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Confidence Intervals for Mean and Difference between Means of Delta-Lognormal Distributions Based on Left-Censored Data

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Abstract: A delta-lognormal distribution consists of zero and positive values. The positive values follow a lognormal distribution, which is an asymmetric distribution. It is well known that the logarithm of these values follows a normal distribution, which is a symmetric distribution. The delta-lognormal distribution is used in medical and environmental sciences. This study considers the challenges of constructing confidence intervals for the mean and difference between means of delta-lognormal distributions containing left-censored data and applies them to compare two daily rainfall average areas in Thailand. Three different approaches for constructing confidence intervals for the mean of the delta-lognormal distribution containing left-censored data, based on the generalized confidence interval approach, the Bayesian approach, and the parametric bootstrap approach, are developed. Moreover, four different approaches for constructing confidence intervals for the difference between means of delta-lognormal distributions containing left-censored data, based on the generalized confidence interval approach, the Bayesian approach, the parametric bootstrap approach, and the method of variance estimates recovery approach, are considered. The performance of the proposed confidence intervals is evaluated by Monte Carlo simulation. The simulation studies indicate that the Bayesian approach can be considered as an alternative to construct a credible interval for the mean of the delta-lognormal distribution containing left-censored data. Additionally, the generalized confidence interval and Bayesian approaches can be recommended as alternatives to estimate the confidence interval for the difference between means of delta-lognormal distributions containing left-censored data. All approaches are illustrated using the daily rainfall data from Chiang Mai and Lampang provinces in Thailand.

Keywords: confidence interval; delta-lognormal distribution; left-censored data; mean; rainfall data



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1. Introduction

The mean is a measure of the center tendency that represents the center point of values in a dataset. Functions of the mean can also be considered, such as the difference between two means, the ratio of two means, and the common of k means. The functions of the mean have been used in many research areas. For example, Zhou et al. [1] presented hypothesis testing of the effect of race on the average medical costs between African American and Caucasian patients with type I diabetes. Zhou and Tu [2], Tian [3], and Li et al. [4] estimated the mean charges for diagnostic tests on patients with unstable chronic medical conditions. Tian and Wu [5] and Krishnamoorthy and Oral [6] estimated the maximum alcohol concentration in men in an alcohol interaction study. Fletcher [7] and Wu and Hsieh [8] established the mean red cod density around New Zealand as an indication of fish abundance. Harvey and van der Merwe [9] estimated the mean of the monthly rainfall totals to compare rainfall in Bloemfontein and Kimberley in South Africa.

Furthermore, Thangjai and Niwitpong [10] presented the relative potency of two drugs using the confidence interval for the ratio of means of normal distributions with unknown coefficients of variation.

Agriculture is the primary occupation of rural people in Thailand, and it heavily relies on water as a critical input for production. Consequently, the success of agriculture mainly depends on rainfall. However, the amount of rainfall varies significantly during the monsoon season. Furthermore, heavy rainfall in certain areas has caused flooding, leading to significant impacts on both the economy and society. Therefore, it is crucial to measure the amount of rainfall. Rainfall data comprises zero and positive values that follow a delta-lognormal (DLN) distribution.

The DLN distribution combines both zero and positive values, where the number of zero observations follows a binomial distribution and the positive observations follow a lognormal (LN) distribution (asymmetric distribution). Furthermore, the logarithm values of the LN random variable follow a normal distribution (symmetric distribution). The DLN distribution has been widely used in rainfall network and flood frequency analyses, as demonstrated in studies by Krstanovic and Singh [11], Singh and Rajagopal [12], and Singh and Singh [13]. Moreover, several researchers have proposed confidence intervals for various functions of the DLN distribution, such as the mean, coefficient of variation, and percentile. For instance, Maneerat et al. [14], Maneerat et al. [15], Yosboonruang et al. [16], and Thangjai et al. [17] have all explored this topic.

A censored dataset contains observations within a restricted range of values that are not otherwise measured [18]. Censored datasets are commonly encountered in water quality-related fields. For instance, laboratory measurements of contaminant concentrations are often reported as less than the detection limit. Moreover, river discharges of less than a given measurement threshold level are reported as zero. Historical river discharge records report over half of the annual minimum flows as zero [19]. These discharges may have been zero and thus reported as such, or they may have been between zero and the measurement threshold and reported as zero. The main concern is how to efficiently estimate moments, quantiles, and other descriptive statistics of the underlying continuous distribution using such censored datasets [20]. Several researchers have studied inference for the parameters from censored data. For instance, Owen and DeRouen [21] proposed estimating the mean for LN data containing zeroes and left-censored values, with applications to the measurement of worker exposure to air contaminants. Glass and Gray [22] estimated mean exposures from censored data. Additionally, Krishnamoorthy et al. [23] presented an inference method for estimating the LN mean and quantiles based on samples with left and right Type I censoring.

It is of interest to make inferences about means of a DLN dataset containing left-censored values. In this paper, the censored LN estimator proposed by Krishnamoorthy et al. [23] was used to construct the confidence interval for the mean and difference between means of DLN distributions based on left-censored data. Moreover, confidence intervals were used to estimate the rainfall data in Thailand. The different approaches were used to compare the interval estimators. First, the generalized confidence interval (GCI), Bayesian (BS), and parametric bootstrap (PB) approaches were used to estimate the confidence intervals for the mean of a DLN distribution containing left-censored data. Second, the GCI, BS, PB, and method of variance estimates recovery (MOVER) approaches were used to construct the confidence intervals for the difference between means of the DLN distributions containing left-censored data. The GCI approach, proposed by Weerahandi [24], utilizes the generalized pivotal quantity (GPQ) to construct the confidence interval. Several researchers have applied the GCI approach to solve various problems, such as Tian and Wu [5], Thangjai and Niwitpong [10], Krishnamoorthy and Lu [25], Tian [26], Ye et al. [27], and Thangjai et al. [28]. One advantage of the GCI approach is its ability to estimate the confidence interval for complex parameters. However, a disadvantage of this approach is that it relies on simulated data. The BS approach utilizes the posterior probability resulting from a prior probability and a likelihood function. This approach has been studied by

several research papers to estimate parameters. Examples of these papers include Harvey and van der Merwe [9], Rao and D’Cunha [29], Ma and Chen [30], and Thangjai et al. [31]. One advantage of the BS approach is its ability to estimate credible intervals for complex parameters, but a disadvantage is that it relies on the prior distribution of the parameter. The PB approach utilizes re-sampling to estimate the sampling distribution of the estimator. This method has been applied by several researchers for parameter inference, such as Padgett and Tomlinson [32] and Zhang [33]. One advantage of the PB approach is its ability to estimate confidence intervals for complex parameters. However, a disadvantage is that it requires a specific assumption about the form of the sampling distribution. The MOVER approach was introduced by Zou and Donner [34] and Zou et al. [35]. This approach utilizes the central limit theorem and assumes independence between two estimators. Many researchers have employed the MOVER approach to establish confidence intervals, including Thangjai et al. [28] and Donner and Zou [36]. An advantage of the MOVER approach is its ease of constructing confidence intervals using an exact formula. However, a disadvantage of this approach is that it requires the initial confidence interval of a single parameter of interest to construct the confidence interval for the difference between two parameters of interest.

This paper is organized as follows. In Section 2, the confidence intervals for the mean of a DLN distribution based on left-censored data are provided. In Section 3, the confidence intervals for the difference between the means of DLN distributions based on left-censored data are presented. In Section 4, simulation results are presented to compare the coverage probabilities (CPs) and average lengths (ALs) of the proposed approaches. In Section 5, the proposed approaches are illustrated using two examples. In Section 6, the discussion is presented. In Section 7, the conclusions are given.

2. Confidence Intervals for the Mean of Delta-Lognormal Distribution Based on Left-Censored Data

The data contain zero values and positive values drawn from a DLN distribution. Suppose that n_0 is the number of zero values, which has a binomial distribution. Moreover, suppose that n_1 is the number of positive values, which has a LN distribution. Suppose that $n = n_0 + n_1$ is the sample size. Let $Z = (Z_1, Z_2, \dots, Z_n)$ be the random variable from a DLN distribution with parameters mean μ , variance σ^2 , and the probability of obtaining a zero observation δ . The distribution of Z is defined by

$$G(z_j; \mu, \sigma^2, \delta) = \begin{cases} \delta; & z_j = 0 \\ \delta + (1 - \delta)F(z_j; \mu, \sigma^2); & z_j > 0 \end{cases} \quad (1)$$

where $F(z_j; \mu, \sigma^2)$ is the LN distribution function and $j = 1, 2, \dots, n$.

The population mean of Z is defined by

$$v = (1 - \delta) \exp\left(\mu + \frac{1}{2}\sigma^2\right). \quad (2)$$

Here, the mean from a censored LN distribution is considered. Let $X = (X_1, X_2, \dots, X_n)$ be the random variable from the LN distribution with parameters μ and σ^2 . The random variable contains n_2 observations greater than some censoring point $\log(\xi)$, and $n_1 = n - n_2$ observations less than or equal to $\log(\xi)$ are not known but they are assumed to be nonzero. Let $Y_i = \log(X_i)$ be the observations above $\log(\xi)$, where $i = 1, 2, \dots, n_2$ since Y_i has a normal distribution. Suppose $h = \frac{n_1}{n}$ is the fraction of observations in the sample that is below $\log(\xi)$. The mean and variance of Y_i are given by

$$\bar{Y} = \frac{1}{n_2} \sum_{i=1}^{n_2} Y_i \quad (3)$$

and

$$S^2 = \frac{1}{n_2} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2. \quad (4)$$

Define

$$a = \frac{\log(\xi) - \mu}{\sigma} \quad (5)$$

$$W(a) = \frac{\phi(a)}{1 - \Phi(a)} \quad (6)$$

and

$$V(h, a) = \frac{hW(-a)}{1 - h}, \quad (7)$$

where ϕ and Φ are the density function and the distribution function of the standard normal distribution.

Following Krishnamoorthy et al. [23], the maximum likelihood estimators of μ and σ^2 are given by

$$\hat{\mu} = \bar{Y} - \psi(h, a)(\bar{Y} - \log(\xi)) \quad (8)$$

and

$$\hat{\sigma}^2 = S^2 + \psi(h, a)(\bar{Y} - \log(\xi))^2. \quad (9)$$

The mean of the censored LN distribution is

$$\theta = \exp\left(\mu + \frac{1}{2}\sigma^2\right). \quad (10)$$

From Equations (2) and (10), it can be seen that $v = \theta$ when $\delta = 0$. Therefore, v will be used to represent both the mean of the delta distribution and the LN distribution.

Therefore, the estimator of the mean of censored DLN distribution is

$$\hat{\omega} = \exp\left(\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\right). \quad (11)$$

Three novel approaches were proposed to construct confidence intervals for the mean of the DLN distribution based on left-censored data.

2.1. Generalized Confidence Interval Approach

For the GCI approach, the generalized pivotal quantity (GPQ) is used to construct the confidence interval.

Definition 1. Let $Z = (Z_1, Z_2, \dots, Z_n)$ be the random variable with the probability density function $f(z; \mu, \sigma^2, \delta)$, where μ , σ^2 , and δ are unknown parameters. Let $z = (z_1, z_2, \dots, z_n)$ be the observed value of $Z = (Z_1, Z_2, \dots, Z_n)$. Suppose that $R(Z; z, \mu, \sigma^2, \delta)$ is the function of Z , z , μ , σ^2 , and δ . It satisfies the following two conditions [24]:

1. For $Z = z$, $R(Z; z, \mu, \sigma^2, \delta)$ has a probability distribution free of unknown parameters.
2. For $Z = z$, the observed value of $R(Z; z, \mu, \sigma^2, \delta)$ does not depend on the nuisance parameter.

The $100(1 - \alpha)\%$ two-sided confidence intervals for the DLN mean based on left-censored data can be constructed using $[R(\alpha/2), R(1 - \alpha/2)]$, where $R(\alpha/2)$ and $R(1 - \alpha/2)$ denote the $(\alpha/2)$ -th and $(1 - \alpha/2)$ -th quantile of $R(Z; z, \mu, \sigma^2, \delta)$, respectively.

The GPQ for μ is given by

$$R_\mu = \hat{\mu} - \frac{\hat{\mu}^*}{\hat{\sigma}^*} \hat{\sigma}, \quad (12)$$

where $\hat{\mu}^*$ and $\hat{\sigma}^*$ are the maximum likelihood estimators based on a censored sample from standard normal distribution and $\hat{\mu}$ and $\hat{\sigma}$ are defined in Equations (8) and (9).

The GPQ for σ is given by

$$R_\sigma = \frac{\hat{\sigma}}{\hat{\sigma}^*}, \quad (13)$$

where $\hat{\sigma}^*$ is the maximum likelihood estimator based on a censored sample from standard normal distribution.

The GPQ for the estimator of the mean of censored DLN distribution is given by

$$R_\omega = \exp\left(R_\mu + \frac{1}{2}(R_\sigma)^2\right), \quad (14)$$

where R_μ and R_σ are defined in Equations (12) and (13), respectively.

Therefore, the $100(1 - \alpha)\%$ two-sided confidence interval for the mean of the DLN distribution based on left-censored data using the GCI approach is given by

$$CI_{\omega.GCI} = [L_{\omega.GCI}, U_{\omega.GCI}] = [R_\omega(\alpha/2), R_\omega(1 - \alpha/2)], \quad (15)$$

where $R_\omega(\alpha/2)$ and $R_\omega(1 - \alpha/2)$ denote the $100(\alpha/2)$ -th and $100(1 - \alpha/2)$ -th percentiles of R_ω , respectively.

Algorithm 1 is used to construct the GCI for the mean of the DLN distribution based on left-censored data.

Algorithm 1:

Step 1: Generate sample from the standard normal distribution and compute $\hat{\mu}^*$ and $\hat{\sigma}^*$

Step 2: Compute R_μ from Equation (12) and compute R_σ from Equation (13)

Step 3: Compute R_ω from Equation (14)

Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of R_ω 's

Step 5: Compute $L_{\omega.GCI}$ and $U_{\omega.GCI}$

2.2. Bayesian Approach

The BS approach is a method for updating probabilities based on Bayes' theorem. The BS inference involves modeling uncertainty about unknown parameters using a prior probability distribution. The joint probability distribution on the parameters and the data is described by the prior distribution and the sampling model. The prior distribution characterizes uncertainty about the parameters before observing data. The prior distribution is based on the experimenter's belief. In this paper, the Jeffreys independence prior is used, which is defined as $p(\mu, \sigma^2) = p(\mu)p(\sigma^2)$.

The prior distribution is updated using Bayes' rule, resulting in the posterior distribution, which contains all relevant information about the unknown parameters based on the observed data. Therefore, the posterior distribution of σ^2 is the inverse gamma distribution, which is defined by

$$\sigma^2|y \sim IG\left(\frac{n_2 - 1}{2}, \frac{(n_2 - 1)s^2}{2}\right), \quad (16)$$

where s^2 is the observed value of S^2 defined in Equation (4).

The posterior distribution of μ given σ^2 is the normal distribution, which is defined by

$$\mu|\sigma^2, y \sim N\left(\bar{y}, \frac{\sigma^2}{n_2}\right), \quad (17)$$

where $\bar{y}$ is observed value of $\bar{Y}$ defined in Equation (3) and σ^2 is defined in Equation (16).

The posterior distribution of ω is defined by

$$\omega_{BS} = \exp\left(\mu + \frac{1}{2}\sigma^2\right), \quad (18)$$

where σ^2 and μ are defined in Equation (16) and Equation (17), respectively.

Therefore, the $100(1 - \alpha)\%$ two-sided credible interval for the mean of DLN distribution based on left-censored data using the BS approach is given by

$$CI_{\omega,BS} = [L_{\omega,BS}, U_{\omega,BS}], \quad (19)$$

where $L_{\omega,BS}$ and $U_{\omega,BS}$ denote the lower and upper limits of the shortest $100(1 - \alpha)\%$ and highest posterior density interval of ω_{BS} , respectively.

Algorithm 2 is used to construct the BS credible interval for the mean of the DLN distribution based on left-censored data.

Algorithm 2:

Step 1: Compute $\sigma^2|y$ from Equation (16)
Step 2: Compute $\mu|\sigma^2, y$ from Equation (17)
Step 3: Compute ω_{BS} from Equation (18)
Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of ω_{BS} 's
Step 5: Compute $L_{\omega,BS}$ and $U_{\omega,BS}$

2.3. Parametric Bootstrap Approach

The PB approach involves random sampling with replacement, which enables the estimation of the sampling distribution for almost any statistic using random sampling methods.

Let $Y_1^*, Y_2^*, \dots, Y_{n_2}^*$ be the sample with replacement from $Y_1, Y_2, \dots, Y_{n_2}$. Moreover, let $y_1^*, y_2^*, \dots, y_{n_2}^*$ be the observed value of $Y_1^*, Y_2^*, \dots, Y_{n_2}^*$. Let $\bar{Y}^*$ be the estimator of the population mean, which is given by

$$\bar{Y}^* = \frac{1}{n_2} \sum_{i=1}^{n_2} Y_i^*. \quad (20)$$

Suppose S^{2*} is the estimator of the population variance, which is given by

$$S^{2*} = \frac{1}{n_2} \sum_{i=1}^{n_2} (Y_i^* - \bar{Y}^*)^2. \quad (21)$$

Therefore, the estimator of the mean of censored DLN distribution is

$$\hat{\omega}^* = \exp\left(\bar{Y}^* + \frac{1}{2}S^{2*}\right), \quad (22)$$

where $\bar{Y}^*$ and S^{2*} are defined in Equation (20) and Equation (21), respectively.

The lower and upper limits of the confidence interval for the DLN mean based on left-censored data are given by

$$L_{\omega,PB} = \widehat{\omega}^* - z_{1-\alpha/2}sd(\hat{\omega}^*) \quad (23)$$

and

$$U_{\omega,PB} = \widehat{\omega}^* + z_{1-\alpha/2}sd(\hat{\omega}^*), \quad (24)$$

where $\widehat{\omega}^*$ is the mean of $\hat{\omega}^*$, $sd(\hat{\omega}^*)$ is the standard deviation of $\hat{\omega}^*$, and $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution.

Therefore, the $100(1 - \alpha)\%$ two-sided confidence interval for the mean of DLN distribution based on left-censored data using the PB approach is given by

$$CI_{\omega,PB} = [L_{\omega,PB}, U_{\omega,PB}], \quad (25)$$

where $L_{\omega,PB}$ and $U_{\omega,PB}$ are defined in Equation (23) and Equation (24), respectively.

Algorithm 3 is used to construct the PB confidence interval for the mean of the DLN distribution based on left-censored data.

Algorithm 3:

- Step 1:** Generate $y_1^*, y_2^*, \dots, y_{n_2}^*$
Step 2: Compute $\bar{y}^*$ from Equation (20) and compute s^{2*} from Equation (21)
Step 3: Compute $\hat{\omega}^*$ from Equation (22)
Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of $\hat{\omega}^*$'s
Step 5: Compute $L_{\omega, PB}$ and $U_{\omega, PB}$
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3. Confidence Intervals for the Difference between Means of Delta-Lognormal Distributions Based on Left-Censored Data

For population 1, let $X_1 = (X_{11}, X_{12}, \dots, X_{1n_1})$ be the random variable from the LN distributions with parameters μ_1 and σ_1^2 . The random variables contain $n_{1(2)}$ observations greater than some censoring point $\log(\xi_1)$ and $n_{1(1)} = n_1 - n_{1(2)}$ observations less than or equal to $\log(\xi_1)$. Suppose $Y_1 = (Y_{11}, Y_{12}, \dots, Y_{1n_{1(2)}})$ are the observations above $\log(\xi_1)$ and are the random variables from the normal distributions with mean μ_1 and variance σ_1^2 . Let $h_1 = \frac{n_{1(1)}}{n_1}$ be the fraction of observations in the sample that is below $\log(\xi_1)$. The mean and variance of Y_1 are given by

$$\bar{Y}_1 = \frac{1}{n_{1(2)}} \sum_{i=1}^{n_{1(2)}} Y_i \quad (26)$$

and

$$S_1^2 = \frac{1}{n_{1(2)}} \sum_{i=1}^{n_{1(2)}} (Y_i - \bar{Y}_1)^2. \quad (27)$$

Define

$$a_1 = \frac{\log(\xi_1) - \mu_1}{\sigma_1} \quad (28)$$

$$W(a_1) = \frac{\phi(a_1)}{1 - \Phi(a_1)} \quad (29)$$

and

$$V(h_1, a_1) = \frac{h_1 W(-a_1)}{1 - h_1}, \quad (30)$$

where ϕ and Φ are the density function and the distributions function of the standard normal distribution.

Following Krishnamoorthy et al. [23], the maximum likelihood estimators of μ_1 and σ_1^2 are given by

$$\hat{\mu}_1 = \bar{Y}_1 - \psi(h_1, a_1)(\bar{Y}_1 - \log(\xi_1)) \quad (31)$$

and

$$\hat{\sigma}_1^2 = S_1^2 + \psi(h_1, a_1)(\bar{Y}_1 - \log(\xi_1))^2. \quad (32)$$

The estimator of the mean of the censored DLN distribution is

$$\hat{\omega}_1 = \exp\left(\hat{\mu}_1 + \frac{1}{2}\hat{\sigma}_1^2\right). \quad (33)$$

Similarly, for population 2, let $X_2 = (X_{21}, X_{22}, \dots, X_{2n_2})$ be the random variable from the LN distributions with parameters μ_2 and σ_2^2 . The random variables contain $n_{2(2)}$ observations greater than some censoring point $\log(\xi_2)$ and $n_{2(1)} = n_2 - n_{2(2)}$ observations less than or equal to $\log(\xi_2)$. Suppose $Y_2 = (Y_{21}, Y_{22}, \dots, Y_{2n_{2(2)}})$ are the observations above $\log(\xi_2)$ and are the random variables from the normal distributions with mean μ_2

and variance σ_2^2 . Let $h_2 = \frac{n_{2(1)}}{n_2}$ be the fraction of observations in the sample that is below $\log(\xi_2)$. The mean and variance of Y_2 are given by

$$\bar{Y}_2 = \frac{1}{n_{2(2)}} \sum_{i=1}^{n_{2(2)}} Y_i \quad (34)$$

and

$$S_2^2 = \frac{1}{n_{2(2)}} \sum_{i=1}^{n_{2(2)}} (Y_i - \bar{Y}_2)^2. \quad (35)$$

Define

$$a_2 = \frac{\log(\xi_2) - \mu_2}{\sigma_2} \quad (36)$$

$$W(a_2) = \frac{\phi(a_2)}{1 - \Phi(a_2)} \quad (37)$$

and

$$V(h_2, a_2) = \frac{h_2 W(-a_2)}{1 - h_2}, \quad (38)$$

where ϕ and Φ are the density function and the distributions function of the standard normal distribution.

Following Krishnamoorthy et al. [23], the maximum likelihood estimators of μ_2 and σ_2^2 are given by

$$\hat{\mu}_2 = \bar{Y}_2 - \psi(h_2, a_2)(\bar{Y}_2 - \log(\xi_2)) \quad (39)$$

and

$$\hat{\sigma}_2^2 = S_2^2 + \psi(h_2, a_2)(\bar{Y}_2 - \log(\xi_2))^2. \quad (40)$$

The estimator of the mean of the censored DLN distribution is

$$\hat{\omega}_2 = \exp\left(\hat{\mu}_2 + \frac{1}{2}\hat{\sigma}_2^2\right). \quad (41)$$

Therefore, the estimator of the difference between two means of censored DLN distributions is

$$\hat{\gamma} = \hat{\omega}_1 - \hat{\omega}_2, \quad (42)$$

where $\hat{\omega}_1$ and $\hat{\omega}_2$ are defined in Equation (33) and Equation (41), respectively.

Four novel approaches were presented to estimate confidence intervals for the difference between means of DLN distributions based on left-censored data.

3.1. Generalized Confidence Interval Approach

For population 1, the GPQ for μ_1 is given by

$$R_{\mu_1} = \hat{\mu}_1 - \frac{\hat{\mu}_1^*}{\hat{\sigma}_1^*} \hat{\sigma}_1, \quad (43)$$

where $\hat{\mu}_1^*$ and $\hat{\sigma}_1^*$ are the maximum likelihood estimators based on a censored sample from a standard normal distribution.

The GPQ for σ_1 is given by

$$R_{\sigma_1} = \frac{\hat{\sigma}_1}{\hat{\sigma}_1^*}. \quad (44)$$

The GPQ for the mean of censored DLN distribution is given by

$$R_{\omega_1} = \exp\left(R_{\mu_1} + \frac{1}{2}(R_{\sigma_1})^2\right), \quad (45)$$

where R_{μ_1} and R_{σ_1} are defined in Equation (43) and Equation (44), respectively.

For population 2, the GPQ for μ_2 is given by

$$R_{\mu_2} = \hat{\mu}_2 - \frac{\hat{\mu}_2^*}{\hat{\sigma}_2^*} \hat{\sigma}_2, \quad (46)$$

where $\hat{\mu}_2^*$ and $\hat{\sigma}_2^*$ are the maximum likelihood estimators based on a censored sample from a standard normal distribution.

The GPQ for σ_2 is given by

$$R_{\sigma_2} = \frac{\hat{\sigma}_2}{\hat{\sigma}_2^*}. \quad (47)$$

The GPQ for the mean of the censored DLN distribution is given by

$$R_{\omega_2} = \exp\left(R_{\mu_2} + \frac{1}{2}(R_{\sigma_2})^2\right), \quad (48)$$

where R_{μ_2} and R_{σ_2} are defined in Equation (46) and Equation (47), respectively.

The GPQ for the difference between two means of censored DLN distributions is given by

$$R_{\gamma} = R_{\omega_1} - R_{\omega_2}, \quad (49)$$

where R_{ω_1} and R_{ω_2} are defined in Equation (45) and Equation (48), respectively.

Therefore, the $100(1 - \alpha)\%$ two-sided confidence interval for the difference between means of DLN distributions based on left-censored data using the GCI approach is given by

$$CI_{\gamma, GCI} = [L_{\gamma, GCI}, U_{\gamma, GCI}] = [R_{\gamma}(\alpha/2), R_{\gamma}(1 - \alpha/2)], \quad (50)$$

where $R_{\gamma}(\alpha/2)$ and $R_{\gamma}(1 - \alpha/2)$ denote the $100(\alpha/2)$ -th and $100(1 - \alpha/2)$ -th percentiles of R_{γ} , respectively.

Algorithm 4 is used to construct the GCI for the difference between means of DLN distributions based on left-censored data.

Algorithm 4:

- Step 1:** Generate sample from the standard normal distribution and compute $\hat{\mu}_1^*$, $\hat{\mu}_2^*$, $\hat{\sigma}_1^*$, and $\hat{\sigma}_2^*$
Step 2: Compute R_{μ_1} and R_{σ_1} from Equations (43) and (44) and compute R_{μ_2} and R_{σ_2} from Equations (46) and (47)
Step 3: Compute R_{ω_1} and R_{ω_2} from Equations (45) and (48) and compute R_{γ} from Equation (49)
Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of R_{γ} 's
Step 5: Compute $L_{\gamma, GCI}$ and $U_{\gamma, GCI}$
-

3.2. Bayesian Approach

For population 1, the posterior distribution of σ_1^2 is defined by

$$\sigma_1^2 | y_1 \sim IG\left(\frac{n_{1(2)} - 1}{2}, \frac{(n_{1(2)} - 1)s_1^2}{2}\right), \quad (51)$$

where s_1^2 is the observed value of S_1^2 defined in Equation (27).

The posterior distribution of μ_1 given σ_1^2 is defined by

$$\mu_1 | \sigma_1^2, y_1 \sim N\left(\bar{y}_1, \frac{\sigma_1^2}{n_{1(2)}}\right), \quad (52)$$

where $\bar{y}_1$ is observed value of $\bar{Y}_1$ defined in Equation (26) and σ_1^2 is defined in Equation (51).

The posterior distribution of the DLN mean based on left-censored data is given by

$$\omega_{1.BS} = \exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right), \quad (53)$$

where σ_1^2 and μ_1 are defined in Equation (51) and Equation (52), respectively.

For population 2, the posterior distribution of σ_2^2 is defined by

$$\sigma_2^2|y_2 \sim IG\left(\frac{n_{2(2)} - 1}{2}, \frac{(n_{2(2)} - 1)s_2^2}{2}\right), \quad (54)$$

where s_2^2 is the observed value of S_2^2 defined in Equation (35).

The posterior distribution of μ_2 given σ_2^2 is defined by

$$\mu_2|\sigma_2^2, y_2 \sim N\left(\bar{y}_2, \frac{\sigma_2^2}{n_{2(2)}}\right), \quad (55)$$

where $\bar{y}_2$ is observed value of $\bar{Y}_2$ defined in Equation (34) and σ_2^2 is defined in Equation (54).

The posterior distribution of the DLN mean based on left-censored data is given by

$$\omega_{2.BS} = \exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right), \quad (56)$$

where σ_2^2 and μ_2 are defined in Equation (54) and Equation (55), respectively.

The posterior distribution of the difference between two means of censored DLN distributions is given by

$$\gamma_{BS} = \omega_{1.BS} - \omega_{2.BS}, \quad (57)$$

where $\omega_{1.BS}$ and $\omega_{2.BS}$ are defined in Equation (53) and Equation (56), respectively.

Therefore, the $100(1 - \alpha)\%$ two-sided credible interval for the difference between means of DLN distributions based on left-censored data using the BS approach is given by

$$CI_{\gamma.BS} = [L_{\gamma.BS}, U_{\gamma.BS}], \quad (58)$$

where $L_{\gamma.BS}$ and $U_{\gamma.BS}$ denote the lower and upper limits of the shortest $100(1 - \alpha)\%$ and highest posterior density interval of γ_{BS} , respectively.

Algorithm 5 is used to construct the BS credible interval for the difference between means of DLN distributions based on left-censored data.

Algorithm 5:

Step 1: Compute $\sigma_1^2|y_1$ from Equation (51) and compute $\mu_1|\sigma_1^2, y_1$ from Equation (52)

Step 2: Compute $\sigma_2^2|y_2$ from Equation (54) and compute $\mu_2|\sigma_2^2, y_2$ from Equation (55)

Step 3: Compute $\omega_{1.BS}$ and $\omega_{2.BS}$ from Equations (53) and (56) and compute γ_{BS} from Equation (57)

Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of γ_{BS} 's

Step 5: Compute $L_{\gamma.BS}$ and $U_{\gamma.BS}$

3.3. Parametric Bootstrap Approach

For population 1, let $Y_1^* = (Y_{11}^*, Y_{12}^*, \dots, Y_{1n_{1(2)}}^*)$ be the sample with replacement from $Y_1 = (Y_{11}, Y_{12}, \dots, Y_{1n_{1(2)}})$. Moreover, let $y_1^* = (y_{11}^*, y_{12}^*, \dots, y_{1n_{1(2)}}^*)$ be the observed value of $Y_1^* = (Y_{11}^*, Y_{12}^*, \dots, Y_{1n_{1(2)}}^*)$. Let $\bar{Y}_1^*$ be the estimator of the population mean, which is given by

$$\bar{Y}_1^* = \frac{1}{n_{1(2)}} \sum_{i=1}^{n_{1(2)}} Y_i^*. \quad (59)$$

Suppose S_1^{2*} is the estimator of the population variance, which is given by

$$S_1^{2*} = \frac{1}{n_{1(2)}} \sum_{i=1}^{n_{1(2)}} (Y_i^* - \bar{Y}_1^*)^2. \quad (60)$$

The estimator of the mean of censored DLN distribution is

$$\hat{\omega}_1^* = \exp\left(\bar{Y}_1^* + \frac{1}{2}S_1^{2*}\right), \quad (61)$$

where $\bar{Y}_1^*$ and S_1^{2*} are defined in Equation (59) and Equation (60), respectively.

For population 2, let $Y_2^* = (Y_{21}^*, Y_{22}^*, \dots, Y_{2n_{2(2)}}^*)$ be the sample with replacement from $Y_2 = (Y_{21}, Y_{22}, \dots, Y_{2n_{2(2)}})$. Moreover, let $y_2^* = (y_{21}^*, y_{22}^*, \dots, y_{2n_{2(2)}}^*)$ be the observed value of $Y_2^* = (Y_{21}^*, Y_{22}^*, \dots, Y_{2n_{2(2)}}^*)$. Let $\bar{Y}_2^*$ be the estimator of the population mean, which is given by

$$\bar{Y}_2^* = \frac{1}{n_{2(2)}} \sum_{i=1}^{n_{2(2)}} Y_i^*. \quad (62)$$

Suppose S_2^{2*} is the estimator of the population variance, which is given by

$$S_2^{2*} = \frac{1}{n_{2(2)}} \sum_{i=1}^{n_{2(2)}} (Y_i^* - \bar{Y}_2^*)^2. \quad (63)$$

The estimator of the mean of censored DLN distribution is

$$\hat{\omega}_2^* = \exp\left(\bar{Y}_2^* + \frac{1}{2}S_2^{2*}\right), \quad (64)$$

where $\bar{Y}_2^*$ and S_2^{2*} are defined in Equation (62) and Equation (63), respectively.

The estimator of the difference between two means of censored DLN distributions is

$$\hat{\gamma}^* = \hat{\omega}_1^* - \hat{\omega}_2^*, \quad (65)$$

where $\hat{\omega}_1^*$ and $\hat{\omega}_2^*$ are defined in Equation (61) and Equation (64), respectively.

The lower and upper limits of the confidence interval for the difference between two means of censored DLN distributions are given by

$$L_{\gamma.PB} = \widehat{\gamma}^* - z_{1-\alpha/2}sd(\hat{\gamma}^*) \quad (66)$$

and

$$U_{\gamma.PB} = \widehat{\gamma}^* + z_{1-\alpha/2}sd(\hat{\gamma}^*), \quad (67)$$

where $\widehat{\gamma}^*$ is the mean of $\hat{\gamma}^*$, $sd(\hat{\gamma}^*)$ is the standard deviation of $\hat{\gamma}^*$, and $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution.

Therefore, the $100(1 - \alpha)\%$ two-sided confidence interval for the difference between means of DLN distributions based on left-censored data using the PB approach is given by

$$CI_{\gamma.PB} = [L_{\gamma.PB}, U_{\gamma.PB}], \quad (68)$$

where $L_{\gamma.PB}$ and $U_{\gamma.PB}$ are defined in Equation (66) and Equation (67), respectively.

Algorithm 6 is used to construct the PB confidence interval for the difference between means of DLN distributions based on left-censored data.

Algorithm 6:

- Step 1:** Generate $y_1^* = (y_{11}^*, y_{12}^*, \dots, y_{1n_1(2)}^*)$ and $y_2^* = (y_{21}^*, y_{22}^*, \dots, y_{2n_2(2)}^*)$
Step 2: Compute $\bar{y}_1^*$ and s_1^{2*} from Equations (59) and (60) and compute $\bar{y}_2^*$ and s_2^{2*} from Equations (62) and (63)
Step 3: Compute $\hat{\omega}_1^*$ and $\hat{\omega}_2^*$ from Equations (61) and (64) and compute $\hat{\gamma}^*$ from Equation (65)
Step 4: Repeat Step 1–Step 3 a total m times and obtain an array of $\hat{\gamma}^*$'s
Step 5: Compute $L_{\gamma, PB}$ and $U_{\gamma, PB}$ from Equations (66) and (67)
-

3.4. Method of Variance Estimates Recovery Approach

Following Maneerat et al. [14], the confidence interval for μ is defined as

$$CI_{\mu} = \left[\hat{\mu} - z_{1-\alpha/2} \sqrt{\frac{(n_2 - 1)\hat{\sigma}^2}{n_2 \chi_{n_2-1}^2}}, \hat{\mu} + z_{1-\alpha/2} \sqrt{\frac{(n_2 - 1)\hat{\sigma}^2}{n_2 \chi_{n_2-1}^2}} \right], \quad (69)$$

where $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution and $\chi_{n_2-1}^2$ is the chi-squared distribution with $n_2 - 1$ degrees of freedom.

Therefore, the confidence interval for μ_i can be written as

$$CI_{\mu_i} = \left[\hat{\mu}_i - z_{1-\alpha/2} \sqrt{\frac{(n_{i(2)} - 1)\hat{\sigma}_i^2}{n_{i(2)} \chi_{n_{i(2)}-1}^2}}, \hat{\mu}_i + z_{1-\alpha/2} \sqrt{\frac{(n_{i(2)} - 1)\hat{\sigma}_i^2}{n_{i(2)} \chi_{n_{i(2)}-1}^2}} \right], \quad (70)$$

where $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution and $\chi_{n_{i(2)}-1}^2$ is the chi-squared distribution with $n_{i(2)} - 1$ degrees of freedom.

For population 1, the lower and upper limits of the confidence interval for μ_1 are defined by

$$l_{\mu_1} = \hat{\mu}_1 - z_{1-\alpha/2} \sqrt{\frac{(n_{1(2)} - 1)\hat{\sigma}_1^2}{n_{1(2)} \chi_{n_{1(2)}-1}^2}} \quad (71)$$

and

$$u_{\mu_1} = \hat{\mu}_1 + z_{1-\alpha/2} \sqrt{\frac{(n_{1(2)} - 1)\hat{\sigma}_1^2}{n_{1(2)} \chi_{n_{1(2)}-1}^2}}, \quad (72)$$

where $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution, $\chi_{n_{1(2)}-1}^2$ is the chi-squared distribution with $n_{1(2)} - 1$ degrees of freedom, and $\hat{\mu}_1$ and $\hat{\sigma}_1^2$ are defined in Equation (31) and Equation (32), respectively.

For population 2, the lower and upper limits of the confidence interval for μ_2 are given by

$$l_{\mu_2} = \hat{\mu}_2 - z_{1-\alpha/2} \sqrt{\frac{(n_{2(2)} - 1)\hat{\sigma}_2^2}{n_{2(2)} \chi_{n_{2(2)}-1}^2}} \quad (73)$$

and

$$u_{\mu_2} = \hat{\mu}_2 + z_{1-\alpha/2} \sqrt{\frac{(n_{2(2)} - 1)\hat{\sigma}_2^2}{n_{2(2)} \chi_{n_{2(2)}-1}^2}}, \quad (74)$$

where $z_{1-\alpha/2}$ is the $100(1 - \alpha/2)$ -th percentile of the standard normal distribution, $\chi_{n_{2(2)}-1}^2$ is the chi-squared distribution with $n_{2(2)} - 1$ degrees of freedom, and $\hat{\mu}_2$ and $\hat{\sigma}_2^2$ are defined in Equation (39) and Equation (40), respectively.

Following Maneerat et al. [14], the confidence interval for σ^2 is defined as

$$CI_{\sigma^2} = \left[\frac{(n_2 - 1)\hat{\sigma}^2}{\chi_{1-\alpha/2, n_2-1}^2}, \frac{(n_2 - 1)\hat{\sigma}^2}{\chi_{\alpha/2, n_2-1}^2} \right], \quad (75)$$

where $\chi_{1-\alpha/2, n_2-1}^2$ and $\chi_{\alpha/2, n_2-1}^2$ are the $100(1 - \alpha/2)$ -th and $100(\alpha/2)$ -th percentiles of the chi-squared distribution with $n_2 - 1$ degrees of freedom.

Therefore, the confidence interval for σ_i^2 can be written as

$$CI_{\sigma_i^2} = \left[\frac{(n_{i(2)} - 1)\hat{\sigma}_i^2}{\chi_{1-\alpha/2, n_{i(2)}-1}^2}, \frac{(n_{i(2)} - 1)\hat{\sigma}_i^2}{\chi_{\alpha/2, n_{i(2)}-1}^2} \right], \quad (76)$$

where $\chi_{1-\alpha/2, n_{i(2)}-1}^2$ and $\chi_{\alpha/2, n_{i(2)}-1}^2$ are the $100(1 - \alpha/2)$ -th and $100(\alpha/2)$ -th percentiles of the chi-squared distribution with $n_{i(2)} - 1$ degrees of freedom.

For population 1, the lower and upper limits of the confidence interval for σ_1^2 are defined by

$$l_{\sigma_1^2} = \frac{(n_{1(2)} - 1)\hat{\sigma}_1^2}{\chi_{1-\alpha/2, n_{1(2)}-1}^2} \quad (77)$$

and

$$u_{\sigma_1^2} = \frac{(n_{1(2)} - 1)\hat{\sigma}_1^2}{\chi_{\alpha/2, n_{1(2)}-1}^2}, \quad (78)$$

where $\chi_{1-\alpha/2, n_{1(2)}-1}^2$ and $\chi_{\alpha/2, n_{1(2)}-1}^2$ are the $100(1 - \alpha/2)$ -th and $100(\alpha/2)$ -th percentiles of the chi-squared distribution with $n_{1(2)} - 1$ degrees of freedom, and $\hat{\sigma}_1^2$ is defined in Equation (32).

For population 2, the lower and upper limits of the confidence interval for σ_2^2 are defined by

$$l_{\sigma_2^2} = \frac{(n_{2(2)} - 1)\hat{\sigma}_2^2}{\chi_{1-\alpha/2, n_{2(2)}-1}^2} \quad (79)$$

and

$$u_{\sigma_2^2} = \frac{(n_{2(2)} - 1)\hat{\sigma}_2^2}{\chi_{\alpha/2, n_{2(2)}-1}^2}, \quad (80)$$

where $\chi_{1-\alpha/2, n_{2(2)}-1}^2$ and $\chi_{\alpha/2, n_{2(2)}-1}^2$ are the $100(1 - \alpha/2)$ -th and $100(\alpha/2)$ -th percentiles of the chi-squared distribution with $n_{2(2)} - 1$ degrees of freedom, and $\hat{\sigma}_2^2$ is defined in Equation (40).

Applying the concept of Donner and Zou [36], the lower and upper limits of confidence interval for $\omega_1 = \exp\left(\mu_1 + \frac{1}{2}\sigma_1^2\right)$ are given by

$$l_{\omega_1} = \exp \left[\left(\hat{\mu}_1 + \frac{1}{2}\hat{\sigma}_1^2 \right) - \sqrt{(\hat{\mu}_1 - l_{\mu_1})^2 + \left(\frac{1}{2}\hat{\sigma}_1^2 - \frac{1}{2}l_{\sigma_1^2} \right)^2} \right] \quad (81)$$

and

$$u_{\omega_1} = \exp \left[\left(\hat{\mu}_1 + \frac{1}{2}\hat{\sigma}_1^2 \right) + \sqrt{(u_{\mu_1} - \hat{\mu}_1)^2 + \left(\frac{1}{2}u_{\sigma_1^2} - \frac{1}{2}\hat{\sigma}_1^2 \right)^2} \right], \quad (82)$$

where $\hat{\mu}_1$, $\hat{\sigma}_1^2$, l_{μ_1} , u_{μ_1} , $l_{\sigma_1^2}$, and $u_{\sigma_1^2}$ are defined in Equation (31), Equation (32), Equation (71), Equation (72), Equation (77), and Equation (78), respectively.

Similarly, the lower and upper limits of confidence interval for $\omega_2 = \exp\left(\mu_2 + \frac{1}{2}\sigma_2^2\right)$ are given by

$$l_{\omega_2} = \exp\left[\left(\hat{\mu}_2 + \frac{1}{2}\hat{\sigma}_2^2\right) - \sqrt{(\hat{\mu}_2 - l_{\mu_2})^2 + \left(\frac{1}{2}\hat{\sigma}_2^2 - \frac{1}{2}l_{\sigma_2^2}\right)^2}\right] \quad (83)$$

and

$$u_{\omega_2} = \exp\left[\left(\hat{\mu}_2 + \frac{1}{2}\hat{\sigma}_2^2\right) + \sqrt{(u_{\mu_2} - \hat{\mu}_2)^2 + \left(\frac{1}{2}u_{\sigma_2^2} - \frac{1}{2}\hat{\sigma}_2^2\right)^2}\right], \quad (84)$$

where $\hat{\mu}_2$, $\hat{\sigma}_2^2$, l_{μ_2} , u_{μ_2} , $l_{\sigma_2^2}$, and $u_{\sigma_2^2}$ are defined in Equation (39), Equation (40), Equation (73), Equation (74), Equation (79), and Equation (80), respectively.

Using the concept of Donner and Zou [36], the lower and upper limits of confidence interval for $\gamma = \omega_1 - \omega_2$ are given by

$$L_{\gamma.MOVER} = \hat{\omega}_1 - \hat{\omega}_2 - \sqrt{(\hat{\omega}_1 - l_{\omega_1})^2 + (u_{\omega_2} - \hat{\omega}_2)^2} \quad (85)$$

and

$$U_{\gamma.MOVER} = \hat{\omega}_1 - \hat{\omega}_2 + \sqrt{(u_{\omega_1} - \hat{\omega}_1)^2 + (\hat{\omega}_2 - l_{\omega_2})^2}, \quad (86)$$

where $\hat{\omega}_1$, $\hat{\omega}_2$, l_{ω_1} , u_{ω_1} , l_{ω_2} , and u_{ω_2} are defined in Equation (33), Equation (41), Equation (81), Equation (82), Equation (83), and Equation (84), respectively.

Therefore, the $100(1 - \alpha)\%$ two-sided confidence interval for the difference between means of DLN distributions based on left-censored data using the MOVER approach is given by

$$CI_{\gamma.MOVER} = [L_{\gamma.MOVER}, U_{\gamma.MOVER}], \quad (87)$$

where $L_{\gamma.MOVER}$ and $U_{\gamma.MOVER}$ are defined in Equation (85) and Equation (86), respectively.

Algorithm 7 is used to construct the MOVER confidence interval for the difference between means of DLN distributions based on left-censored data.

Algorithm 7:

- Step 1:** Compute l_{μ_1} from Equation (71) and compute u_{μ_1} from Equation (72)
 - Step 2:** Compute l_{μ_2} from Equation (73) and compute u_{μ_2} from Equation (74)
 - Step 3:** Compute $l_{\sigma_1^2}$ from Equation (77) and compute $u_{\sigma_1^2}$ from Equation (78)
 - Step 4:** Compute $l_{\sigma_2^2}$ from Equation (79) and compute $u_{\sigma_2^2}$ from Equation (80)
 - Step 5:** Compute l_{ω_1} from Equation (81) and compute u_{ω_1} from Equation (82)
 - Step 6:** Compute l_{ω_2} from Equation (83) and compute u_{ω_2} from Equation (84)
 - Step 7:** Compute $L_{\gamma.MOVER}$ from Equation (85) and compute $U_{\gamma.MOVER}$ from Equation (86)
-

4. Results

The performance of the proposed confidence intervals was evaluated using the R statistical program through a Monte Carlo simulation. CP and AL were used to compare the performance of the proposed confidence intervals at a nominal confidence level of 0.95. The best confidence interval was defined as the one with a CP greater than or equal to 0.95 and the shortest AL.

For the mean of the DLN distribution based on left-censored data, consider the iteration number as $M = 5000$ and take $m = 2500$ in the algorithms of the GCI, BS, and PB estimations. The sample size was set as $n = 20, 30, 50$, and 100 . Following Owen and DeRouen [21], the parameter values used in the simulations are given in Table 1. Note that the estimate of μ is the maximum likelihood estimator for the censored lognormal distribution. The estimate of μ is the range between $-\infty$ and ∞ . If $\mu + \frac{1}{2}\sigma^2$ is greater than zero, then $\theta = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$ is positive value.

Table 1. Values selected for the population mean, population standard deviation, probability of obtaining zero observation, and censoring point for the mean of DLN distribution based on left-censored data.

Run Number	μ	σ	δ	ξ
1	0.0326	0.3815	0.10	0.25
2	−0.0742	0.5992	0.10	0.10
3	−0.1810	0.7568	0.10	0.05
4	0.1971	0.4257	0.25	0.25
5	0.0821	0.6412	0.25	0.10
6	−0.0302	0.7974	0.25	0.05
7	−0.2061	0.7175	0.05	0.25
8	−0.2722	0.8690	0.10	0.25
9	−0.1656	0.9522	0.25	0.25
10	−0.5909	1.1801	0.10	0.25

Algorithm 8 is used to construct the proposed confidence intervals for the mean of the DLN distribution based on left-censored data.

Algorithm 8:

Step 1: Generate z from DLN distribution with parameters μ , σ , and δ and set x from LN distribution with parameters μ and σ
Step 2: Compute $y = \log(x)$ and select $y > \log(\xi)$
Step 3: Compute n_1 , n_2 , $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\omega}$
Step 4: Use Algorithms 1–3 to construct the confidence intervals
Step 5: If $L \leq \omega \leq U$, set $p = 1$; else, set $p = 0$
Step 6: Compute $U - L$
Step 7: Repeat Step 1–Step 6 a total M times
Step 8: Compute mean of p defined by the CP
Step 9: Compute mean of $U - L$ defined by the AL

The results presented in Table 2 indicate that the CPs of the confidence interval based on the GCI approach were greater than 0.95, except for run 6 and $n = 50$ and 100, where the CPs of the confidence intervals were less than 0.95. In addition, the CPs of the confidence interval based on the BS approach were greater than 0.95 for all cases. Moreover, the CPs of the confidence interval based on the PB approach were less than 0.95 for all cases. When considering the AL, it is observed that the ALs of the confidence interval based on the BS approach were shorter than those of the confidence interval based on the GCI approach. However, for run 5, the ALs of the confidence interval based on the GCI approach were shorter than those of the confidence interval based on the BS approach. Thus, the results suggest that the BS approach performed satisfactorily, while the GCI approach is recommended for run 5.

For the difference between means of DLN distributions based on left-censored data, for each set of parameter settings, $M = 5000$ simulation runs were generated for the GCI, BS, PB, and MOVER estimations. Moreover, $m = 2500$ runs were fixed for the GCI, BS, and PB estimations. The random sample sizes $(n_1, n_2) = (20, 20), (30, 30), (20, 30), (50, 50), (30, 50), (100, 100)$, and $(50, 100)$ were generated with specific parameters. According to Owen and DeRouen [21], the parameter values used in the simulations were applied in Table 3.

Table 2. The CPs and ALs of 95% two-sided confidence intervals for the mean of DLN distribution based on left-censored data.

<i>n</i>	Run Number	CP (AL)		
		<i>CI_{ω.GCI}</i>	<i>CI_{ω.BS}</i>	<i>CI_{ω.PB}</i>
20	1	0.9912 (0.8372)	0.9872 (0.7328)	0.8692 (0.3881)
	2	0.9902 (1.7040)	0.9904 (1.6223)	0.9224 (0.7408)
	3	0.9806 (3.1942)	0.9888 (3.1780)	0.9054 (1.1575)
	4	0.9990 (2.2617)	0.9984 (2.0191)	0.8184 (0.6068)
	5	0.9982 (6.4150)	0.9998 (9.3597)	0.9052 (1.4474)
	6	0.9828 (18.7733)	0.9994 (42.5334)	0.8090 (2.8766)
	7	0.9788 (1.3040)	0.9650 (1.0321)	0.8832 (0.7072)
	8	0.9818 (2.3494)	0.9740 (1.8102)	0.8630 (0.9981)
	9	0.9800 (5.8573)	0.9796 (6.4620)	0.8232 (1.5802)
	10	0.9742 (6.9770)	0.9798 (6.7278)	0.8446 (1.7690)
30	1	0.9918 (0.6294)	0.9904 (0.5688)	0.8774 (0.3223)
	2	0.9928 (1.2028)	0.9924 (1.1615)	0.9268 (0.6090)
	3	0.9824 (2.0167)	0.9908 (2.0038)	0.9012 (0.9383)
	4	0.9992 (1.5172)	0.9994 (1.3733)	0.8188 (0.4992)
	5	0.9974 (3.6072)	0.9996 (3.9979)	0.9050 (1.1628)
	6	0.9684 (8.2978)	0.9988 (10.9098)	0.7400 (2.2252)
	7	0.9770 (0.9274)	0.9652 (0.7750)	0.8986 (0.5868)
	8	0.9766 (1.5154)	0.9686 (1.2409)	0.8736 (0.8145)
	9	0.9770 (3.0594)	0.9756 (2.6148)	0.8372 (1.2530)
	10	0.9722 (3.2162)	0.9764 (2.6407)	0.8728 (1.3501)

Table 2. Cont.

n	Run Number	CP (AL)		
		$CI_{\omega.GCI}$	$CI_{\omega.BS}$	$CI_{\omega.PB}$
50	1	0.9754 (0.4543)	0.9884 (0.4208)	0.8504 (0.2501)
	2	0.9942 (0.8482)	0.9952 (0.8277)	0.9380 (0.4770)
	3	0.9740 (1.3171)	0.9906 (1.3268)	0.8792 (0.7227)
	4	0.9980 (1.0337)	0.9980 (0.9580)	0.7780 (0.3907)
	5	0.9938 (2.2628)	0.9998 (2.4167)	0.8826 (0.8943)
	6	0.9144 (4.5057)	0.9924 (5.2494)	0.5864 (1.6769)
	7	0.9692 (0.6525)	0.9598 (0.5668)	0.9070 (0.4596)
	8	0.9696 (1.0010)	0.9620 (0.8609)	0.8784 (0.6282)
	9	0.9780 (1.8114)	0.9730 (1.5904)	0.8340 (0.9488)
	10	0.9698 (1.8249)	0.9728 (1.5558)	0.8828 (1.0062)
100	1	0.9122 (0.3077)	0.9754 (0.2897)	0.7966 (0.1789)
	2	0.9942 (0.5614)	0.9962 (0.5531)	0.9356 (0.3382)
	3	0.9484 (0.8421)	0.9746 (0.8586)	0.8010 (0.5120)
	4	0.9928 (0.6693)	0.9958 (0.6333)	0.6548 (0.2777)
	5	0.9820 (1.3874)	0.9980 (1.4675)	0.8164 (0.6278)
	6	0.7512 (2.5295)	0.9358 (2.8640)	0.2938 (1.1597)
	7	0.9524 (0.4352)	0.9572 (0.3878)	0.9090 (0.3298)
	8	0.9618 (0.6394)	0.9590 (0.5676)	0.8856 (0.4452)
	9	0.9678 (1.0887)	0.9582 (0.9860)	0.8214 (0.6666)
	10	0.9720 (1.0669)	0.9688 (0.9445)	0.8932 (0.6959)

Table 3. Values selected for the population means, population standard deviations, probabilities of obtaining zero observation, and censoring points for the difference between means of DLN distributions based on left-censored data.

Run Number	(μ_1, μ_2)	(σ_1, σ_2)	(δ_1, δ_2)	(ξ_1, ξ_2)
1	(0.00, 0.00)	(0.3815, 0.3815)	(0.10, 0.10)	(0.10, 0.10)
2	(0.00, 0.00)	(0.3815, 0.3815)	(0.10, 0.10)	(0.10, 0.25)
3	(0.00, 0.00)	(0.3815, 0.3815)	(0.10, 0.25)	(0.10, 0.10)
4	(0.00, 0.00)	(0.3815, 0.3815)	(0.10, 0.25)	(0.10, 0.25)
5	(0.00, 0.00)	(0.3815, 0.5992)	(0.10, 0.10)	(0.10, 0.10)
6	(0.00, 0.00)	(0.3815, 0.5992)	(0.10, 0.10)	(0.10, 0.25)
7	(0.00, 0.00)	(0.3815, 0.5992)	(0.10, 0.25)	(0.10, 0.10)
8	(0.00, 0.00)	(0.3815, 0.5992)	(0.10, 0.25)	(0.10, 0.25)

Algorithm 9 is used to construct the proposed confidence intervals for the difference between means of DLN distributions based on left-censored data.

Algorithm 9:

Step 1: Generate z_1 from DLN distribution with parameters μ_1 , σ_1 , and δ_1 , and set x_1 from LN distribution with parameters μ_1 and σ_1
Step 2: Generate z_2 from DLN distribution with parameters μ_2 , σ_2 , and δ_2 , and set x_2 from LN distribution with parameters μ_2 and σ_2
Step 3: Compute $y_1 = \log(x_1)$ and select $y_1 > \log(\xi_1)$
Step 4: Compute $y_2 = \log(x_2)$ and select $y_2 > \log(\xi_2)$
Step 5: Compute $n_{1(1)}, n_{1(2)}, n_{2(1)}, n_{2(2)}, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\omega}_1, \hat{\omega}_2$, and $\hat{\gamma}$
Step 6: Use Algorithms 4–7 to construct the confidence intervals
Step 7: If $L \leq \gamma \leq U$, set $p = 1$; else, set $p = 0$
Step 8: Compute $U - L$
Step 9: Repeat Step 1–Step 8 a total M times
Step 10: Compute mean of p defined by the CP
Step 11: Compute mean of $U - L$ defined by the AL

The estimated CPs and ALs of 95% confidence intervals for the difference between means of DLN distributions based on left-censored data are presented in Table 4. For all sample sizes, the results indicated that the CPs of the confidence intervals based on the GCI, BS, and MOVER approaches were greater than 0.95. The CPs of the confidence interval based on the PB approach were less than 0.95 for all cases, except for $(n_1, n_2) = (50, 50)$ in run 7 and $(n_1, n_2) = (30, 50)$ in run 5, where the CPs of the confidence interval based on the PB approach were greater than 0.95. For $(n_1, n_2) = (20, 20), (30, 30), (20, 30)$, the ALs of the confidence interval based on the GCI approach were shorter than the ALs of the other confidence intervals in runs 1, 3, 4, 5, and 7, whereas the ALs of the credible interval based on the BS approach were shorter than the ALs of the other confidence intervals in runs 2, 6, and 8. For $(n_1, n_2) = (50, 50)$, the ALs of the confidence interval based on the GCI approach were shorter than the others in runs 1, 3, 4, and 5, while the ALs of the credible interval based on the BS approach were shorter than the others in runs 2, 6, and 8. For run 7, the ALs of the confidence interval based on the PB approach were the shortest. For $(n_1, n_2) = (30, 50)$, the GCI approach had the shortest ALs in runs 1, 3, 4, and 7, whereas the BS approach had the shortest ALs in runs 2, 6, and 8. The PB approach had the shortest ALs in run 5. In addition, for $(n_1, n_2) = (100, 100)$, the ALs of the confidence interval based on the GCI approach were shorter than the others in runs 1, 3, and 7, whereas the ALs of the confidence interval based on the BS approach were shorter than the others in runs 2, 4, 5, 6, and 8. Moreover, for $(n_1, n_2) = (50, 100)$, the GCI approach had the shortest ALs in runs 1, 3, 5, and 7, while the BS approach had the shortest ALs in runs 2, 4, 6, and 8. Therefore, the GCI and BS approaches were satisfactory. Moreover, the PB approach performed satisfactorily for $(n_1, n_2) = (50, 50)$ in run 7 and $(n_1, n_2) = (30, 50)$ in run 5.

Table 4. The CPs and ALs of 95% two-sided confidence intervals for the difference between means of DLN distributions based on left-censored data.

(n_1, n_2)	Run Number	CP (AL)			
		$CI_{\gamma.GCI}$	$CI_{\gamma.BS}$	$CI_{\gamma.PB}$	$CI_{\gamma.MOVER}$
(20, 20)	1	0.9990 (2.1168)	0.9996 (2.2141)	0.9340 (0.6374)	0.9990 (2.3162)
	2	0.9966 (1.6883)	0.9978 (1.6417)	0.8996 (0.5923)	0.9956 (1.7354)
	3	1.0000 (3.9405)	1.0000 (5.1912)	0.9266 (0.7716)	1.0000 (7.6454)
	4	0.9986 (2.0779)	0.9996 (2.1431)	0.8128 (0.6094)	0.9996 (2.2208)
	5	0.9962 (2.6335)	0.9984 (2.7108)	0.9302 (0.9446)	0.9958 (2.8697)
	6	0.9916 (2.1284)	0.9958 (2.0000)	0.8800 (0.8267)	0.9912 (2.0541)
	7	0.9996 (5.2628)	1.0000 (6.9590)	0.9386 (1.2488)	1.0000 (12.2312)
	8	0.9934 (2.7956)	0.9966 (2.7810)	0.8072 (0.8828)	0.9942 (2.9778)
(30, 30)	1	0.9994 (1.4633)	0.9998 (1.4972)	0.9372 (0.5237)	0.9994 (1.5127)
	2	0.9950 (1.1725)	0.9984 (1.1409)	0.8886 (0.4819)	0.9974 (1.1625)
	3	0.9998 (2.4588)	1.0000 (2.7804)	0.9318 (0.6282)	1.0000 (3.0647)
	4	0.9976 (1.4264)	0.9990 (1.4370)	0.7602 (0.5009)	0.9986 (1.4470)
	5	0.9974 (1.8046)	0.9986 (1.8281)	0.9382 (0.7750)	0.9982 (1.8711)
	6	0.9888 (1.4737)	0.9930 (1.3893)	0.8828 (0.6837)	0.9908 (1.4006)
	7	0.9994 (3.1278)	1.0000 (3.4931)	0.9414 (1.0036)	0.9998 (4.0365)
	8	0.9940 (1.8611)	0.9974 (1.8076)	0.7866 (0.7303)	0.9946 (1.8445)
(20, 30)	1	0.9992 (1.7747)	0.9992 (1.8158)	0.9276 (0.5816)	0.9992 (1.8873)
	2	0.9954 (1.5513)	0.9978 (1.5031)	0.9038 (0.5497)	0.9972 (1.6174)
	3	1.0000 (2.7087)	1.0000 (3.1494)	0.9282 (0.6747)	1.0000 (3.3786)
	4	0.9984 (1.7824)	0.9998 (1.8087)	0.8114 (0.5643)	0.9990 (1.8763)
	5	0.9966 (2.1119)	0.9990 (2.1777)	0.9440 (0.8188)	0.9976 (2.2600)
	6	0.9906 (1.8202)	0.9948 (1.7418)	0.8912 (0.7325)	0.9918 (1.8165)
	7	0.9996 (3.4029)	1.0000 (3.9400)	0.9458 (1.0435)	1.0000 (4.3992)
	8	0.9916 (2.1869)	0.9960 (2.1775)	0.8104 (0.7755)	0.9940 (2.2339)

Table 4. Cont.

(n_1, n_2)	Run Number	CP (AL)			
		$CI_{\gamma.GCI}$	$CI_{\gamma.BS}$	$CI_{\gamma.PB}$	$CI_{\gamma.MOVER}$
(50, 50)	1	0.9996	0.9998	0.9448	1.0000
		(1.0123)	(1.0245)	(0.4072)	(1.0278)
	2	0.9882	0.9986	0.8382	0.9966
		(0.8374)	(0.8199)	(0.3782)	(0.8273)
	3	1.0000	1.0000	0.9236	1.0000
		(1.5939)	(1.7293)	(0.4847)	(1.8045)
	4	0.9920	0.9984	0.6556	0.9970
		(0.9874)	(0.9881)	(0.3916)	(0.9902)
	5	0.9984	0.9996	0.9408	0.9990
		(1.2249)	(1.2283)	(0.6016)	(1.2408)
	6	0.9786	0.9894	0.8558	0.9860
		(1.0218)	(0.9703)	(0.5325)	(0.9718)
	7	0.9992	1.0000	0.9532	1.0000
		(1.9803)	(2.1144)	(0.7749)	(2.2552)
	8	0.9798	0.9884	0.7294	0.9862
		(1.2322)	(1.1939)	(0.5631)	(1.2042)
(30, 50)	1	0.9994	0.9996	0.9386	0.9996
		(1.2356)	(1.2480)	(0.4675)	(1.2666)
	2	0.9912	0.9984	0.8862	0.9968
		(1.0900)	(1.0604)	(0.4424)	(1.0960)
	3	1.0000	1.0000	0.9156	1.0000
		(1.7892)	(1.9630)	(0.5385)	(2.0156)
	4	0.9932	0.9984	0.7366	0.9968
		(1.2141)	(1.2149)	(0.4537)	(1.2314)
	5	0.9990	0.9998	0.9520	0.9994
		(1.4403)	(1.4596)	(0.6444)	(1.4757)
	6	0.9852	0.9948	0.8862	0.9894
		(1.2451)	(1.1966)	(0.5814)	(1.2150)
	7	0.9978	1.0000	0.9446	0.9994
		(2.1593)	(2.3543)	(0.8129)	(2.4570)
	8	0.9864	0.9946	0.7740	0.9910
		(1.4486)	(1.4275)	(0.6113)	(1.4375)
(100, 100)	1	0.9998	0.9998	0.9422	0.9998
		(0.6640)	(0.6678)	(0.2888)	(0.6702)
	2	0.9488	0.9940	0.7474	0.9912
		(0.5514)	(0.5422)	(0.2670)	(0.5467)
	3	1.0000	1.0000	0.8930	1.0000
		(1.0056)	(1.0697)	(0.3430)	(1.0928)
	4	0.9450	0.9858	0.4352	0.9850
		(0.6428)	(0.6420)	(0.2777)	(0.6438)
	5	0.9986	0.9988	0.9456	0.9984
		(0.7958)	(0.7956)	(0.4256)	(0.7998)
	6	0.9574	0.9852	0.8098	0.9814
		(0.6671)	(0.6384)	(0.3765)	(0.6407)
	7	0.9990	1.0000	0.9258	0.9996
		(1.2338)	(1.3014)	(0.5453)	(1.3453)
	8	0.9540	0.9664	0.5948	0.9672
		(0.7916)	(0.7697)	(0.3999)	(0.7739)

Table 4. Cont.

(n_1, n_2)	Run Number	CP (AL)			
		$CI_{\gamma.GCI}$	$CI_{\gamma.BS}$	$CI_{\gamma.PB}$	$CI_{\gamma.MOVER}$
(50, 100)	1	0.9996	0.9998	0.9428	0.9998
		(0.8511)	(0.8540)	(0.3532)	(0.8623)
	2	0.9766	0.9968	0.8262	0.9954
		(0.7631)	(0.7493)	(0.3361)	(0.7651)
	3	1.0000	1.0000	0.8872	1.0000
		(1.1519)	(1.2302)	(0.3992)	(1.2417)
	4	0.9688	0.9928	0.5956	0.9902
		(0.8307)	(0.8290)	(0.3432)	(0.8377)
	5	0.9990	0.9994	0.9480	0.9996
		(0.9611)	(0.9662)	(0.4709)	(0.9725)
	6	0.9778	0.9936	0.8508	0.9904
		(0.8517)	(0.8253)	(0.4291)	(0.8343)
	7	0.9990	1.0000	0.9330	0.9996
		(1.3531)	(1.4418)	(0.5792)	(1.4654)
	8	0.9680	0.9876	0.6782	0.9844
		(0.9544)	(0.9381)	(0.4474)	(0.9431)

5. Empirical Applications

The GCI, BS, and PB approaches discussed in Section 2 for constructing the confidence interval for the mean of the DLN distribution based on left-censored data can be applied to estimate the average daily rainfall of Chiang Mai and Lampang provinces. Additionally, the GCI, BS, PB, and MOVER approaches presented in Section 3 for constructing the confidence interval for the difference between the means of the DLN distributions based on left-censored data can be used to compare the average daily rainfall in Chiang Mai and Lampang provinces. Table 5 shows the daily rainfall data from 1 to 25 July 2022, reported by the Thai Meteorological Department. The table includes 50 observations, out of which 20 of 25 (80.00%) and 14 of 25 (56.00%) represent positive observed values in Chiang Mai and Lampang provinces, respectively. The log-transformed positive daily rainfall values in Chiang Mai and Lampang provinces follow normal distributions. Thus, the daily rainfall datasets in Chiang Mai and Lampang provinces fit the DLN distributions.

Table 5. The daily rainfall data of Chiang Mai and Lampang provinces.

Chiang Mai Province					Lampang Province				
2.0	14.2	2.6	0.3	13.3	1.3	0.1	0.0	0.0	7.7
0.2	1.6	0.5	0.0	45.7	0.0	0.0	1.6	0.0	23.6
0.0	10.9	18.6	0.0	7.1	0.0	0.4	5.0	0.0	0.4
0.0	1.7	16.8	4.6	0.0	0.0	2.8	36.6	38.3	0.0
7.7	0.5	2.0	0.3	0.8	1.4	29.2	1.2	0.0	0.0

Source: Thai Meteorological Department.

For Chiang Mai province, the statistics are $n_1 = 25$, $n_{1(1)} = 6$, $n_{1(2)} = 19$, $\hat{\mu}_1 = 0.42$, $\hat{\sigma}_1^2 = 4.07$, and $\hat{\omega}_1 = 11.67$. The 95% two-sided confidence intervals for the mean of the daily rainfall data in Chiang Mai province were estimated using the GCI, BS, and PB approaches. For the GCI approach, $CI_{\omega.GCI} = [4.1935, 326.4762]$ with an interval length of 322.2827. For the BS approach, $CI_{\omega.BS} = [2.6455, 81.6410]$ with an interval length of 78.9955. For the PB approach, $CI_{\omega.PB} = [3.7505, 31.3524]$ with an interval length of 27.6019.

For Lampang province, the statistics are $n_2 = 25$, $n_{2(1)} = 12$, $n_{2(2)} = 13$, $\hat{\mu}_2 = -0.16$, $\hat{\sigma}_2^2 = 6.78$, and $\hat{\omega}_2 = 25.24$. The 95% two-sided confidence intervals for the mean of the daily rainfall data in Lampang province were constructed using the GCI, BS, and PB approaches. For the GCI approach, $CI_{\omega.GCI} = [5.9458, 4363.6820]$ with an interval length of 4357.7362. For the BS approach, $CI_{\omega.BS} = [2.1516, 1971.9490]$ with an interval length of 1969.7974. For the PB approach, $CI_{\omega.PB} = [3.6736, 150.9010]$ with an interval length of 147.2274.

Based on the daily rainfall data from Chiang Mai and Lampang provinces, it was found that the interval lengths of the PB confidence intervals were shorter compared to the interval lengths of the GCI and BS credible intervals. This confirms the results of simulation studies in terms of interval length. However, it should be noted that the confidence intervals for the means of the daily rainfall data were computed using one sample, while the ALs in the simulation studies were computed using 5000 random samples. Additionally, the CPs of the PB confidence intervals in the simulation studies were less than 0.95. On the other hand, the BS credible intervals had shorter interval lengths than the GCI, with CPs greater than 0.95 in the simulation studies. Therefore, it is recommended to use the BS approach for constructing the credible interval for the mean of the DLN distribution based on left-censored data.

The difference between the means of the daily rainfall data in Chiang Mai province and Lampang province is $\hat{\gamma} = -13.57$. The 95% two-sided confidence intervals for the difference between the means of the daily rainfall data in Chiang Mai province and Lampang province are estimated based on GCI, BS, PB, and MOVER approaches. For the GCI approach, $CI_{\gamma, GCI} = [-7757.1400, 222.2289]$ with an interval length of 7979.3689. For the BS approach, $CI_{\gamma, BS} = [-2071.8670, 422.0152]$ with an interval length of 2493.8822. For the PB approach, $CI_{\gamma, PB} = [-108.4549, 59.8140]$ with an interval length of 168.2689. For the MOVER approach, $CI_{\gamma, MOVER} = [-10,463.6900, 133.1321]$ with an interval length of 10,596.8221. Therefore, the PB confidence interval has the shortest interval length. These results confirm the simulation studies. However, the CPs of the PB confidence interval were less than 0.95. Therefore, the BS approach was suggested for estimating the credible interval for the difference between means of DLN distributions based on left-censored data.

6. Discussion

Maneerat et al. [15] proposed credible intervals for the single mean and the difference between two means of DLN distributions using the BS approach. However, in some situations, environmental and medical data may contain left-censored data, such as particulate matter data and rainfall data. Owen and DeRouen [21] proposed point estimators for the mean of LN distributions containing zeroes and left-censored values. This study extends to confidence intervals for the mean and the difference between means of DLN distributions that include left-censored data. First, the confidence intervals for the mean of the DLN distribution containing left-censored data are constructed based on the GCI, BS, and PB approaches. Second, the confidence intervals for the difference between the means of DLN distributions containing left-censored data are estimated using the GCI, BS, PB, and MOVER approaches. The GCI approach computes the GPQ for the parameter of interest; the BS approach obtains the posterior distribution of the parameter of interest; the PB approach resamples bootstrap samples for computing the estimator of the parameter of interest; and the MOVER approach obtains the single confidence interval for the parameter of interest.

The GCI and BS approaches for constructing confidence intervals for the mean yielded CPs greater than 0.95. However, the BS approach had shorter ALs than the GCI approach.

The results for the confidence intervals of the difference between means showed that the GCI, BS, and MOVER approaches had CPs greater than 0.95. The ALs of the GCI and BS approaches were shorter than those of the MOVER approach. In addition, the PB approach had CPs greater than 0.95, and its ALs were shorter than those of the other approaches for $(n_1, n_2) = (50, 50)$ in run 7 and $(n_1, n_2) = (30, 50)$ in run 5.

The BS approach is recommended for constructing credible intervals for the mean and the difference between means of DLN distributions containing left-censored data. This result is consistent with those reported by Thangjai et al. [17,31]. Additionally, the GCI approach can be used to estimate the confidence intervals for the difference between means of DLN distributions based on left-censored data, which is similar to the findings reported by Thangjai and Niwitpong [10] and Thangjai et al. [28].

7. Conclusions

The confidence intervals for the single mean of the DLN distribution based on left-censored data were estimated using the GCI, BS, and PB approaches. The BS approach was found to perform the best in terms of CPs and ALs in all cases.

The confidence intervals for the difference between means of DLN distributions based on left-censored data were estimated using the GCI, BS, PB, and MOVER approaches. The GCI and BS approaches performed better than the PB and MOVER approaches in terms of both CPs and ALs.

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