ii) *if for some $u \in X$ and some index $\mathbf{n} \in \mathbb{Z}^2$, which the equality $y_{\mathbf{n}} = u$ holds, then the set*

$$\left\{\mathbf{j} \in \mathbb{Z}^2 : y_{\mathbf{j}} = u, \mathbf{j} >^k \mathbf{l}\right\}$$

is infinite for any $l \in \mathbb{Z}^2$.

Then, the notion of weak chain recurrence is given as below.

Definition 4 ([8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). Let $\delta > 0$, $x, y \in X$ and $k \in \{1,2,3,4\}$. A sequence $\{x_{t_0}, x_{t_1}, \ldots, x_{t_n}, x_{t_{n+1}}\} \subset X$ with $\{\boldsymbol{t}_i\}_{i=0}^{n+1} \subset \mathbb{Z}^2$ is said to be a weak k-type δ-pseudo orbit from x to y if $\boldsymbol{t}_{s+1} >^k \boldsymbol{t}_s >^k \boldsymbol{0}$ and $\rho(T^{\boldsymbol{t}_s}(x_{t_s}), x_{t_{s+1}}) < \delta$ for $s = 0, 1, \ldots, n$ and additionally $x_{t_0} = x$, $x_{t_{n+1}} = y$.*

Definition 5 ([8])**.** *Let $x \in X$, $k \in \{1,2,3,4\}$ and $\delta > 0$. Any weak k-type δ-pseudo orbit from x to x is called a weak k-type δ-chain for x. A point x is said to be a weak k-type chain recurrent point if, for every $\delta > 0$, there exists a weak k-type δ-chain for x. The set of all weak k-type chain recurrent points will be denoted by $WCR^k(T)$.*

Definition 6 ([8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ) and $k \in \{1,2,3,4\}$. For $x, y \in X$ and $\delta > 0$, we define x as k-type δ-related to y (written as $x \sim^{k(\delta)} y$) if there exist two weak k-type δ-pseudo orbits ξ_1 and ξ_2 from x to y and from y to x, respectively, through $\xi_1 = \{x_{\mathbf{t}_0} = x, \ldots, x_{\mathbf{t}_{n+1}} = y\}$ and $\xi_2 = \{y_{\mathbf{s}_0} = y, \ldots, y_{\mathbf{s}_{m+1}} = x\}$ for some $n, m \in \mathbb{N}$. If $x \sim^{k(\delta)} y$ for any $\delta > 0$, then x is k-type related to y (written as $x \sim^k y$).*

The relation $\sim^k$ is an equivalence relation on $WCR^k(T)$ [8]. For an index set Λ, we denote D_λ for $\lambda \in \Lambda$ as an equivalence class of this relation. The equivalence class is called a k-type chain component of T. Next, we want to define a k-type weak extending property for a $\mathbb{Z}^2$-action. First, we define an infinite weak k-type δ-pseudo orbit.

Definition 7 ([8])**.** *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). Let $\delta > 0$, $x, y \in X$ and $k \in \{1,2,3,4\}$. A bi-infinite sequence $\{x_{\boldsymbol{t}_i}\}_{i\in\mathbb{Z}} \subset X$, which $\{\boldsymbol{t}_i\}_{i\in\mathbb{Z}} \subset \mathbb{Z}^2$ with $\boldsymbol{t}_i \geq^k \boldsymbol{0}$ whenever $i \geq 0$ and $\boldsymbol{t}_i <^k \boldsymbol{0}$ whenever $i < 0$, is said to be an infinite weak k-type δ-pseudo orbit if $\boldsymbol{t}_{s+1} >^k \boldsymbol{t}_s$ and $\rho\left(T^{\boldsymbol{t}_s}(x_{\boldsymbol{t}_s}), x_{\boldsymbol{t}_{s+1}}\right) < \delta$ for any $s \in \mathbb{Z}$.*

Then, we define a box and k-type chain of boxes, which were introduced earlier by Oprocha (2008) as in the following.

Definition 8 ([8])**.** *For any $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^2$ with $\boldsymbol{n} >^k \boldsymbol{m}$ where $k \in \{1,2,3,4\}$, a set $B^k(\boldsymbol{n}, \boldsymbol{m}) \subset \mathbb{Z}^2$ is said to be a box given by the formula*

$$B^k(\mathbf{n}, \mathbf{m}) = \left\{\mathbf{b} \in \mathbb{Z}^2 : \mathbf{n} \leq^k \mathbf{b} \leq^k \mathbf{m}\right\}$$

Definition 9 ([8])**.** *A sequence $\{B_i\}_{i\in\mathbb{Z}} \subset \mathbb{Z}^2$ is called a k-type chain of boxes if it fulfills the following conditions:*

(*i*) *for any $i \in \mathbb{Z}$, set B_i is a box and $B_i \cap B_{i+1} \neq \varnothing$,*
(*ii*) *for any $i \in \mathbb{Z}$, $\boldsymbol{a} \in B_i$ and $\boldsymbol{b} \in B_{i+1}$, the inequality $\boldsymbol{a} \leq^k \boldsymbol{b}$ holds.*

The set $C = \cup_{i\in\mathbb{Z}} B_i \subset \mathbb{Z}^2$ is said to be a realization of k-type chain of boxes $\{B_i\}_{i\in\mathbb{Z}}$.

Remark 1 ([8])**.** *Observe that for any given k-type chain of boxes $\left\{B_i = B^k(\boldsymbol{p}_i, \boldsymbol{q}_i)\right\}_{i\in\mathbb{Z}}$ and any $i \in \mathbb{Z}$, it holds that*

$$B_i \cap B_{i+1} = \{\mathbf{q}_i\} = \{\mathbf{p}_{i+1}\}.$$

Furthermore, $B_i \cap B_j \cap B_l = \varnothing$ for any $i, j, l \in \mathbb{Z}$.

For an infinite weak k-type δ-pseudo orbit $\{x_{\mathbf{t}_i}\}_{i\in\mathbb{Z}}$, we let

$$\mathbf{s}_j = \begin{cases} \sum_{i=0}^{j-1} \mathbf{t}_i, \text{ if } j > 0, \\ \mathbf{0}, \text{ if } j = 0, \\ -\sum_{i=j}^{-1} \mathbf{t}_i, \text{ if } j < 0. \end{cases}$$

For each $i \in \mathbb{Z}$, we let $A_i = B^k(\mathbf{0}, \mathbf{t}_i)$ whenever $i \geq 0$ and $A_i = B^k(\mathbf{t}_i, \mathbf{0})$ whenever $i < 0$. We recall that if $F \subset \mathbb{Z}^2$ and $\mathbf{n} \in \mathbb{Z}^2$, then $\mathbf{n} + F = \{\mathbf{n} + \mathbf{m} : \mathbf{m} \in F\} \subset \mathbb{Z}^2$ is a set. Let $s \in \mathbb{N}$. A set $\overline{\Lambda}(s) = \{-s+1, \ldots, s-1\}^2$ is said to be a symmetric s-cube centered at 0.

Then, we let $B_i = s_i + A_i$ for each $i \in \mathbb{Z}$. Since $\mathbf{s}_i + \mathbf{t}_i = \mathbf{s}_{i+1}$ for each $i \in \mathbb{Z}$, it implies that $\{B_i\}_{i\in\mathbb{Z}}$ is a k-type chain of boxes, and let $C = \cup_{i\in\mathbb{Z}} B_i$ be its realization. Then, we define a sequence $\{z_{\mathbf{n}}\}_{\mathbf{n}\in C}$ by the formula $z_{\mathbf{n}} = T^{\mathbf{n}-\mathbf{s}_i}(x_{\mathbf{t}_i})$ for every $\mathbf{n} \in B_i \backslash B_{i+1}$ and $i \in \mathbb{Z}$. The sequence $\{z_{\mathbf{n}}\}_{\mathbf{n}\in C}$ is said to be a box realization of an infinite weak k-type δ-pseudo orbit $\{x_{\mathbf{t}_i}\}_{i\in\mathbb{Z}}$. Then, the definition of a k-type weak extending property is described as follows.

Definition 10 ([8]). *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ) and $k \in \{1,2,3,4\}$. It is said that T has a k-type weak extending property if, for any $\varepsilon > 0$, there exist $\delta > 0$ and $s \in \mathbb{N}_+$ such that if $\{x_{\mathbf{t}_i}\}_{i\in\mathbb{Z}}$ is any infinite weak k-type δ-pseudo orbit and $\{z_{\mathbf{n}}\}_{\mathbf{n}\in C}$ is its box realization, then there exists ε-pseudo orbit $\{y_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^2}$ fulfilling $z_{\mathbf{i}} = y_{\mathbf{i}}$ for any $\mathbf{i} \in \mathbb{Z}^2$ such that $\mathbf{i} + \overline{\Lambda}(s) \subset C$.*

Next, the definition of a k-type non-wandering point is described as below.

Definition 11 ([6,8]). *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). A point $x \in X$ is said to be a k-type non-wandering point of T if, for any open ball $U_\delta(x)$ and $\mathbf{m} \in \mathbb{Z}^2$, there exists $\mathbf{n} \in \mathbb{Z}^2$ such that $\mathbf{n} >^k \mathbf{m}$ and $T^{\mathbf{n}}(U_\delta(x)) \cap U_\delta(x) \neq \varnothing$. Then, we denote $\Omega^k(T)$ as the set of all k-type non-wandering points of T.*

According to Kim and Lee [6], we know that every k-type non-wandering point $x \in \Omega^k(T)$ is k-type δ-related to $T^{\mathbf{e}_i}(x)$ for every $\delta > 0$ and $i \in \{1,2\}$ by the nature of its definition. Next, we introduce a δ-neighborhood of a k-type chain component D_λ in $\Omega^k(T)$ as the set $U_\delta(D_\lambda) = \left\{y \in \Omega^k(T) \middle| \rho(y, D_\lambda) < \delta\right\}$ for $\delta > 0$. Then, we prove that the set $U_\delta(D_\lambda)$ is an open set of $\Omega^k(T)$ as follows.

Lemma 5. *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). Let $U_\delta(D_\lambda) = \left\{y \in \Omega^k(T) \middle| \rho(y, D_\lambda) < \delta\right\}$ be a δ-neighborhood of a k-type chain component D_λ in $\Omega^k(T)$ for $\delta > 0$. Then, $U_\delta(D_\lambda)$ is an open set of $\Omega^k(T)$.*

Proof. Clearly, $U_\delta(D_\lambda) = \left\{y \in \Omega^k(T) \middle| \rho(y, D_\lambda) < \delta\right\} = \{y \in X | \rho(y, D_\lambda) < \delta\} \cap \Omega^k(T)$. Since $\rho(y, D_\lambda) = \inf_{b\in D_\lambda} \rho(a,b)$, then $\{y \in X | \rho(y, D_\lambda) < \delta\} = \bigcup_{\delta'<\delta} \bigcup_{a\in D_\lambda} U_{\delta'}(a)$ is an open set. Therefore, $U_\delta(D_\lambda)$ is open. □

Then, we recall the definition of topologically k-type transitive as follows.

Definition 12 ([6,8]). *A $\mathbb{Z}^2$-action $T : \mathbb{Z}^2 \times X \to X$ is said to be topologically k-type transitive if, for every open set U and V of X, there exists $\mathbf{n} >^k \mathbf{0}$ such that $T^{\mathbf{n}}(U) \cap V \neq \varnothing$ where $\mathbf{n} \in \mathbb{Z}^2$.*

3. Shadowing Property and *k*-Type Weak Extending Property

Throughout this section, we explore some related results of shadowing property and k-type weak extending property of a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). The shadowing property is one of the main ingredients to prove the spectral decomposition theorem of a $\mathbb{Z}^2$-action. Therefore, we recall some lemmas and theorems from Kim and Lee [6] as follows.

Theorem 1 ([6]). *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). If T has the shadowing property, then*

(i) *$Per(T)$ is dense in $\Omega^k(T)$ for any $k \in \{1,2,3,4\}$,*
(ii) *$WCR^k(T) = \Omega^k(T)$ for each $k \in \{1,2,3,4\}$,*
(iii) *Each k-type chain component D_λ is open in $\Omega^k(T)$ for any $k \in \{1,2,3,4\}$, and*
(iv) *$T|_{\Omega^k(T)}$ has the shadowing property.*

Proof. Refer to Theorem 3.6, Lemma 4.1, Lemma 4.2 and Theorem 4.1 in [6]. □

Next, we recall a theorem from [8], which described a property under k-type weak extending property as in the following.

Theorem 2 ([8]). *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ) and $k \in \{1,2,3,4\}$. If T has a k-type weak extending property, then for any $\delta > 0$ and $x \in WCR^k(T)$, there exists a k-type δ-chain for x.*

Proof. Refer to Theorem 7.5 in [8]. □

In this study, we also prove some interesting implications of shadowing property. First, we show that each k-type chain component D_λ is closed in $\Omega^k(T)$ and T-invariant as in the following.

Lemma 6. *Each k-type chain component D_λ is closed in $\Omega^k(T)$ and T-invariant if T has shadowing property.*

Proof. Let $\lambda' \in \Lambda$ and $D_{\lambda'}$ be a k-type chain component of T. Then, $\Omega^k(T) \backslash D_{\lambda'} = \cup_{\lambda \in \Lambda \backslash \{\lambda'\}} D_\lambda$. By Theorem 1, then $\Omega^k(T) \backslash D_{\lambda'}$ is open. Thus, $D_{\lambda'}$ is closed. Next, we let $y \in D_\lambda$. By Theorem 1, $y \in \Omega^k(T)$. By the definition of k-type non-wandering point of T, we know that y is k-type δ-related to $T^{\mathbf{e}_i}(y)$ for every $\delta > 0$ and $i \in \{1,2\}$. Then, $y \sim^k T^{\mathbf{e}_i}(y)$ for any $i \in \{1,2\}$. Therefore, $T^{\mathbf{e}_i}(y) \in D_\lambda$ for any $i \in \{1,2\}$. Thus, D_λ is T-invariant. □

We then show the existence of periodic points in any neighborhood of any point in k-type chain component of T.

Lemma 7. *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). Let D_λ be a k-type chain component of T. Suppose that T has shadowing property. Then, for any $y \in D_\lambda$ and $\delta > 0$, there exists a periodic point p_δ such that $\rho(p_\delta, y) < \delta$.*

Proof. Let $\delta > 0$, and let $y \in D_\lambda$. Since T has shadowing property, by Theorem 1, $Per(T)$ is dense in $\Omega^k(T)$. Let $U_\delta(y)$ be an open ball of y with radius δ and $U'_\delta(y)$ be a set such that $U'_\delta(y) = U_\delta(y) \cap \Omega^k(T)$. Clearly, $U'_\delta(y)$ is open since it is an intersection of the two open sets and, therefore, $Per(T) \cap U'_\delta(y) \neq \varnothing$. Hence, there exists $p_\delta \in Per(T)$ such that $\rho(p_\delta, y) < \delta$. □

All of the results above are necessary to prove the last main theorem of this study that will be discussed in the last section of this article.

4. Measure Expanding

In this section, we discuss some basic notions in measure theory for a $\mathbb{Z}^2$-action. For a given compact metric space (X,ρ), we let $\beta(X)$ be the Borel σ-algebra on X, which is the σ-algebra generated by the open sets of X. Then, a Borel measure on X is a non-negative σ-additive map μ defined on $\beta(X)$. We also assume that a Borel measure on X implies a Borel probability measure, that is, $\mu(X) = 1$.

We denote $M(X,T)$ as the collection of all invariant Borel probability measures on X. For a given $\delta > 0$, let $\tau_\delta^T(x)$ be a dynamical δ-ball centered at $x \in X$, which is defined by

$$\tau_\delta^T(x) = \left\{ y \in X \middle| \rho(T^{\mathbf{m}}(x), T^{\mathbf{m}}(y)) \leq \delta \text{ for all } \mathbf{m} \in \mathbb{Z}^2 \right\}.$$

Then, it is said that a $\mathbb{Z}^2$-action T is expansive if there is an $\varepsilon > 0$ such that $\tau_\varepsilon^T(x) = \{x\}$ for all $x \in X$ [28].

We introduce the definition of measure expanding, which is given as in the following.

Definition 13. *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X,ρ). A Borel measure μ on X is said to be expanding for T if there is $\delta > 0$ such that $\mu(\tau_\delta^T(x) \backslash O_T(x)) = 0$ for all $x \in X$. Then,* *we say that a $\mathbb{Z}^2$-action T is measure expanding (resp. invariantly measure expanding) if every Borel measure (resp. invariant Borel measures) μ on X expanding for T.*

In this study, we found a lemma that is important to prove the measurable version of the spectral decomposition theorem for a $\mathbb{Z}^2$-action. To prove the lemma, we first introduce a Dirac measure defined for a given $T^{\mathbf{n}}(x) \in X$ where $\mathbf{n} \in \mathbb{Z}^2$ and for any $A \in \beta(X)$ by

$$\delta_{T^{\mathbf{n}}(x)}(A) = \begin{cases} 1, \text{ if } T^{\mathbf{n}}(x) \in A, \\ 0, \text{ if } T^{\mathbf{n}}(x) \notin A. \end{cases}$$

We assume $B^k(\mathbf{0}, \mathbf{n}) \subset \mathbb{Z}^2$ is a box as introduced in Definition 8 where $\mathbf{n} = (n_1, n_2) >^k \mathbf{0}$. The cardinality of the box $B^k(\mathbf{0}, \mathbf{n})$ is the number of its elements, which is given by $n\Big(B^k(\mathbf{0}, \mathbf{n})\Big) = |(n_1+1)(n_2+1)|$. Next, we prove a lemma that describes a property of measure expanding as follows.

Lemma 8. *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ) and suppose that it is invariantly measure expanding. Then, for any $k \in \{1, 2, 3, 4\}$, there is a constant $e > 0$ such that if*

$$\rho(T^{\mathbf{n}}(x), T^{\mathbf{n}}(p)) \le e \textit{ for all } \mathbf{n} >^k \mathbf{0} \ \Big(\textit{resp. for all } \mathbf{n} <^k \mathbf{0}\Big)$$

for some $x \in X$ and $p \in Per(T)$, then

$$\lim_{s \to +\infty} \rho\big(T^{\mathbf{t}_s}(x), O_T(p)\big) = 0 \ \left(\textit{resp. } \lim_{s \to +\infty} \rho\big(T^{-\mathbf{t}_s}(x), O_T(p)\big) = 0\right)$$

or any infinite sequence $\{\mathbf{t}_s\}_{s \in \mathbb{N}}$ in $\mathbb{Z}^2$ with $\mathbf{t}_{s+1} >^k \mathbf{t}_s$.

Proof. Suppose that T is a $\mathbb{Z}^2$-action that is invariantly measure expanding on a compact metric space (X, ρ). Then, every Borel measure is expanding. Let $e > 0$ be a constant such that $\mu\big(\tau_e^T(a) \backslash O_T(a)\big) = 0$ for all $a \in X$ and $\mu \in M(X, T)$.

Suppose, by contradiction, that there are $x \in X$, a periodic point $p \in Per(T)$ and $r > 0$ such that for any $k \in \{1, 2, 3, 4\}$, $\rho(T^{\mathbf{n}}(x), T^{\mathbf{n}}(p)) \le e$ for all $\mathbf{n} >^k \mathbf{0}$, but there is a sequence $\{\mathbf{u}_s\}_{s \in \mathbb{N}} \subset \mathbb{Z}^2$ with $\mathbf{u}_{s+1} >^k \mathbf{u}_s$, which converge to infinity as $s \to +\infty$ and $\rho(T^{\mathbf{u}_s}(x), O_T(p)) > r$ for all $s \in \mathbb{N}$. Since X is compact, we suppose that $\lim\limits_{s \to +\infty} T^{\mathbf{u}_s}(x) = x_0$ and $\lim\limits_{s \to +\infty} T^{\mathbf{u}_s}(p) = p_0$ for some $x_0, p_0 \in X$. Obviously, p_0 is a periodic point. Since

$$\rho(x_0, O_T(p)) = \rho\left(\lim_{s \to +\infty} T^{\mathbf{u}_s}(x), O_T(p)\right) = \lim_{s \to +\infty} \rho(T^{\mathbf{u}_s}(x), O_T(p)) > r,$$

then $x_0 \notin O_T(p)$. Next, for each $\mathbf{m} \in \mathbb{Z}^2$, we construct a sequence $\{\mathbf{w}_s\}_{s \in \mathbb{N}} \subset \mathbb{Z}^2$ defined by $\mathbf{w}_s = \mathbf{u}_s + \mathbf{m}$ for all $s \in \mathbb{N}$. By Lemma 3, then $\mathbf{w}_{s+1} >^k \mathbf{w}_s$ for each $s \in \mathbb{N}$. Then, by Lemma 4, there exists $S \in \mathbb{N}$ such that $\mathbf{w}_t >^k \mathbf{0}$ for all $t > S$. Thus, we have $\rho(T^{\mathbf{w}_t}(x), T^{\mathbf{w}_t}(p)) \le e$ for all $t > S$ and so

$$\lim_{s \to +\infty} \rho(T^{\mathbf{w}_s}(x), T^{\mathbf{w}_s}(p)) \le e.$$

Then, we have

$$\begin{aligned} \rho(T^{\mathbf{m}}(x_0), T^{\mathbf{m}}(p_0)) &= \rho\left(T^{\mathbf{m}}\left(\lim_{s \to +\infty} T^{\mathbf{u}_s}(x)\right), T^{\mathbf{m}}\left(\lim_{s \to +\infty} T^{\mathbf{u}_s}(p)\right)\right) \\ &= \rho\left(\lim_{s \to +\infty} T^{\mathbf{m}+\mathbf{u}_s}(x), \lim_{s \to +\infty} T^{\mathbf{m}+\mathbf{u}_s}(p)\right) \\ &= \lim_{s \to +\infty} \rho(T^{\mathbf{w}_s}(x), T^{\mathbf{w}_s}(p)) \le e \end{aligned}$$

for all $\mathbf{m} \in \mathbb{Z}^2$ and, therefore, $p_0 \in \tau_e^T(x_0)$. Let $\mathbf{n}_{p_0} >^k \mathbf{0}$ be the period of p_0 and, obviously, $\mathbf{n}_{p_0} \in B^k(\mathbf{0}, \mathbf{n}_{p_0})$. Then, we define an invariant Borel measure μ_j of T by

$$\mu_j(A) = \frac{1}{n\big(B^k(\mathbf{0}, \mathbf{n}_{p_0})\big)} \sum_{\mathbf{i} \in B^k(\mathbf{0}, \mathbf{n}_{p_0})} \delta_{T^{\mathbf{i}}(p_0)}(A)$$

where $A \in \beta(X)$ and $n\Big(B^k(\mathbf{0}, \mathbf{n}_{p_0})\Big)$ is the cardinality of the box $B^k(\mathbf{0}, \mathbf{n}_{p_0})$. Since $p_0 \in \tau_e^T(x_0)$, then $\delta_{p_0}(\tau_e^T(x_0) \backslash O_T(x_0)) = \delta_{T^{\mathbf{0}}(p_0)}(\tau_e^T(x_0) \backslash O_T(x_0)) = 1$.

Thus, $\mu_j(\tau_e^T(x_0) \backslash O_T(x_0)) \geq \mu_j(\{p_0\}) > 0$, and this is a contradiction since T is invariantly measure expanding. Therefore, $\lim_{s \to +\infty} \rho(T^{\mathbf{t}_s}(x), O_T(p)) = 0$ for any sequence $\{\mathbf{t}_s\}_{s \in \mathbb{N}}$ in $\mathbb{Z}^2$ with $\mathbf{t}_{s+1} >^k \mathbf{t}_s$.

Similarly, we can show that if $\rho(T^{\mathbf{n}}(x), T^{\mathbf{n}}(p)) \leq e$ for all $\mathbf{n} <^k \mathbf{0}$ for some $x \in X$ and $p \in Per(T)$, then $\lim_{s \to +\infty} \rho(T^{-\mathbf{t}_s}(x), O_T(p)) = 0$ for any infinite sequence $\{\mathbf{t}_s\}_{s \in \mathbb{N}}$ in $\mathbb{Z}^2$ with $\mathbf{t}_{s+1} >^k \mathbf{t}_s$. □

5. Measurable Spectral Decomposition Theorem

The main highlight of this paper is to prove the measurable version of the spectral decomposition theorem for a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). More precisely, we show that if a $\mathbb{Z}^2$-action is invariantly measure expanding, has shadowing property and has k-type weak extending property, then the set of all k-type non-wandering points can be decomposed into a disjoint union of closed and invariant sets such that the restriction map on each of the sets is topologically k-type transitive. Now, we present the theorem and its proof in this section.

Theorem 3. *Let T be a $\mathbb{Z}^2$-action on a compact metric space (X, ρ). Let $k \in \{1, 2, 3, 4\}$. Suppose that T is invariantly measure expanding, has shadowing property and has k-type weak extending property. Then, there exists closed, pairwise disjoint and invariant sets $S_1, \ldots, S_l \subset \Omega^k(T)$, which additionally fulfill the following conditions:*

(*i*) $\Omega^k(T) = S_1 \cup \ldots \cup S_l$,
(*ii*) $T|_{S_i}$ *is topologically k-type transitive for each $i = 1, 2, \ldots, l$.*

Proof. (i). Suppose that T is invariantly measure expanding, has shadowing property and has k-type weak extending property. By properties (i) and (ii) in Theorem 1, we have $\overline{Per(T)} = \Omega^k(T) = WCR^k(T)$. Thus,

$$\Omega^k(T) = WCR^k(T) = \cup_{\lambda \in \Lambda} D_\lambda$$

where D_λ is the k-type chain components of T. By Lemma 6, k-type chain component D_λ is closed and T-invariant. By property (iv) in Theorem 1, $T|_{\Omega^k(T)}$ has the shadowing property.

Now, let us claim that each D_λ is open in $\Omega^k(T)$. Since T is invariantly measure expanding on X, then every Borel measure in $M(X, T)$ is expanding. Let $e > 0$ be a constant as given in Lemma 8, which $\mu(\tau_e^T(x) \backslash O_T(x)) = 0$ for all $x \in X$ and $\mu \in M(X, T)$. By the shadowing property of $T|_{\Omega^k(T)}$, take $\delta_e > 0$ corresponding to e, which satisfy that every δ_e-pseudo orbit of $\Omega^k(T)$ is e-shadowed by a point in $\Omega^k(T)$.

We fix $\lambda \in \Lambda$ and let $U_{\delta_e}(D_\lambda) = \Big\{y \in \Omega^k(T) \Big| \rho(y, D_\lambda) < \delta_e\Big\}$ be a δ_e-neighborhood of D_λ in $\Omega^k(T)$. By Lemma 5, $U_{\delta_e}(D_\lambda)$ is an open set of $\Omega^k(T)$. Since $\overline{Per(T)} = \Omega^k(T)$, then $U_{\delta_e}(D_\lambda) \cap Per(T) \neq \varnothing$. Then, we can take $p \in U_{\delta_e}(D_\lambda) \cap Per(T)$ and $y \in D_\lambda$ such that $\rho(p, y) < \delta_e$. We claim that p is k-type related to y, $p \sim^k y$, i.e., $p \sim^{k(\tau)} y$ for any $\tau > 0$.

Let $\tau > 0$. Then, for $y \in D_\lambda$ and $\frac{\tau}{2} > 0$, by Lemma 7, we let $p_\tau \in Per(T)$ such that $\rho(p_\tau, y) < \frac{\tau}{2}$. We first show that $p_\tau \sim^{k(\tau)} y$. Since p_τ is a periodic point, which is also a weak k-type chain recurrent point, then we let $\left\{x_{\mathbf{t}_0} = p_\tau, x_{\mathbf{t}_1}, \ldots, x_{\mathbf{t}_j}, x_{\mathbf{t}_{j+1}} = p_\tau\right\}$ be a weak k-type $\frac{\tau}{2}$-chain for p_τ such that $\{\mathbf{t}_i\}_{i=0}^{j+1} \subset \mathbb{Z}^2$ with $\mathbf{t}_{s+1} >^k \mathbf{t}_s >^k \mathbf{0}$ and $\rho\left(T^{\mathbf{t}_s}(x_{\mathbf{t}_s}), x_{\mathbf{t}_{s+1}}\right) < \frac{\tau}{2}$ for $s = 0, 1, \ldots, j$. Since $\rho\left(T^{\mathbf{t}_j}\left(x_{\mathbf{t}_j}\right), p_\tau\right) < \frac{\tau}{2}$ and $\rho(p_\tau, y) < \frac{\tau}{2}$, then

$$\rho\left(T^{\mathbf{t}_j}\left(x_{\mathbf{t}_j}\right), y\right) \leq \rho\left(T^{\mathbf{t}_j}\left(x_{\mathbf{t}_j}\right), p_\tau\right) + \rho(p_\tau, y) < \tau.$$

By letting a new sequence $\omega = \left\{w_{\mathbf{t}_0}, \ldots, w_{\mathbf{t}_{j+1}}\right\}$ such that $w_{\mathbf{t}_s} = x_{\mathbf{t}_s}$ for all $s \in \{0, 1, \ldots, j\}$ and $w_{\mathbf{t}_{j+1}} = y$, then ω is a weak k-type τ-pseudo orbit from p_τ to y. Similarly, we can construct a weak k-type τ-pseudo orbit from y to p_τ. Therefore, $p_\tau \sim^{k(\tau)} y$.

Next, we claim that $p \sim^{k(\tau)} p_\tau$. Let a sequence $\gamma = \{v_\mathbf{i}\}_{\mathbf{i} \in \mathbb{Z}^2}$ be a k-type δ_e-pseudo orbit for p, which is defined $v_\mathbf{j} = T^\mathbf{j}(p)$ if $\mathbf{j} \geq^k \mathbf{0}$ and $v_\mathbf{j} = T^\mathbf{j}(p_\tau)$ if $\mathbf{j} <^k \mathbf{0}$ for some index $\mathbf{j} \in \mathbb{Z}^2$. By the shadowing property of $T|_{\Omega^k(T)}$, there exists $z \in \Omega^k(T)$ such that γ is e-shadowed by it. That is, $\rho(T^\mathbf{i}(z), v_\mathbf{i}) \leq e$ for every $\mathbf{i} \in \mathbb{Z}^2$. Thus, we have

$$\rho\left(T^\mathbf{j}(z), T^\mathbf{j}(p)\right) \leq e \text{ for all } \mathbf{j} >^k \mathbf{0}$$

and

$$\rho\left(T^\mathbf{j}(z), T^\mathbf{j}(p_\tau)\right) \leq e \text{ for all } \mathbf{j} <^k \mathbf{0}.$$

By Lemma 8, it implies that

$$\lim_{s \to +\infty} \rho\left(T^{\mathbf{t}_s}(z), O_T(p)\right) = 0$$

and

$$\lim_{m \to +\infty} \rho\left(T^{-\mathbf{u}_m}(z), O_T(p_\tau)\right) = 0$$

for two infinite sequences $\{\mathbf{t}_s\}_{s \in \mathbb{N}}, \{\mathbf{u}_m\}_{m \in \mathbb{N}} \subset \mathbb{Z}^2$ with $\mathbf{t}_{s+1} >^k \mathbf{t}_s$ and $\mathbf{u}_{m+1} >^k \mathbf{u}_m$, respectively. Next, let $S > 0$ and $M > 0$ such that

$$\rho\left(T^{\mathbf{t}_{s'}}(z), O_T(p)\right) < \tau$$

and

$$\rho\left(T^{-\mathbf{u}_{m'}}(z), O_T(p_\tau)\right) < \tau$$

for all $s' > S$ and $m' > M$. By Lemma 4, we can choose $s_1 > S$ and $m_1 > M$ such that $\mathbf{t}_{s_1} >^k \mathbf{0}$ and $\mathbf{u}_{m_1} >^k \mathbf{0}$. Since $T^{-\mathbf{u}_{m_1}}(p_\tau) \in O_T(p_\tau)$ and $p \in O_T(p)$, then we can have

$$\rho\left(T^{-\mathbf{u}_{m_1}}(z), T^{-\mathbf{u}_{m_1}}(p_\tau)\right) < \tau \tag{1}$$

and

$$\rho\left(T^{\mathbf{t}_{s_1}}(z), p\right) < \tau \tag{2}$$

Since $-\mathbf{u}_{m_1} <^k \mathbf{0}$ and $\mathbf{t}_{s_1} >^k \mathbf{0}$, then we can let a sequence $\{\mathbf{r}_i\}_{i=0}^{n+1} \subset \mathbb{Z}^2$ for some $n \in \mathbb{N}$ such that $\mathbf{r}_0 = -\mathbf{u}_{m_1}, \mathbf{r}_{n+1} = \mathbf{t}_{s_1}$ and $\mathbf{r}_{i+1} >^k \mathbf{r}_i$ for each $i \in \{0, 1, \ldots, n\}$. Then, for the sequence $\{\mathbf{r}_t\}_{t=0}^{n+2} \subset \mathbb{Z}^2$, we let $\alpha = \left\{u_{\mathbf{r}_0}, u_{\mathbf{r}_1}, \ldots, u_{\mathbf{r}_n}, u_{\mathbf{r}_{n+1}}, u_{\mathbf{r}_{n+2}}\right\}$ be a sequence of X such that $u_{\mathbf{r}_0} = p_\tau, u_{\mathbf{r}_1} = T^{\mathbf{r}_0}(z), u_{\mathbf{r}_i} = T^{\mathbf{r}_{i-1}}\left(u_{\mathbf{r}_{i-1}}\right)$ for all $i \in \{2, 3, \ldots, n\}, u_{\mathbf{r}_{n+1}} = z$ and $u_{\mathbf{r}_{n+2}} = p$. Then, we have these observations:

1. $$\begin{aligned} \rho(T^{\mathbf{r}_0}(u_{\mathbf{r}_0}), u_{\mathbf{r}_1}) &= \rho\left(T^{-\mathbf{u}_{m_1}}(p_\tau), T^{\mathbf{r}_0}(z)\right) \\ &= \rho\left(T^{-\mathbf{u}_{m_1}}(p_\tau), T^{-\mathbf{u}_{m_1}}(z)\right) < \tau \text{ by (1).} \end{aligned}$$
2. $$\begin{aligned} \rho(T^{\mathbf{r}_1}(u_{\mathbf{r}_1}), u_{\mathbf{r}_2}) &= \rho(T^{\mathbf{r}_1}(T^{\mathbf{r}_0}(z)), T^{\mathbf{r}_1}(u_{\mathbf{r}_1})) \\ &= \rho(T^{\mathbf{r}_1}(T^{\mathbf{r}_0}(z)), T^{\mathbf{r}_1}(T^{\mathbf{r}_0}(z))) \\ &= 0 < \tau. \end{aligned}$$

3. $$\begin{aligned}\rho\big(T^{\mathbf{r}_s}(u_{\mathbf{r}_s}), u_{\mathbf{r}_{s+1}}\big) &= \rho\Big(T^{\mathbf{r}_s}(u_{\mathbf{r}_s}), T^{\mathbf{r}(s+1)-1}\Big(u_{\mathbf{r}_{(s+1)-1}}\Big)\Big)\\ &= \rho\big(T^{\mathbf{r}_s}(u_{\mathbf{r}_s}), T^{\mathbf{r}_s}(u_{\mathbf{r}_s})\big)\\ &= 0 < \tau\end{aligned}$$
   for each $s \in \{2, 3, \ldots, n\}$
4. $\rho\big(T^{\mathbf{r}_{n+1}}(u_{\mathbf{r}_{n+1}}), u_{\mathbf{r}_{n+2}}\big) = \rho\big(T^{\mathbf{t}_{s_1}}(z), p\big) < \tau$ by (2).

Then, α is a weak k-type τ-pseudo orbit from p_τ to p. Similarly, we can construct a weak k-type τ-pseudo orbit from p to p_τ. Then, $p \sim^{k(\tau)} p_\tau \sim^{k(\tau)} y$. Since τ is arbitrary, then we obtain $p \sim^k y$, and this means $p \in D_\lambda$. Thus, $U_{\delta_e}(D_\lambda) \cap Per(T) \subset D_\lambda$.

Since D_λ is closed in $\Omega^k(T)$, then $D_\lambda = \overline{D_\lambda}$. Since $U_{\delta_e}(D_\lambda) \cap Per(T) \subset D_\lambda$, then $\overline{U_{\delta_e}(D_\lambda) \cap Per(T)} \subset \overline{D_\lambda}$. Then, we have

$$D_\lambda = \overline{D_\lambda} \supset \overline{U_{\delta_e}(D_\lambda) \cap Per(T)}.$$

Let $c \in U_{\delta_e}(D_\lambda) \cap \overline{Per(T)}$. Since $c \in U_{\delta_e}(D_\lambda)$, then $c \in \Omega^k(T)$ such that $\rho(c, D_\lambda) < \delta_e$. Since $c \in \overline{Per(T)}$, then $c \in Per(T)$ or c is a limit point of $Per(T)$. The first case is when $c \in Per(T)$. Then, $c \in U_{\delta_e}(D_\lambda) \cap Per(T)$. Therefore,

$$\overline{U_{\delta_e}(D_\lambda) \cap Per(T)} \supset U_{\delta_e}(D_\lambda) \cap \overline{Per(T)}.$$

The second case is when $c \notin Per(T)$. Then, c is a limit point of $Per(T)$. Let $U_r(c)$ be an open ball of c with radius $r > 0$. Then, $U_r(c) \cap Per(T) \backslash \{c\} \neq \varnothing$. Let $\varepsilon' > 0$ and $\delta_c > 0$ such that $\rho(c, D_\lambda) = \delta_c < \delta_e$. Then, let $\beta = \min\{\delta_e - \delta_c, \varepsilon'\}$. Then, we have $U_\beta(c) \cap Per(T) \backslash \{c\} \neq \varnothing$. Thus, there exists $\widetilde{c} \in Per(T)$ such that $\rho(\widetilde{c}, c) < \beta$. Then, $\rho(\widetilde{c}, D_\lambda) \leq \rho(\widetilde{c}, c) + \rho(c, D_\lambda) < \beta + \delta_c < (\delta_e - \delta_c) + \delta_c = \delta_e$. Therefore, $\widetilde{c} \in U_{\delta_e}(D_\lambda)$. Since $\widetilde{c} \in U_{\varepsilon'}(c) \cap [U_{\delta_e}(D_\lambda) \cap Per(T)]$, then

$$U_{\varepsilon'}(c) \cap [U_{\delta_e}(D_\lambda) \cap Per(T)] \backslash \{c\} \neq \varnothing$$

for any $\varepsilon' > 0$. Thus, c is a limit point of $U_{\delta_e}(D_\lambda) \cap Per(T)$.

Since $c \in \overline{U_{\delta_e}(D_\lambda) \cap Per(T)}$, then we have

$$\overline{U_{\delta_e}(D_\lambda) \cap Per(T)} \supset U_{\delta_e}(D_\lambda) \cap \overline{Per(T)}.$$

We know obviously that $U_{\delta_e}(D_\lambda) \cap \overline{Per(T)} \subset U_{\delta_e}(D_\lambda)$. Now, we let $b \in U_{\delta_e}(D_\lambda)$. Then, $b \in \Omega^k(T)$ such that $\rho(b, D_\lambda) < \delta_e$. Since $\Omega^k(T) = \overline{Per(T)}$, then $b \in \overline{Per(T)}$. Hence, $U_{\delta_e}(D_\lambda) = U_{\delta_e} \cap \overline{Per(T)}$. It implies that

$$D_\lambda \supset \overline{U_{\delta_e}(D_\lambda) \cap Per(T)} \supset U_{\delta_e}(D_\lambda) \cap \overline{Per(T)} = U_{\delta_e}(D_\lambda).$$

Thus, each D_λ is open in $\Omega^k(T)$. Therefore, $\{D_\lambda\}_{\lambda \in \Lambda}$ is an open cover of $\Omega^k(T)$. By compactness of $\Omega^k(T)$, $\{D_\lambda\}_{\lambda \in \Lambda}$ has a finite subcover, $\{D_i\}_{i=1}^l$. Let us denote $S_i = D_i$ for each $i \in \{1, 2, \ldots, l\}$. Hence, $\Omega^k(T)$ can be expressed as a union of a finite set of $\{S_i\}_{i=1}^l$, that is,

$$\Omega^k(T) = \bigcup_{i=1}^{l} S_i.$$

(ii). Next, we prove that $T|_{S_i}$ for each $i \in \{1, 2, \ldots, l\}$ is topologically k-type transitive. Let $t \in \{1, 2, \ldots, l\}$ and $S_t = D_t$, which is one of the k-type chain components of T. Let U and V be two nonempty open sets of D_t. We want to show that there exists $\mathbf{m} >^k \mathbf{0}$ such that $T^{\mathbf{m}}(U) \cap V \neq \varnothing$.

Let $x \in U$ and $y \in V$. Then, let $U_r(x) \cap D_t \subset U$ and $U_r(y) \cap D_t \subset V$ for some $r > 0$. Since $r > 0$ and T has shadowing property, take δ_r corresponding to r, which satisfy that every δ_r-pseudo orbit of T in X is r-shadowed by a point in X. Since $x, y \in D_t$, then x must be k-type related to y. Then, we can say that $x \sim^{k(\delta)} y$ for any $\delta > 0$. Let

$\xi_1 = \left\{x'_{\mathbf{t}_0} = x, \ldots, x'_{\mathbf{t}_{n+1}} = y\right\}$ and $\xi_2 = \left\{y'_{\mathbf{s}_0} = y, \ldots, y'_{\mathbf{s}_{m+1}} = x\right\}$ be the two weak k-type δ-pseudo orbits from x to y and from y to x, respectively. By joining the two sequences ξ_1 and ξ_2, we obtain a weak k-type δ-pseudo orbit from x to x, or it is known as a weak k-type δ-chain for x. Since T has k-type weak extending property and $x \in WCR^k(T)$, then by Theorem 2, there exists a k-type δ-chain for x, which contains y at some position. Since $\delta > 0$ is arbitrary, then we can let $\{u_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^2}$ be a k-type δ_r-chain for x and let $\mathbf{l} >^k \mathbf{0}$ be an index such that $u_{\mathbf{l}} = y$. Since $\{u_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^2}$ is a δ_r-pseudo orbit of T, then by shadowing property of T, there exists $z \in X$ such that $\rho(T^{\mathbf{n}}(z), u_{\mathbf{n}}) < r$ for all $\mathbf{n} \in \mathbb{Z}^2$. Since $u_{\mathbf{0}} = x$ and $u_{\mathbf{l}} = y$, then we have $\rho(z, x) < r$ and $\rho\left(T^{\mathbf{l}}(z), y\right) < r$. Then, $z \in U_r(x) \cap D_t \subset U$ and $T^{\mathbf{l}}(z) \in U_r(y) \cap D_t \subset V$. Therefore, we have $\mathbf{m} = \mathbf{l} >^k \mathbf{0}$ such that $T^{\mathbf{m}}(U) \cap V \neq \varnothing$, and it implies that $T\big|_{D_i} = T\big|_{S_i}$ for each $i \in \{1, 2, \ldots, l\}$ is topologically k-type transitive. □

6. Discussion and Conclusions

By referring to the discussions from Aoki [3] and Das et al. [4], the spectral decomposition theorem for a homeomorphism had already been proven with two sufficient conditions, which are the shadowing property and expansivity. While Dong et al. [5] presented an alternative to the theorem by introducing the concept of measure expanding using a fundamental concept of measure theory and had distributed a measurable version of the spectral decomposition theorem for a homeomorphism. The study from Oprocha [8] presented the spectral decomposition theorem for a $\mathbb{Z}^d$-action with three sufficient conditions, which are the shadowing property, k-type weak extending property and expansivity. Kim and Lee [6] focused on $\mathbb{Z}^2$-action and proved the spectral decomposition theorem with two sufficient conditions, which are the shadowing property and expansivity. The concept of expansivity of $\mathbb{Z}^d$-action in [6,8] were described by a topological approach. Our study focused on $\mathbb{Z}^2$-action for a measurable version of the spectral decomposition theorem. The three conditions are required to prove the result, which are the measure expanding, shadowing property and k-type weak extending property. As a future direction of the research, this study can be extended by proving a measurable version of the spectral decomposition theorem for a $\mathbb{Z}^d$-action. Furthermore, there are many different kinds of shadowing property and expansivity, which can be considered as other future works to prove different versions of the spectral decomposition theorem.

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