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Review

Fluxbrane Polynomials and Melvin-like Solutions for Simple Lie Algebras

Sergey V. Bolokhov [1] and Vladimir D. Ivashchuk [1,2,*]

[1] Institute of Gravitation and Cosmology, RUDN University, 6 Miklukho-Maklaya Str., 117198 Moscow, Russia
[2] Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya Str., 119361 Moscow, Russia
* Correspondence: ivas@vniims.ru

Abstract: This review dealt with generalized Melvin solutions for simple finite-dimensional Lie algebras. Each solution appears in a model which includes a metric and n scalar fields coupled to n Abelian 2-forms with dilatonic coupling vectors determined by simple Lie algebra of rank n. The set of n moduli functions $H_s(z)$ comply with n non-linear (ordinary) differential equations (of second order) with certain boundary conditions set. Earlier, it was hypothesized that these moduli functions should be polynomials in z (so-called "fluxbrane" polynomials) depending upon certain parameters $p_s > 0$, $s = 1, \ldots, n$. Here, we presented explicit relations for the polynomials corresponding to Lie algebras of ranks $n = 1, 2, 3, 4, 5$ and exceptional algebra E_6. Certain relations for the polynomials (e.g., symmetry and duality ones) were outlined. In a general case where polynomial conjecture holds, 2-form flux integrals are finite. The use of fluxbrane polynomials to dilatonic black hole solutions was also explored.

Keywords: Melvin solution; polynomials; Lie algebras; duality; black holes

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Citation: Bolokhov, S.V.; Ivashchuk, V.D. Fluxbrane Polynomials and Melvin-like Solutions for Simple Lie Algebras. *Symmetry* **2023**, *15*, 1199. https://doi.org/10.3390/sym15061199

Academic Editor: Ignatios Antoniadis

Received: 12 April 2023
Revised: 25 May 2023
Accepted: 25 May 2023
Published: 3 June 2023

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1. Introduction

In this review article, we dealt with a certain generalization of the Melvin solution [1], which was studied earlier in Ref. [2]. It occurs in the model which contains metric, n Abelian 2-forms $F^s = dA^s$ and l scalar fields φ^α ($l \geq n$). This solution is governed by a certain non-degenerate matrix (A_{sl}) ("quasi-Cartan" matrix), $s, l = 1, \ldots, n$. It is a particular case of the so-called generalized fluxbrane solutions presented earlier in Ref. [3].

The original $4d$ Melvin's solution, which describes the gravitational field of a magnetic flux tube, has numerous multidimensional analogs, supported by certain configurations of form fields. These analogs are usually referred to as fluxbranes. The fluxbrane solutions originally appear in superstring/brane models. For generalized Melvin and fluxbrane solutions, see [4–26] and references therein.

It is important to note that the solutions from Ref. [3] are governed by a set of moduli functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$. Here, $z = \rho^2$ and ρ is a radial variable. These functions obey a set of n non-linear ordinary differential equations of second order—so-called master equations, governed by the matrix $(A_{ss'})$ (in fact, they are equivalent to Toda-like equations). The moduli functions should also obey the boundary conditions: $H_s(+0) = 1$, $s = 1, ..., n$.

Here, we assume that the matrix $(A_{ss'})$ is just a Cartan matrix of a simple finite-dimensional Lie algebra $\mathcal{G}$ of rank n. (Obviously, $A_{ss} = 2$ for all s). Due to "polynomial conjecture" from Ref. [3], the solutions to master equations with the boundary conditions imposed have a polynomial structure:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \tag{1}$$

where $P_s^{(k)}$ are constants ($P_s^{(n_s)} \neq 0$) and

$$n_s = 2 \sum_{s'=1}^{n} A^{ss'}. \tag{2}$$

Here, we denote $(A^{ss'}) = (A_{ss'})^{-1}$. The parameters n_s are integers, which are called components of a twice-dual Weyl vector in the basis of simple roots [27].

In Refs. [2,28], a program (in Maple) for the calculation of these (fluxbrane) polynomials for a classical series of Lie algebras (A_n, B_n, C_n, D_n) was presented.

The fluxbrane polynomials H_s define special solutions to open Toda chain equations [29–32], corresponding to simple finite-dimensional Lie algebra $\mathcal{G}$,

$$\frac{d^2 y^s}{du^2} = -4P_s \exp(\sum_{l=1}^{n} A_{sl} y^l), \tag{3}$$

where

$$H_s = \exp(-y^s(u) - n_s u), \tag{4}$$

$P_s > 0$, $s = 1, ..., n$, and $z = e^{-2u}$. These special solutions obey

$$y^s(u) = -n_s u + o(1), \tag{5}$$

as $u \to +\infty$.

In Section 2, we describe the generalized Melvin solution related to a simple finite-dimensional Lie algebra $\mathcal{G}$ [2]. In Section 3 and in Appendices A and B, we present fluxbrane polynomials for Lie algebras of ranks $n = 1, 2, 3, 4, 5$ and also for E_6. Here, we also outline the so-called symmetry and duality identities for these polynomials and consider certain relations between them. In Section 4, we present calculations of 2-form flux integrals $\Phi^s = \int_{M_*} F^s$ over a certain $2d$ submanifold M_*. It is amazing that these integrals (fluxes) are finite for all parameters of fluxbrane polynomials. In Section 5, we outline possible applications of fluxbrane polynomials to dilatonic black hole solutions.

It should be noted that definitions of fluxbrane polynomials can be easily extended to (finite dimensional) semisimple Lie algebras.

2. The Solutions

We consider a model governed by the action

$$S = \int d^D x \sqrt{|g|} \Big\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^{n} \exp[2\lambda_s(\varphi)](F^s)^2 \Big\}, \tag{6}$$

where $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a set of scalar fields, and $(h_{\alpha\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $F^s = dA^s = \frac{1}{2} F^s_{MN} dx^M \wedge dx^N$ is a 2-form, λ_s is a 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha}\varphi^\alpha$, $s = 1, ..., n$; $\alpha = 1, \dots, l$. Here, $(\lambda_{s\alpha})$, $s = 1, \dots, n$, are dilatonic coupling vectors. In (6) we denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{M_1 M_2} F^s_{N_1 N_2} g^{M_1 N_1} g^{M_2 N_2}$, $s = 1, \dots, n$.

Here, we deal with a family of exact solutions to field equations which correspond to the action (6) and depend on one variable ρ. These solutions are defined on the manifold,

$$M = (0, +\infty) \times M_1 \times M_2, \tag{7}$$

where M_1 is a one-dimensional manifold (say S^1 or $\mathbb{R}$) and M_2 is a $(D-2)$-dimensional Ricci-flat manifold. The solution (from the family under consideration) reads [2]

$$g = \Big(\prod_{s=1}^{n} H_s^{2h_s/(D-2)} \Big) \Big\{ w d\rho \otimes d\rho + \Big(\prod_{s=1}^{n} H_s^{-2h_s} \Big) \rho^2 d\phi \otimes d\phi + g^2 \Big\}, \tag{8}$$

$$\exp(\varphi^\alpha) = \prod_{s=1}^{n} H_s^{h_s \lambda_s^\alpha}, \tag{9}$$

$$F^s = q_s \left(\prod_{l=1}^{n} H_l^{-A_{sl}} \right) \rho d\rho \wedge d\phi, \tag{10}$$

$s = 1, \dots, n$; $\alpha = 1, \dots, l$, where $w = \pm 1$, $g^1 = d\phi \otimes d\phi$ is a metric on M_1 and g^2 is a Ricci-flat metric on M_2. Here, $q_s \neq 0$ are integration constants ($q_s = -Q_s$ in notations of Ref. [2]), $s = 1, \dots, n$.

The moduli functions $H_s(z) > 0$, $z = \rho^2$ obey the master equations,

$$\frac{d}{dz} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = P_s \prod_{l=1}^{n} H_l^{-A_{sl}}, \tag{11}$$

with the following boundary conditions:

$$H_s(+0) = 1, \tag{12}$$

where

$$P_s = \frac{1}{4} K_s q_s^2, \tag{13}$$

$s = 1, \dots, n$. For $w = +1$, the boundary conditions (12) are necessary to avoid a conic singularity (for our metric (8)) in the limit $\rho = +0$.

The parameters h_s obey the relations,

$$h_s = K_s^{-1}, \qquad K_s = B_{ss} > 0, \tag{14}$$

where

$$B_{ss'} \equiv 1 + \frac{1}{2-D} + \lambda_{s\alpha} \lambda_{s'\beta} h^{\alpha\beta}, \tag{15}$$

$s, s' = 1, ..., n$, with $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$. In the relations above we denote $\lambda_s^\alpha = h^{\alpha\beta} \lambda_{s\beta}$ and

$$(A_{ss'}) = (2B_{ss'}/B_{s's'}). \tag{16}$$

This is the so-called quasi-Cartan matrix.

The constants $B_{ss'}$ and $K_s = B_{ss}$ are related to scalar products of so-called "brane vectors" U^s, belonging to a certain linear space (in our case it has dimension $l + 2$). We have $B_{ss'} = (U^s, U^{s'})$ and $K_s = (U^s, U^s)$, with certain scalar product $(.,.)$ defined in Refs. [33,34]. Such scalar products appear in various solutions with branes (black branes, fluxbranes, S-branes etc), e.g., in calculations of certain physical parameters (Hawking temperature, black hole/brane entropy, PPN parameters etc), see Ref. [34].

Product relation (16) defines generalized intersection rules for branes [33], while numbers K_s are invariant under dimensional reductions with typical value $K_s = 2$ for brane U-vectors which appear in numerous supergravity models, e.g., for $D = 10, 11$ [35].

It can be readily shown that if the matrix $(h_{\alpha\beta})$ is of Euclidean signature, $l \geq n$, and $(A_{ss'})$ is a Cartan matrix of certain simple Lie algebra of rank n, then there exist co-vectors $\lambda_1, \dots, \lambda_n$ obeying (16).

Our solution is nothing more than a special case of the fluxbrane (for $w = +1$, $M_1 = S^1$) and S-brane ($w = -1$) solutions from Refs. [3] and [25], respectively.

If we put $w = +1$ and choose Ricci-flat metric g^2 of pseudo-Euclidean signature on manifold M_2 of dimension $d_2 > 2$, we obtain a higher dimensional generalization of the Melvin's solution [1].

The Melvin's solution does not contain scalar fields. It corresponds (in our notations) to $D = 4$, $n = 1$, $w = +1$, $l = 0$, $M_1 = S^1$ $(0 < \phi < 2\pi)$, $M_2 = \mathbb{R}^2$, $g^2 = -dt \otimes dt + dx \otimes dx$, and Lie algebra $\mathcal{G} = A_1 = \mathrm{sl}(2)$.

For the case of $w = -1$ and g^2 of Euclidean signature, one can obtain a cosmological solution with a horizon (as $\rho = +0$) if $M_1 = \mathbb{R}$ $(-\infty < \phi < +\infty)$.

3. Examples of Solutions for Certain Lie Algebras

Here, we deal with the generalized Melvin-like solution for $n = l$, $w = +1$ and $M_1 = S^1$, which corresponds to simple (finite dimensional) Lie algebra of certain rank n with the Cartan matrix $A = (A_{sl})$.

We put here $h_{\alpha\beta} = \delta_{\alpha\beta}$ and denote $(\lambda_{s\alpha}) = (\lambda_s^\alpha) = \vec{\lambda}_s$, $s = 1, \ldots, n$.

Due to (14)–(16) we get

$$K_s = \frac{D-3}{D-2} + \vec{\lambda}_s^2, \tag{17}$$

$h_s = K_s^{-1}$, and

$$\vec{\lambda}_s \vec{\lambda}_l = \frac{1}{2} K_l A_{sl} - \frac{D-3}{D-2} \equiv G_{sl}, \tag{18}$$

$s, l = 1, \ldots, n$; (17) just follows from (18).

Remark 1. *For large enough K_s in (18) (or large enough $\vec{\lambda}_s^2$), there exist vectors $\vec{\lambda}_s$ obeying (18) (and hence (17)). Indeed, the matrix $G = (G_{sl})$ is positive definite if $K_s > K_*$, where K_* is some positive number. Hence, there exists a matrix Λ, such that $\Lambda^T \Lambda = G$. We put $(\Lambda_{as}) = (\lambda_s^a)$ and get a set of vectors obeying (18).*

Here, as in [36], we use another parameter p_s instead of P_s:

$$p_s = P_s / n_s, \tag{19}$$

$s = 1, ..., n$. This is carried out to avoid big denominators for $P_s^{(k)}$ in relation (1).

Remark 2. *The parameters p_s give us the coefficients $P_s^{(1)}$ in (1):*

$$P_s^{(1)} = P_s = p_s n_s, \tag{20}$$

$s = 1, ..., n$. These relations can be readily obtained by putting $z = +0$ into master Equation (11) and using the boundary conditions (12). Moreover, for given Lie algebra one can deduce recurrent relations for higher coefficients $P_s^{(k+1)}$ in (1) as functions of $P_s^{(k)}, \ldots, P_s^{(1)} = P_s$ and solve the chain of recurrent relations, justifying that $P_s^{(n_s+1)} = 0$, for all $s = 1, ..., n$.

We note also that, for a special choice of p_s parameters: $p_s = p > 0$, the polynomials have the following simple form [3]:

$$H_s(z) = (1 + pz)^{n_s}, \tag{21}$$

$s = 1, ..., n$. This relation may be considered a nice tool for the verification of general solutions for polynomials.

3.1. Rank-1 Case

***A_1*-case.** We start with the simplest example, which occurs for the Lie algebra $A_1 = sl(2)$. We obtain [3]

$$H_1 = 1 + p_1 z. \tag{22}$$

Here, $n_1 = 1$.

In this case (due to (14)–(16)), we have

$$K_1 = \frac{D-3}{D-2} + \vec{\lambda}_1^2, \tag{23}$$

$h_1 = K_1^{-1}$.

Due to (22), we get the following asymptotical behavior:

$$H_1 = H_1(z, p_1) \sim p_1 z \equiv H_1^{as}(z, p_1), \tag{24}$$

as $z \to \infty$.

Relations (22) and (24) imply the following identity.

Duality Relation

Proposition 1. *The fluxbrane polynomial corresponding to Lie algebra A_1 obeys for all $p_1 \neq 0$ and $z \neq 0$ the identity*

$$H_1(z, p_1) = H_1^{as}(z, p_1) H_1(z^{-1}, p_1^{-1}). \tag{25}$$

3.2. Rank-2 Case

Now, we proceed with the solutions which correspond to simple Lie algebras $\mathcal{G}$ of rank 2, i.e., the matrix $A = (A_{sl})$ is just a Cartan matrix,

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -k & 2 \end{pmatrix}, \tag{26}$$

where $k = 1, 2, 3$ for $\mathcal{G} = A_2, C_2 \cong B_2, G_2$, respectively [37].

The matrix A is described graphically by the Dynkin diagrams presented in Figure 1 (for any of these three Lie algebras).

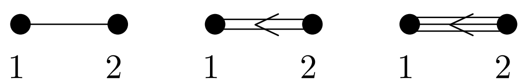

Figure 1. The Dynkin diagrams for the Lie algebras A_2, C_2, G_2, respectively.

It follows from (14)–(16) that

$$\frac{h_1}{h_2} = \frac{K_2}{K_1} = \frac{A_{21}}{A_{12}} = k, \tag{27}$$

where $k = 1, 2, 3$ for $\mathcal{G} = A_2, C_2, G_2$, respectively.

3.2.1. Polynomials

***A_2*-case.** For the Lie algebra $A_2 = sl(3)$, we have [3,25,36]

$$H_1 = 1 + 2p_1 z + p_1 p_2 z^2, \tag{28}$$
$$H_2 = 1 + 2p_2 z + p_1 p_2 z^2. \tag{29}$$

***C_2*-case.** In the case of Lie algebra $C_2 = so(5)$, we get the following polynomials [25,36]:

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1^2 p_2 z^3, \tag{30}$$
$$H_2 = 1 + 4p_2 z + 6p_1 p_2 z^2 + 4p_1^2 p_2 z^3 + p_1^2 p_2^2 z^4. \tag{31}$$

***G_2*-case.** For the Lie algebra G_2 , the fluxbrane polynomials read [25,36]:

$$H_1 = 1 + 6p_1 z + 15p_1 p_2 z^2 + 20p_1^2 p_2 z^3 + \\ 15p_1^3 p_2 z^4 + 6p_1^3 p_2^2 z^5 + p_1^4 p_2^2 z^6, \tag{32}$$

$$H_2 = 1 + 10p_2 z + 45p_1 p_2 z^2 + 120p_1^2 p_2 z^3 + p_1^2 p_2 (135p_1 + 75p_2) z^4 \\ +252p_1^3 p_2^2 z^5 + p_1^3 p_2^2 \Big(75p_1 + 135p_2\Big) z^6 + 120p_1^4 p_2^3 z^7 \\ +45p_1^5 p_2^3 z^8 + 10p_1^6 p_2^3 z^9 + p_1^6 p_2^4 z^{10}. \tag{33}$$

Let us denote

$$H_s = H_s(z) = H_s(z, (p_i)), \tag{34}$$

$s = 1, 2$; where $(p_i) = (p_1, p_2)$.

We have the following asymptotical relations for polynomials:

$$H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{2} (p_l)^{\nu^{sl}}\right) z^{n_s} \equiv H_s^{as}(z, (p_i)), \tag{35}$$

$s = 1, 2$, as $z \to \infty$.

Here, $\nu = (\nu^{sl})$ is the integer valued matrix,

$$\nu = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix}, \tag{36}$$

for Lie algebras A_2, C_2, G_2, respectively.

For last two cases (C_2 and G_2), we have $\nu = 2A^{-1}$ (A^{-1} is inverse Cartan matrix). For the A_2-case, the matrix ν reads

$$\nu = A^{-1}(I + P), \tag{37}$$

where I is a 2×2 identity matrix and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{38}$$

is a permutation matrix. It corresponds to the permutation $\sigma \in S_2$ (S_2 is symmetric group)

$$\sigma : (1, 2) \mapsto (2, 1), \tag{39}$$

by the following relation $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here, σ is the generator of the group $S_2 = \{\sigma, id\}$—the group of symmetry of the Dynkin diagram (for A_2), which is isomorphic to the group Z_2.

Here, in all cases we get

$$\sum_{l=1}^{2} \nu^{sl} = n_s, \tag{40}$$

$s = 1, 2$.

Now, we denote $\hat{p}_i = p_{\sigma(i)}$ for the A_2-case and $\hat{p}_i = p_i$ for C_2 and G_2 cases, $i = 1, 2$. We call the ordered set $(\hat{p}_i)$ a dual one to the ordered set (p_i). Using the relations for polynomials, we obtain the following identities (which can be readily verified just "by hands").

3.2.2. Symmetry Relations

Proposition 2. *The fluxbrane polynomials for A_2 obey for all p_i and z the identities:*

$$H_{\sigma(s)}(z, (p_i)) = H_s(z, (\hat{p}_i)), \tag{41}$$

$s = 1, 2$.

3.2.3. Duality Relations

Proposition 3. *The fluxbranes polynomials corresponding to Lie algebras A_2, C_2 and G_2 obey for all $p_i > 0$ and $z > 0$ the identities*

$$H_s(z,(p_i)) = H_s^{as}(z,(p_i))H_s(z^{-1},(\hat{p}_i^{-1})), \tag{42}$$

$s = 1, 2$.

We call relations (42) duality ones.

3.3. Rank-3 Algebras

Now, we deal with polynomials which correspond to simple Lie algebras $\mathcal{G}$ of rank 3, when the matrix $A = (A_{sl})$ coincides with one of the Cartan matrices,

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \tag{43}$$

for $\mathcal{G} = A_3, B_3, C_3$, respectively [38].

Any of these matrices is described graphically by a Dynkin diagram pictured in Figure 2.

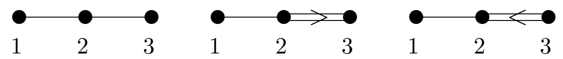

Figure 2. The Dynkin diagrams for the Lie algebras A_3, B_3, C_3, respectively.

It follows from (14), (16) that

$$\frac{h_s}{h_l} = \frac{K_l}{K_s} = \frac{B_{ll}}{B_{ss}} = \frac{B_{ls}}{B_{ss}}\frac{B_{ll}}{B_{sl}} = \frac{A_{ls}}{A_{sl}}, \tag{44}$$

for any $s \neq l$ obeying $A_{sl}, A_{ls} \neq 0$. This implies

$$K_1 = K_2 = K, \qquad K_3 = K, \ \frac{1}{2}K, \ 2K \tag{45}$$

or

$$h_1 = h_2 = h, \qquad h_3 = h, \ 2h, \ \frac{1}{2}h, \tag{46}$$

($h = K^{-1}$) for $\mathcal{G} = A_3, B_3, C_3$, respectively.

3.3.1. Polynomials

The set of moduli functions $(H_1(z), H_2(z), H_3(z))$, obeying Equations (11) and (12) with the matrix $A = (A_{sl})$ from (43) are polynomials with powers $(n_1, n_2, n_3) = (3, 4, 3)$, $(6, 10, 6)$, $(5, 8, 9)$ for $\mathcal{G} = A_3, B_3, C_3$, respectively. We get the following polynomials [38].

***A*₃-case.** For the Lie algebra $A_3 \cong sl(4)$ we have [28,36]

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1 p_2 p_3 z^3, \tag{47}$$
$$H_2 = 1 + 4p_2 z + (3p_1 p_2 + 3p_2 p_3)z^2 + 4p_1 p_2 p_3 z^3 + p_1 p_2^2 p_3 z^4, \tag{48}$$
$$H_3 = 1 + 3p_3 z + 3p_2 p_3 z^2 + p_1 p_2 p_3 z^3. \tag{49}$$

B_3-case. In the case of Lie algebra $B_3 \cong so(7)$, the fluxbrane polynomials read [28]

$$
\begin{aligned}
H_1 &= 1 + 6p_1 z + 15p_1 p_2 z^2 + 20p_1 p_2 p_3 z^3 + 15p_1 p_2 p_3^2 z^4 + 6p_1 p_2^2 p_3^2 z^5 + p_1^2 p_2^2 p_3^2 z^6, && (50)\\
H_2 &= 1 + 10p_2 z + (15p_1 p_2 + 30p_2 p_3) z^2 + \left(80p_1 p_2 p_3 + 40p_2 p_3^2\right) z^3 \\
&\quad + \left(50p_1 p_2^2 p_3 + 135p_1 p_2 p_3^2 + 25p_2^2 p_3^2\right) z^4 + 252p_1 p_2^2 p_3^2 z^5 + \left(25p_1^2 p_2^2 p_3^2 + 135p_1 p_2^3 p_3^2 + 50p_1 p_2^2 p_3^3\right) z^6 && (51)\\
&\quad + \left(40p_1^2 p_2^3 p_3^2 + 80p_1 p_2^3 p_3^3\right) z^7 + \left(30p_1^2 p_2^3 p_3^3 + 15p_1 p_2^3 p_3^4\right) z^8 + 10p_1^2 p_2^3 p_3^4 z^9 + p_1^2 p_2^4 p_3^4 z^{10},\\
H_3 &= 1 + 6p_3 z + 15p_2 p_3 z^2 + \left(10p_1 p_2 p_3 + 10p_2 p_3^2\right) z^3 + 15p_1 p_2 p_3^2 z^4 + 6p_1 p_2^2 p_3^2 z^5 + p_1 p_2^2 p_3^3 z^6. && (52)
\end{aligned}
$$

***C_3*-case.** For the Lie algebra $C_3 \cong sp(3)$, we obtain (with the use of MATHEMATICA) the following polynomials:

$$
\begin{aligned}
H_1 &= 1 + 5p_1 z + 10p_1 p_2 z^2 + 10p_1 p_2 p_3 z^3 + 5p_1 p_2^2 p_3 z^4 + p_1^2 p_2^2 p_3 z^5, && (53)\\
H_2 &= 1 + 8p_2 z + (10p_1 p_2 + 18p_2 p_3) z^2 + \left(40p_1 p_2 p_3 + 16p_2^2 p_3\right) z^3 + 70p_1 p_2^2 p_3 z^4 \\
&\quad + \left(16p_1^2 p_2^2 p_3 + 40p_1 p_2^3 p_3\right) z^5 + \left(18p_1^2 p_2^3 p_3 + 10p_1 p_2^3 p_3^2\right) z^6 && (54)\\
&\quad + 8p_1^2 p_2^3 p_3^2 z^7 + p_1^2 p_2^4 p_3^2 z^8,\\
H_3 &= 1 + 9p_3 z + 36p_2 p_3 z^2 + \left(20p_1 p_2 p_3 + 64p_2^2 p_3\right) z^3 + \left(90p_1 p_2^2 p_3 + 36p_2^2 p_3^2\right) z^4 \\
&\quad + \left(36p_1^2 p_2^2 p_3 + 90p_1 p_2^2 p_3^2\right) z^5 + \left(64p_1^2 p_2^2 p_3^2 + 20p_1 p_2^3 p_3^2\right) z^6 && (55)\\
&\quad + 36p_1^2 p_2^3 p_3^2 z^7 + 9p_1^2 p_2^4 p_3^2 z^8 + p_1^2 p_2^4 p_3^3 z^9.
\end{aligned}
$$

We denote

$$
H_s = H_s(z) = H_s(z, (p_i)), \tag{56}
$$

where $(p_i) = (p_1, p_2, p_3)$.

Due to obtained relations for polynomials, we get the asymptotical behavior

$$
H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{3} (p_l)^{\nu^{sl}} \right) z^{n_s} \equiv H_s^{as}(z, (p_i)), \tag{57}
$$

as $z \to \infty$.

Here, $\nu = (\nu^{sl})$ is the integer valued matrix

$$
\nu = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}, \tag{58}
$$

for Lie algebras A_3, B_3, C_3, respectively.

For Lie algebras B_3 and C_3 we have

$$
\nu = 2A^{-1}, \tag{59}
$$

where A^{-1} is inverse Cartan matrix. For the A_3-case the matrix ν reads as follows:

$$
\nu = A^{-1}(I + P), \tag{60}
$$

where I is 3×3 identity matrix and

$$
P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{61}
$$

is a permutation matrix, which corresponds to the permutation $\sigma \in S_3$ (S_3 is symmetric group)

$$\sigma : (1,2,3) \mapsto (3,2,1), \tag{62}$$

by the following formula $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here, σ is the generator of the group $G = \{\sigma, \mathrm{id}\}$—the group of symmetry of the Dynkin diagram (for A_3), which is isomorphic to the group $\mathbb{Z}_2$.

Here, in all three cases we have

$$\sum_{l=1}^{3} \nu^{sl} = n_s, \tag{63}$$

$s = 1,2,3.$

Now, we introduce notations: $\hat{p}_i = p_{\sigma(i)}$ for the A_3 and $\hat{p}_i = p_i$ for B_3 and C_3 algebras, $i = 1,2,3$. Using relations for rank-3 polynomials, we obtain (with a help of MATHEMATICA) the following identities.

3.3.2. Symmetry Relations

Proposition 4. *The fluxbrane polynomials for A_3 algebra obey for all p_i and z the identities:*

$$H_{\sigma(s)}(z,(p_i)) = H_s(z,(\hat{p}_i)), \tag{64}$$

$s = 1,2,3.$

3.3.3. Duality Relations

Proposition 5. *The fluxbranes polynomials corresponding to Lie algebras A_3, B_3 and C_3 obey for all $p_i > 0$ and $z > 0$ the identities*

$$H_s(z,(p_i)) = H_s^{as}(z,(p_i)) H_s(z^{-1},(\hat{p}_i^{-1})), \tag{65}$$

$s = 1,2,3.$

3.4. Rank-4 Algebras

In this subsection, we deal with the solutions related to Lie algebras $\mathcal{G}$ of rank 4, i.e., the matrix $A = (A_{sl})$ coincides with one of the Cartan matrices.

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{66}$$

for $\mathcal{G} = A_4$, B_4, C_4, D_4, F_4, respectively.

These matrices are graphically described using the Dynkin diagrams pictured in Figure 3.

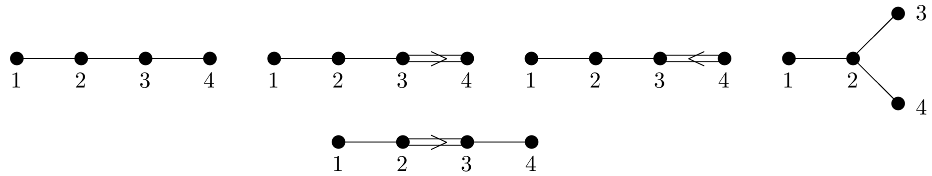

Figure 3. The Dynkin diagrams for the Lie algebras A_4, B_4, C_4, D_4, F_4, respectively.

It follows from (14), (16) that

$$\frac{h_s}{h_l} = \frac{K_l}{K_s} = \frac{B_{ll}}{B_{ss}} = \frac{B_{ls}}{B_{ss}}\frac{B_{ll}}{B_{sl}} = \frac{A_{ls}}{A_{sl}} \tag{67}$$

for any $s \neq l$ obeying $A_{sl}, A_{ls} \neq 0$. This implies

$$K_1 = K_2 = K_3 = K, \qquad K_4 = K,\ \frac{1}{2}K,\ 2K,\ K, \tag{68}$$

or

$$h_1 = h_2 = h_3 = h, \qquad h_4 = h,\ 2h,\ \frac{1}{2}h,\ h, \tag{69}$$

($h = K^{-1}$) for $\mathcal{G} = A_4, B_4, C_4, D_4$, respectively, and

$$K_1 = K_2 = K, \qquad K_3 = K_4 = \frac{1}{2}K, \tag{70}$$

or

$$h_1 = h_2 = h, \qquad h_3 = h_4 = 2h, \tag{71}$$

($h = K^{-1}$) for $\mathcal{G} = F_4$.

3.4.1. Polynomials

Due to polynomial conjecture, the functions $H_1(z), \ldots, H_4(z)$, obeying Equations (11) and (12) with the Cartan matrix $A = (A_{sl})$ from (66) should be polynomials with powers $(n_1, n_2, n_3, n_4) = (4,6,6,4)$, $(8,14,18,10)$, $(7,12,15,16)$, $(6,10,6,6)$, $(22,42,30,16)$ (see (2)) for Lie algebras A_4, B_4, C_4, D_4, F_4, respectively.

One can verify this conjecture by using appropriate MATHEMATICA algorithm, which follows from master Equation (11). Below we present a list of the obtained polynomials [39,40].

***A*₄-case.** For the Lie algebra $A_4 \cong sl(5)$ we find

$$H_1 = 1 + 4p_1 z + 6p_1p_2 z^2 + 4p_1p_2p_3 z^3 + p_1p_2p_3p_4 z^4, \tag{72}$$

$$\begin{aligned} H_2 = {} & 1 + 6p_2 z + (6p_1p_2 + 9p_2p_3) z^2 + (16p_1p_2p_3 + 4p_2p_3p_4) z^3 \\ & + \left(6p_1p_2^2p_3 + 9p_1p_2p_3p_4\right) z^4 + 6p_1p_2^2p_3p_4 z^5 + p_1p_2^2p_3^2p_4 z^6, \end{aligned} \tag{73}$$

$$\begin{aligned} H_3 = {} & 1 + 6p_3 z + (9p_2p_3 + 6p_3p_4) z^2 + (4p_1p_2p_3 + 16p_2p_3p_4) z^3 \\ & + \left(9p_1p_2p_3p_4 + 6p_2p_3^2p_4\right) z^4 + 6p_1p_2p_3^2p_4 z^5 + p_1p_2^2p_3^2p_4 z^6, \end{aligned} \tag{74}$$

$$H_4 = 1 + 4p_4 z + 6p_3p_4 z^2 + 4p_2p_3p_4 z^3 + p_1p_2p_3p_4 z^4. \tag{75}$$

***B*₄-case.** In the case of Lie algebra $B_4 \cong so(9)$, the fluxbrane polynomials read:

$$
\begin{aligned}
H_1 = {}& 1 + 8p_1z + 28p_1p_2z^2 + 56p_1p_2p_3z^3 + 70p_1p_2p_3p_4z^4 + 56p_1p_2p_3p_4^2z^5 \\
& + 28p_1p_2p_3^2p_4^2z^6 + 8p_1p_2^2p_3^2p_4^2z^7 + p_1^2p_2^2p_3^2p_4^2z^8,
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
H_2 = {}& 1 + 14p_2z + (28p_1p_2 + 63p_2p_3)z^2 + (224p_1p_2p_3 + 140p_2p_3p_4)z^3 \\
& + (196p_1p_2^2p_3 + 630p_1p_2p_3p_4 + 175p_2p_3p_4^2)z^4 \\
& + \left(980p_1p_2^2p_3p_4 + 896p_1p_2p_3p_4^2 + 126p_2p_3^2p_4^2\right)z^5 \\
& + \left(490p_1p_2^2p_3^2p_4 + 1764p_1p_2^2p_3p_4^2 + 700p_1p_2p_3^2p_4^2 + 49p_2^2p_3^2p_4^2\right)z^6 + 3432p_1p_2^2p_3^2p_4^2z^7 \\
& + \left(49p_1^2p_2^2p_3^2p_4^2 + 700p_1p_2^3p_3^2p_4^2 + 1764p_1p_2^2p_3^3p_4^2 + 490p_1p_2^2p_3^2p_4^3\right)z^8 \\
& + \left(126p_1^2p_2^3p_3^2p_4^2 + 896p_1p_2^3p_3^3p_4^2 + 980p_1p_2^2p_3^3p_4^3\right)z^9 \\
& + \left(175p_1^2p_2^3p_3^3p_4^2 + 630p_1p_2^3p_3^3p_4^3 + 196p_1p_2^2p_3^3p_4^4\right)z^{10} + \left(140p_1^2p_2^3p_3^3p_4^3 + 224p_1p_2^3p_3^3p_4^4\right)z^{11} \\
& + \left(63p_1^2p_2^3p_3^3p_4^4 + 28p_1p_2^3p_3^4p_4^4\right)z^{12} + 14p_1^2p_2^3p_3^4p_4^4z^{13} + p_1^2p_2^4p_3^4p_4^4z^{14},
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
H_3 = {}& 1 + 18p_3z + (63p_2p_3 + 90p_3p_4)z^2 + \left(56p_1p_2p_3 + 560p_2p_3p_4 + 200p_3p_4^2\right)z^3 \\
& + \left(630p_1p_2p_3p_4 + 630p_2p_3^2p_4 + 1575p_2p_3p_4^2 + 225p_3^2p_4^2\right)z^4 \\
& + \left(1260p_1p_2p_3^2p_4 + 2016p_1p_2p_3p_4^2 + 5292p_2p_3^2p_4^2\right)z^5 \\
& + \left(490p_1p_2^2p_3^2p_4 + 9996p_1p_2p_3^2p_4^2 + 1225p_2^2p_3^2p_4^2 + 5103p_2p_3^3p_4^2 + 1750p_2p_3^2p_4^3\right)z^6 \\
& + \left(5616p_1p_2^2p_3^2p_4^2 + 12600p_1p_2p_3^3p_4^2 + 3528p_2^2p_3^3p_4^2 + 5040p_1p_2p_3^2p_4^3 + 5040p_2p_3^3p_4^3\right)z^7 \\
& + \left(441p_1^2p_2^2p_3^2p_4^2 + 17172p_1p_2^2p_3^3p_4^2 + 4410p_1p_2^2p_3^2p_4^3 + 15750p_1p_2p_3^3p_4^3 + 4410p_2^2p_3^3p_4^3 + 1575p_2p_3^3p_4^4\right)z^8 \\
& + \left(2450p_1^2p_2^2p_3^3p_4^2 + 5600p_1p_2^3p_3^3p_4^2 + 32520p_1p_2^2p_3^3p_4^3 + 5600p_1p_2p_3^3p_4^4 + 2450p_2^2p_3^3p_4^4\right)z^9 \\
& + \left(1575p_1^2p_2^3p_3^3p_4^2 + 4410p_1^2p_2^2p_3^3p_4^3 + 15750p_1p_2^3p_3^3p_4^3 + 4410p_1p_2^2p_3^4p_4^3 + 17172p_1p_2^2p_3^3p_4^4 + 441p_2^2p_3^4p_4^4\right)z^{10} \\
& + \left(5040p_1^2p_2^3p_3^3p_4^3 + 5040p_1p_2^3p_3^4p_4^3 + 3528p_1^2p_2^2p_3^3p_4^4 + 12600p_1p_2^3p_3^3p_4^4 + 5616p_1p_2^2p_3^4p_4^4\right)z^{11} \\
& + \left(1750p_1^2p_2^3p_3^4p_4^3 + 5103p_1^2p_2^3p_3^3p_4^4 + 1225p_1^2p_2^2p_3^4p_4^4 + 9996p_1p_2^3p_3^4p_4^4 + 490p_1p_2^2p_3^4p_4^5\right)z^{12} \\
& + \left(5292p_1^2p_2^3p_3^4p_4^4 + 2016p_1p_2^3p_3^5p_4^4 + 1260p_1p_2^3p_3^4p_4^5\right)z^{13} \\
& + \left(225p_1^2p_2^4p_3^4p_4^4 + 1575p_1^2p_2^3p_3^5p_4^4 + 630p_1^2p_2^3p_3^4p_4^5 + 630p_1p_2^3p_3^5p_4^5\right)z^{14} \\
& + \left(200p_1^2p_2^4p_3^5p_4^4 + 560p_1^2p_2^3p_3^5p_4^5 + 56p_1p_2^3p_3^5p_4^6\right)z^{15} \\
& + \left(90p_1^2p_2^4p_3^5p_4^5 + 63p_1^2p_2^3p_3^5p_4^6\right)z^{16} + 18p_1^2p_2^4p_3^5p_4^6z^{17} + p_1^2p_2^4p_3^6p_4^6z^{18},
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
H_4 = {}& 1 + 10p_4z + 45p_3p_4z^2 + \left(70p_2p_3p_4 + 50p_3p_4^2\right)z^3 + \left(35p_1p_2p_3p_4 + 175p_2p_3p_4^2\right)z^4 \\
& + \left(126p_1p_2p_3p_4^2 + 126p_2p_3^2p_4^2\right)z^5 + \left(175p_1p_2p_3^2p_4^2 + 35p_2p_3^2p_4^3\right)z^6 \\
& + \left(50p_1p_2^2p_3^2p_4^2 + 70p_1p_2p_3^2p_4^3\right)z^7 + 45p_1p_2^2p_3^2p_4^3z^8 + 10p_1p_2^2p_3^3p_4^3z^9 + p_1p_2^2p_3^3p_4^4z^{10}.
\end{aligned} \tag{79}
$$

***C*$_4$-case.** For the Lie algebra $C_4 \cong sp(6)$, we have the following polynomials:

$$
\begin{aligned}
H_1 = {} & 1 + 7p_1 z + 21p_1p_2z^2 + 35p_1p_2p_3z^3 + 35p_1p_2p_3p_4z^4 \\
& + 21p_1p_2p_3^2p_4z^5 + 7p_1p_2^2p_3^2p_4z^6 + p_1^2p_2^2p_3^2p_4z^7,
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
H_2 = {} & 1 + 12p_2z + (21p_1p_2 + 45p_2p_3)z^2 + (140p_1p_2p_3 + 80p_2p_3p_4)z^3 \\
& + \left(105p_1p_2^2p_3 + 315p_1p_2p_3p_4 + 75p_2p_3^2p_4\right)z^4 + \left(420p_1p_2^2p_3p_4 + 336p_1p_2p_3^2p_4 + 36p_2^2p_3^2p_4\right)z^5 \\
& + 924p_1p_2^2p_3^2p_4z^6 + \left(36p_1^2p_2^2p_3^2p_4 + 336p_1p_2^3p_3^2p_4 + 420p_1p_2^2p_3^3p_4\right)z^7 \\
& + \left(75p_1^2p_2^3p_3^2p_4 + 315p_1p_2^3p_3^3p_4 + 105p_1p_2^2p_3^3p_4^2\right)z^8 + \left(80p_1^2p_2^3p_3^3p_4 + 140p_1p_2^3p_3^3p_4^2\right)z^9 \\
& + \left(45p_1^2p_2^3p_3^3p_4^2 + 21p_1p_2^3p_3^4p_4^2\right)z^{10} + 12p_1^2p_2^3p_3^4p_4^2z^{11} + p_1^2p_2^4p_3^4p_4^2z^{12},
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
H_3 = {} & 1 + 15p_3z + (45p_2p_3 + 60p_3p_4)z^2 + \left(35p_1p_2p_3 + 320p_2p_3p_4 + 100p_3^2p_4\right)z^3 \\
& + \left(315p_1p_2p_3p_4 + 1050p_2p_3^2p_4\right)z^4 + \left(1302p_1p_2p_3^2p_4 + 576p_2^2p_3^2p_4 + 1125p_2p_3^3p_4\right)z^5 \\
& + \left(1050p_1p_2^2p_3^2p_4 + 2240p_1p_2p_3^3p_4 + 1215p_2^2p_3^3p_4 + 500p_2p_3^3p_4^2\right)z^6 \\
& + \left(225p_1^2p_2^2p_3^2p_4 + 3990p_1p_2^2p_3^3p_4 + 1260p_1p_2p_3^3p_4^2 + 960p_2^2p_3^3p_4^2\right)z^7 \\
& + \left(960p_1^2p_2^2p_3^3p_4 + 1260p_1p_2^3p_3^3p_4 + 3990p_1p_2^2p_3^3p_4^2 + 225p_2^2p_3^4p_4^2\right)z^8 \\
& + \left(500p_1^2p_2^3p_3^3p_4 + 1215p_1^2p_2^2p_3^3p_4^2 + 2240p_1p_2^3p_3^3p_4^2 + 1050p_1p_2^2p_3^4p_4^2\right)z^9 \\
& + \left(1125p_1^2p_2^3p_3^3p_4^2 + 576p_1^2p_2^2p_3^4p_4^2 + 1302p_1p_2^3p_3^4p_4^2\right)z^{10} + \left(1050p_1^2p_2^3p_3^4p_4^2 + 315p_1p_2^3p_3^5p_4^2\right)z^{11} \\
& + \left(100p_1^2p_2^4p_3^4p_4^2 + 320p_1^2p_2^3p_3^5p_4^2 + 35p_1p_2^3p_3^5p_4^3\right)z^{12} + \left(60p_1^2p_2^4p_3^5p_4^2 + 45p_1^2p_2^3p_3^5p_4^3\right)z^{13} \\
& + 15p_1^2p_2^4p_3^5p_4^3z^{14} + p_1^2p_2^4p_3^6p_4^3z^{15},
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
H_4 = {} & 1 + 16p_4z + 120p_3p_4z^2 + \left(160p_2p_3p_4 + 400p_3^2p_4\right)z^3 \\
& + \left(70p_1p_2p_3p_4 + 1350p_2p_3^2p_4 + 400p_3^2p_4^2\right)z^4 + \left(672p_1p_2p_3^2p_4 + 1296p_2^2p_3^2p_4 + 2400p_2p_3^2p_4^2\right)z^5 \\
& + \left(1400p_1p_2^2p_3^2p_4 + 1512p_1p_2p_3^2p_4^2 + 4096p_2^2p_3^2p_4^2 + 1000p_2p_3^3p_4^2\right)z^6 \\
& + \left(400p_1^2p_2^2p_3^2p_4 + 5600p_1p_2^2p_3^2p_4^2 + 1120p_1p_2p_3^3p_4^2 + 4320p_2^2p_3^3p_4^2\right)z^7 \\
& + \left(2025p_1^2p_2^2p_3^2p_4^2 + 8820p_1p_2^2p_3^3p_4^2 + 2025p_2^2p_3^4p_4^2\right)z^8 \\
& + \left(4320p_1^2p_2^2p_3^3p_4^2 + 1120p_1p_2^3p_3^3p_4^2 + 5600p_1p_2^2p_3^4p_4^2 + 400p_2^2p_3^4p_4^3\right)z^9 \\
& + \left(1000p_1^2p_2^3p_3^3p_4^2 + 4096p_1^2p_2^2p_3^4p_4^2 + 1512p_1p_2^3p_3^4p_4^2 + 1400p_1p_2^2p_3^4p_4^3\right)z^{10} \\
& + \left(2400p_1^2p_2^3p_3^4p_4^2 + 1296p_1^2p_2^2p_3^4p_4^3 + 672p_1p_2^3p_3^4p_4^3\right)z^{11} \\
& + \left(400p_1^2p_2^4p_3^4p_4^2 + 1350p_1^2p_2^3p_3^4p_4^3 + 70p_1p_2^3p_3^5p_4^3\right)z^{12} + \left(400p_1^2p_2^4p_3^4p_4^3 + 160p_1^2p_2^3p_3^5p_4^3\right)z^{13} \\
& + 120p_1^2p_2^4p_3^5p_4^3z^{14} + 16p_1^2p_2^4p_3^6p_4^3z^{15} + p_1^2p_2^4p_3^6p_4^4z^{16}.
\end{aligned} \tag{83}
$$

***D_4*-case.** In the case of Lie algebra $D_4 \cong so(8)$, we obtain the following polynomials

$$\begin{aligned} H_1 &= 1 + 6p_1z + 15p_1p_2z^2 + (10p_1p_2p_3 + 10p_1p_2p_4)z^3 + 15p_1p_2p_3p_4z^4 \\ &\quad + 6p_1p_2^2p_3p_4z^5 + p_1^2p_2^2p_3p_4z^6, \end{aligned} \tag{84}$$

$$\begin{aligned} H_2 &= 1 + 10p_2z + (15p_1p_2 + 15p_2p_3 + 15p_2p_4)z^2 + (40p_1p_2p_3 + 40p_1p_2p_4 + 40p_2p_3p_4)z^3 \\ &\quad + \left(25p_1p_2^2p_3 + 25p_1p_2^2p_4 + 135p_1p_2p_3p_4 + 25p_2^2p_3p_4\right)z^4 + 252p_1p_2^2p_3p_4z^5 \\ &\quad + \left(25p_1^2p_2^2p_3p_4 + 135p_1p_2^3p_3p_4 + 25p_1p_2^2p_3^2p_4 + 25p_1p_2^2p_3p_4^2\right)z^6 \\ &\quad + \left(40p_1^2p_2^3p_3p_4 + 40p_1p_2^3p_3^2p_4 + 40p_1p_2^3p_3p_4^2\right)z^7 + \left(15p_1^2p_2^3p_3^2p_4 + 15p_1^2p_2^3p_3p_4^2 + 15p_1p_2^3p_3^2p_4^2\right)z^8 \\ &\quad + 10p_1^2p_2^3p_3^2p_4^2z^9 + p_1^2p_2^4p_3^2p_4^2z^{10}, \end{aligned} \tag{85}$$

$$\begin{aligned} H_3 &= 1 + 6p_3z + 15p_2p_3z^2 + (10p_1p_2p_3 + 10p_2p_3p_4)z^3 + 15p_1p_2p_3p_4z^4 \\ &\quad + 6p_1p_2^2p_3p_4z^5 + p_1p_2^2p_3^2p_4z^6, \end{aligned} \tag{86}$$

$$\begin{aligned} H_4 &= 1 + 6p_4z + 15p_2p_4z^2 + (10p_1p_2p_4 + 10p_2p_3p_4)z^3 + 15p_1p_2p_3p_4z^4 \\ &\quad + 6p_1p_2^2p_3p_4z^5 + p_1p_2^2p_3p_4^2z^6. \end{aligned} \tag{87}$$

***F_4*-case.** Now we consider the exceptional Lie algebra F_4. We obtain the following polynomials

$$\begin{aligned} H_1 &= 1 + 22p_1z + 231p_1p_2z^2 + 1540p_1p_2p_3z^3 + (5775p_1p_2p_3^2 + 1540p_1p_2p_3p_4)z^4 + (9702p_1p_2^2p_3^2 + 16632p_1p_2p_3^2p_4)z^5 \\ &\quad + (5929p_1^2p_2^2p_3^2 + 53900p_1p_2^2p_3^2p_4 + 14784p_1p_2p_3^2p_4^2)z^6 + (47432p_1^2p_2^2p_3^2p_4 + 33000p_1p_2^2p_3^3p_4 + 90112p_1p_2^2p_3^2p_4^2)z^7 \\ &\quad + (65340p_1^2p_2^2p_3^3p_4 + 108900p_1^2p_2^2p_3^2p_4^2 + 145530p_1p_2^2p_3^3p_4^2)z^8 + (33880p_1^2p_2^3p_3^3p_4 + 355740p_1^2p_2^2p_3^3p_4^2 + 107800p_1p_2^2p_3^4p_4^2)z^9 \\ &\quad + (10164p_1^2p_2^3p_3^4p_4 + 211750p_1^2p_2^3p_3^3p_4^2 + 379456p_1^2p_2^2p_3^4p_4^2 + 45276p_1p_2^3p_3^4p_4^2)z^{10} + 705432p_1^2p_2^3p_3^4p_4^2z^{11} \\ &\quad + (45276p_1^3p_2^3p_3^4p_4^2 + 379456p_1^2p_2^4p_3^4p_4^2 + 211750p_1^2p_2^3p_3^5p_4^2 + 10164p_1^2p_2^3p_3^4p_4^3)z^{12} \\ &\quad + (107800p_1^3p_2^4p_3^4p_4^2 + 355740p_1^2p_2^4p_3^5p_4^2 + 33880p_1^2p_2^3p_3^5p_4^3)z^{13} \\ &\quad + (145530p_1^3p_2^4p_3^5p_4^2 + 108900p_1^2p_2^4p_3^6p_4^2 + 65340p_1^2p_2^4p_3^5p_4^3)z^{14} \\ &\quad + (90112p_1^3p_2^4p_3^6p_4^2 + 33000p_1^3p_2^4p_3^5p_4^3 + 47432p_1^2p_2^4p_3^6p_4^3)z^{15} + (14784p_1^3p_2^5p_3^6p_4^2 + 53900p_1^3p_2^4p_3^6p_4^3 + 5929p_1^2p_2^4p_3^6p_4^4)z^{16} \\ &\quad + (16632p_1^3p_2^5p_3^6p_4^3 + 9702p_1^3p_2^4p_3^6p_4^4)z^{17} + (1540p_1^3p_2^5p_3^7p_4^3 + 5775p_1^3p_2^5p_3^6p_4^4)z^{18} + 1540p_1^3p_2^5p_3^7p_4^4z^{19} + 231p_1^3p_2^5p_3^8p_4^4z^{20} \\ &\quad + 22p_1^3p_2^6p_3^8p_4^4z^{21} + p_1^4p_2^6p_3^8p_4^4z^{22}, \end{aligned} \tag{88}$$

$$
\begin{aligned}
H_2 = {} & 1 + 42p_2z + (231p_1p_2 + 630p_2p_3)z^2 + (6160p_1p_2p_3 + 4200p_2p_3^2 + 1120p_2p_3p_4)z^3 \\
& + (16170p_1p_2^2p_3 + 51975p_1p_2p_3^2 + 11025p_2^2p_3^2 + 13860p_1p_2p_3p_4 + 18900p_2p_3^2p_4)z^4 \\
& + (407484p_1p_2^2p_3^2 + 64680p_1p_2^2p_3p_4 + 266112p_1p_2p_3^2p_4 + 88200p_2^2p_3^2p_4 + 24192p_2p_3^2p_4^2)z^5 \\
& + (148225p_1^2p_2^2p_3^2 + 916839p_1p_2^3p_3^2 + 404250p_1p_2^2p_3^3 + 3132668p_1p_2^2p_3^2p_4 + 73500p_2^2p_3^3p_4 + 369600p_1p_2p_3^2p_4^2 + 200704p_2^2p_3^2p_4^2)z^6 \\
& + (996072p_1^2p_2^3p_3^2 + 2716560p_1p_2^3p_3^3 + 1707552p_1^2p_2^2p_3^2p_4 + 9055200p_1p_2^3p_3^2p_4 + 6035040p_1p_2^2p_3^3p_4 + 6044544p_1p_2^2p_3^2p_4^2 + 423360p_2^2p_3^3p_4^2)z^7 \\
& + (3735270p_1^2p_2^3p_3^3 + 2546775p_1p_2^3p_3^4 + 12450900p_1^2p_2^3p_3^2p_4 + 3201660p_1^2p_2^2p_3^3p_4 + 43423380p_1p_2^3p_3^3p_4 + 4365900p_1p_2^2p_3^4p_4 + 5336100p_1^2p_2^2p_3^2p_4^2 + 23654400p_1p_2^3p_3^2p_4^2 \\
& + 18918900p_1p_2^2p_3^3p_4^2 + 396900p_2^2p_3^4p_4^2)z^8 \\
& + (6225450p_1^2p_2^3p_3^4 + 81650800p_1^2p_2^3p_3^3p_4 + 93601200p_1p_2^3p_3^4p_4 + 41164200p_1^2p_2^3p_3^2p_4^2 + 22767360p_1^2p_2^2p_3^3p_4^2 + 171990280p_1p_2^3p_3^3p_4^2 + 24147200p_1p_2^2p_3^4p_4^2 \\
& + 205800p_2^3p_3^4p_4^2 + 4139520p_1p_2^2p_3^3p_4^3)z^9 \\
& + (2614689p_1^2p_2^4p_3^4 + 17431260p_1^2p_2^4p_3^3p_4 + 231708708p_1^2p_2^3p_3^4p_4 + 23769900p_1p_2^4p_3^4p_4 + 77962500p_1p_2^3p_3^5p_4 + 420637140p_1^2p_2^3p_3^3p_4^2 \\
& + 30735936p_1^2p_2^2p_3^4p_4^2 + 598635576p_1p_2^3p_3^4p_4^2 + 56770560p_1p_2^3p_3^3p_4^3 + 11176704p_1p_2^2p_3^4p_4^3)z^{10} \\
& + (175877856p_1^2p_2^4p_3^4p_4 + 274428000p_1^2p_2^3p_3^5p_4 + 58212000p_1p_2^4p_3^5p_4 + 142296000p_1^2p_2^4p_3^3p_4^2 + 1896293952p_1^2p_2^3p_3^4p_4^2 + 191866752p_1p_2^4p_3^4p_4^2 \\
& + 984060000p_1p_2^3p_3^5p_4^2 + 121968000p_1^2p_2^3p_3^3p_4^3 + 435558816p_1p_2^3p_3^4p_4^3)z^{11} \\
& + (12782924p_1^3p_2^4p_3^4p_4 + 525427980p_1^2p_2^4p_3^5p_4 + 5478396p_1^3p_2^3p_3^4p_4^2 + 2005022376p_1^2p_2^4p_3^4p_4^2 + 4106272940p_1^2p_2^3p_3^5p_4^2 + 816487980p_1p_2^4p_3^5p_4^2 + 707437500p_1p_2^3p_3^6p_4^2 \\
& + 1396604748p_1^2p_2^3p_3^4p_4^3 + 220774400p_1p_2^4p_3^4p_4^3 + 1201272380p_1p_2^3p_3^5p_4^3 + 60555264p_1p_2^3p_3^4p_4^4)z^{12} \\
& + (70436520p_1^3p_2^4p_3^5p_4 + 239057280p_1^2p_2^5p_3^5p_4 + 96049800p_1^2p_2^4p_3^6p_4 + 180457200p_1^3p_2^4p_3^4p_4^2 \\
& + 398428800p_1^2p_2^5p_3^4p_4^2 + 9178974000p_1^2p_2^4p_3^5p_4^2 + 3585859200p_1^2p_2^3p_3^6p_4^2 + 1189465200p_1p_2^4p_3^6p_4^2 + 1611502200p_1^2p_2^4p_3^4p_4^3 + 5439772800p_1^2p_2^3p_3^5p_4^3 + 1540871640p_1p_2^4p_3^5p_4^3 \\
& + 1303948800p_1p_2^3p_3^6p_4^3 + 292723200p_1^2p_2^3p_3^4p_4^4 + 391184640p_1p_2^3p_3^5p_4^4)z^{13} \\
& + (82175940p_1^3p_2^5p_3^5p_4 + 112058100p_1^2p_2^5p_3^6p_4 + 136959900p_1^3p_2^5p_3^4p_4^2 + 1285029900p_1^3p_2^4p_3^5p_4^2 + 5685080940p_1^2p_2^5p_3^5p_4^2 + 15028648200p_1^2p_2^4p_3^6p_4^2 + 499167900p_1p_2^5p_3^6p_4^2 \\
& + 234788400p_1^3p_2^4p_3^4p_4^3 + 15327479700p_1^2p_2^4p_3^5p_4^3 + 7171718400p_1^2p_2^3p_3^6p_4^3 + 3451486500p_1p_2^4p_3^6p_4^3 + 446054400p_1^2p_2^4p_3^4p_4^4 + 2151515520p_1^2p_2^3p_3^5p_4^4 + 596090880p_1p_2^4p_3^5p_4^4 \\
& + 651974400p_1p_2^3p_3^6p_4^4)z^{14} \\
& + (43827168p_1^3p_2^5p_3^6p_4 + 2179888480p_1^3p_2^5p_3^5p_4^2 + 2414513024p_1^3p_2^4p_3^6p_4^2 + 21026246976p_1^2p_2^5p_3^6p_4^2 + 3557400000p_1^2p_2^4p_3^7p_4^2 + 3277206240p_1^3p_2^4p_3^5p_4^3 + 10654446880p_1^2p_2^5p_3^5p_4^3 \\
& + 38613582112p_1^2p_2^4p_3^6p_4^3 + 1774819200p_1p_2^5p_3^6p_4^3 + 646800000p_1p_2^4p_3^7p_4^3 + 8150714880p_1^2p_2^4p_3^5p_4^4 + 4079910912p_1^2p_2^3p_3^6p_4^4 + 2253071744p_1p_2^4p_3^6p_4^4)z^{15} \\
& + (9717029784p_1^3p_2^5p_3^6p_4^2 + 8199664704p_1^2p_2^6p_3^6p_4^2 + 13199224500p_1^2p_2^5p_3^7p_4^2 + 4946287500p_1^3p_2^5p_3^5p_4^3 + 10108843668p_1^3p_2^4p_3^6p_4^3 + 64474736508p_1^2p_2^5p_3^6p_4^3 + 14007262500p_1^2p_2^4p_3^7p_4^3 \\
& + 611226000p_1p_2^5p_3^7p_4^3 + 1760913000p_1^3p_2^4p_3^5p_4^4 + 7805952000p_1^2p_2^5p_3^5p_4^4 + 29296429974p_1^2p_2^4p_3^6p_4^4 + 1669054464p_1p_2^5p_3^6p_4^4 + 713097000p_1p_2^4p_3^7p_4^4)z^{16} \\
& + (439267752p_1^4p_2^5p_3^6p_4^2 + 6754454784p_1^3p_2^6p_3^6p_4^2 + 6903638280p_1^3p_2^5p_3^7p_4^2 + 10040405760p_1^2p_2^6p_3^7p_4^2 + 2858625000p_1^2p_2^5p_3^8p_4^2 + 37825702992p_1^3p_2^5p_3^6p_4^3 + 33468019200p_1^2p_2^6p_3^6p_4^3 \\
& + 4507937280p_1^3p_2^4p_3^7p_4^3 + 57537501840p_1^2p_2^5p_3^7p_4^3 + 4192650000p_1^3p_2^5p_3^5p_4^4 + 8611029504p_1^3p_2^4p_3^6p_4^4 + 63276492636p_1^2p_2^5p_3^6p_4^4 + 16802311680p_1^2p_2^4p_3^7p_4^4 + 1198002960p_1p_2^5p_3^7p_4^4 \\
& + 245887488p_1^2p_2^4p_3^6p_4^5)z^{17} \\
& + (1423552900p_1^4p_2^6p_3^6p_4^2 + 10086748980p_1^3p_2^6p_3^7p_4^2 + 2862182400p_1^3p_2^5p_3^8p_4^2 + 3890016900p_1^2p_2^6p_3^8p_4^2 + 2440376400p_1^4p_2^5p_3^6p_4^3 + 33759456500p_1^3p_2^6p_3^6p_4^3 + 44524657100p_1^3p_2^5p_3^7p_4^3 \\
& + 59339922180p_1^2p_2^6p_3^7p_4^3 + 16165587900p_1^2p_2^5p_3^8p_4^3 + 43888833450p_1^3p_2^5p_3^6p_4^4 + 38856294400p_1^2p_2^6p_3^6p_4^4 + 6135803520p_1^3p_2^4p_3^7p_4^4 + 86086107380p_1^2p_2^5p_3^7p_4^4 + 1859334400p_1^2p_2^4p_3^8p_4^4 \\
& + 221852400p_1p_2^5p_3^8p_4^4 + 1040793600p_1^2p_2^5p_3^6p_4^5 + 1115600640p_1^2p_2^4p_3^7p_4^5)z^{18} \\
& + (2510101440p_1^4p_2^6p_3^7p_4^2 + 6411081600p_1^3p_2^6p_3^8p_4^2 + 8367004800p_1^4p_2^6p_3^6p_4^3 + 2151515520p_1^4p_2^5p_3^7p_4^3 + 81592267680p_1^3p_2^6p_3^7p_4^3 + 18912247200p_1^3p_2^5p_3^8p_4^3 + 38377231200p_1^2p_2^6p_3^8p_4^3 \\
& + 3585859200p_1^4p_2^5p_3^6p_4^4 + 45964195200p_1^3p_2^6p_3^6p_4^4 + 79733253600p_1^3p_2^5p_3^7p_4^4 + 102862932480p_1^2p_2^6p_3^7p_4^4 + 46561158000p_1^2p_2^5p_3^8p_4^4 + 804988800p_1^3p_2^5p_3^6p_4^5 + 8941474080p_1^2p_2^5p_3^7p_4^5)z^{19} \\
& + (1967099904p_1^4p_2^6p_3^8p_4^2 + 788889024p_1^3p_2^7p_3^8p_4^2 + 24726420180p_1^4p_2^6p_3^7p_4^3 + 5259260160p_1^3p_2^7p_3^7p_4^3 + 75784320612p_1^3p_2^6p_3^8p_4^3 + 8004150000p_1^2p_2^6p_3^9p_4^3 + 13340250000p_1^4p_2^6p_3^6p_4^4 \\
& + 6589016280p_1^4p_2^5p_3^7p_4^4 + 166955605740p_1^3p_2^6p_3^7p_4^4 + 57761551386p_1^3p_2^5p_3^8p_4^4 + 113404704966p_1^2p_2^6p_3^8p_4^4 + 9338175000p_1^2p_2^5p_3^9p_4^4 + 7582847580p_1^3p_2^5p_3^7p_4^5 + 13113999360p_1^2p_2^6p_3^7p_4^5 \\
& + 9175317228p_1^2p_2^5p_3^8p_4^5)z^{20} \\
& + (398428800p_1^4p_2^7p_3^8p_4^2 + 2656192000p_1^4p_2^7p_3^7p_4^3 + 29530356856p_1^4p_2^6p_3^8p_4^3 + 14144946816p_1^3p_2^7p_3^8p_4^3 + 20764887000p_1^3p_2^6p_3^9p_4^3 + 60120060000p_1^4p_2^6p_3^7p_4^4 \\
& + 14609056000p_1^3p_2^7p_3^7p_4^4 + 3123681792p_1^4p_2^5p_3^8p_4^4 \\
& + 247562655912p_1^3p_2^6p_3^8p_4^4 + 3123681792p_1^2p_2^7p_3^8p_4^4 + 14609056000p_1^3p_2^5p_3^9p_4^4 + 60120060000p_1^2p_2^6p_3^9p_4^4 + 20764887000p_1^3p_2^6p_3^7p_4^5 + 14144946816p_1^3p_2^5p_3^8p_4^5 \\
& + 29530356856p_1^2p_2^6p_3^8p_4^5 + 2656192000p_1^2p_2^5p_3^9p_4^5 + 398428800p_1^2p_2^5p_3^8p_4^6)z^{21} \\
& + (9175317228p_1^4p_2^7p_3^8p_4^3 + 13113999360p_1^4p_2^6p_3^9p_4^3 + 7582847580p_1^3p_2^7p_3^9p_4^3 + 9338175000p_1^4p_2^7p_3^7p_4^4 + 113404704966p_1^4p_2^6p_3^8p_4^4 + 57761551386p_1^3p_2^7p_3^8p_4^4 \\
& + 166955605740p_1^3p_2^6p_3^9p_4^4 + 6589016280p_1^2p_2^7p_3^9p_4^4 + 13340250000p_1^2p_2^6p_3^{10}p_4^4 + 8004150000p_1^4p_2^6p_3^7p_4^5 + 75784320612p_1^3p_2^6p_3^8p_4^5 + 5259260160p_1^3p_2^5p_3^9p_4^5 + 24726420180p_1^2p_2^6p_3^9p_4^5 \\
& + 788889024p_1^3p_2^5p_3^8p_4^6 + 1967099904p_1^2p_2^6p_3^8p_4^6)z^{22} \\
& + (8941474080p_1^4p_2^7p_3^9p_4^3 + 804988800p_1^3p_2^7p_3^{10}p_4^3 + 46561158000p_1^4p_2^7p_3^8p_4^4 + 102862932480p_1^4p_2^6p_3^9p_4^4 + 79733253600p_1^3p_2^7p_3^9p_4^4 + 45964195200p_1^3p_2^6p_3^{10}p_4^4 + 3585859200p_1^2p_2^7p_3^{10}p_4^4 \\
& + 38377231200p_1^4p_2^6p_3^8p_4^5 + 18912247200p_1^3p_2^7p_3^8p_4^5 + 81592267680p_1^3p_2^6p_3^9p_4^5 + 2151515520p_1^2p_2^7p_3^9p_4^5 + 8367004800p_1^2p_2^6p_3^{10}p_4^5 + 6411081600p_1^3p_2^6p_3^8p_4^6 + 2510101440p_1^2p_2^6p_3^9p_4^6)z^{23} \\
& + (1115600640p_1^4p_2^8p_3^9p_4^3 + 1040793600p_1^4p_2^7p_3^{10}p_4^3 + 221852400p_1^5p_2^7p_3^8p_4^4 + 1859334400p_1^4p_2^8p_3^8p_4^4 + 86086107380p_1^4p_2^7p_3^9p_4^4 + 6135803520p_1^3p_2^8p_3^9p_4^4 + 38856294400p_1^4p_2^6p_3^{10}p_4^4 \\
& + 43888833450p_1^3p_2^7p_3^{10}p_4^4 + 16165587900p_1^4p_2^7p_3^8p_4^5 + 59339922180p_1^4p_2^6p_3^9p_4^5 + 44524657100p_1^3p_2^7p_3^9p_4^5 + 33759456500p_1^3p_2^6p_3^{10}p_4^5 + 2440376400p_1^2p_2^7p_3^{10}p_4^5 + 3890016900p_1^4p_2^6p_3^8p_4^6 \\
& + 2862182400p_1^3p_2^7p_3^8p_4^6 + 10086748980p_1^3p_2^6p_3^9p_4^6 + 1423552900p_1^2p_2^6p_3^{10}p_4^6)z^{24} \\
& + (245887488p_1^4p_2^8p_3^{10}p_4^3 + 1198002960p_1^5p_2^7p_3^9p_4^4 + 16802311680p_1^4p_2^8p_3^9p_4^4 + 63276492636p_1^4p_2^7p_3^{10}p_4^4 + 8611029504p_1^3p_2^8p_3^{10}p_4^4 + 4192650000p_1^3p_2^7p_3^{11}p_4^4 + 57537501840p_1^4p_2^7p_3^9p_4^5 \\
& + 4507937280p_1^3p_2^8p_3^9p_4^5 + 33468019200p_1^4p_2^6p_3^{10}p_4^5 + 37825702992p_1^3p_2^7p_3^{10}p_4^5 + 2858625000p_1^4p_2^7p_3^8p_4^6 + 10040405760p_1^4p_2^6p_3^9p_4^6 + 6903638280p_1^3p_2^7p_3^9p_4^6 + 6754454784p_1^3p_2^6p_3^{10}p_4^6 \\
& + 439267752p_1^2p_2^7p_3^{10}p_4^6)z^{25} \\
& + (713097000p_1^5p_2^8p_3^9p_4^4 + 1669054464p_1^5p_2^7p_3^{10}p_4^4 + 29296429974p_1^4p_2^8p_3^{10}p_4^4 + 7805952000p_1^4p_2^7p_3^{11}p_4^4 + 1760913000p_1^3p_2^8p_3^{11}p_4^4 + 611226000p_1^5p_2^7p_3^9p_4^5 + 14007262500p_1^4p_2^8p_3^9p_4^5 \\
& + 64474736508p_1^4p_2^7p_3^{10}p_4^5 + 10108843668p_1^3p_2^8p_3^{10}p_4^5 + 4946287500p_1^3p_2^7p_3^{11}p_4^5 + 13199224500p_1^4p_2^7p_3^9p_4^6 + 8199664704p_1^4p_2^6p_3^{10}p_4^6 + 9717029784p_1^3p_2^7p_3^{10}p_4^6)z^{26} \\
& + (2253071744p_1^5p_2^8p_3^{10}p_4^4 + 4079910912p_1^4p_2^9p_3^{10}p_4^4 + 8150714880p_1^4p_2^8p_3^{11}p_4^4 + 646800000p_1^5p_2^8p_3^9p_4^5 + 1774819200p_1^5p_2^7p_3^{10}p_4^5 + 38613582112p_1^4p_2^8p_3^{10}p_4^5 + 10654446880p_1^4p_2^7p_3^{11}p_4^5 \\
& + 3277206240p_1^3p_2^8p_3^{11}p_4^5 + 3557400000p_1^4p_2^8p_3^9p_4^6 + 21026246976p_1^4p_2^7p_3^{10}p_4^6 + 2414513024p_1^3p_2^8p_3^{10}p_4^6 + 2179888480p_1^3p_2^7p_3^{11}p_4^6 + 43827168p_1^3p_2^7p_3^{10}p_4^7)z^{27} \\
& + (651974400p_1^5p_2^9p_3^{10}p_4^4 + 596090880p_1^5p_2^8p_3^{11}p_4^4 + 2151515520p_1^4p_2^9p_3^{11}p_4^4 + 446054400p_1^4p_2^8p_3^{12}p_4^4 + 3451486500p_1^5p_2^8p_3^{10}p_4^5 + 7171718400p_1^4p_2^9p_3^{10}p_4^5 + 15327479700p_1^4p_2^8p_3^{11}p_4^5 \\
& + 234788400p_1^3p_2^8p_3^{12}p_4^5 + 499167900p_1^5p_2^7p_3^{10}p_4^6 + 15028648200p_1^4p_2^8p_3^{10}p_4^6 + 5685080940p_1^4p_2^7p_3^{11}p_4^6 + 1285029900p_1^3p_2^8p_3^{11}p_4^6 + 136959900p_1^3p_2^7p_3^{12}p_4^6 + 112058100p_1^4p_2^7p_3^{10}p_4^7 \\
& + 82175940p_1^3p_2^7p_3^{11}p_4^7)z^{28} \\
& + (391184640p_1^5p_2^9p_3^{11}p_4^4 + 292723200p_1^4p_2^9p_3^{12}p_4^4 + 1303948800p_1^5p_2^9p_3^{10}p_4^5 + 1540871640p_1^5p_2^8p_3^{11}p_4^5 + 5439772800p_1^4p_2^9p_3^{11}p_4^5 + 1611502200p_1^4p_2^8p_3^{12}p_4^5 + 1189465200p_1^5p_2^8p_3^{10}p_4^6 \\
& + 3585859200p_1^4p_2^9p_3^{10}p_4^6 + 9178974000p_1^4p_2^8p_3^{11}p_4^6 + 398428800p_1^4p_2^7p_3^{12}p_4^6 + 180457200p_1^3p_2^8p_3^{12}p_4^6 + 96049800p_1^4p_2^8p_3^{10}p_4^7 + 239057280p_1^4p_2^7p_3^{11}p_4^7 + 70436520p_1^3p_2^8p_3^{11}p_4^7)z^{29} \\
& + (60555264p_1^5p_2^9p_3^{12}p_4^4 + 1201272380p_1^5p_2^9p_3^{11}p_4^5 + 220774400p_1^5p_2^8p_3^{12}p_4^5 + 1396604748p_1^4p_2^9p_3^{12}p_4^5 + 707437500p_1^5p_2^9p_3^{10}p_4^6 + 816487980p_1^5p_2^8p_3^{11}p_4^6 + 4106272940p_1^4p_2^9p_3^{11}p_4^6 \\
& + 2005022376p_1^4p_2^8p_3^{12}p_4^6 + 5478396p_1^3p_2^9p_3^{12}p_4^6 + 525427980p_1^4p_2^8p_3^{11}p_4^7 + 12782924p_1^3p_2^8p_3^{12}p_4^7)z^{30} \\
& + (435558816p_1^5p_2^9p_3^{12}p_4^5 + 121968000p_1^4p_2^9p_3^{13}p_4^5 + 984060000p_1^5p_2^9p_3^{11}p_4^6 + 191866752p_1^5p_2^8p_3^{12}p_4^6 + 1896293952p_1^4p_2^9p_3^{12}p_4^6 + 142296000p_1^4p_2^8p_3^{13}p_4^6 + 58212000p_1^5p_2^8p_3^{11}p_4^7 \\
& + 274428000p_1^4p_2^9p_3^{11}p_4^7 + 175877856p_1^4p_2^8p_3^{12}p_4^7)z^{31} \\
& + (11176704p_1^5p_2^{10}p_3^{12}p_4^5 + 56770560p_1^5p_2^9p_3^{13}p_4^5 + 598635576p_1^5p_2^9p_3^{12}p_4^6 + 30735936p_1^4p_2^{10}p_3^{12}p_4^6 + 420637140p_1^4p_2^9p_3^{13}p_4^6 + 77962500p_1^5p_2^9p_3^{11}p_4^7 + 23769900p_1^5p_2^8p_3^{12}p_4^7 \\
& + 231708708p_1^4p_2^9p_3^{12}p_4^7 + 17431260p_1^4p_2^8p_3^{13}p_4^7 + 2614689p_1^4p_2^8p_3^{12}p_4^8)z^{32} \\
& + (4139520p_1^5p_2^{10}p_3^{13}p_4^5 + 205800p_1^6p_2^9p_3^{12}p_4^6 + 24147200p_1^5p_2^{10}p_3^{12}p_4^6 + 171990280p_1^5p_2^9p_3^{13}p_4^6 + 22767360p_1^4p_2^{10}p_3^{13}p_4^6 + 41164200p_1^4p_2^9p_3^{14}p_4^6 + 93601200p_1^5p_2^9p_3^{12}p_4^7 + 81650800p_1^4p_2^9p_3^{13}p_4^7 \\
& + 6225450p_1^4p_2^9p_3^{12}p_4^8)z^{33} \\
& + (396900p_1^6p_2^{10}p_3^{12}p_4^6 + 18918900p_1^5p_2^{10}p_3^{13}p_4^6 + 23654400p_1^5p_2^9p_3^{14}p_4^6 + 5336100p_1^4p_2^{10}p_3^{14}p_4^6 + 4365900p_1^5p_2^{10}p_3^{12}p_4^7 + 43423380p_1^5p_2^9p_3^{13}p_4^7 + 3201660p_1^4p_2^{10}p_3^{13}p_4^7 \\
& + 12450900p_1^4p_2^9p_3^{14}p_4^7 + 2546775p_1^5p_2^9p_3^{12}p_4^8 + 3735270p_1^4p_2^9p_3^{13}p_4^8)z^{34} \\
& + (423360p_1^6p_2^{10}p_3^{13}p_4^6 + 6044544p_1^5p_2^{10}p_3^{14}p_4^6 + 6035040p_1^5p_2^{10}p_3^{13}p_4^7 + 9055200p_1^5p_2^9p_3^{14}p_4^7 + 1707552p_1^4p_2^{10}p_3^{14}p_4^7 + 2716560p_1^5p_2^9p_3^{13}p_4^8 + 996072p_1^4p_2^9p_3^{14}p_4^8)z^{35} \\
& + (200704p_1^6p_2^{10}p_3^{14}p_4^6 + 369600p_1^5p_2^{11}p_3^{14}p_4^6 + 73500p_1^6p_2^{10}p_3^{13}p_4^7 + 3132668p_1^5p_2^{10}p_3^{14}p_4^7 + 404250p_1^5p_2^{10}p_3^{13}p_4^8 + 916839p_1^5p_2^9p_3^{14}p_4^8 + 148225p_1^4p_2^{10}p_3^{14}p_4^8)z^{36} \\
& + (24192p_1^6p_2^{11}p_3^{14}p_4^6 + 88200p_1^6p_2^{10}p_3^{14}p_4^7 + 266112p_1^5p_2^{11}p_3^{14}p_4^7 + 64680p_1^5p_2^{10}p_3^{15}p_4^7 + 407484p_1^5p_2^{10}p_3^{14}p_4^8)z^{37} \\
& + (18900p_1^6p_2^{11}p_3^{14}p_4^7 + 13860p_1^5p_2^{11}p_3^{15}p_4^7 + 11025p_1^6p_2^{10}p_3^{14}p_4^8 + 51975p_1^5p_2^{11}p_3^{14}p_4^8 + 16170p_1^5p_2^{10}p_3^{15}p_4^8)z^{38} \\
& + (1120p_1^6p_2^{11}p_3^{15}p_4^7 + 4200p_1^6p_2^{11}p_3^{14}p_4^8 + 6160p_1^5p_2^{11}p_3^{15}p_4^8)z^{39} \\
& + (630p_1^6p_2^{11}p_3^{15}p_4^8 + 231p_1^5p_2^{11}p_3^{16}p_4^8)z^{40} + 42p_1^6p_2^{11}p_3^{16}p_4^8z^{41} + p_1^6p_2^{12}p_3^{16}p_4^8z^{42},
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
H_3 = {} & 1 + 30p_3 z + (315p_2p_3 + 120p_3p_4)z^2 + (770p_1p_2p_3 + 1050p_2p_3^2 + 2240p_2p_3p_4)z^3 \\
& + (5775p_1p_2p_3^2 + 6930p_1p_2p_3p_4 + 14700p_2p_3^2p_4)z^4 + (9702p_1p_2^2p_3^2 + 90552p_1p_2p_3^2p_4 + 31500p_2p_3^3p_4 + 10752p_2p_3^2p_4^2)z^5 \\
& + (8085p_1p_2^2p_3^3 + 161700p_1p_2^2p_3^2p_4 + 249480p_1p_2p_3^3p_4 + 36750p_2^2p_3^3p_4 + 92400p_1p_2p_3^2p_4^2 + 45360p_2p_3^3p_4^2)z^6 \\
& + (1181400p_1p_2^2p_3^3p_4 + 316800p_1p_2^2p_3^2p_4^2 + 443520p_1p_2p_3^3p_4^2 + 94080p_2^2p_3^3p_4^2)z^7 \\
& + (177870p_1^2p_2^2p_3^3p_4 + 1358280p_1p_2^3p_3^3p_4 + 782100p_1p_2^2p_3^4p_4 + 3490575p_1p_2^2p_3^3p_4^2 + 44100p_2^2p_3^4p_4^2)z^8 \\
& + (830060p_1^2p_2^3p_3^3p_4 + 2633400p_1p_2^3p_3^4p_4 + 711480p_1^2p_2^2p_3^3p_4^2 + 4928000p_1p_2^3p_3^3p_4^2 + 5035250p_1p_2^2p_3^4p_4^2 + 168960p_1p_2^2p_3^3p_4^3)z^9 \\
& + (2144604p_1^2p_2^3p_3^4p_4 + 1559250p_1p_2^3p_3^5p_4 + 3811500p_1^2p_2^3p_3^3p_4^2 + 853776p_1^2p_2^2p_3^4p_4^2 + 16967181p_1p_2^3p_3^4p_4^2 \\
& + 3234000p_1p_2^2p_3^5p_4^2 + 1474704p_1p_2^2p_3^4p_4^3)z^{10} \\
& + (2439360p_1^2p_2^3p_3^5p_4 + 18117750p_1^2p_2^3p_3^4p_4^2 + 26826030p_1p_2^3p_3^5p_4^2 + 5174400p_1p_2^3p_3^4p_4^3 + 2069760p_1p_2^2p_3^5p_4^3)z^{11} \\
& + (711480p_1^2p_2^4p_3^5p_4 + 2371600p_1^2p_2^4p_3^4p_4^2 + 38368225p_1^2p_2^3p_3^5p_4^2 + 6338640p_1p_2^4p_3^5p_4^2 \\
& + 14437500p_1p_2^3p_3^6p_4^2 + 5336100p_1^2p_2^3p_3^4p_4^3 + 18929680p_1p_2^3p_3^5p_4^3)z^{12} \\
& + (21783930p_1^2p_2^4p_3^5p_4^2 + 32524800p_1^2p_2^3p_3^6p_4^2 + 8731800p_1p_2^4p_3^6p_4^2 + 29988000p_1^2p_2^3p_3^5p_4^3 + 8279040p_1p_2^4p_3^5p_4^3 \\
& + 16678200p_1p_2^3p_3^6p_4^3 + 1774080p_1p_2^3p_3^5p_4^4)z^{13} \\
& + (1584660p_1^3p_2^4p_3^5p_4^2 + 46973475p_1^2p_2^4p_3^6p_4^2 + 25194480p_1^2p_2^4p_3^5p_4^3 + 43705200p_1^2p_2^3p_3^6p_4^3 \\
& + 17948700p_1p_2^4p_3^6p_4^3 + 5488560p_1^2p_2^3p_3^5p_4^4 + 4527600p_1p_2^3p_3^6p_4^4)z^{14} \\
& + (5588352p_1^3p_2^4p_3^6p_4^2 + 15937152p_1^2p_2^5p_3^6p_4^2 + 5808000p_1^2p_2^4p_3^7p_4^2 + 3234000p_1^3p_2^4p_3^5p_4^3 + 93982512p_1^2p_2^4p_3^6p_4^3 \\
& + 3234000p_1p_2^4p_3^7p_4^3 + 5808000p_1^2p_2^4p_3^5p_4^4 + 15937152p_1^2p_2^3p_3^6p_4^4 + 5588352p_1p_2^4p_3^6p_4^4)z^{15} \\
& + (4527600p_1^3p_2^5p_3^6p_4^2 + 5488560p_1^2p_2^5p_3^7p_4^2 + 17948700p_1^3p_2^4p_3^6p_4^3 + 43705200p_1^2p_2^5p_3^6p_4^3 \\
& + 25194480p_1^2p_2^4p_3^7p_4^3 + 46973475p_1^2p_2^4p_3^6p_4^4 + 1584660p_1p_2^4p_3^7p_4^4)z^{16} \\
& + (1774080p_1^3p_2^5p_3^7p_4^2 + 16678200p_1^3p_2^5p_3^6p_4^3 + 8279040p_1^3p_2^4p_3^7p_4^3 + 29988000p_1^2p_2^5p_3^7p_4^3 \\
& + 8731800p_1^3p_2^4p_3^6p_4^4 + 32524800p_1^2p_2^5p_3^6p_4^4 + 21783930p_1^2p_2^4p_3^7p_4^4)z^{17} \\
& + (18929680p_1^3p_2^5p_3^7p_4^3 + 5336100p_1^2p_2^5p_3^8p_4^3 + 14437500p_1^3p_2^5p_3^6p_4^4 + 6338640p_1^3p_2^4p_3^7p_4^4 \\
& + 38368225p_1^2p_2^5p_3^7p_4^4 + 2371600p_1^2p_2^4p_3^8p_4^4 + 711480p_1^2p_2^4p_3^7p_4^5)z^{18} \\
& + (2069760p_1^3p_2^6p_3^7p_4^3 + 5174400p_1^3p_2^5p_3^8p_4^3 + 26826030p_1^3p_2^5p_3^7p_4^4 + 18117750p_1^2p_2^5p_3^8p_4^4 + 2439360p_1^2p_2^5p_3^7p_4^5)z^{19} \\
& + (1474704p_1^3p_2^6p_3^8p_4^3 + 3234000p_1^3p_2^6p_3^7p_4^4 + 16967181p_1^3p_2^5p_3^8p_4^4 + 853776p_1^2p_2^6p_3^8p_4^4 \\
& + 3811500p_1^2p_2^5p_3^9p_4^4 + 1559250p_1^3p_2^5p_3^7p_4^5 + 2144604p_1^2p_2^5p_3^8p_4^5)z^{20} \\
& + (168960p_1^3p_2^6p_3^9p_4^3 + 5035250p_1^3p_2^6p_3^8p_4^4 + 4928000p_1^3p_2^5p_3^9p_4^4 + 711480p_1^2p_2^6p_3^9p_4^4 + 2633400p_1^3p_2^5p_3^8p_4^5 + 830060p_1^2p_2^5p_3^9p_4^5)z^{21} \\
& + (44100p_1^4p_2^6p_3^8p_4^4 + 3490575p_1^3p_2^6p_3^9p_4^4 + 782100p_1^3p_2^6p_3^8p_4^5 + 1358280p_1^3p_2^5p_3^9p_4^5 + 177870p_1^2p_2^6p_3^9p_4^5)z^{22} \\
& + (94080p_1^4p_2^6p_3^9p_4^4 + 443520p_1^3p_2^7p_3^9p_4^4 + 316800p_1^3p_2^6p_3^{10}p_4^4 + 1181400p_1^3p_2^6p_3^9p_4^5)z^{23} \\
& + (45360p_1^4p_2^7p_3^9p_4^4 + 92400p_1^3p_2^7p_3^{10}p_4^4 + 36750p_1^4p_2^6p_3^9p_4^5 + 249480p_1^3p_2^7p_3^9p_4^5 + 161700p_1^3p_2^6p_3^{10}p_4^5 + 8085p_1^3p_2^6p_3^9p_4^6)z^{24} \\
& + (10752p_1^4p_2^7p_3^{10}p_4^4 + 31500p_1^4p_2^7p_3^9p_4^5 + 90552p_1^3p_2^7p_3^{10}p_4^5 + 9702p_1^3p_2^6p_3^{10}p_4^6)z^{25} \\
& + (14700p_1^4p_2^7p_3^{10}p_4^5 + 6930p_1^3p_2^7p_3^{11}p_4^5 + 5775p_1^3p_2^7p_3^{10}p_4^6)z^{26} \qquad + (2240p_1^4p_2^7p_3^{11}p_4^5 + 1050p_1^4p_2^7p_3^{10}p_4^6 + 770p_1^3p_2^7p_3^{11}p_4^6)z^{27} \\
& + (120p_1^4p_2^8p_3^{11}p_4^5 + 315p_1^4p_2^7p_3^{11}p_4^6)z^{28} + 30p_1^4p_2^8p_3^{11}p_4^6z^{29} + p_1^4p_2^8p_3^{12}p_4^6z^{30},
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
H_4 = {} & 1 + 16p_4z + 120p_3p_4z^2 + 560p_2p_3p_4z^3 + (770p_1p_2p_3p_4 + 1050p_2p_3^2p_4)z^4 + (3696p_1p_2p_3^2p_4 + 672p_2p_3^2p_4^2)z^5 \\
& + (4312p_1p_2^2p_3^2p_4 + 3696p_1p_2p_3^2p_4^2)z^6 + (2640p_1p_2^2p_3^3p_4 + 8800p_1p_2^2p_3^2p_4^2)z^7 + 12870p_1p_2^2p_3^3p_4^2z^8 \\
& + (8800p_1p_2^2p_3^4p_4^2 + 2640p_1p_2^2p_3^3p_4^3)z^9 + (3696p_1p_2^3p_3^4p_4^2 + 4312p_1p_2^2p_3^4p_4^3)z^{10} + (672p_1^2p_2^3p_3^4p_4^2 + 3696p_1p_2^3p_3^4p_4^3)z^{11} \\
& + (1050p_1^2p_2^3p_3^4p_4^3 + 770p_1p_2^3p_3^5p_4^3)z^{12} + 560p_1^2p_2^3p_3^5p_4^3z^{13} + 120p_1^2p_2^4p_3^5p_4^3z^{14} + 16p_1^2p_2^4p_3^6p_4^3z^{15} + p_1^2p_2^4p_3^6p_4^4z^{16}.
\end{aligned}
\tag{91}
$$

Now we denote

$$
H_s \equiv H_s(z) = H_s(z,(p_i)), \quad (p_i) \equiv (p_1, p_2, p_3, p_4). \tag{92}
$$

The asymptotic formulae for the polynomials read:

$$
H_s = H_s(z,(p_i)) \sim \left(\prod_{l=1}^{4} (p_l)^{\nu^{sl}} \right) z^{n_s} \equiv H_s^{as}(z,(p_i)), \quad \text{as } z \to \infty, \tag{93}
$$

where the matricies $\nu = (\nu^{sl})$ have the following form

$$
\nu = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 2 & 4 & 6 & 4 \end{pmatrix},
$$
$$
\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 6 & 8 & 4 \\ 6 & 12 & 16 & 8 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 4 \end{pmatrix} \tag{94}
$$

for Lie algebras A_4, B_4, C_4, D_4, F_4, respectively. In all these cases we are led to the relations

$$\sum_{l=1}^{4} \nu^{sl} = n_s, \quad s = 1,2,3,4. \tag{95}$$

In the case of Lie algebras B_4, C_4, D_4 and F_4 we have

$$\nu(\mathcal{G}) = 2A^{-1}, \quad \mathcal{G} = B_4, C_4, D_4, F_4, \tag{96}$$

where A^{-1} is inverse Cartan matrix, whereas in the A_4-case the matrix ν reads as follows

$$\nu(\mathcal{G}) = A^{-1}(I + P), \quad \mathcal{G} = A_4. \tag{97}$$

Here, I is 4×4 identity matrix and

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{98}$$

is a matrix which corresponds to the permutation $\sigma \in S_4$ (S_4 is symmetric group)

$$\sigma : (1,2,3,4) \mapsto (4,3,2,1), \tag{99}$$

due to relation $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here, σ is the generator of the group of symmetry of the Dynkin diagram for A_4: $G = \{\sigma, \mathrm{id}\}$. This group is isomorphic to the group $\mathbb{Z}_2$.

For the Lie algebra D_4, we are led to the group of symmetry of the Dynkin diagram G', which is isomorphic to the symmetric group S_3. This group is acting on the set of three vertices of the diagram $\{1,3,4\}$ by their permutations. The groups symmetry $G \cong \mathbb{Z}_2$ and $G' \cong S_3$ imply certain identity properties for the polynomials $H_s(z)$.

Now, we introduce the dual (ordered) set: $(\hat{p}_i) = (p_{\sigma(i)})$ for the algebra A_4 , and $(\hat{p}_i) = (p_i)$ for algebras B_4, C_4, D_4, F_4 ($i = 1,2,3,4$). The dual set for A_4 case is a result of action of the generator σ of the group $G \cong \mathbb{Z}_2$ on vertices of the Dynkin diagram.

Afterwards we obtain symmetry and duality identities. They were verified by using certain MATHEMATICA algorithm.

3.4.2. Symmetry Relations

Proposition 6. *The fluxbrane polynomials satisfy for all p_i and z the following identities:*

$$\begin{aligned} H_{\sigma(s)}(z,(p_i)) &= H_s(z,(\hat{p}_i)) \qquad \text{for } A_4 \text{ case}, \\ H_{\sigma'(s)}(z,(p_i)) &= H_s(z,(p_{\sigma'(i)})) \qquad \text{for } D_4 \text{ case}, \end{aligned} \tag{100}$$

for any $\sigma' \in S_3$, $s = 1,\dots,4$.

3.4.3. Duality Relations

Proposition 7. *The fluxbrane polynomials corresponding to Lie algebras A_4, B_4, C_4, D_4 and F_4 obey for all $p_i > 0$ and $z > 0$ the identities:*

$$H_s(z,(p_i)) = H_s^{as}(z,(p_i)) H_s(z^{-1},(\hat{p}_i^{-1})), \tag{101}$$

$s = 1,2,3,4.$

3.5. Rank-5 Algebras

Now turn our attention to the solutions corresponding to Lie algebras of rank 5 when the matrix $A = (A_{sl})$ is coinciding with one of the Cartan matrices

$$
\begin{aligned}
(A_{ss'}) = &\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \\
&\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}.
\end{aligned} \tag{102}
$$

for $\mathcal{G} = A_5$, B_5, C_5, D_5, respectively.

Figure 4 gives us graphical presentations of these matrices by Dynkin diagrams.

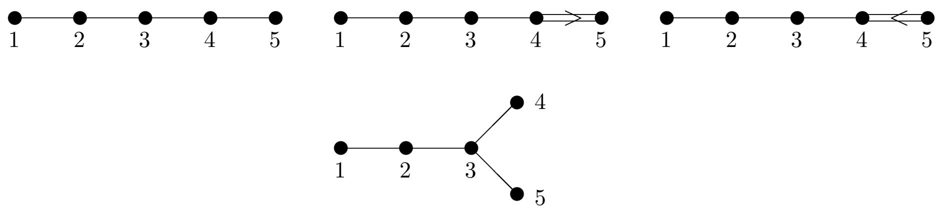

Figure 4. Dynkin diagrams for the Lie algebras A_5, B_5, C_5, D_5, respectively.

3.5.1. Polynomials

According to the conjecture from Ref. [3], the functions $H_1(z), ..., H_5(z)$ obeying Equations (11) and (12) with any matrix $A = (A_{sl})$ from (102), should be polynomials. Relation (2) gives us the powers of these polynomials: $(n_1, n_2, n_3, n_4, n_5) = (5, 8, 9, 8, 5)$, $(10, 18, 24, 28, 15)$, $(9, 16, 21, 24, 25)$, $(8, 14, 18, 10, 10)$ for Lie algebras A_5, B_5, C_5, D_5, respectively.

In this case the verification (or proof) of the polynomial conjecture [3] is following: we solve the set of algebraic equations for the coefficients of the polynomials (1) which follow from the master Equation (11).

In this subsection, we present the structures (or "truncated versions") of these polynomials. The total list of the polynomials is presented in Appendix A. The polynomials were obtained by using a certain MATHEMATICA algorithm in Ref. [41].

***A*$_5$-case.** For the Lie algebra A_5 the polynomials have the following structure

$$
\begin{aligned}
H_1 =& 1 + 5p_1 z + 10p_1p_2z^2 + 10p_1p_2p_3z^3 + 5p_1p_2p_3p_4z^4 + p_1p_2p_3p_4p_5z^5, \\
H_2 =& 1 + 8p_2z + (10p_1p_2 + 18p_2p_3)z^2 + \cdots + (10p_1p_2^2p_3^2p_4 + 18p_1p_2^2p_3p_4p_5)z^6 + 8p_1p_2^2p_3^2p_4p_5z^7 + p_1p_2^2p_3^2p_4^2p_5z^8, \\
H_3 =& 1 + 9p_3z + (18p_2p_3 + 18p_3p_4)z^2 + \cdots + (18p_1p_2^2p_3^2p_4p_5 + 18p_1p_2p_3^2p_4^2p_5)z^7 + 9p_1p_2^2p_3^2p_4^2p_5z^8 + p_1p_2^2p_3^3p_4^2p_5z^9, \\
H_4 =& 1 + 8p_4z + (18p_3p_4 + 10p_4p_5)z^2 + \cdots + (18p_1p_2p_3p_4^2p_5 + 10p_2p_3^2p_4^2p_5)z^6 + 8p_1p_2p_3^2p_4^2p_5z^7 + p_1p_2^2p_3^2p_4^2p_5z^8, \\
H_5 =& 1 + 5p_5z + 10p_4p_5z^2 + 10p_3p_4p_5z^3 + 5p_2p_3p_4p_5z^4 + p_1p_2p_3p_4p_5z^5.
\end{aligned}
$$

***B*$_5$-case.** In the case of Lie algebra B_5 the polynomial structure is following

$$
\begin{aligned}
H_1 =& 1 + 10p_1z + 45p_1p_2z^2 + \cdots + 45p_1p_2p_3^2p_4^2p_5^2z^8 + 10p_1p_2^2p_3^2p_4^2p_5^2z^9 + p_1^2p_2^2p_3^2p_4^2p_5^2z^{10}, \\
H_2 =& 1 + 18p_2z + (45p_1p_2 + 108p_2p_3)z^2 + \cdots + (108p_1^2p_2^3p_3^3p_4^4p_5^4 + 45p_1p_2^3p_3^4p_4^4p_5^4)z^{16} + 18p_1^2p_2^3p_3^4p_4^4p_5^4z^{17} + p_1^2p_2^4p_3^4p_4^4p_5^4z^{18}, \\
H_3 =& 1 + 24p_3z + (108p_2p_3 + 168p_3p_4)z^2 + \cdots + (168p_1^2p_2^4p_3^5p_4^5p_5^6 + 108p_1^2p_2^3p_3^5p_4^6p_5^6)z^{22} + 24p_1^2p_2^4p_3^5p_4^6p_5^6z^{23} \\
&+ p_1^2p_2^4p_3^6p_4^6p_5^6z^{24},
\end{aligned}
$$

$$\begin{aligned}
H_4 =& 1 + 28p_4z + (168p_3p_4 + 210p_4p_5)z^2 + \cdots + (210p_1^2p_2^4p_3^6p_4^7p_5^7 + 168p_1^2p_2^4p_3^5p_4^7p_5^8)z^{26} + 28p_1^2p_2^4p_3^6p_4^7p_5^8z^{27} \\
&+ p_1^2p_2^4p_3^6p_4^8p_5^8z^{28}, \\
H_5 =& 1 + 15p_5z + 105p_4p_5z^2 + \cdots + 105p_1p_2^2p_3^3p_4^3p_5^4z^{13} + 15p_1p_2^2p_3^3p_4^4p_5^4z^{14} + p_1p_2^2p_3^3p_4^4p_5^5z^{15}.
\end{aligned}$$

***C_5*-case.** The polynomials corresponding to Lie algebra C_5 have the following structure:

$$\begin{aligned}
H_1 =& 1 + 9p_1z + 36p_1p_2z^2 + \cdots + 36p_1p_2p_3^2p_4^2p_5z^7 + 9p_1p_2^2p_3^2p_4^2p_5z^8 + p_1^2p_2^2p_3^2p_4^2p_5z^9, \\
H_2 =& 1 + 16p_2z + (36p_1p_2 + 84p_2p_3)z^2 + \cdots + (84p_1^2p_2^3p_3^3p_4^4p_5^2 + 36p_1p_2^3p_3^4p_4^4p_5^2)z^{14} + 16p_1^2p_2^3p_3^4p_4^4p_5^2z^{15} + p_1^2p_2^4p_3^4p_4^4p_5^2z^{16}, \\
H_3 =& 1 + 21p_3z + (84p_2p_3 + 126p_3p_4)z^2 + \cdots + (126p_1^2p_2^4p_3^5p_4^5p_5^3 + 84p_1^2p_2^3p_3^5p_4^6p_5^3)z^{19} + 21p_1^2p_2^4p_3^5p_4^6p_5^3z^{20} + p_1^2p_2^4p_3^6p_4^6p_5^3z^{21}, \\
H_4 =& 1 + 24p_4z + (126p_3p_4 + 150p_4p_5)z^2 + \cdots + (150p_1^2p_2^4p_3^6p_4^7p_5^3 + 126p_1^2p_2^4p_3^5p_4^7p_5^4)z^{22} + 24p_1^2p_2^4p_3^6p_4^7p_5^4z^{23} \\
&+ p_1^2p_2^4p_3^6p_4^8p_5^4z^{24}, \\
H_5 =& 1 + 25p_5z + 300p_4p_5z^2 + \cdots + 300p_1^2p_2^4p_3^6p_4^7p_5^4z^{23} + 25p_1^2p_2^4p_3^6p_4^8p_5^4z^{24} + p_1^2p_2^4p_3^6p_4^8p_5^5z^{25}.
\end{aligned}$$

***D_5*-case.** In the case of Lie algebra D_5, we are led to the the following structure of polynomials

$$\begin{aligned}
H_1 =& 1 + 8p_1z + 28p_1p_2z^2 + \cdots + 28p_1p_2p_3^2p_4p_5z^6 + 8p_1p_2^2p_3^2p_4p_5z^7 + p_1^2p_2^2p_3^2p_4p_5z^8, \\
H_2 =& 1 + 14p_2z + (28p_1p_2 + 63p_2p_3)z^2 + \cdots + (63p_1^2p_2^3p_3^3p_4^2p_5^2 + 28p_1p_2^3p_3^4p_4^2p_5^2)z^{12} + 14p_1^2p_2^3p_3^4p_4^2p_5^2z^{13} + p_1^2p_2^4p_3^4p_4^2p_5^2z^{14}, \\
H_3 =& 1 + 18p_3z + (63p_2p_3 + 45p_3p_4 + 45p_3p_5)z^2 + \cdots + (45p_1^2p_2^4p_3^5p_4^3p_5^2 + 45p_1^2p_2^4p_3^5p_4^2p_5^3 + 63p_1^2p_2^3p_3^5p_4^3p_5^3)z^{16} \\
&+ 18p_1^2p_2^4p_3^5p_4^3p_5^3z^{17} + p_1^2p_2^4p_3^6p_4^3p_5^3z^{18}, \\
H_4 =& 1 + 10p_4z + 45p_3p_4z^2 + \cdots + 45p_1p_2^2p_3^2p_4^2p_5z^8 + 10p_1p_2^2p_3^3p_4^2p_5z^9 + p_1p_2^2p_3^3p_4^2p_5^2z^{10}, \\
H_5 =& 1 + 10p_5z + 45p_3p_5z^2 + \cdots + 45p_1p_2^2p_3^2p_4p_5^2z^8 + 10p_1p_2^2p_3^3p_4p_5^2z^9 + p_1p_2^2p_3^3p_4^2p_5^2z^{10}.
\end{aligned}$$

By using notations

$$H_s \equiv H_s(z) = H_s(z, (p_i)), \quad (p_i) \equiv (p_1, p_2, p_3, p_4, p_5) \tag{103}$$

we write asymptotic relations for polynomials

$$H_s = H_s(z, (p_i)) \sim \left(\prod_{l=1}^{5} (p_l)^{\nu^{sl}} \right) z^{n_s} \equiv H_s^{as}(z, (p_i)), \quad \text{as } z \to \infty. \tag{104}$$

Here, we denote by $\nu = (\nu^{sl})$ the integer valued matrix. It has the form

$$\nu = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & 8 & 8 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 & 2 & 1 \\ 2 & 4 & 4 & 4 & 2 \\ 2 & 4 & 6 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 2 & 4 & 6 & 8 & 5 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 4 & 4 & 2 & 2 \\ 2 & 4 & 6 & 3 & 3 \\ 1 & 2 & 3 & 2 & 2 \\ 1 & 2 & 3 & 2 & 2 \end{pmatrix} \tag{105}$$

for Lie algebras A_5, B_5, C_5, D_5, respectively.

One can readily verify that any matrix $\nu = (\nu^{sl})$ obeys the following identity

$$\sum_{l=1}^{5} \nu^{sl} = n_s, \quad s = 1, 2, 3, 4, 5. \tag{106}$$

For Lie algebras B_5, C_5, the matrix ν is twice inverse Cartan matrix A^{-1}:

$$\nu(\mathcal{G}) = 2A^{-1}, \quad \mathcal{G} = B_5, C_5, \tag{107}$$

and in the A_5 and D_5 cases we are led to another relation:

$$\nu(\mathcal{G}) = A^{-1}(I + P(\mathcal{G})), \quad \mathcal{G} = A_5, D_5. \tag{108}$$

Here, we use the notations: I—for 5×5 identity matrix and $P(\mathcal{G})$—for a matrix corresponding to a certain permutation $\sigma \in S_5$ (S_5 is symmetric group), $P = (P^i_j) = (\delta^i_{\sigma(j)})$, where σ is the generator of the group $G = \{\sigma, \mathrm{id}\}$. The group G is isomorphic to the group $\mathbb{Z}_2$. For A_5 and D_5 the group of symmetry of the Dynkin diagram acts on the set of corresponding five vertices via their permutations.

The explicit forms for the permutation matrix P and the generator σ for both Lie algebras A_5, D_5 read as follows:

$$P(A_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma : (1,2,3,4,5) \mapsto (5,4,3,2,1); \tag{109}$$

$$P(D_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma : (1,2,3,4,5) \mapsto (1,2,3,5,4). \tag{110}$$

The above symmetry groups control certain identity properties for polynomials $H_s(z)$.

We also denote $\hat{p}_i = p_{\sigma(i)}$ for the A_5 and D_5 cases, and $\hat{p}_i = p_i$ for B_5 and C_5 cases ($i = 1,2,3,4,5$).

By using MATHEMATICA algorithms we were able to verify the validity of the following identities.

3.5.2. Symmetry Relations

Proposition 8. *The fluxbrane polynomials, corresponding to Lie algebras A_5 and D_5, obey for all p_i and z the following identities:*

$$H_{\sigma(s)}(z,(p_i)) = H_s(z,(\hat{p}_i)), \tag{111}$$

where $\sigma \in S_5$, $s = 1,\ldots,5$, is defined for each algebra by Equations (109) *and* (110)*.*

3.5.3. Duality Relations

Proposition 9. *The fluxbrane polynomials which correspond to Lie algebras A_5, B_5, C_5, D_5, satisfy for all $p_i > 0$ and $z > 0$ the following identities*

$$H_s(z,(p_i)) = H_s^{as}(z,(p_i)) H_s(z^{-1},(\hat{p}_i^{-1})), \tag{112}$$

$s = 1,2,3,4,5.$

3.6. *E_6 Algebra*

Now we deal with the exceptional Lie algebra E_6. The matrix A is coinciding with the Cartan matrix (for E_6)

$$A = (A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{113}$$

This matrix is graphically depicted by the Dynkin diagram in Figure 5.

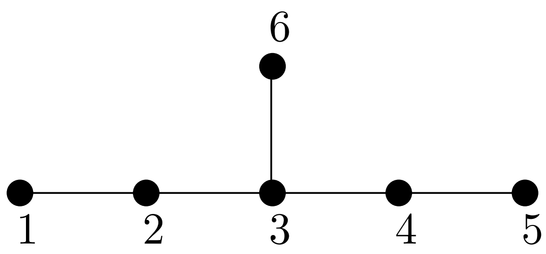

Figure 5. The Dynkin diagram for the Lie algebra E_6.

The inverse Cartan matrix for E_6

$$A^{-1} = (A^{ss'}) = \begin{pmatrix} \frac{4}{3} & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} & 1 \\ \frac{5}{3} & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} & 2 \\ 2 & 4 & 6 & 4 & 2 & 3 \\ \frac{4}{3} & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} & 1 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix} \tag{114}$$

implies due to (2)

$$(n_1, n_2, n_3, n_4, n_5, n_6) = (16, 30, 42, 30, 16, 22). \tag{115}$$

For the Lie algebra E_6 we find the set of six fluxbrane polynomials [42], which are listed in Appendix B.

The polynomials have the following structure:

$$\begin{aligned}
H_1 &= 1 + 16p_1 z + 120p_1p_2z^2 + \cdots + 120p_1^2p_2^3p_3^4p_4^2p_5p_6^2z^{14} \\
&\qquad + 16p_1^2p_2^3p_3^4p_4^3p_5p_6^2z^{15} + p_1^2p_2^3p_3^4p_4^3p_5^2p_6^2z^{16}, \\
H_2 &= 1 + 30p_2z + (120p_1p_2 + 315p_3p_2)z^2 \\
&\qquad \cdots + (120p_1^3p_2^6p_4^5p_5^2p_6^4p_3^8 + 315p_1^3p_2^6p_4^5p_5^3p_6^4p_3^7)z^{28} \\
&\qquad + 30p_1^3p_2^6p_3^8p_4^5p_5^3p_6^4z^{29} + p_1^3p_2^6p_3^8p_4^6p_5^3p_6^4z^{30}, \\
H_3 &= 1 + 42p_3z + (315p_2p_3 + 315p_4p_3 + 231p_6p_3)z^2 \\
&\qquad \cdots + (315p_1^4p_2^7p_4^8p_5^4p_6^6p_3^{11} + 315p_1^4p_2^8p_4^7p_5^4p_6^6p_3^{11} + 231p_1^4p_2^8p_4^8p_5^4p_6^5p_3^{11})z^{40} \\
&\qquad + 42p_1^4p_2^8p_3^{11}p_4^8p_5^4p_6^6z^{41} + p_1^4p_2^8p_3^{12}p_4^8p_5^4p_6^6z^{42}, \\
H_4 &= 1 + 30p_4z + (315p_3p_4 + 120p_5p_4)z^2 \\
&\qquad \cdots + (120p_1^2p_2^5p_4^6p_5^3p_6^4p_3^8 + 315p_1^3p_2^5p_4^6p_5^3p_6^4p_3^7)z^{28} \\
&\qquad + 30p_1^3p_2^5p_3^8p_4^6p_5^3p_6^4z^{29} + p_1^3p_2^6p_3^8p_4^6p_5^3p_6^4z^{30}, \\
H_5 &= 1 + 16p_5z + 120p_4p_5z^2 + \cdots + 120p_1p_2^2p_3^4p_4^3p_5^2p_6^2z^{14} \\
&\qquad + 16p_1p_2^3p_3^4p_4^3p_5^2p_6^2z^{15} + p_1^2p_2^3p_3^4p_4^3p_5^2p_6^2z^{16}, \\
H_6 &= 1 + 22p_6z + 231p_3p_6z^2 + \cdots + 231p_1^2p_2^4p_3^5p_4^4p_5^2p_6^3z^{20} \\
&\qquad + 22p_1^2p_2^4p_3^6p_4^4p_5^2p_6^3z^{21} + p_1^2p_2^4p_3^6p_4^4p_5^2p_6^4z^{22}.
\end{aligned} \tag{116}$$

The powers of polynomials are in agreement with the relation (115). In what follows we denote

$$H_s = H_s(z) = H_s(z,(p_i)), \tag{117}$$

$s = 1,\dots,6$; where $(p_i) = (p_1, p_2, p_3, p_4, p_5, p_6)$.

Due to (116) the asymptotical relations for the polynomials read as follows

$$H_s = H_s(z,(p_i)) \sim \left(\prod_{l=1}^{6}(p_l)^{\nu^{sl}}\right) z^{n_s} \equiv H_s^{as}(z,(p_i)), \tag{118}$$

$s = 1,\dots,6$, as $z \to \infty$. Here,

$$\nu = (\nu^{sl}) = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 4 & 8 & 12 & 8 & 4 & 6 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 2 & 3 & 4 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 2 & 4 \end{pmatrix}. \tag{119}$$

This matrix reads

$$\nu = A^{-1}(I + P), \tag{120}$$

where A^{-1} is inverse Cartan matrix, I is 6×6 identity matrix and

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{121}$$

is permutation matrix. This matrix is related to the permutation $\sigma \in S_6$ (S_6 is symmetric group)

$$\sigma : (1,2,3,4,5,6) \mapsto (5,4,3,2,1,6), \tag{122}$$

by the following formula $P = (P^i_j) = (\delta^i_{\sigma(j)})$. Here, σ is the generator of the group of symmetry of the Dynkin diagram $G = \{\sigma, id\}$. (G is isomorphic to the group Z_2.) σ is a composition of two transpositions: $(1 \mapsto 5)$ and $(2 \mapsto 4)$.

We note that the matrix ν is symmetric one and

$$\sum_{s=1}^{6} \nu^{sl} = n_l, \tag{123}$$

$l = 1,\dots,6$.

Now we introduce the dual (ordered) set $(\hat{p}_i) = (p_{\sigma(i)})$, $i = 1,\dots,6$. By using the relations for polynomials from Appendix B we are led to the following two identities which are verified with the aid of MATHEMATICA.

3.6.1. Symmetry Relations

Proposition 10. *For all p_i and z*

$$H_{\sigma(s)}(z,(p_i)) = H_s(z,(\hat{p}_i)), \tag{124}$$

$s = 1,\dots,6$.

3.6.2. Duality Relations

Proposition 11. *For all* $p_i > 0$ *and* $z > 0$

$$H_s(z,(p_i)) = H_s^{as}(z,(p_i))H_s(z^{-1},(\hat{p}_i^{-1})), \tag{125}$$

$s = 1,\ldots,6.$

The solution (8)–(10) reads in our case

$$g = \left(\prod_{s=1}^{6} H_s^{2h/(D-2)}\right)\left\{d\rho\otimes d\rho + \left(\prod_{s=1}^{6} H_s^{-2h}\right)\rho^2 d\phi\otimes d\phi + g^2\right\}, \tag{126}$$

$$\exp(\varphi^a) = \prod_{s=1}^{6} H_s^{h\lambda_s^a}, \tag{127}$$

$$F^s = \mathcal{B}^s \rho d\rho\wedge d\phi, \tag{128}$$

$a, s = 1,\ldots,6$, where $g^1 = d\phi\otimes d\phi$ is a metric on $M_1 = S^1$ ($0 < \phi < 2\pi$), g^2 is a Ricci-flat metric on M_2 of signature $(-,+,\ldots,+)$. Here,

$$\mathcal{B}^s = -Q_s\left(\prod_{l=1}^{6} H_l^{-A_{sl}}\right) \tag{129}$$

and due to (14)–(16)

$$K = K_s = \frac{D-3}{D-2} + \vec{\lambda}_s^2, \tag{130}$$

$h_s = h = K^{-1}$,

$$\vec{\lambda}_s\vec{\lambda}_{s'} = \frac{1}{2}KA_{ss'} - \frac{D-3}{D-2} \equiv G_{ss'}, \tag{131}$$

$s, s' = 1,\ldots,6.$

3.7. Some Relations between Polynomials

Here, we denote the set of polynomials corresponding to a set of parameters $p_1 > 0$, ..., $p_n > 0$ as following

$$H_s = H_s(z, p_1, ..., p_n; A), \tag{132}$$

$s = 1,\ldots,n$, where $A = A[\mathcal{G}]$ is the Cartan matrix corresponding to a (semi)simple Lie algebra $\mathcal{G}$ of rank n.

3.7.1. C_{n+1}-Polynomials from A_{2n+1}-Ones

It was conjectured in Ref. [28] that the set of polynomials corresponding to the Lie algebra C_{n+1} may be obtained from the set of polynomials corresponding to the Lie algebra A_{2n+1} according to the following relations

$$H_s(z, p_1, ..., p_{n+1}; A[C_{n+1}]) = H_s(z, p_1, ..., p_{n+1}, p_{n+2} = p_n, ..., p_{2n+1} = p_1; A[A_{2n+1}]), \tag{133}$$

$s = 1,\ldots,n+1$, i.e., the parameters $p_1, ..., p_{n+1}, p_{n+2}, ..., p_{2n+1}$ are identified symmetrically with respect to p_{n+1}.

Relation (133) may be readily verified at least for $n = 1, 2$ by using explicit relations for corresponding polynomials presented above.

3.7.2. B_n-Polynomials from D_{n+1}-Ones

Due to the conjecture from Ref. [28], the set polynomials corresponding to the Lie algebra B_n can be obtained from the set of polynomials corresponding to the Lie algebra D_{n+1} according to the following relation

$$H_s(z, p_1, ..., p_n; A[B_n]) = H_s(z, p_1, ..., p_n, p_{n+1} = p_n; A[D_{n+1}]), \tag{134}$$

$s = 1, \ldots, n$, i.e., the parameters p_n and p_{n+1} are identified.

Relation (134) can be readily verified at least for $n = 3, 4$ by using explicit relations for corresponding polynomials presented above.

3.7.3. G_2-Polynomials from D_4-Ones

It can be readily checked that G_2-polynomials may be obtained just by imposing the following relations on parameters of D_4-polynomials: $p_1 = p_3 = p_4$. We obtain

$$H_1(z, p_1, p_2, p_1, p_1; A[D_4]) = H_1(z, p_1, p_2; A[G_2]), \tag{135}$$

$$H_2(z, p_1, p_2, p_1, p_1; A[D_4]) = H_2(z, p_1, p_2; A[G_2]). \tag{136}$$

It looks like we glue the symmetric points (1, 3 and 4) at the Dynkin graph for D_4 algebra (see Figure 3) in order to obtain the Dynkin graph for G_2 algebra (see Figure 1).

3.7.4. F_4-Polynomials from E_6-Ones

It can be verified that F_4-polynomials may be obtained by imposing the following relations on parameters of E_6-polynomials: $p_1 = p_5 = \bar{p}_4$, $p_2 = p_4 = \bar{p}_3$, $p_3 = \bar{p}_2$, $p_6 = \bar{p}_1$. We obtain

$$H_1(z, p_1, p_2, p_3, p_2, p_1, p_6; A[E_6]) = H_4(z, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; A[F_4]), \tag{137}$$

$$H_2(z, p_1, p_2, p_3, p_2, p_1, p_6; A[E_6]) = H_3(z, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; A[F_4]), \tag{138}$$

$$H_3(z, p_1, p_2, p_3, p_2, p_1, p_6; A[E_6]) = H_2(z, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; A[F_4]), \tag{139}$$

$$H_6(z, p_1, p_2, p_3, p_2, p_1, p_6; A[E_6]) = H_1(z, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; A[F_4]). \tag{140}$$

It looks like we glue the symmetric points (1 and 5, 2 and 4) at the Dynkin graph for E_6 algebra (see Figure 5) in order to obtain the Dynkin graph for F_4 algebra (see Figure 3).

3.7.5. Reduction Formulas

Here, we denote the Cartan matrix in the following way: $A = A_\Gamma$, where Γ is the related Dynkin graph. Let i be a node of Γ. We denote by Γ_i a Dynkin graph (which corresponds to a certain semi-simple Lie algebra) that is obtained from Γ by erasing all lines that have endpoints at i. It can be verified (e.g., by using MATHEMATICA) that for the polynomials presented above the following reduction formulae hold

$$H_s(z, p_1, ..., p_i = 0, ..., p_n; A_\Gamma) = H_s(z, p_1, ..., p_{i-1}, p_{i+1}, ..., p_n; A_{\Gamma_i}), \tag{141}$$

$s = 1, \ldots, i-1, i+1, \ldots, n$, and

$$H_i(z, p_1, ..., p_i = 0, ..., p_n; A_\Gamma) = 1. \tag{142}$$

This means that by setting $p_i = 0$ we reduce the set of polynomials by replacing the Cartan matrix A_Γ with the Cartan matrix A_{Γ_i}. In this case the polynomial $H_i = 1$ corresponds to A_1-subalgebra (represented by the node i) and the parameter $p_i = 0$.

As an example of reduction formulas we present the following relations

$$H_s(z, p_1, ..., p_n, p_{n+1} = 0; A[\mathcal{G}]) = H_s(z, p_1, ..., p_n; A[A_n]), \tag{143}$$

$s = 1, \ldots, n$, for $\mathcal{G} = A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$ with suitable restrictions on n. These relations are valid at least for all $ABCD$-polynomials presented above. In writing relation (141) we use the numbering of the nodes in accordance with the Dynkin diagrams shown in the figures presented above.

The reduction formulas (142) for A_5-polynomials with $p_3 = 0$ are shown in Figure 6. The reduced polynomials coincide with those corresponding to semisimple Lie algebra $A_2 \bigoplus A_1 \bigoplus A_2$.

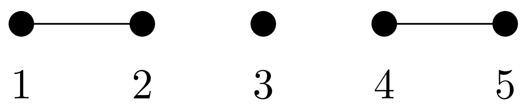

Figure 6. Dynkin diagram for semisimple Lie algebra $A_2 \bigoplus A_1 \bigoplus A_2$ describing the set of A_5-polynomials with $p_3 = 0$ [28].

4. Flux Integrals

As in the previous section, we deal here with the solution (8)–(10) with $n = l$, $w = +1$ and $M_1 = S^1$ and $h_{\alpha\beta} = \delta_{\alpha\beta}$.

We denote $(\lambda_{sa}) = (\lambda_s^a) = \vec{\lambda}_s$, $s = 1, \dots, n$. The solution corresponds to a simple finite-dimensional Lie algebra $\mathcal{G}$, i.e., the matrix $A = (A_{ss'})$ is coinciding with the Cartan matrix of this Lie algebra.

In this case, we get relations (17), (18) for $h_s = K_s^{-1} > 0$ and scalar products $\vec{\lambda}_s \vec{\lambda}_l$ which follow from (14)–(16).

Due to (14)–(16) that

$$\frac{h_i}{h_j} = \frac{K_j}{K_i} = \frac{B_{jj}}{B_{ii}} = \frac{B_{ji}}{B_{ii}} \frac{B_{jj}}{B_{ij}} = \frac{A_{ji}}{A_{ij}} \tag{144}$$

for any $i \neq j$ obeying $A_{ij} = A_{ji} \neq 0$; $i, j = 1, \dots, n$.

It follows from (144) and the connectedness of the Dynkin diagram for a simple Lie algebra that

$$\frac{h_i}{h_j} = \frac{K_j}{K_i} = \frac{r_j}{r_i} \tag{145}$$

$i \neq j$, where $r_i = (\alpha_i, \alpha_i)$ is length squared of a simple root α_i of the Lie algebra $\mathcal{G}$. Here, $((.,.)$ is dual Killing–Cartan form and

$$A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j), \tag{146}$$

$i, j = 1, \dots, n$. Due to (145) we obtain

$$K_i = \frac{1}{2} K r_i, \tag{147}$$

$i = 1, \dots, n$, where $K > 0$.

Now we consider the oriented 2-dimensional manifold $M_* = (0, +\infty) \times S^1$. The flux integrals read

$$\Phi^s = \int_{M_*} F^s = \int_0^{+\infty} d\rho \int_0^{2\pi} d\phi \, \rho \mathcal{B}^s(\rho^2) = 2\pi \int_0^{+\infty} d\rho \, \rho \mathcal{B}^s(\rho^2), \tag{148}$$

where

$$\mathcal{B}^s(\rho^2) = q_s \prod_{l=1}^{n} (H_l(\rho^2))^{-A_{sl}}. \tag{149}$$

Due to (13) and (19) we have

$$p_s = \frac{P_s}{n_s} = \frac{K_s}{4 n_s} q_s^2. \tag{150}$$

The integrals (148) are convergent for all $s = 1, \dots, n$, if the conjecture on polynomial structure of moduli functions H_s is satisfied for the Lie algebra $\mathcal{G}$ under consideration.

Indeed, polynomial assumption (1) implies

$$H_s(\rho^2) \sim C_s \rho^{2n_s}, \qquad C_s = P_s^{(n_s)}, \tag{151}$$

as $\rho \to +\infty$; $s = 1, \dots, n$. From (149), (151) and the identities $\sum_1^n A_{sl} n_l = 2$, following from (2), we obtain

$$\mathcal{B}^s(\rho^2) \sim q_s C^s \rho^{-4}, \quad C^s = \prod_{l=1}^{n} C_l^{-A_{sl}}. \tag{152}$$

Hence the integral (148) is convergent for any $s = 1, \dots, n$.

From master Equation (11), we obtain

$$\begin{aligned} \int_0^{+\infty} d\rho \rho \mathcal{B}^s(\rho^2) &= q_s P_s^{-1} \frac{1}{2} \int_0^{+\infty} dz \frac{d}{dz}\left(\frac{z}{H_s} \frac{d}{dz} H_s \right) \\ &= \frac{1}{2} q_s P_s^{-1} \lim_{z \to +\infty} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = \frac{1}{2} n_s q_s P_s^{-1}, \end{aligned} \tag{153}$$

which implies (see (13)) [43]

$$\Phi^s = 4\pi n_s q_s^{-1} h_s, \tag{154}$$

$s = 1, \dots, n$.

Thus, we see that any flux Φ^s depends only upon one integration constant $q_s \neq 0$. This is a nontrivial fact since the integrand form F^s depends upon all constants: $q_1, \dots, q_n$.

It should be noted that, for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, q_s is coinciding with the value of the x-component of the magnetic field on the axis of symmetry.

For the case of Gibbons–Maeda dilatonic generalization of the Melvin solution, corresponding to $D = 4$, $n = l = 1$ and $\mathcal{G} = A_1$ [6], the flux from (154) ($s = 1$) is in agreement with that obtained in Ref. [26]. For the Melvin's solution and some higher dimensional extensions (with $\mathcal{G} = A_1$) see also Ref. [15].

Owing to (145) the ratios

$$\frac{q_i \Phi^i}{q_j \Phi^j} = \frac{n_i h_i}{n_j h_j} = \frac{n_i r_j}{n_j r_i} \tag{155}$$

are just fixed numbers which depend upon the Cartan matrix (A_{ij}) of a simple finite-dimensional Lie algebra $\mathcal{G}$.

Since the manifold $M_* = (0, +\infty) \times S^1$ is isomorphic to the manifold $\mathbb{R}^2_* = \mathbb{R}^2 \setminus \{0\}$, the solution (8)–(10) may be rewritten (by pull-backs) onto manifold $\mathbb{R}^2_* \times M_2$. In this new presentation of the solution coordinates ρ, ϕ are understood as polar coordinates on $\mathbb{R}^2_*$. Since they are not globally defined one may consider two charts with coordinates ρ, $\phi = \phi_1$ and ρ, $\phi = \phi_2$, where $\rho > 0$, $0 < \phi_1 < 2\pi$ and $-\pi < \phi_2 < \pi$. Here, $\exp(i\phi_1) = \exp(i\phi_2)$. For both charts we have $x = \rho \cos\phi$ and $y = \rho \sin\phi$, where x, y are standard (Euclidean) coordinates of $\mathbb{R}^2$. By using the identity $\rho d\rho \wedge d\phi = dx \wedge dy$ we obtain

$$F^s = q_s \prod_{l=1}^{n} (H_l(x^2 + y^2))^{-A_{sl}} dx \wedge dy, \tag{156}$$

$s = 1, \dots, n$.

Now, let us show that 2-forms (156) are well-defined on $\mathbb{R}^2$. Indeed, due to conjecture from Ref. [3] any polynomial $H_s(z)$ is a smooth function on $\mathbb{R} = (-\infty, +\infty)$ obeying $H_s(z) > 0$ for $z \in (-\varepsilon_s, +\infty)$, where $\varepsilon_s > 0$. This is so since due to polynomial conjecture from Ref. [3] we have $H_s(z) > 0$ for $z > 0$ and $H_s(+0) = 1$. Hence, $\left(\prod_{l=1}^{n} (H_l(x^2 + y^2))^{-A_{sl}}\right)$ is a smooth function since it is just a composition of two well-defined smooth functions: $\left(\prod_{l=1}^{n} (H_l(z))^{-A_{sl}}\right)$ and $z = x^2 + y^2$.

Now, we show that there exist 1-forms A^s obeying $F^s = dA^s$ which are globally defined on $\mathbb{R}^2$. Let us consider the open submanifold $\mathbb{R}^2_*$. The 1-forms

$$A^s = \left(\int_0^{\rho} d\bar{\rho}\bar{\rho}\mathcal{B}^s(\bar{\rho}^2) \right) d\phi = \frac{1}{2} \left(\int_0^{\rho^2} d\bar{z}\mathcal{B}^s(\bar{z}) \right) d\phi \tag{157}$$

are well defined on $\mathbb{R}^2_*$ and obey $F^s = dA^s$, $s = 1, \dots, n$. It is obvious that here

$$d\phi = (x^2 + y^2)^{-1}(-ydx + xdy). \tag{158}$$

It follows from the master Equation (11) that

$$\begin{aligned} A^s &= \frac{q_s}{2P_s} \left(\int_0^{\rho^2} d\bar{z} \frac{d}{d\bar{z}} \left(\frac{\bar{z}}{H_s(\bar{z})} \frac{d}{d\bar{z}} H_s(\bar{z}) \right) \right) d\phi \\ &= \frac{2h_s}{q_s} \frac{H_s'(\rho^2)}{H_s(\rho^2)} \rho^2 d\phi, \end{aligned} \tag{159}$$

$s = 1, \dots, n$. Here, $H'_s = \frac{d}{dz} H_s$. Due to relation $\rho^2 d\phi = -ydx + xdy$, we obtain

$$A^s = \frac{2h_s}{q_s} \frac{H_s'(x^2+y^2)}{H_s(x^2+y^2)} (-ydx + xdy), \tag{160}$$

$s = 1, \dots, n$. Thus, we are led to 1-forms (160) which are well-defined (smooth) 1-forms on $\mathbb{R}^2$.

It should be noted that in case of Gibbons–Maeda solution [6] (with $D = 4$, $n = l = 1$ and $\mathcal{G} = A_1$) the gauge potential from (159) coincides (up to notations) with that considered in Ref. [8].

Now we verify the formula (154) for flux integrals by using the relation (160) for 1-forms A^s. Let us consider a 2d oriented manifold (disk)

$$D_R = \{(x, y) : x^2 + y^2 \leq R^2\} \tag{161}$$

with the boundary

$$\partial D_R = C_R = \{(x, y) : x^2 + y^2 = R^2\}. \tag{162}$$

Here, C_R is a circle of radius R. It is just an 1d oriented manifold with the orientation (inherited from that of D_R) obeying the relation $\int_{C_R} d\phi = 2\pi$. Using the Stokes–Cartan theorem we obtain

$$\Phi^s(R) = \int_{D_R} F^s = \int_{D_R} dA^s = \int_{C_R} A^s = \frac{4\pi h_s}{q_s} \frac{H_s'(R^2)}{H_s(R^2)} R^2, \tag{163}$$

$s = 1, \dots, n$. By using the asymptotic relation (151) we get

$$\lim_{R \to +\infty} \Phi^s(R) = \lim_{R \to +\infty} \int_{D_R} F^s = \frac{4\pi h_s n_s}{q_s} = \Phi^s, \tag{164}$$

$s = 1, \dots, n$, in agreement with (154).

We note that, according to the definition of Abelian Wilson loop (factor), we have

$$W^s(C_R) = \exp(i \int_{C_R} A^s) = \exp(i\Phi^s(R)), \tag{165}$$

$s = 1, \dots, n$. As a consequence (see (164)), we obtain finite limits

$$\lim_{R \to +\infty} W^s(C_R) = \exp(i\Phi^s), \tag{166}$$

$s = 1, \ldots, n$.

Finally, we note that the metric and scalar fields for our solution with $w = +1$ and $l = n$ can be extended to the manifold $\mathbb{R}^2 \times M_2$.

Indeed, in standard x, y coordinates on $\mathbb{R}^2$ the metric (8) and scalar fields (9) read as follows [43]

$$g = \Big(\prod_{s=1}^{n} H_s^{2h_s/(D-2)}\Big)\Big\{dx \otimes dx + dy \otimes dy + F(-ydx + xdy)^2_{\otimes} + g^2\Big\}, \tag{167}$$

$$\varphi^a = \sum_{s=1}^{n} h_s \lambda_s^a \ln H_s, \tag{168}$$

$a = 1, \ldots, n$. Here, $H_s = H_s(x^2 + y^2)$, $s = 1, \ldots, n$, $\omega^2_{\otimes} = \omega \otimes \omega$ and $F = F(x^2 + y^2)$, where

$$F(z) = \left(\Big(\prod_{s=1}^{n}(H_s(z))^{-2h_s}\Big) - 1\right) z^{-1}, \tag{169}$$

for $z \neq 0$ and

$$F(0) = \lim_{z \to 0} F(z) = \sum_{s=1}^{n}(-2h_s p_s). \tag{170}$$

The metric and scalar fields are smooth on the manifold $\mathbb{R}^2 \times M_2$ [43].

5. Dilatonic Black Holes

Relations on dilatonic coupling vectors (14)–(16) also appear for dilatonic black hole (DBH) solutions defined on the manifold

$$M = (R_0, +\infty) \times (M_0 = S^2) \times (M_1 = \mathbb{R}) \times M_2, \tag{171}$$

where $R_0 = 2\mu > 0$ and M_2 is a Ricci-flat manifold. These DBH solutions on M from (171) for the model under consideration may be extracted just from more general black brane solutions, see Refs. [35,36,44]. They read:

$$g = \Big(\prod_{s=1}^{n} \mathbf{H}_s^{2h_s/(D-2)}\Big)\Big\{f^{-1} dR \otimes dR + R^2 g^0 - \Big(\prod_{s=1}^{n} \mathbf{H}^{-2h_s}\Big) f dt \otimes dt + g^2\Big\}, \tag{172}$$

$$\exp(\varphi^a) = \prod_{s=1}^{n} \mathbf{H}^{h_s \lambda_s^a}, \tag{173}$$

$$F^s = -Q_s R^{-2} \left(\prod_{l=1}^{n} \mathbf{H}_l^{-A_{sl}}\right) dR \wedge dt, \tag{174}$$

$s, a = 1, \ldots, n$, where $f = 1 - 2\mu R^{-1}$, g^0 is the standard metric on $M_0 = S^2$ and g^2 is a Ricci-flat metric of signature $(+, \ldots, +)$ on M_2. Here, $Q_s \neq 0$ are integration constants (charges).

The functions $\mathbf{H}_s = \mathbf{H}_s(R) > 0$ obey the master equations

$$R^2 \frac{d}{dR}\left(f \frac{R^2}{\mathbf{H}_s} \frac{d}{dR} \mathbf{H}_s\right) = B_s \prod_{l=1}^{n} \mathbf{H}_l^{-A_{sl}}, \tag{175}$$

with the following boundary conditions (on the horizon and at infinity) imposed:

$$\mathbf{H}_s(R_0 + 0) = \mathbf{H}_{s0} > 0, \qquad \mathbf{H}_s(+\infty) = 1, \tag{176}$$

where

$$B_s = -K_s Q_s^2, \tag{177}$$

$s = 1, \ldots, n$.

It was shown in Ref. [36] that these polynomials may be obtained (at least for small enough Q_s) from fluxbrane polynomials $H_s(z)$ presented in this paper.

Indeed, let us denote $f = f(z) = 1 - 2\mu z$, $z = 1/R$. Then, the relations (175) may be rewritten as follows

$$\frac{d}{df}\left(\frac{f}{\mathbf{H}_s}\frac{d}{df}\mathbf{H}_s\right) = B_s(2\mu)^{-2}\prod_{l=1}^{n}\mathbf{H}_l^{-A_{sl}}, \tag{178}$$

$s = 1, \dots, n$. These relations could be solved (at least for small enough Q_s) by using fluxbrane polynomials $H_s(f) = H_s(f;p)$, given by $n \times n$ Cartan matrix (A_{sl}), where $p = (p_1, \dots, p_n)$ is the set of parameters [36]. Here we impose the restrictions $p_s \neq 0$ for all s.

Due to approach of Ref. [36] we put

$$\mathbf{H}_s(z) = H_s(f(z);p)/H_s(1;p) \tag{179}$$

for $s = 1, \dots, n$. Then, the relations (178), are satisfied identically if [36]

$$n_s p_s \prod_{l=1}^{n}(H_l(1;p))^{-A_{sl}} = B_s/(2\mu)^2, \tag{180}$$

$s = 1, ..., n$.

We call the set of parameters $p = (p_1, \dots, p_n)$ $(p_i \neq 0)$ a proper one if [36]

$$H_s(f;p) > 0 \tag{181}$$

for all $f \in [0,1]$ and $s = 1, \dots, n$. Here, we consider only proper p. Due to relations (180) and $B_s < 0$, we have $p_s < 0$ for all $s = 1, \dots, n$, i.e., one should use fluxbrane polynomials with negative parameters p_s for a description of black hole solutions under consideration.

The boundary conditions (176) are valid, since

$$\mathbf{H}_s((2\mu)^{-1} - 0) = 1/H_s(1;p) > 0, \tag{182}$$

$s = 1, \dots, n$ (see definition (179)).

For small enough p_i, the set $(p_1, \dots, p_n)$ is proper and relation (180) defines one-to-one correspondence between the sets of parameters $(p_1, \dots, p_n)$ and $(Q_1^2, \dots, Q_n^2)$.

Relations (182) imply the following formula for the Hawking temperature [36]

$$T_H = \frac{1}{8\pi\mu}\prod_{s=1}^{n}(H_s(1;p))^{h_s}. \tag{183}$$

It should be noted that fluxbrane polynomials were used in Refs. [45–47] in the context of dyon-like black hole solutions. (To a certain extent these papers were inspired by Ref. [48].)

6. Conclusions

Here, we have explored a multidimensional generalization of the Melvin's solution corresponding to a simple finite-dimensional Lie algebra $\mathcal{G}$. It takes place in a D-dimensional model which contains metric n Abelian 2-forms $F^s = dA^s$ and $l \geq n$ scalar fields φ^α. (Here, we have put $l = n$ for simplicity.)

The solution is governed by a set of n moduli functions $H_s(z)$, $s = 1, \dots, n$, which were assumed earlier to be polynomials—so-called fluxbrane polynomials. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $\mathcal{G}$. The polynomials $H_s(z)$ also depend upon parameters $p_s \sim q_s^2$, where q_s are coinciding for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

Here, we have presented examples of polynomials corresponding to Lie algebras of rank $r = 1, 2, 3, 4, 5$ and exceptional Lie algebra E_6 and have outlined the symmetry and duality relations for these polynomials. The so-called duality relations for fluxbrane polynomials describe a behavior of the solutions under the inversion: $\rho \to 1/\rho$, which makes the model in tune with so-called T-duality in string models. These relations can be also mathematically understood in terms of the discrete groups of symmetry of Dynkin diagrams for the corresponding Lie algebras.

We have presented calculations of $2d$ flux integrals $\Phi^s = \int F^s$, where F^s are 2-forms, $s = 1, \dots, n$. It is remarkable that any flux Φ^s depends only upon one parameter q_s, while the integrand F^s depends upon all parameters $q_1, \dots, q_n$.

Here, we have also outlined possible applications of fluxbrane polynomials for seeking dilatonic black hole solutions (e.g., $4d$ ones) in the model under consideration.

It should be noted that fluxbrane polynomials may also describe certain subclass of self-dual solutions to Yang-Mills equations in a flat space of signature $(+,+,-,-)$ [49]. This subclass belong to a more general Toda chain class of self-dual solutions.

Here, an interesting question occurs: how can the nice properties of polynomials (symmetries, duality relations, reduction formulas) be used for a description of geometries of Melvin-like solutions (and other fluxbrane ones) and possible generating of new solutions? This and other related questions may be a subject of future publications.

Author Contributions: Conceptualization, S.V.B. and V.D.I.; methodology, S.V.B. and V.D.I.; software, S.V.B.; validation, S.V.B. and V.D.I.; formal analysis, S.V.B. and V.D.I.; investigation, S.V. and V.D.I.; resources, S.V.B. and V.D.I.; data curation, S.V.B. ; writing—original draft preparation, S.V.B. and V.D.I.; writing—review and editing, S.V.B. and V.D.I.; visualization, S.V.B.; supervision, V.D.I.; project administration, S.V.B.; funding acquisition, S.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by RUDN University, scientific project number FSSF-2023-0003.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Polynomials for A_5, B_5, C_5, D_5

Here, we present the polynomials corresponding to Lie algebras of rank 5 [41].

***A_5*-case.** For the Lie algebra $A_5 \cong sl(6)$ the polynomials read

$$
\begin{aligned}
H_1 =& 1 + 5p_1z + 10p_1p_2z^2 + 10p_1p_2p_3z^3 + 5p_1p_2p_3p_4z^4 + p_1p_2p_3p_4p_5z^5 \\
H_2 =& 1 + 8p_2z + (10p_1p_2 + 18p_2p_3)z^2 + (40p_1p_2p_3 + 16p_2p_3p_4)z^3 + (20p_1p_2^2p_3 + 45p_1p_2p_3p_4 + 5p_2p_3p_4p_5)z^4 \\
&+ (40p_1p_2^2p_3p_4 + 16p_1p_2p_3p_4p_5)z^5 + (10p_1p_2^2p_3^2p_4 + 18p_1p_2^2p_3p_4p_5)z^6 + 8p_1p_2^2p_3^2p_4p_5z^7 + p_1p_2^2p_3^2p_4^2p_5z^8 \\
H_3 =& 1 + 9p_3z + (18p_2p_3 + 18p_3p_4)z^2 + (10p_1p_2p_3 + 64p_2p_3p_4 + 10p_3p_4p_5)z^3 + (45p_1p_2p_3p_4 + 36p_2p_3^2p_4 + 45p_2p_3p_4p_5)z^4 \\
&+ (45p_1p_2p_3^2p_4 + 36p_1p_2p_3p_4p_5 + 45p_2p_3^2p_4p_5)z^5 + (10p_1p_2^2p_3^2p_4 + 64p_1p_2p_3^2p_4p_5 + 10p_2p_3^2p_4^2p_5)z^6 + (18p_1p_2^2p_3^2p_4p_5 \\
&+ 18p_1p_2p_3^2p_4^2p_5)z^7 + 9p_1p_2^2p_3^2p_4^2p_5z^8 + p_1p_2^2p_3^3p_4^2p_5z^9 \\
H_4 =& 1 + 8p_4z + (18p_3p_4 + 10p_4p_5)z^2 + (16p_2p_3p_4 + 40p_3p_4p_5)z^3 + (5p_1p_2p_3p_4 + 45p_2p_3p_4p_5 + 20p_3p_4^2p_5)z^4 \\
&+ (16p_1p_2p_3p_4p_5 + 40p_2p_3p_4^2p_5)z^5 + (18p_1p_2p_3p_4^2p_5 + 10p_2p_3^2p_4^2p_5)z^6 + 8p_1p_2p_3^2p_4^2p_5z^7 + p_1p_2^2p_3^2p_4^2p_5z^8 \\
H_5 =& 1 + 5p_5z + 10p_4p_5z^2 + 10p_3p_4p_5z^3 + 5p_2p_3p_4p_5z^4 + p_1p_2p_3p_4p_5z^5
\end{aligned}
$$

***B_5*-case.** For the Lie algebra $B_5 \cong so(11)$ we find

$$
\begin{aligned}
H_1 =& 1 + 10p_1z + 45p_1p_2z^2 + 120p_1p_2p_3z^3 + 210p_1p_2p_3p_4z^4 + 252p_1p_2p_3p_4p_5z^5 + 210p_1p_2p_3p_4p_5^2z^6 + 120p_1p_2p_3p_4^2p_5^2z^7 \\
&+ 45p_1p_2p_3^2p_4^2p_5^2z^8 + 10p_1p_2^2p_3^2p_4^2p_5^2z^9 + p_1^2p_2^2p_3^2p_4^2p_5^2z^{10} \\
H_2 =& 1 + 18p_2z + (45p_1p_2 + 108p_2p_3)z^2 + (480p_1p_2p_3 + 336p_2p_3p_4)z^3 + (540p_1p_2^2p_3 + 1890p_1p_2p_3p_4 + 630p_2p_3p_4p_5)z^4 \\
&+ (3780p_1p_2^2p_3p_4 + 4032p_1p_2p_3p_4p_5 + 756p_2p_3p_4p_5^2)z^5 + (2520p_1p_2^2p_3^2p_4 + 10206p_1p_2^2p_3p_4p_5 + 5250p_1p_2p_3p_4p_5^2 \\
&+ 588p_2p_3p_4^2p_5^2)z^6 + (12096p_1p_2^2p_3^2p_4p_5 + 15120p_1p_2^2p_3p_4p_5^2 + 4320p_1p_2p_3p_4^2p_5^2 + 288p_2p_3^2p_4^2p_5^2)z^7 \\
&+ (5292p_1p_2^2p_3^2p_4^2p_5 + 22680p_1p_2^2p_3^2p_4p_5^2 + 13500p_1p_2^2p_3p_4^2p_5^2 + 2205p_1p_2p_3^2p_4^2p_5^2 + 81p_2^2p_3^2p_4^2p_5^2)z^8 \\
&+ 48620p_1p_2^2p_3^2p_4^2p_5^2z^9 + (81p_1^2p_2^2p_3^2p_4^2p_5^2 + 2205p_1p_2^3p_3^2p_4^2p_5^2 + 13500p_1p_2^2p_3^3p_4^2p_5^2 + 22680p_1p_2^2p_3^2p_4^3p_5^2 \\
&+ 5292p_1p_2^2p_3^2p_4^2p_5^3)z^{10} + (288p_1^2p_2^3p_3^2p_4^2p_5^2 + 4320p_1p_2^3p_3^3p_4^2p_5^2 + 15120p_1p_2^2p_3^3p_4^3p_5^2 + 12096p_1p_2^2p_3^2p_4^3p_5^3)z^{11} \\
&+ (588p_1^2p_2^3p_3^3p_4^2p_5^2 + 5250p_1p_2^3p_3^3p_4^3p_5^2 + 10206p_1p_2^2p_3^3p_4^3p_5^3 + 2520p_1p_2^2p_3^2p_4^3p_5^4)z^{12} + (756p_1^2p_2^3p_3^3p_4^3p_5^2 \\
&+ 4032p_1p_2^3p_3^3p_4^3p_5^3 + 3780p_1p_2^2p_3^3p_4^3p_5^4)z^{13} + (630p_1^2p_2^3p_3^3p_4^3p_5^3 + 1890p_1p_2^3p_3^3p_4^3p_5^4 + 540p_1p_2^2p_3^3p_4^4p_5^4)z^{14} \\
&+ (336p_1^2p_2^3p_3^3p_4^3p_5^4 + 480p_1p_2^3p_3^3p_4^4p_5^4)z^{15} + (108p_1^2p_2^3p_3^3p_4^4p_5^4 + 45p_1p_2^3p_3^4p_4^4p_5^4)z^{16} + 18p_1^2p_2^3p_3^4p_4^4p_5^4z^{17} + p_1^2p_2^4p_3^4p_4^4p_5^4z^{18} \\
H_3 =& 1 + 24p_3z + (108p_2p_3 + 168p_3p_4)z^2 + (120p_1p_2p_3 + 1344p_2p_3p_4 + 560p_3p_4p_5)z^3 + (1890p_1p_2p_3p_4 + 2016p_2p_3^2p_4 \\
&+ 5670p_2p_3p_4p_5 + 1050p_3p_4p_5^2)z^4 + (5040p_1p_2p_3^2p_4 + 9072p_1p_2p_3p_4p_5 + 15120p_2p_3^2p_4p_5 + 12096p_2p_3p_4p_5^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 1176p_3p_4^2p_5^2)z^5 + (2520p_1p_2^2p_3^2p_4 + 43008p_1p_2p_3^2p_4p_5 + 11760p_2p_3^2p_4^2p_5 + 21000p_1p_2p_3p_4p_5^2 + 40824p_2p_3^2p_4p_5^2\\
&+ 14700p_2p_3p_4^2p_5^2 + 784p_3^2p_4^2p_5^2)z^6 + (27216p_1p_2^2p_3^2p_4p_5 + 42336p_1p_2p_3^2p_4^2p_5 + 126000p_1p_2p_3^2p_4p_5^2\\
&+ 27000p_1p_2p_3p_4^2p_5^2 + 123552p_2p_3^2p_4^2p_5^2)z^7 + (47628p_1p_2^2p_3^2p_4^2p_5 + 90720p_1p_2^2p_3^2p_4p_5^2 + 424710p_1p_2p_3^2p_4^2p_5^2\\
&+ 3969p_2^2p_3^2p_4^2p_5^2 + 43200p_2p_3^3p_4^2p_5^2 + 98784p_2p_3^2p_4^3p_5^2 + 26460p_2p_3^2p_4^2p_5^3)z^8 + (14112p_1p_2^2p_3^3p_4^2p_5 + 434720p_1p_2^2p_3^2p_4^2p_5^2\\
&+ 147000p_1p_2p_3^3p_4^2p_5^2 + 17496p_2^2p_3^3p_4^2p_5^2 + 408240p_1p_2p_3^2p_4^3p_5^2 + 86016p_2p_3^3p_4^3p_5^2 + 117600p_1p_2p_3^2p_4^2p_5^3\\
&+ 82320p_2p_3^2p_4^3p_5^3)z^9 + (1296p_1^2p_2^2p_3^2p_4^2p_5^2 + 291720p_1p_2^2p_3^3p_4^2p_5^2 + 567000p_1p_2^2p_3^2p_4^3p_5^2 + 370440p_1p_2p_3^3p_4^3p_5^2\\
&+ 37800p_2^2p_3^3p_4^3p_5^2 + 190512p_1p_2^2p_3^2p_4^2p_5^3 + 387072p_1p_2p_3^2p_4^3p_5^3 + 90720p_2p_3^3p_4^3p_5^3 + 24696p_2p_3^2p_4^3p_5^4)z^{10}\\
&+ (10584p_1^2p_2^2p_3^3p_4^2p_5^2 + 52920p_1p_2^3p_3^3p_4^2p_5^2 + 960960p_1p_2^2p_3^3p_4^3p_5^2 + 127008p_1p_2^2p_3^3p_4^2p_5^3 + 680400p_1p_2^2p_3^2p_4^3p_5^3\\
&+ 444528p_1p_2p_3^3p_4^3p_5^3 + 45360p_2^2p_3^3p_4^3p_5^3 + 126000p_1p_2p_3^2p_4^3p_5^4 + 48384p_2p_3^3p_4^3p_5^4)z^{11} + (9408p_1^2p_2^3p_3^3p_4^2p_5^2\\
&+ 30618p_1^2p_2^2p_3^3p_4^3p_5^2 + 257250p_1p_2^3p_3^3p_4^3p_5^2 + 252000p_1p_2^2p_3^4p_4^3p_5^2 + 1605604p_1p_2^2p_3^3p_4^3p_5^3 + 252000p_1p_2^2p_3^2p_4^3p_5^4\\
&+ 257250p_1p_2p_3^3p_4^3p_5^4 + 30618p_2^2p_3^3p_4^3p_5^4 + 9408p_2p_3^3p_4^4p_5^4)z^{12} + (48384p_1^2p_2^3p_3^3p_4^3p_5^2 + 126000p_1p_2^3p_3^4p_4^3p_5^2\\
&+ 45360p_1^2p_2^2p_3^3p_4^3p_5^3 + 444528p_1p_2^3p_3^3p_4^3p_5^3 + 680400p_1p_2^2p_3^4p_4^3p_5^3 + 127008p_1p_2^2p_3^3p_4^4p_5^3 + 960960p_1p_2^2p_3^3p_4^3p_5^4\\
&+ 52920p_1p_2p_3^3p_4^4p_5^4 + 10584p_2^2p_3^3p_4^4p_5^4)z^{13} + (24696p_1^2p_2^3p_3^4p_4^3p_5^2 + 90720p_1^2p_2^3p_3^3p_4^3p_5^3 + 387072p_1p_2^3p_3^4p_4^3p_5^3\\
&+ 190512p_1p_2^2p_3^4p_4^4p_5^3 + 37800p_1^2p_2^2p_3^3p_4^3p_5^4 + 370440p_1p_2^3p_3^3p_4^3p_5^4 + 567000p_1p_2^2p_3^4p_4^3p_5^4 + 291720p_1p_2^2p_3^3p_4^4p_5^4\\
&+ 1296p_2^2p_3^4p_4^4p_5^4)z^{14} + (82320p_1^2p_2^3p_3^4p_4^3p_5^3 + 117600p_1p_2^3p_3^4p_4^4p_5^3 + 86016p_1^2p_2^3p_3^3p_4^3p_5^4 + 408240p_1p_2^3p_3^4p_4^3p_5^4\\
&+ 17496p_1^2p_2^2p_3^3p_4^4p_5^4 + 147000p_1p_2^3p_3^3p_4^4p_5^4 + 434720p_1p_2^2p_3^4p_4^4p_5^4 + 14112p_1p_2^2p_3^3p_4^4p_5^5)z^{15} + (26460p_1^2p_2^3p_3^4p_4^4p_5^3\\
&+ 98784p_1^2p_2^3p_3^4p_4^3p_5^4 + 43200p_1^2p_2^3p_3^3p_4^4p_5^4 + 3969p_1^2p_2^2p_3^4p_4^4p_5^4 + 424710p_1p_2^3p_3^4p_4^4p_5^4 + 90720p_1p_2^2p_3^4p_4^5p_5^4\\
&+ 47628p_1p_2^2p_3^4p_4^4p_5^5)z^{16} + (123552p_1^2p_2^3p_3^4p_4^4p_5^4 + 27000p_1p_2^3p_3^5p_4^4p_5^4 + 126000p_1p_2^3p_3^4p_4^5p_5^4 + 42336p_1p_2^3p_3^4p_4^4p_5^5\\
&+ 27216p_1p_2^2p_3^4p_4^5p_5^5)z^{17} + (784p_1^2p_2^4p_3^4p_4^4p_5^4 + 14700p_1^2p_2^3p_3^5p_4^4p_5^4 + 40824p_1^2p_2^3p_3^4p_4^5p_5^4 + 21000p_1p_2^3p_3^5p_4^5p_5^4\\
&+ 11760p_1^2p_2^3p_3^4p_4^4p_5^5 + 43008p_1p_2^3p_3^4p_4^5p_5^5 + 2520p_1p_2^2p_3^4p_4^5p_5^6)z^{18} + (1176p_1^2p_2^4p_3^5p_4^4p_5^4 + 12096p_1^2p_2^3p_3^5p_4^5p_5^4\\
&+ 15120p_1^2p_2^3p_3^4p_4^5p_5^5 + 9072p_1p_2^3p_3^5p_4^5p_5^5 + 5040p_1p_2^3p_3^4p_4^5p_5^6)z^{19} + (1050p_1^2p_2^4p_3^5p_4^5p_5^4 + 5670p_1^2p_2^3p_3^5p_4^5p_5^5\\
&+ 2016p_1^2p_2^3p_3^4p_4^5p_5^6 + 1890p_1p_2^3p_3^5p_4^5p_5^6)z^{20} + (560p_1^2p_2^4p_3^5p_4^5p_5^5 + 1344p_1^2p_2^3p_3^5p_4^5p_5^6 + 120p_1p_2^3p_3^5p_4^6p_5^6)z^{21}\\
&+ (168p_1^2p_2^4p_3^5p_4^5p_5^6 + 108p_1^2p_2^3p_3^5p_4^6p_5^6)z^{22} + 24p_1^2p_2^4p_3^5p_4^6p_5^6z^{23} + p_1^2p_2^4p_3^6p_4^6p_5^6z^{24}\\
H_4 =& 1 + 28p_4z + (168p_3p_4 + 210p_4p_5)z^2 + (336p_2p_3p_4 + 2240p_3p_4p_5 + 700p_4p_5^2)z^3 + (210p_1p_2p_3p_4 + 5670p_2p_3p_4p_5\\
&+ 3920p_3p_4^2p_5 + 9450p_3p_4p_5^2 + 1225p_4^2p_5^2)z^4 + (4032p_1p_2p_3p_4p_5 + 17640p_2p_3p_4^2p_5 + 27216p_2p_3p_4p_5^2 + 49392p_3p_4^2p_5^2)z^5\\
&+ (15876p_1p_2p_3p_4^2p_5 + 11760p_2p_3^2p_4^2p_5 + 21000p_1p_2p_3p_4p_5^2 + 209916p_2p_3p_4^2p_5^2 + 19600p_3^2p_4^2p_5^2 + 74088p_3p_4^3p_5^2\\
&+ 24500p_3p_4^2p_5^3)z^6 + (18816p_1p_2p_3^2p_4^2p_5 + 195120p_1p_2p_3p_4^2p_5^2 + 202176p_2p_3^2p_4^2p_5^2 + 411600p_2p_3p_4^3p_5^2\\
&+ 87808p_3^2p_4^3p_5^2 + 158760p_2p_3p_4^2p_5^3 + 109760p_3p_4^3p_5^3)z^7 + (5292p_1p_2^2p_3^2p_4^2p_5 + 277830p_1p_2p_3^2p_4^2p_5^2 + 35721p_2^2p_3^2p_4^2p_5^2\\
&+ 425250p_1p_2p_3p_4^3p_5^2 + 961632p_2p_3^2p_4^3p_5^2 + 176400p_1p_2p_3p_4^2p_5^3 + 238140p_2p_3^2p_4^2p_5^3 + 771750p_2p_3p_4^3p_5^3 + 164640p_3^2p_4^3p_5^3\\
&+ 51450p_3p_4^3p_5^4)z^8 + (109760p_1p_2^2p_3^2p_4^2p_5^2 + 1292760p_1p_2p_3^2p_4^3p_5^2 + 308700p_2^2p_3^2p_4^3p_5^2 + 537600p_2p_3^3p_4^3p_5^2\\
&+ 470400p_1p_2p_3^2p_4^2p_5^3 + 907200p_1p_2p_3p_4^3p_5^3 + 2731680p_2p_3^2p_4^3p_5^3 + 411600p_2p_3p_4^3p_5^4 + 137200p_3^2p_4^3p_5^4)z^9\\
&+ (7056p_1^2p_2^2p_3^2p_4^2p_5^2 + 666680p_1p_2^2p_3^2p_4^3p_5^2 + 1029000p_1p_2p_3^3p_4^3p_5^2 + 340200p_2^2p_3^3p_4^3p_5^2 + 190512p_1p_2^2p_3^2p_4^2p_5^3\\
&+ 4484844p_1p_2p_3^2p_4^3p_5^3 + 833490p_2^2p_3^2p_4^3p_5^3 + 2268000p_2p_3^3p_4^3p_5^3 + 576240p_2p_3^2p_4^4p_5^3 + 525000p_1p_2p_3p_4^3p_5^4\\
&+ 2163672p_2p_3^2p_4^3p_5^4 + 38416p_3^2p_4^4p_5^4)z^{10} + (81648p_1^2p_2^2p_3^2p_4^3p_5^2 + 1132320p_1p_2^2p_3^3p_4^3p_5^2 + 2621472p_1p_2^2p_3^2p_4^3p_5^3\\
&+ 4939200p_1p_2p_3^3p_4^3p_5^3 + 1632960p_2^2p_3^3p_4^3p_5^3 + 1524096p_1p_2p_3^2p_4^4p_5^3 + 1128960p_2p_3^3p_4^4p_5^3 + 3591000p_1p_2p_3^2p_4^3p_5^4\\
&+ 1000188p_2^2p_3^2p_4^3p_5^4 + 2721600p_2p_3^3p_4^3p_5^4 + 1100736p_2p_3^2p_4^4p_5^4)z^{11} + (166698p_1^2p_2^2p_3^3p_4^3p_5^2 + 257250p_1p_2^3p_3^3p_4^3p_5^2\\
&+ 272160p_1^2p_2^2p_3^2p_4^3p_5^3 + 6419812p_1p_2^2p_3^3p_4^3p_5^3 + 1190700p_1p_2^2p_3^2p_4^4p_5^3 + 3111696p_1p_2p_3^3p_4^4p_5^3 + 882000p_2^2p_3^3p_4^4p_5^3\\
&+ 2666720p_1p_2^2p_3^2p_4^3p_5^4 + 6431250p_1p_2p_3^3p_4^3p_5^4 + 2480058p_2^2p_3^3p_4^3p_5^4 + 2500470p_1p_2p_3^2p_4^4p_5^4 + 540225p_2^2p_3^2p_4^4p_5^4\\
&+ 3358656p_2p_3^3p_4^4p_5^4 + 144060p_2p_3^2p_4^4p_5^5)z^{12} + (65856p_1^2p_2^3p_3^3p_4^3p_5^2 + 987840p_1^2p_2^2p_3^3p_4^3p_5^3 + 1778112p_1p_2^3p_3^3p_4^3p_5^3\\
&+ 5551504p_1p_2^2p_3^3p_4^4p_5^3 + 403200p_1^2p_2^2p_3^2p_4^3p_5^4 + 10190880p_1p_2^2p_3^3p_4^3p_5^4 + 2744000p_1p_2^2p_3^2p_4^4p_5^4 + 9560880p_1p_2p_3^3p_4^4p_5^4\\
&+ 4000752p_2^2p_3^3p_4^4p_5^4 + 1053696p_2p_3^3p_4^5p_5^4 + 470400p_1p_2p_3^2p_4^4p_5^5 + 635040p_2p_3^3p_4^4p_5^5)z^{13} + (493920p_1^2p_2^3p_3^3p_4^3p_5^3\\
&+ 714420p_1^2p_2^2p_3^3p_4^4p_5^3 + 2160900p_1p_2^3p_3^3p_4^4p_5^3 + 529200p_1p_2^2p_3^4p_4^4p_5^3 + 1852200p_1^2p_2^2p_3^3p_4^3p_5^4 + 3333960p_1p_2^3p_3^3p_4^3p_5^4\\
&+ 291600p_1^2p_2^2p_3^2p_4^4p_5^4 + 21364200p_1p_2^2p_3^3p_4^4p_5^4 + 291600p_2^2p_3^4p_4^4p_5^4 + 3333960p_1p_2p_3^3p_4^5p_5^4 + 1852200p_2^2p_3^3p_4^5p_5^4\\
&+ 529200p_1p_2^2p_3^2p_4^4p_5^5 + 2160900p_1p_2p_3^3p_4^4p_5^5 + 714420p_2^2p_3^3p_4^4p_5^5 + 493920p_2p_3^3p_4^5p_5^5)z^{14} + (635040p_1^2p_2^3p_3^3p_4^4p_5^3\\
&+ 470400p_1p_2^3p_3^4p_4^4p_5^3 + 1053696p_1^2p_2^3p_3^3p_4^3p_5^4 + 4000752p_1^2p_2^2p_3^3p_4^4p_5^4 + 9560880p_1p_2^3p_3^3p_4^4p_5^4 + 2744000p_1p_2^2p_3^4p_4^4p_5^4\\
&+ 10190880p_1p_2^2p_3^3p_4^5p_5^4 + 403200p_2^2p_3^4p_4^5p_5^4 + 5551504p_1p_2^2p_3^3p_4^4p_5^5 + 1778112p_1p_2p_3^3p_4^5p_5^5 + 987840p_2^2p_3^3p_4^5p_5^5\\
&+ 65856p_2p_3^3p_4^5p_5^6)z^{15} + (144060p_1^2p_2^3p_3^4p_4^4p_5^3 + 3358656p_1^2p_2^3p_3^3p_4^4p_5^4 + 540225p_1^2p_2^2p_3^4p_4^4p_5^4 + 2500470p_1p_2^3p_3^4p_4^4p_5^4\\
&+ 2480058p_1^2p_2^2p_3^3p_4^5p_5^4 + 6431250p_1p_2^3p_3^3p_4^5p_5^4 + 2666720p_1p_2^2p_3^4p_4^5p_5^4 + 882000p_1^2p_2^2p_3^3p_4^4p_5^5 + 3111696p_1p_2^3p_3^3p_4^4p_5^5\\
&+ 1190700p_1p_2^2p_3^4p_4^4p_5^5 + 6419812p_1p_2^2p_3^3p_4^5p_5^5 + 272160p_2^2p_3^4p_4^5p_5^5 + 257250p_1p_2p_3^3p_4^5p_5^6 + 166698p_2^2p_3^3p_4^5p_5^6)z^{16}\\
&+ (1100736p_1^2p_2^3p_3^4p_4^4p_5^4 + 2721600p_1^2p_2^3p_3^3p_4^5p_5^4 + 1000188p_1^2p_2^2p_3^4p_4^5p_5^4 + 3591000p_1p_2^3p_3^4p_4^5p_5^4 + 1128960p_1^2p_2^3p_3^3p_4^4p_5^5\\
&+ 1524096p_1p_2^3p_3^4p_4^4p_5^5 + 1632960p_1^2p_2^2p_3^3p_4^5p_5^5 + 4939200p_1p_2^3p_3^3p_4^5p_5^5 + 2621472p_1p_2^2p_3^4p_4^5p_5^5 + 1132320p_1p_2^2p_3^3p_4^5p_5^6\\
&+ 81648p_2^2p_3^4p_4^5p_5^6)z^{17} + (38416p_1^2p_2^4p_3^4p_4^4p_5^4 + 2163672p_1^2p_2^3p_3^4p_4^5p_5^4 + 525000p_1p_2^3p_3^5p_4^5p_5^4 + 576240p_1^2p_2^3p_3^4p_4^4p_5^5\\
&+ 2268000p_1^2p_2^3p_3^3p_4^5p_5^5 + 833490p_1^2p_2^2p_3^4p_4^5p_5^5 + 4484844p_1p_2^3p_3^4p_4^5p_5^5 + 190512p_1p_2^2p_3^4p_4^6p_5^5 + 340200p_1^2p_2^2p_3^3p_4^5p_5^6\\
&+ 1029000p_1p_2^3p_3^3p_4^5p_5^6 + 666680p_1p_2^2p_3^4p_4^5p_5^6 + 7056p_2^2p_3^4p_4^6p_5^6)z^{18} + (137200p_1^2p_2^4p_3^4p_4^5p_5^4 + 411600p_1^2p_2^3p_3^5p_4^5p_5^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 2731680p_1^2p_2^3p_3^4p_4^5p_5^5 + 907200p_1p_2^3p_3^5p_4^5p_5^5 + 470400p_1p_2^3p_3^4p_4^6p_5^5 + 537600p_1^2p_2^3p_3^3p_4^5p_5^6 + 308700p_1^2p_2^2p_3^4p_4^5p_5^6 \\
&+ 1292760p_1p_2^3p_3^4p_4^5p_5^6 + 109760p_1p_2^2p_3^4p_4^6p_5^6)z^{19} + (51450p_1^2p_2^4p_3^5p_4^5p_5^4 + 164640p_1^2p_2^4p_3^4p_4^5p_5^5 + 771750p_1^2p_2^3p_3^5p_4^5p_5^5 \\
&+ 238140p_1^2p_2^3p_3^4p_4^6p_5^5 + 176400p_1p_2^3p_3^5p_4^6p_5^5 + 961632p_1^2p_2^3p_3^4p_4^5p_5^6 + 425250p_1p_2^3p_3^5p_4^5p_5^6 + 35721p_1^2p_2^2p_3^4p_4^6p_5^6 \\
&+ 277830p_1p_2^3p_3^4p_4^6p_5^6 + 5292p_1p_2^2p_3^4p_4^6p_5^7)z^{20} + (109760p_1^2p_2^4p_3^5p_4^5p_5^5 + 158760p_1^2p_2^3p_3^5p_4^6p_5^5 + 87808p_1^2p_2^4p_3^4p_4^5p_5^6 \\
&+ 411600p_1^2p_2^3p_3^5p_4^5p_5^6 + 202176p_1^2p_2^3p_3^4p_4^6p_5^6 + 195120p_1p_2^3p_3^5p_4^6p_5^6 + 18816p_1p_2^3p_3^4p_4^6p_5^7)z^{21} + (24500p_1^2p_2^4p_3^5p_4^6p_5^5 \\
&+ 74088p_1^2p_2^4p_3^5p_4^5p_5^6 + 19600p_1^2p_2^4p_3^4p_4^6p_5^6 + 209916p_1^2p_2^3p_3^5p_4^6p_5^6 + 21000p_1p_2^3p_3^5p_4^7p_5^6 + 11760p_1^2p_2^3p_3^4p_4^6p_5^7 \\
&+ 15876p_1p_2^3p_3^5p_4^6p_5^7)z^{22} + (49392p_1^2p_2^4p_3^5p_4^6p_5^6 + 27216p_1^2p_2^3p_3^5p_4^7p_5^6 + 17640p_1^2p_2^3p_3^5p_4^6p_5^7 + 4032p_1p_2^3p_3^5p_4^7p_5^7)z^{23} \\
&+ (1225p_1^2p_2^4p_3^6p_4^6p_5^6 + 9450p_1^2p_2^4p_3^5p_4^7p_5^6 + 3920p_1^2p_2^4p_3^5p_4^6p_5^7 + 5670p_1^2p_2^3p_3^5p_4^7p_5^7 + 210p_1p_2^3p_3^5p_4^7p_5^8)z^{24} + (700p_1^2p_2^4p_3^6p_4^7p_5^6 \\
&+ 2240p_1^2p_2^4p_3^5p_4^7p_5^7 + 336p_1^2p_2^3p_3^5p_4^7p_5^8)z^{25} + (210p_1^2p_2^4p_3^6p_4^7p_5^7 + 168p_1^2p_2^4p_3^5p_4^7p_5^8)z^{26} + 28p_1^2p_2^4p_3^6p_4^7p_5^8z^{27} + p_1^2p_2^4p_3^6p_4^8p_5^8z^{28}
\end{aligned}
$$

$$
\begin{aligned}
H_5 =&1 + 15p_5z + 105p_4p_5z^2 + (280p_3p_4p_5 + 175p_4p_5^2)z^3 + (315p_2p_3p_4p_5 + 1050p_3p_4p_5^2)z^4 + (126p_1p_2p_3p_4p_5 \\
&+ 1701p_2p_3p_4p_5^2 + 1176p_3p_4^2p_5^2)z^5 + (840p_1p_2p_3p_4p_5^2 + 3675p_2p_3p_4^2p_5^2 + 490p_3p_4^2p_5^3)z^6 + (2430p_1p_2p_3p_4^2p_5^2 \\
&+ 1800p_2p_3^2p_4^2p_5^2 + 2205p_2p_3p_4^2p_5^3)z^7 + (2205p_1p_2p_3^2p_4^2p_5^2 + 1800p_1p_2p_3p_4^2p_5^3 + 2430p_2p_3^2p_4^2p_5^3)z^8 + (490p_1p_2^2p_3^2p_4^2p_5^2 \\
&+ 3675p_1p_2p_3^2p_4^2p_5^3 + 840p_2p_3^2p_4^3p_5^3)z^9 + (1176p_1p_2^2p_3^2p_4^2p_5^3 + 1701p_1p_2p_3^2p_4^3p_5^3 + 126p_2p_3^2p_4^3p_5^4)z^{10} + (1050p_1p_2^2p_3^2p_4^3p_5^3 \\
&+ 315p_1p_2p_3^2p_4^3p_5^4)z^{11} + (175p_1p_2^2p_3^3p_4^3p_5^3 + 280p_1p_2^2p_3^2p_4^3p_5^4)z^{12} + 105p_1p_2^2p_3^3p_4^3p_5^4z^{13} + 15p_1p_2^2p_3^3p_4^4p_5^4z^{14} \\
&+ p_1p_2^2p_3^3p_4^4p_5^5z^{15}
\end{aligned}
$$

***C_5*-case.** For the Lie algebra $C_5 \cong sp(5)$ we obtain

$$
\begin{aligned}
H_1 =&1 + 9p_1z + 36p_1p_2z^2 + 84p_1p_2p_3z^3 + 126p_1p_2p_3p_4z^4 + 126p_1p_2p_3p_4p_5z^5 + 84p_1p_2p_3p_4^2p_5z^6 + 36p_1p_2p_3^2p_4^2p_5z^7 \\
&+ 9p_1p_2^2p_3^2p_4^2p_5z^8 + p_1^2p_2^2p_3^2p_4^2p_5z^9 \\
H_2 =&1 + 16p_2z + (36p_1p_2 + 84p_2p_3)z^2 + (336p_1p_2p_3 + 224p_2p_3p_4)z^3 + (336p_1p_2^2p_3 + 1134p_1p_2p_3p_4 + 350p_2p_3p_4p_5)z^4 \\
&+ (2016p_1p_2^2p_3p_4 + 2016p_1p_2p_3p_4p_5 + 336p_2p_3p_4^2p_5)z^5 + (1176p_1p_2^2p_3^2p_4 + 4536p_1p_2^2p_3p_4p_5 + 2100p_1p_2p_3p_4^2p_5 \\
&+ 196p_2p_3^2p_4^2p_5)z^6 + (4704p_1p_2^2p_3^2p_4p_5 + 5376p_1p_2^2p_3p_4^2p_5 + 1296p_1p_2p_3^2p_4^2p_5 + 64p_2^2p_3^2p_4^2p_5)z^7 \\
&+ 12870p_1p_2^2p_3^2p_4^2p_5z^8 + (64p_1^2p_2^2p_3^2p_4^2p_5 + 1296p_1p_2^3p_3^2p_4^2p_5 + 5376p_1p_2^2p_3^3p_4^2p_5 + 4704p_1p_2^2p_3^2p_4^3p_5)z^9 \\
&+ (196p_1^2p_2^3p_3^2p_4^2p_5 + 2100p_1p_2^3p_3^3p_4^2p_5 + 4536p_1p_2^2p_3^3p_4^3p_5 + 1176p_1p_2^2p_3^2p_4^3p_5^2)z^{10} + (336p_1^2p_2^3p_3^3p_4^2p_5 \\
&+ 2016p_1p_2^3p_3^3p_4^3p_5 + 2016p_1p_2^2p_3^3p_4^3p_5^2)z^{11} + (350p_1^2p_2^3p_3^3p_4^3p_5 + 1134p_1p_2^3p_3^3p_4^3p_5^2 + 336p_1p_2^2p_3^3p_4^4p_5^2)z^{12} \\
&+ (224p_1^2p_2^3p_3^3p_4^3p_5^2 + 336p_1p_2^3p_3^3p_4^4p_5^2)z^{13} + (84p_1^2p_2^3p_3^3p_4^4p_5^2 + 36p_1p_2^3p_3^4p_4^4p_5^2)z^{14} + 16p_1^2p_2^3p_3^4p_4^4p_5^2z^{15} + p_1^2p_2^4p_3^4p_4^4p_5^2z^{16} \\
H_3 =&1 + 21p_3z + (84p_2p_3 + 126p_3p_4)z^2 + (84p_1p_2p_3 + 896p_2p_3p_4 + 350p_3p_4p_5)z^3 + (1134p_1p_2p_3p_4 + 1176p_2p_3^2p_4 \\
&+ 3150p_2p_3p_4p_5 + 525p_3p_4^2p_5)z^4 + (2646p_1p_2p_3^2p_4 + 4536p_1p_2p_3p_4p_5 + 7350p_2p_3^2p_4p_5 + 5376p_2p_3p_4^2p_5 + 441p_3^2p_4^2p_5)z^5 \\
&+ (1176p_1p_2^2p_3^2p_4 + 18816p_1p_2p_3^2p_4p_5 + 8400p_1p_2p_3p_4^2p_5 + 25872p_2p_3^2p_4^2p_5)z^6 + (10584p_1p_2^2p_3^2p_4p_5 \\
&+ 68112p_1p_2p_3^2p_4^2p_5 + 2304p_2^2p_3^2p_4^2p_5 + 16464p_2p_3^3p_4^2p_5 + 18816p_2p_3^2p_4^3p_5)z^7 + (48510p_1p_2^2p_3^2p_4^2p_5 + 48384p_1p_2p_3^3p_4^2p_5 \\
&+ 8400p_2^2p_3^3p_4^2p_5 + 66150p_1p_2p_3^2p_4^3p_5 + 24696p_2p_3^3p_4^3p_5 + 7350p_2p_3^2p_4^3p_5^2)z^8 + (784p_1^2p_2^2p_3^2p_4^2p_5 + 65142p_1p_2^2p_3^3p_4^2p_5 \\
&+ 75264p_1p_2^2p_3^2p_4^3p_5 + 91854p_1p_2p_3^3p_4^3p_5 + 14336p_2^2p_3^3p_4^3p_5 + 29400p_1p_2p_3^2p_4^3p_5^2 + 17150p_2p_3^3p_4^3p_5^2)z^9 \\
&+ (5376p_1^2p_2^2p_3^3p_4^2p_5 + 18900p_1p_2^3p_3^3p_4^2p_5 + 196812p_1p_2^2p_3^3p_4^3p_5 + 42336p_1p_2^2p_3^2p_4^3p_5^2 + 72576p_1p_2p_3^3p_4^3p_5^2 \\
&+ 12600p_2^2p_3^3p_4^3p_5^2 + 4116p_2p_3^3p_4^4p_5^2)z^{10} + (4116p_1^2p_2^3p_3^3p_4^2p_5 + 12600p_1^2p_2^2p_3^3p_4^3p_5 + 72576p_1p_2^3p_3^3p_4^3p_5 \\
&+ 42336p_1p_2^2p_3^4p_4^3p_5 + 196812p_1p_2^2p_3^3p_4^3p_5^2 + 18900p_1p_2p_3^3p_4^4p_5^2 + 5376p_2^2p_3^3p_4^4p_5^2)z^{11} + (17150p_1^2p_2^3p_3^3p_4^3p_5 \\
&+ 29400p_1p_2^3p_3^4p_4^3p_5 + 14336p_1^2p_2^2p_3^3p_4^3p_5^2 + 91854p_1p_2^3p_3^3p_4^3p_5^2 + 75264p_1p_2^2p_3^4p_4^3p_5^2 + 65142p_1p_2^2p_3^3p_4^4p_5^2 \\
&+ 784p_2^2p_3^4p_4^4p_5^2)z^{12} + (7350p_1^2p_2^3p_3^4p_4^3p_5 + 24696p_1^2p_2^3p_3^3p_4^3p_5^2 + 66150p_1p_2^3p_3^4p_4^3p_5^2 + 8400p_1^2p_2^2p_3^3p_4^4p_5^2 \\
&+ 48384p_1p_2^3p_3^3p_4^4p_5^2 + 48510p_1p_2^2p_3^4p_4^4p_5^2)z^{13} + (18816p_1^2p_2^3p_3^4p_4^3p_5^2 + 16464p_1^2p_2^3p_3^3p_4^4p_5^2 + 2304p_1^2p_2^2p_3^4p_4^4p_5^2 \\
&+ 68112p_1p_2^3p_3^4p_4^4p_5^2 + 10584p_1p_2^2p_3^4p_4^5p_5^2)z^{14} + (25872p_1^2p_2^3p_3^4p_4^4p_5^2 + 8400p_1p_2^3p_3^5p_4^4p_5^2 + 18816p_1p_2^3p_3^4p_4^5p_5^2 \\
&+ 1176p_1p_2^2p_3^4p_4^5p_5^3)z^{15} + (441p_1^2p_2^4p_3^4p_4^4p_5^2 + 5376p_1^2p_2^3p_3^5p_4^4p_5^2 + 7350p_1^2p_2^3p_3^4p_4^5p_5^2 + 4536p_1p_2^3p_3^5p_4^5p_5^2 \\
&+ 2646p_1p_2^3p_3^4p_4^5p_5^3)z^{16} + (525p_1^2p_2^4p_3^5p_4^4p_5^2 + 3150p_1^2p_2^3p_3^5p_4^5p_5^2 + 1176p_1^2p_2^3p_3^4p_4^5p_5^3 + 1134p_1p_2^3p_3^5p_4^5p_5^3)z^{17} \\
&+ (350p_1^2p_2^4p_3^5p_4^5p_5^2 + 896p_1^2p_2^3p_3^5p_4^5p_5^3 + 84p_1p_2^3p_3^5p_4^6p_5^3)z^{18} + (126p_1^2p_2^4p_3^5p_4^5p_5^3 + 84p_1^2p_2^3p_3^5p_4^6p_5^3)z^{19} + 21p_1^2p_2^4p_3^5p_4^6p_5^3z^{20} \\
&+ p_1^2p_2^4p_3^6p_4^6p_5^3z^{21} \\
H_4 =&1 + 24p_4z + (126p_3p_4 + 150p_4p_5)z^2 + (224p_2p_3p_4 + 1400p_3p_4p_5 + 400p_4^2p_5)z^3 + (126p_1p_2p_3p_4 + 3150p_2p_3p_4p_5 \\
&+ 7350p_3p_4^2p_5)z^4 + (2016p_1p_2p_3p_4p_5 + 20832p_2p_3p_4^2p_5 + 7056p_3^2p_4^2p_5 + 12600p_3p_4^3p_5)z^5 + (15288p_1p_2p_3p_4^2p_5 \\
&+ 29400p_2p_3^2p_4^2p_5 + 57344p_2p_3p_4^3p_5 + 23814p_3^2p_4^3p_5 + 8750p_3p_4^3p_5^2)z^6 + (22752p_1p_2p_3^2p_4^2p_5 + 14400p_2^2p_3^2p_4^2p_5 \\
&+ 50400p_1p_2p_3p_4^3p_5 + 178752p_2p_3^2p_4^3p_5 + 50400p_2p_3p_4^3p_5^2 + 29400p_3^2p_4^3p_5^2)z^7 + (16758p_1p_2^2p_3^2p_4^2p_5 \\
&+ 180900p_1p_2p_3^2p_4^3p_5 + 98304p_2^2p_3^2p_4^3p_5 + 98784p_2p_3^3p_4^3p_5 + 50400p_1p_2p_3p_4^3p_5^2 + 279300p_2p_3^2p_4^3p_5^2 + 11025p_3^2p_4^4p_5^2)z^8 \\
&+ (3136p_1^2p_2^2p_3^2p_4^2p_5 + 143472p_1p_2^2p_3^2p_4^3p_5 + 163296p_1p_2p_3^3p_4^3p_5 + 89600p_2^2p_3^3p_4^3p_5 + 321600p_1p_2p_3^2p_4^3p_5^2 \\
&+ 194400p_2^2p_3^2p_4^3p_5^2 + 274400p_2p_3^3p_4^3p_5^2 + 117600p_2p_3^2p_4^4p_5^2)z^9 + (29400p_1^2p_2^2p_3^2p_4^3p_5 + 233100p_1p_2^2p_3^3p_4^3p_5 \\
&+ 322812p_1p_2^2p_3^2p_4^3p_5^2 + 516096p_1p_2p_3^3p_4^3p_5^2 + 315000p_2^2p_3^3p_4^3p_5^2 + 142200p_1p_2p_3^2p_4^4p_5^2 + 147456p_2^2p_3^2p_4^4p_5^2 \\
&+ 255192p_2p_3^3p_4^4p_5^2)z^{10} + (50400p_1^2p_2^2p_3^3p_4^3p_5 + 50400p_1p_2^3p_3^3p_4^3p_5 + 75264p_1^2p_2^2p_3^2p_4^3p_5^2 + 932400p_1p_2^2p_3^3p_4^3p_5^2 \\
&+ 268128p_1p_2^2p_3^2p_4^4p_5^2 + 550368p_1p_2p_3^3p_4^4p_5^2 + 470400p_2^2p_3^3p_4^4p_5^2 + 98784p_2p_3^3p_4^5p_5^2)z^{11} + (17150p_1^2p_2^3p_3^3p_4^3p_5 \\
&+ 229376p_1^2p_2^2p_3^3p_4^3p_5^2 + 255150p_1p_2^3p_3^3p_4^3p_5^2 + 78400p_1^2p_2^2p_3^2p_4^4p_5^2 + 1544004p_1p_2^2p_3^3p_4^4p_5^2 + 78400p_2^2p_3^4p_4^4p_5^2 \\
&+ 255150p_1p_2p_3^3p_4^5p_5^2 + 229376p_2^2p_3^3p_4^5p_5^2 + 17150p_2p_3^3p_4^5p_5^3)z^{12} + (98784p_1^2p_2^3p_3^3p_4^3p_5^2 + 470400p_1^2p_2^2p_3^3p_4^4p_5^2
\end{aligned}
$$

$$
\begin{aligned}
&+550368p_1p_2^3p_3^3p_4^4p_5^2+268128p_1p_2^2p_3^4p_4^4p_5^2+932400p_1p_2^2p_3^3p_4^5p_5^2+75264p_2^2p_3^4p_4^5p_5^2+50400p_1p_2p_3^3p_4^5p_5^3\\
&+50400p_2^2p_3^3p_4^5p_5^3)z^{13}+(255192p_1^2p_2^3p_3^3p_4^4p_5^2+147456p_1^2p_2^2p_3^4p_4^4p_5^2+142200p_1p_2^3p_3^4p_4^4p_5^2+315000p_1^2p_2^2p_3^3p_4^5p_5^2\\
&+516096p_1p_2^3p_3^3p_4^5p_5^2+322812p_1p_2^2p_3^4p_4^5p_5^2+233100p_1p_2^2p_3^3p_4^5p_5^3+29400p_2^2p_3^4p_4^5p_5^3)z^{14}+(117600p_1^2p_2^3p_3^4p_4^4p_5^2\\
&+274400p_1^2p_2^3p_3^3p_4^5p_5^2+194400p_1^2p_2^2p_3^4p_4^5p_5^2+321600p_1p_2^3p_3^4p_4^5p_5^2+89600p_1^2p_2^2p_3^3p_4^5p_5^3+163296p_1p_2^3p_3^3p_4^5p_5^3\\
&+143472p_1p_2^2p_3^4p_4^5p_5^3+3136p_2^2p_3^4p_4^6p_5^3)z^{15}+(11025p_1^2p_2^4p_3^4p_4^4p_5^2+279300p_1^2p_2^3p_3^4p_4^5p_5^2+50400p_1p_2^3p_3^5p_4^5p_5^2\\
&+98784p_1^2p_2^3p_3^3p_4^5p_5^3+98304p_1^2p_2^2p_3^4p_4^5p_5^3+180900p_1p_2^3p_3^4p_4^5p_5^3+16758p_1p_2^2p_3^4p_4^6p_5^3)z^{16}+(29400p_1^2p_2^4p_3^4p_4^5p_5^2\\
&+50400p_1^2p_2^3p_3^5p_4^5p_5^2+178752p_1^2p_2^3p_3^4p_4^5p_5^3+50400p_1p_2^3p_3^5p_4^5p_5^3+14400p_1^2p_2^2p_3^4p_4^6p_5^3+22752p_1p_2^3p_3^4p_4^6p_5^3)z^{17}\\
&+(8750p_1^2p_2^4p_3^5p_4^5p_5^2+23814p_1^2p_2^4p_3^4p_4^5p_5^3+57344p_1^2p_2^3p_3^5p_4^5p_5^3+29400p_1^2p_2^3p_3^4p_4^6p_5^3+15288p_1p_2^3p_3^5p_4^6p_5^3)z^{18}\\
&+(12600p_1^2p_2^4p_3^5p_4^5p_5^3+7056p_1^2p_2^4p_3^4p_4^6p_5^3+20832p_1^2p_2^3p_3^5p_4^6p_5^3+2016p_1p_2^3p_3^5p_4^7p_5^3)z^{19}+(7350p_1^2p_2^4p_3^5p_4^6p_5^3\\
&+3150p_1^2p_2^3p_3^5p_4^7p_5^3+126p_1p_2^3p_3^5p_4^7p_5^4)z^{20}+(400p_1^2p_2^4p_3^6p_4^6p_5^3+1400p_1^2p_2^4p_3^5p_4^7p_5^3+224p_1^2p_2^3p_3^5p_4^7p_5^4)z^{21}\\
&+(150p_1^2p_2^4p_3^6p_4^7p_5^3+126p_1^2p_2^4p_3^5p_4^7p_5^4)z^{22}+24p_1^2p_2^4p_3^6p_4^7p_5^4z^{23}+p_1^2p_2^4p_3^6p_4^8p_5^4z^{24}
\end{aligned}
$$

$$
\begin{aligned}
H_5 =&1+25p_5z+300p_4p_5z^2+(700p_3p_4p_5+1600p_4^2p_5)z^3+(700p_2p_3p_4p_5+9450p_3p_4^2p_5+2500p_4^2p_5^2)z^4+(252p_1p_2p_3p_4p_5\\
&+10752p_2p_3p_4^2p_5+15876p_3^2p_4^2p_5+26250p_3p_4^2p_5^2)z^5+(4200p_1p_2p_3p_4^2p_5+39200p_2p_3^2p_4^2p_5+37800p_2p_3p_4^2p_5^2\\
&+78400p_3^2p_4^2p_5^2+17500p_3p_4^3p_5^2)z^6+(16200p_1p_2p_3^2p_4^2p_5+25600p_2^2p_3^2p_4^2p_5+16800p_1p_2p_3p_4^2p_5^2+245000p_2p_3^2p_4^2p_5^2\\
&+44800p_2p_3p_4^3p_5^2+132300p_3^2p_4^3p_5^2)z^7+(22050p_1p_2^2p_3^2p_4^2p_5+115200p_1p_2p_3^2p_4^2p_5^2+202500p_2^2p_3^2p_4^2p_5^2+25200p_1p_2p_3p_4^3p_5^2\\
&+617400p_2p_3^2p_4^3p_5^2+99225p_3^2p_4^4p_5^2)z^8+(4900p_1^2p_2^2p_3^2p_4^2p_5+198450p_1p_2^2p_3^2p_4^2p_5^2+353400p_1p_2p_3^2p_4^3p_5^2\\
&+691200p_2^2p_3^2p_4^3p_5^2+137200p_2p_3^3p_4^3p_5^2+627200p_2p_3^2p_4^4p_5^2+30625p_3^2p_4^4p_5^3)z^9+(50176p_1^2p_2^2p_3^2p_4^2p_5^2+798504p_1p_2^2p_3^2p_4^3p_5^2\\
&+145152p_1p_2p_3^3p_4^3p_5^2+280000p_2^2p_3^3p_4^3p_5^2+405000p_1p_2p_3^2p_4^4p_5^2+1048576p_2^2p_3^2p_4^4p_5^2+296352p_2p_3^3p_4^4p_5^2\\
&+245000p_2p_3^2p_4^4p_5^3)z^{10}+(235200p_1^2p_2^2p_3^2p_4^3p_5^2+491400p_1p_2^2p_3^3p_4^3p_5^2+1411200p_1p_2^2p_3^2p_4^4p_5^2+340200p_1p_2p_3^3p_4^4p_5^2\\
&+1075200p_2^2p_3^3p_4^4p_5^2+180000p_1p_2p_3^2p_4^4p_5^3+518400p_2^2p_3^2p_4^4p_5^3+205800p_2p_3^3p_4^4p_5^3)z^{11}+(179200p_1^2p_2^2p_3^3p_4^3p_5^2\\
&+56700p_1p_2^3p_3^3p_4^3p_5^2+490000p_1^2p_2^2p_3^2p_4^4p_5^2+2118900p_1p_2^2p_3^3p_4^4p_5^2+313600p_2^2p_3^4p_4^4p_5^2+793800p_1p_2^2p_3^2p_4^4p_5^3\\
&+268800p_1p_2p_3^3p_4^4p_5^3+945000p_2^2p_3^3p_4^4p_5^3+34300p_2p_3^3p_4^5p_5^3)z^{12}+(34300p_1^2p_2^3p_3^3p_4^3p_5^2+945000p_1^2p_2^2p_3^3p_4^4p_5^2\\
&+268800p_1p_2^3p_3^3p_4^4p_5^2+793800p_1p_2^2p_3^4p_4^4p_5^2+313600p_1^2p_2^2p_3^2p_4^4p_5^3+2118900p_1p_2^2p_3^3p_4^4p_5^3+490000p_2^2p_3^4p_4^4p_5^3\\
&+56700p_1p_2p_3^3p_4^5p_5^3+179200p_2^2p_3^3p_4^5p_5^3)z^{13}+(205800p_1^2p_2^3p_3^3p_4^4p_5^2+518400p_1^2p_2^2p_3^4p_4^4p_5^2+180000p_1p_2^3p_3^4p_4^4p_5^2\\
&+1075200p_1^2p_2^2p_3^3p_4^4p_5^3+340200p_1p_2^3p_3^3p_4^4p_5^3+1411200p_1p_2^2p_3^4p_4^4p_5^3+491400p_1p_2^2p_3^3p_4^5p_5^3+235200p_2^2p_3^4p_4^5p_5^3)z^{14}\\
&+(245000p_1^2p_2^3p_3^4p_4^4p_5^2+296352p_1^2p_2^3p_3^3p_4^4p_5^3+1048576p_1^2p_2^2p_3^4p_4^4p_5^3+405000p_1p_2^3p_3^4p_4^4p_5^3+280000p_1^2p_2^2p_3^3p_4^5p_5^3\\
&+145152p_1p_2^3p_3^3p_4^5p_5^3+798504p_1p_2^2p_3^4p_4^5p_5^3+50176p_2^2p_3^4p_4^6p_5^3)z^{15}+(30625p_1^2p_2^4p_3^4p_4^4p_5^2+627200p_1^2p_2^3p_3^4p_4^4p_5^3\\
&+137200p_1^2p_2^3p_3^3p_4^5p_5^3+691200p_1^2p_2^2p_3^4p_4^5p_5^3+353400p_1p_2^3p_3^4p_4^5p_5^3+198450p_1p_2^2p_3^4p_4^6p_5^3+4900p_2^2p_3^4p_4^6p_5^4)z^{16}\\
&+(99225p_1^2p_2^4p_3^4p_4^4p_5^3+617400p_1^2p_2^3p_3^4p_4^5p_5^3+25200p_1p_2^3p_3^5p_4^5p_5^3+202500p_1^2p_2^2p_3^4p_4^6p_5^3+115200p_1p_2^3p_3^4p_4^6p_5^3\\
&+22050p_1p_2^2p_3^4p_4^6p_5^4)z^{17}+(132300p_1^2p_2^4p_3^4p_4^5p_5^3+44800p_1^2p_2^3p_3^5p_4^5p_5^3+245000p_1^2p_2^3p_3^4p_4^6p_5^3+16800p_1p_2^3p_3^5p_4^6p_5^3\\
&+25600p_1^2p_2^2p_3^4p_4^6p_5^4+16200p_1p_2^3p_3^4p_4^6p_5^4)z^{18}+(17500p_1^2p_2^4p_3^5p_4^5p_5^3+78400p_1^2p_2^4p_3^4p_4^6p_5^3+37800p_1^2p_2^3p_3^5p_4^6p_5^3\\
&+39200p_1^2p_2^3p_3^4p_4^6p_5^4+4200p_1p_2^3p_3^5p_4^6p_5^4)z^{19}+(26250p_1^2p_2^4p_3^5p_4^6p_5^3+15876p_1^2p_2^4p_3^4p_4^6p_5^4+10752p_1^2p_2^3p_3^5p_4^6p_5^4\\
&+252p_1p_2^3p_3^5p_4^7p_5^4)z^{20}+(2500p_1^2p_2^4p_3^6p_4^6p_5^3+9450p_1^2p_2^4p_3^5p_4^6p_5^4+700p_1^2p_2^3p_3^5p_4^7p_5^4)z^{21}+(1600p_1^2p_2^4p_3^6p_4^6p_5^4\\
&+700p_1^2p_2^4p_3^5p_4^7p_5^4)z^{22}+300p_1^2p_2^4p_3^6p_4^7p_5^4z^{23}+25p_1^2p_2^4p_3^6p_4^8p_5^4z^{24}+p_1^2p_2^4p_3^6p_4^8p_5^5z^{25}
\end{aligned}
$$

***D_5*-case.** In the case of the Lie algebra $D_5 \cong so(10)$ we get the following polynomials:

$$
\begin{aligned}
H_1 =&1+8p_1z+28p_1p_2z^2+56p_1p_2p_3z^3+(35p_1p_2p_3p_4+35p_1p_2p_3p_5)z^4+56p_1p_2p_3p_4p_5z^5+28p_1p_2p_3^2p_4p_5z^6\\
&+8p_1p_2^2p_3^2p_4p_5z^7+p_1^2p_2^2p_3^2p_4p_5z^8\\
H_2 =&1+14p_2z+(28p_1p_2+63p_2p_3)z^2+(224p_1p_2p_3+70p_2p_3p_4+70p_2p_3p_5)z^3+(196p_1p_2^2p_3+315p_1p_2p_3p_4\\
&+315p_1p_2p_3p_5+175p_2p_3p_4p_5)z^4+(490p_1p_2^2p_3p_4+490p_1p_2^2p_3p_5+896p_1p_2p_3p_4p_5+126p_2p_3^2p_4p_5)z^5\\
&+(245p_1p_2^2p_3^2p_4+245p_1p_2^2p_3^2p_5+1764p_1p_2^2p_3p_4p_5+700p_1p_2p_3^2p_4p_5+49p_2^2p_3^2p_4p_5)z^6+3432p_1p_2^2p_3^2p_4p_5z^7\\
&+(49p_1^2p_2^2p_3^2p_4p_5+700p_1p_2^3p_3^2p_4p_5+1764p_1p_2^2p_3^3p_4p_5+245p_1p_2^2p_3^2p_4^2p_5+245p_1p_2^2p_3^2p_4p_5^2)z^8+(126p_1^2p_2^3p_3^2p_4p_5\\
&+896p_1p_2^3p_3^3p_4p_5+490p_1p_2^2p_3^3p_4^2p_5+490p_1p_2^2p_3^3p_4p_5^2)z^9+(175p_1^2p_2^3p_3^3p_4p_5+315p_1p_2^3p_3^3p_4^2p_5+315p_1p_2^3p_3^3p_4p_5^2\\
&+196p_1p_2^2p_3^3p_4^2p_5^2)z^{10}+(70p_1^2p_2^3p_3^3p_4^2p_5+70p_1^2p_2^3p_3^3p_4p_5^2+224p_1p_2^3p_3^3p_4^2p_5^2)z^{11}+(63p_1^2p_2^3p_3^3p_4^2p_5^2+28p_1p_2^3p_3^4p_4^2p_5^2)z^{12}\\
&+14p_1^2p_2^3p_3^4p_4^2p_5^2z^{13}+p_1^2p_2^4p_3^4p_4^2p_5^2z^{14}\\
H_3 =&1+18p_3z+(63p_2p_3+45p_3p_4+45p_3p_5)z^2+(56p_1p_2p_3+280p_2p_3p_4+280p_2p_3p_5+200p_3p_4p_5)z^3+(315p_1p_2p_3p_4\\
&+315p_2p_3^2p_4+315p_1p_2p_3p_5+315p_2p_3^2p_5+1575p_2p_3p_4p_5+225p_3^2p_4p_5)z^4+(630p_1p_2p_3^2p_4+630p_1p_2p_3^2p_5\\
&+2016p_1p_2p_3p_4p_5+5292p_2p_3^2p_4p_5)z^5+(245p_1p_2^2p_3^2p_4+245p_1p_2^2p_3^2p_5+9996p_1p_2p_3^2p_4p_5+1225p_2^2p_3^2p_4p_5\\
&+5103p_2p_3^3p_4p_5+875p_2p_3^2p_4^2p_5+875p_2p_3^2p_4p_5^2)z^6+(5616p_1p_2^2p_3^2p_4p_5+12600p_1p_2p_3^3p_4p_5+3528p_2^2p_3^3p_4p_5\\
&+2520p_1p_2p_3^2p_4^2p_5+2520p_2p_3^3p_4^2p_5+2520p_1p_2p_3^2p_4p_5^2+2520p_2p_3^3p_4p_5^2)z^7+(441p_1^2p_2^2p_3^2p_4p_5+17172p_1p_2^2p_3^3p_4p_5\\
&+2205p_1p_2^2p_3^2p_4^2p_5+7875p_1p_2p_3^3p_4^2p_5+2205p_2^2p_3^3p_4^2p_5+2205p_1p_2^2p_3^2p_4p_5^2+7875p_1p_2p_3^3p_4p_5^2+2205p_2^2p_3^3p_4p_5^2\\
&+1575p_2p_3^3p_4^2p_5^2)z^8+(2450p_1^2p_2^2p_3^3p_4p_5+5600p_1p_2^3p_3^3p_4p_5+16260p_1p_2^2p_3^3p_4^2p_5+16260p_1p_2^2p_3^3p_4p_5^2+5600p_1p_2p_3^3p_4^2p_5^2\\
&+2450p_2^2p_3^3p_4^2p_5^2)z^9+(1575p_1^2p_2^3p_3^3p_4p_5+2205p_1^2p_2^2p_3^3p_4^2p_5+7875p_1p_2^3p_3^3p_4^2p_5+2205p_1p_2^2p_3^4p_4^2p_5+2205p_1^2p_2^2p_3^3p_4p_5^2\\
&+7875p_1p_2^3p_3^3p_4p_5^2+2205p_1p_2^2p_3^4p_4p_5^2+17172p_1p_2^2p_3^3p_4^2p_5^2+441p_2^2p_3^4p_4^2p_5^2)z^{10}+(2520p_1^2p_2^3p_3^3p_4^2p_5+2520p_1p_2^3p_3^4p_4^2p_5
\end{aligned}
$$

$$
\begin{aligned}
&+2520p_1^2p_2^3p_3^3p_4p_5^2+2520p_1p_2^3p_3^4p_4p_5^2+3528p_1^2p_2^2p_3^3p_4^2p_5^2+12600p_1p_2^3p_3^3p_4^2p_5^2+5616p_1p_2^2p_3^4p_4^2p_5^2)z^{11}+(875p_1^2p_2^3p_3^4p_4^2p_5\\
&+875p_1^2p_2^3p_3^4p_4p_5^2+5103p_1^2p_2^3p_3^3p_4^2p_5^2+1225p_1^2p_2^2p_3^4p_4^2p_5^2+9996p_1p_2^3p_3^4p_4^2p_5^2+245p_1p_2^2p_3^4p_4^3p_5^2+245p_1p_2^2p_3^4p_4^2p_5^3)z^{12}\\
&+(5292p_1^2p_2^3p_3^4p_4^2p_5^2+2016p_1p_2^3p_3^5p_4^2p_5^2+630p_1p_2^3p_3^4p_4^3p_5^2+630p_1p_2^3p_3^4p_4^2p_5^3)z^{13}+(225p_1^2p_2^4p_3^4p_4^2p_5^2+1575p_1^2p_2^3p_3^5p_4^2p_5^2\\
&+315p_1^2p_2^3p_3^4p_4^3p_5^2+315p_1p_2^3p_3^5p_4^3p_5^2+315p_1^2p_2^3p_3^4p_4^2p_5^3+315p_1p_2^3p_3^5p_4^2p_5^3)z^{14}+(200p_1^2p_2^4p_3^5p_4^2p_5^2\\
&+280p_1^2p_2^3p_3^5p_4^3p_5^2+280p_1^2p_2^3p_3^5p_4^2p_5^3+56p_1p_2^3p_3^5p_4^3p_5^3)z^{15}+(45p_1^2p_2^4p_3^5p_4^3p_5^2+45p_1^2p_2^4p_3^5p_4^2p_5^3+63p_1^2p_2^3p_3^5p_4^3p_5^3)z^{16}\\
&+18p_1^2p_2^4p_3^5p_4^3p_5^3z^{17}+p_1^2p_2^4p_3^6p_4^3p_5^3z^{18}\\
H_4=&1+10p_4z+45p_3p_4z^2+(70p_2p_3p_4+50p_3p_4p_5)z^3+(35p_1p_2p_3p_4+175p_2p_3p_4p_5)z^4+(126p_1p_2p_3p_4p_5\\
&+126p_2p_3^2p_4p_5)z^5+(175p_1p_2p_3^2p_4p_5+35p_2p_3^2p_4^2p_5)z^6+(50p_1p_2^2p_3^2p_4p_5+70p_1p_2p_3^2p_4^2p_5)z^7+45p_1p_2^2p_3^2p_4^2p_5z^8\\
&+10p_1p_2^2p_3^3p_4^2p_5z^9+p_1p_2^2p_3^3p_4^2p_5^2z^{10}\\
H_5=&1+10p_5z+45p_3p_5z^2+(70p_2p_3p_5+50p_3p_4p_5)z^3+(35p_1p_2p_3p_5+175p_2p_3p_4p_5)z^4+(126p_1p_2p_3p_4p_5\\
&+126p_2p_3^2p_4p_5)z^5+(175p_1p_2p_3^2p_4p_5+35p_2p_3^2p_4p_5^2)z^6+(50p_1p_2^2p_3^2p_4p_5+70p_1p_2p_3^2p_4p_5^2)z^7+45p_1p_2^2p_3^2p_4p_5^2z^8\\
&+10p_1p_2^2p_3^3p_4p_5^2z^9+p_1p_2^2p_3^3p_4^2p_5^2z^{10}
\end{aligned}
$$

Appendix B. Polynomials for E_6

In this subsection we present polynomials corresponding to the Lie algebra E_6 [42].

$$
\begin{aligned}
H_1=&p_1^2p_2^3p_3^4p_4^3p_5^2p_6^2z^{16}+16p_1^2p_2^3p_3^4p_4^3p_5p_6^2z^{15}+120p_1^2p_2^3p_3^4p_4^2p_5p_6^2z^{14}+560p_1^2p_2^3p_3^3p_4^2p_5p_6^2z^{13}+(1050p_1^2p_2^2p_4^2p_5p_6^2p_3^3\\
&+770p_1^2p_2^3p_4^2p_5p_6p_3^3)z^{12}+(672p_1p_2^2p_4^2p_5p_6^2p_3^3+3696p_1^2p_2^2p_4^2p_5p_6p_3^3)z^{11}+(3696p_1p_2^2p_4^2p_5p_6p_3^3+4312p_1^2p_2^2p_4^2p_5p_6p_3^2)z^{10}\\
&+(8800p_1p_2^2p_4^2p_5p_6p_3^2+2640p_1^2p_2^2p_4p_5p_6p_3^2)z^9+(660p_1^2p_2^2p_4p_6p_3^2+4125p_1p_2p_4^2p_5p_6p_3^2+8085p_1p_2^2p_4p_5p_6p_3^2)z^8\\
&+(2640p_1p_2^2p_4p_6p_3^2+8800p_1p_2p_4p_5p_6p_3^2)z^7+(4312p_1p_2p_4p_6p_3^2+3696p_1p_2p_4p_5p_6p_3)z^6+(672p_1p_2p_3p_4p_5+3696p_1p_2p_3p_4p_6)z^5\\
&+(1050p_1p_2p_3p_4+770p_1p_2p_3p_6)z^4+560p_1p_2p_3z^3+120p_1p_2z^2+16p_1z+1
\end{aligned}
$$

$$
\begin{aligned}
H_2=&p_1^3p_2^6p_3^8p_4^6p_5^3p_6^4z^{30}+30p_1^3p_2^6p_3^8p_4^5p_5^3p_6^4z^{29}+(120p_1^3p_2^6p_4^5p_5^2p_6^4p_3^8+315p_1^3p_2^6p_4^5p_5^3p_6^4p_3^7)z^{28}+(1050p_1^3p_2^5p_4^5p_5^3p_6^4p_3^7+2240p_1^3p_2^6p_4^5p_5^2p_6^4p_3^7\\
&+770p_1^3p_2^6p_4^5p_5^3p_6^3p_3^7)z^{27}+(1050p_1^2p_2^5p_4^5p_5^3p_6^4p_3^7+9450p_1^3p_2^5p_4^5p_5^2p_6^4p_3^7+4200p_1^3p_2^6p_4^4p_5^2p_6^4p_3^7+5775p_1^3p_2^5p_4^5p_5^3p_6^3p_3^7\\
&+6930p_1^3p_2^6p_4^5p_5^2p_6^3p_3^7)z^{26}+(10752p_1^2p_2^5p_4^5p_5^2p_6^4p_3^7+31500p_1^3p_2^5p_4^4p_5^2p_6^4p_3^7+8316p_1^2p_2^5p_4^5p_5^3p_6^3p_3^7+59136p_1^3p_2^5p_4^5p_5^2p_6^3p_3^7\\
&+23100p_1^3p_2^6p_4^4p_5^2p_6^3p_3^7+9702p_1^3p_2^5p_4^5p_5^3p_6^3p_3^6)z^{25}+(45360p_1^2p_2^5p_4^4p_5^2p_6^4p_3^7+92400p_1^2p_2^5p_4^5p_5^2p_6^3p_3^7+249480p_1^3p_2^5p_4^4p_5^2p_6^3p_3^7\\
&+36750p_1^3p_2^5p_4^4p_5^2p_6^4p_3^6+26950p_1^2p_2^5p_4^5p_5^3p_6^3p_3^6+8085p_1^3p_2^5p_4^4p_5^3p_6^3p_3^6+107800p_1^3p_2^5p_4^5p_5^2p_6^3p_3^6+26950p_1^3p_2^6p_4^4p_5^2p_6^3p_3^6)z^{24}\\
&+(443520p_1^2p_2^5p_4^4p_5^2p_6^3p_3^7+94080p_1^2p_2^5p_4^4p_5^2p_6^4p_3^6+16500p_1^2p_2^4p_4^5p_5^3p_6^3p_3^6+32340p_1^2p_2^5p_4^4p_5^3p_6^3p_3^6+316800p_1^2p_2^5p_4^5p_5^2p_6^3p_3^6\\
&+1132560p_1^3p_2^5p_4^4p_5^2p_6^3p_3^6)z^{23}+(44100p_1^2p_2^4p_4^4p_5^2p_6^4p_3^6+44550p_1^2p_2^4p_4^4p_5^3p_6^3p_3^6+202125p_1^2p_2^4p_4^5p_5^2p_6^3p_3^6+3256110p_1^2p_2^5p_4^4p_5^2p_6^3p_3^6\\
&+242550p_1^3p_2^4p_4^4p_5^2p_6^3p_3^6+495000p_1^3p_2^5p_4^3p_5^2p_6^3p_3^6+32340p_1^3p_2^5p_4^4p_5p_6^3p_3^6+177870p_1^3p_2^5p_4^4p_5^2p_6^2p_3^6+1358280p_1^3p_2^5p_4^4p_5^2p_6^3p_3^5)z^{22}\\
&+(2674100p_1^2p_2^4p_4^4p_5^2p_6^3p_3^6+2182950p_1^2p_2^5p_4^3p_5^2p_6^3p_3^6+168960p_1^2p_2^5p_4^4p_5p_6^3p_3^6+178200p_1^3p_2^5p_4^3p_5p_6^3p_3^6+711480p_1^2p_2^5p_4^4p_5^2p_6^2p_3^6\\
&+23100p_1^2p_2^4p_4^4p_5^3p_6^3p_3^5+4928000p_1^2p_2^5p_4^4p_5^2p_6^3p_3^5+1131900p_1^3p_2^4p_4^4p_5^2p_6^3p_3^5+1478400p_1^3p_2^5p_4^3p_5^2p_6^3p_3^5+830060p_1^3p_2^5p_4^4p_5^2p_6^2p_3^5)z^{21}\\
&+(155232p_1p_2^4p_4^4p_5^2p_6^3p_3^6+3234000p_1^2p_2^4p_4^3p_5^2p_6^3p_3^6+349272p_1^2p_2^4p_4^4p_5p_6^3p_3^6+970200p_1^2p_2^5p_4^3p_5p_6^3p_3^6+853776p_1^2p_2^4p_4^4p_5^2p_6^2p_3^6\\
&+9315306p_1^2p_2^4p_4^4p_5^2p_6^3p_3^5+7074375p_1^2p_2^5p_4^3p_5^2p_6^3p_3^5+1559250p_1^3p_2^4p_4^3p_5^2p_6^3p_3^5+577500p_1^3p_2^5p_4^3p_5p_6^3p_3^5+5082p_1^2p_2^4p_4^4p_5^3p_6^2p_3^5\\
&+3811500p_1^2p_2^5p_4^4p_5^2p_6^2p_3^5+996072p_1^3p_2^4p_4^4p_5^2p_6^2p_3^5+1143450p_1^3p_2^5p_4^3p_5^2p_6^2p_3^5)z^{20}+(2069760p_1^2p_2^4p_4^3p_5p_6^3p_3^6+1478400p_1p_2^4p_4^4p_5^2p_6^3p_3^5\\
&+4331250p_1^2p_2^3p_4^4p_5^2p_6^3p_3^5+21801780p_1^2p_2^4p_4^3p_5^2p_6^3p_3^5+369600p_1^2p_2^4p_4^4p_5p_6^3p_3^5+3326400p_1^2p_2^5p_4^3p_5p_6^3p_3^5+693000p_1^3p_2^4p_4^3p_5p_6^3p_3^5\\
&+11384100p_1^2p_2^4p_4^4p_5^2p_6^2p_3^5+6225450p_1^2p_2^5p_4^3p_5^2p_6^2p_3^5+2439360p_1^3p_2^4p_4^4p_5^2p_6^2p_3^5+508200p_1^3p_2^5p_4^3p_5p_6^2p_3^5)z^{19}+(1559250p_1p_2^3p_4^4p_5^2p_6^3p_3^5\\
&+3056130p_1p_2^4p_4^3p_5^2p_6^3p_3^5+14437500p_1^2p_2^3p_4^3p_5^2p_6^3p_3^5+14314300p_1^2p_2^4p_4^3p_5p_6^3p_3^5+2032800p_1p_2^4p_4^4p_5^2p_6^2p_3^5+8575875p_1^2p_2^3p_4^4p_5^2p_6^2p_3^5\\
&+28420210p_1^2p_2^4p_4^3p_5^2p_6^2p_3^5+127050p_1^2p_2^4p_4^4p_5p_6^2p_3^5+3176250p_1^2p_2^5p_4^3p_5p_6^2p_3^5+1372140p_1^3p_2^4p_4^3p_5p_6^2p_3^5+6338640p_1^2p_2^4p_4^3p_5^2p_6^3p_3^4\\
&+2371600p_1^2p_2^4p_4^4p_5^2p_6^2p_3^4+711480p_1^3p_2^4p_4^3p_5^2p_6^2p_3^4)z^{18}+(5913600p_1p_2^3p_4^3p_5^2p_6^3p_3^5+1774080p_1p_2^4p_4^3p_5p_6^3p_3^5+10187100p_1^2p_2^3p_4^3p_5p_6^3p_3^5\\
&+577500p_1^2p_2^4p_4^2p_5p_6^3p_3^5+3811500p_1p_2^3p_4^4p_5^2p_6^2p_3^5+7470540p_1p_2^4p_4^3p_5^2p_6^2p_3^5+32524800p_1^2p_2^3p_4^3p_5^2p_6^2p_3^5+18705960p_1^2p_2^4p_4^3p_5p_6^2p_3^5\\
&+8731800p_1^2p_2^3p_4^3p_5^2p_6^3p_3^4+8279040p_1^2p_2^4p_4^3p_5p_6^3p_3^4+4446750p_1^2p_2^3p_4^4p_5^2p_6^2p_3^4+16625700p_1^2p_2^4p_4^3p_5^2p_6^2p_3^4+711480p_1^3p_2^4p_4^3p_5p_6^2p_3^4)z^{17}\\
&+(4527600p_1p_2^3p_4^3p_5p_6^3p_3^5+18295200p_1p_2^3p_4^3p_5^2p_6^2p_3^5+5488560p_1p_2^4p_4^3p_5p_6^2p_3^5+24901800p_1^2p_2^3p_4^3p_5p_6^2p_3^5+508200p_1^2p_2^4p_4^2p_5p_6^2p_3^5\\
&+3880800p_1p_2^3p_4^3p_5^2p_6^3p_3^4+11884950p_1^2p_2^3p_4^3p_5p_6^3p_3^4+2182950p_1^2p_2^4p_4^2p_5p_6^3p_3^4+2268750p_1p_2^3p_4^4p_5^2p_6^2p_3^4+4446750p_1p_2^4p_4^3p_5^2p_6^2p_3^4\\
&+45530550p_1^2p_2^3p_4^3p_5^2p_6^2p_3^4+1334025p_1^2p_2^4p_4^2p_5^2p_6^2p_3^4+18478980p_1^2p_2^4p_4^3p_5p_6^2p_3^4+108900p_1^3p_2^4p_4^2p_5p_6^2p_3^4+1584660p_1^2p_2^4p_4^3p_5^2p_6p_3^4)z^{16}\\
&+(15937152p_1p_2^3p_4^3p_5p_6^2p_3^5+5588352p_1p_2^3p_4^3p_5p_6^3p_3^4+3234000p_1^2p_2^3p_4^2p_5p_6^3p_3^4+34036496p_1p_2^3p_4^3p_5^2p_6^2p_3^4+5808000p_1^2p_2^3p_4^2p_5^2p_6^2p_3^4\\
&+5808000p_1p_2^4p_4^3p_5p_6^2p_3^4+53742416p_1^2p_2^3p_4^3p_5p_6^2p_3^4+6203600p_1^2p_2^4p_4^2p_5p_6^2p_3^4+5588352p_1^2p_2^3p_4^3p_5^2p_6p_3^4+3234000p_1^2p_2^4p_4^3p_5p_6p_3^4\\
&+15937152p_1^2p_2^3p_4^3p_5^2p_6^2p_3^3)z^{15}+(108900p_1^2p_3^4p_4^2p_6^2p_2^4+1334025p_1p_3^4p_4^2p_5p_6^2p_2^4+508200p_1^2p_3^3p_4^2p_5p_6^2p_2^4+2182950p_1^2p_3^4p_4^2p_5p_6p_2^4\\
&+1584660p_1p_3^4p_4^2p_5p_6^2p_2^3+18295200p_1p_3^3p_4^3p_5^2p_6p_2^3+4446750p_1p_3^4p_4^2p_5^2p_6p_2^2+5488560p_1^2p_3^3p_4^2p_5^2p_6p_2^3+45530550p_1p_3^4p_4^3p_5p_6^2p_2^3\\
&+24901800p_1^2p_3^3p_4^3p_5p_6^2p_2^3+18478980p_1^2p_3^4p_4^2p_5p_6p_2^3+3880800p_1p_3^3p_4^3p_5^2p_6p_2^3+4527600p_1^2p_3^3p_4^3p_5p_6p_2^3+11884950p_1^2p_3^4p_4^3p_5p_6p_2^3\\
&+2268750p_1p_3^4p_4^3p_5^2p_6^2p_2^2)z^{14}+(577500p_1^2p_3^3p_4^2p_5p_6p_2^4+711480p_1^2p_3^4p_4^2p_6^2p_2^3+7470540p_1p_3^3p_4^2p_5^2p_6^2p_2^3+32524800p_1p_3^3p_4^3p_5p_6^2p_2^3\\
&+16625700p_1p_3^4p_4^2p_5p_6^2p_2^3+18705960p_1^2p_3^3p_4^2p_5p_6^2p_2^3+5913600p_1p_3^3p_4^3p_5^2p_6p_2^3+1774080p_1^2p_3^3p_4^2p_5^2p_6p_2^3+8731800p_1p_3^4p_4^3p_5p_6p_2^3\\
&+10187100p_1^2p_3^3p_4^3p_5p_6p_2^3+8279040p_1^2p_3^4p_4^2p_5p_6p_2^3+3811500p_1p_3^3p_4^3p_5^2p_6^2p_2^2+4446750p_1p_3^4p_4^3p_5p_6^2p_2^2)z^{13}+(711480p_1p_3^3p_4^2p_6^2p_2^4\\
&+2371600p_1p_2^2p_4^2p_5p_6^2p_3^4+6338640p_1p_2^3p_4^2p_5p_6p_3^4+1372140p_1^2p_2^3p_4^2p_6^2p_3^3+2032800p_1p_2^2p_4^2p_5^2p_6^2p_3^3+8575875p_1p_2^2p_4^3p_5p_6^2p_3^3\\
&+28420210p_1p_2^3p_4^2p_5p_6^2p_3^3+127050p_1^2p_2^2p_4^2p_5p_6^2p_3^3+3176250p_1^2p_2^3p_4p_5p_6^2p_3^3+1559250p_1p_2^2p_4^3p_5^2p_6p_3^3+3056130p_1p_2^3p_4^2p_5^2p_6p_3^3\\
&+14437500p_1p_2^3p_4^3p_5p_6p_3^3+14314300p_1^2p_2^3p_4^2p_5p_6p_3^3)z^{12}+(2439360p_1p_3^3p_4^2p_6^2p_2^3+508200p_1^2p_3^3p_4p_6^2p_2^3+6225450p_1p_3^3p_4p_5p_6^2p_2^3\\
&+693000p_1^2p_3^3p_4^2p_6p_2^3+21801780p_1p_3^3p_4^2p_5p_6p_2^3+2069760p_1^2p_3^2p_4^2p_5p_6p_2^3+3326400p_1^2p_3^3p_4p_5p_6p_2^3+11384100p_1p_3^3p_4^2p_5p_6^2p_2^2\\
&+1478400p_1p_3^3p_4^2p_5^2p_6p_2^2+4331250p_1p_3^3p_4^3p_5p_6p_2^2+369600p_1^2p_3^3p_4^2p_5p_6p_2^2)z^{11}+(1143450p_1p_3^3p_4p_6^2p_2^3+1559250p_1p_3^3p_4^2p_6p_2^3
\end{aligned}
$$

$$+ 577500p_1^2p_3^3p_4p_6p_2^3 + 3234000p_1p_3^2p_4^2p_5p_6p_2^3 + 7074375p_1p_3^3p_4p_5p_6p_2^3 + 970200p_1^2p_3^2p_4p_5p_6p_2^3 + 996072p_1p_3^3p_4^2p_6^2p_2^2$$
$$+ 5082p_3^3p_4^2p_5p_6^2p_2^2 + 853776p_1p_3^2p_4^2p_5p_6^2p_2^2 + 3811500p_1p_3^3p_4p_5p_6^2p_2^2 + 155232p_1p_3^2p_4^2p_5^2p_6p_2^2 + 9315306p_1p_3^3p_4^2p_5p_6p_2^2$$
$$+ 349272p_1^2p_3^2p_4^2p_5p_6p_2^2)z^{10} + (1478400p_1p_3^3p_4p_6p_2^3 + 178200p_1^2p_3^2p_4p_6p_2^3 + 2182950p_1p_3^2p_4p_5p_6p_2^3 + 830060p_1p_3^3p_4p_6^2p_2^2$$
$$+ 711480p_1p_3^2p_4p_5p_6^2p_2^2 + 1131900p_1p_3^3p_4^2p_6p_2^2 + 23100p_3^3p_4^2p_5p_6p_2^2 + 2674100p_1p_3^2p_4^2p_5p_6p_2^2 + 4928000p_1p_3^3p_4p_5p_6p_2^2$$
$$+ 168960p_1^2p_3^2p_4p_5p_6p_2^2)z^9 + (495000p_1p_3^2p_4p_6p_2^3 + 177870p_1p_3^2p_4p_6^2p_2^2 + 44100p_1p_3^2p_4^2p_5p_2^2 + 242550p_1p_3^2p_4^2p_6p_2^2$$
$$+ 1358280p_1p_3^3p_4p_6p_2^2 + 32340p_1^2p_3^2p_4p_6p_2^2 + 44550p_3^2p_4^2p_5p_6p_2^2 + 3256110p_1p_3^2p_4p_5p_6p_2^2 + 202125p_1p_3^2p_4^2p_5p_6p_2)z^8$$
$$+ (94080p_1p_3^2p_4p_5p_2^2 + 1132560p_1p_3^2p_4p_6p_2^2 + 32340p_3^2p_4p_5p_6p_2^2 + 443520p_1p_3p_4p_5p_6p_2^2 + 16500p_3^2p_4^2p_5p_6p_2$$
$$+ 316800p_1p_3^2p_4p_5p_6p_2)z^7 + (36750p_1p_3^2p_4p_2^2 + 45360p_1p_3p_4p_5p_2^2 + 26950p_1p_3^2p_6p_2^2 + 8085p_3^2p_4p_6p_2^2 + 249480p_1p_3p_4p_6p_2^2$$
$$+ 107800p_1p_3^2p_4p_6p_2 + 26950p_3^2p_4p_5p_6p_2 + 92400p_1p_3p_4p_5p_6p_2)z^6 + (31500p_1p_3p_4p_2^2 + 23100p_1p_3p_6p_2^2 + 10752p_1p_3p_4p_5p_2$$
$$+ 9702p_3^2p_4p_6p_2 + 59136p_1p_3p_4p_6p_2 + 8316p_3p_4p_5p_6p_2)z^5 + (4200p_1p_3p_2^2 + 9450p_1p_3p_4p_2 + 1050p_3p_4p_5p_2 + 6930p_1p_3p_6p_2$$
$$+ 5775p_3p_4p_6p_2)z^4 + (2240p_1p_2p_3 + 1050p_2p_4p_3 + 770p_2p_6p_3)z^3 + (120p_1p_2 + 315p_3p_2)z^2 + 30p_2z + 1$$

$$H_3 = p_1^4p_2^8p_3^{12}p_4^8p_5^4p_6^6z^{42} + 42p_1^4p_2^8p_3^{11}p_4^8p_5^4p_6^6z^{41} + (315p_1^4p_2^7p_4^8p_5^4p_6^6p_3^{11} + 315p_1^4p_2^8p_4^7p_5^4p_6^6p_3^{11} + 231p_1^4p_2^8p_4^8p_5^4p_6^5p_3^{11})z^{40}$$
$$+ (560p_1^3p_2^7p_4^8p_5^4p_6^6p_3^{11} + 4200p_1^4p_2^7p_4^7p_5^4p_6^6p_3^{11} + 560p_1^4p_2^8p_4^7p_5^3p_6^6p_3^{11} + 3080p_1^4p_2^7p_4^8p_5^4p_6^5p_3^{11} + 3080p_1^4p_2^8p_4^7p_5^4p_6^5p_3^{11})z^{39}$$
$$+ (9450p_1^3p_2^7p_4^7p_5^4p_6^6p_3^{11} + 9450p_1^4p_2^7p_4^7p_5^3p_6^6p_3^{11} + 6930p_1^3p_2^7p_4^8p_5^4p_6^5p_3^{11} + 51975p_1^4p_2^7p_4^7p_5^4p_6^5p_3^{11} + 6930p_1^4p_2^8p_4^7p_5^3p_6^5p_3^{11}$$
$$+ 11025p_1^4p_2^7p_4^7p_5^4p_6^6p_3^{10} + 8085p_1^4p_2^7p_4^8p_5^4p_6^5p_3^{10} + 8085p_1^4p_2^8p_4^7p_5^4p_6^5p_3^{10})z^{38} + (24192p_1^3p_2^7p_4^7p_5^3p_6^6p_3^{11}$$
$$+ 133056p_1^3p_2^7p_4^7p_5^4p_6^5p_3^{11} + 133056p_1^4p_2^7p_4^7p_5^3p_6^5p_3^{11} + 44100p_1^3p_2^7p_4^7p_5^4p_6^6p_3^{10} + 44100p_1^4p_2^7p_4^7p_5^3p_6^6p_3^{10} + 32340p_1^3p_2^7p_4^8p_5^4p_6^5p_3^{10}$$
$$+ 407484p_1^4p_2^7p_4^7p_5^4p_6^5p_3^{10} + 32340p_1^4p_2^8p_4^7p_5^3p_6^5p_3^{10})z^{37} + (369600p_1^3p_2^7p_4^7p_5^3p_6^5p_3^{11} + 36750p_1^3p_2^6p_4^7p_5^4p_6^6p_3^{10}$$
$$+ 200704p_1^3p_2^7p_4^7p_5^3p_6^6p_3^{10} + 36750p_1^4p_2^7p_4^6p_5^3p_6^6p_3^{10} + 26950p_1^3p_2^6p_4^8p_5^4p_6^5p_3^{10} + 1539384p_1^3p_2^7p_4^7p_5^4p_6^5p_3^{10}$$
$$+ 202125p_1^4p_2^6p_4^7p_5^4p_6^5p_3^{10} + 202125p_1^4p_2^7p_4^6p_5^4p_6^5p_3^{10} + 1539384p_1^4p_2^7p_4^7p_5^3p_6^5p_3^{10} + 26950p_1^4p_2^8p_4^6p_5^3p_6^5p_3^{10} + 148225p_1^4p_2^7p_4^7p_5^4p_6^4p_3^{10}$$
$$+ 916839p_1^4p_2^7p_4^7p_5^4p_6^5p_3^9)z^{36} + (211680p_1^3p_2^6p_4^7p_5^3p_6^6p_3^{10} + 211680p_1^3p_2^7p_4^6p_5^3p_6^6p_3^{10} + 1853280p_1^3p_2^6p_4^7p_5^4p_6^5p_3^{10} + 1164240p_1^3p_2^7p_4^6p_5^4p_6^5p_3^{10}$$
$$+ 6044544p_1^3p_2^7p_4^7p_5^3p_6^5p_3^{10} + 1164240p_1^4p_2^6p_4^7p_5^3p_6^5p_3^{10} + 1853280p_1^4p_2^7p_4^6p_5^3p_6^5p_3^{10} + 853776p_1^3p_2^7p_4^7p_5^4p_6^4p_3^{10} + 853776p_1^4p_2^7p_4^7p_5^3p_6^4p_3^{10}$$
$$+ 4527600p_1^3p_2^7p_4^7p_5^4p_6^5p_3^9 + 1358280p_1^4p_2^6p_4^7p_5^4p_6^5p_3^9 + 1358280p_1^4p_2^7p_4^6p_5^4p_6^5p_3^9 + 4527600p_1^4p_2^7p_4^7p_5^3p_6^5p_3^9 + 996072p_1^4p_2^7p_4^7p_5^4p_6^4p_3^9)z^{35}$$
$$+ (396900p_1^3p_2^6p_4^6p_5^3p_6^6p_3^{10} + 291060p_1^2p_2^6p_4^7p_5^4p_6^5p_3^{10} + 2182950p_1^3p_2^6p_4^6p_5^4p_6^5p_3^{10} + 9168390p_1^3p_2^6p_4^7p_5^3p_6^5p_3^{10} + 9168390p_1^3p_2^7p_4^6p_5^3p_6^5p_3^{10}$$
$$+ 2182950p_1^4p_2^6p_4^6p_5^3p_6^5p_3^{10} + 291060p_1^4p_2^7p_4^6p_5^2p_6^5p_3^{10} + 1600830p_1^3p_2^6p_4^7p_5^4p_6^4p_3^{10} + 5336100p_1^3p_2^7p_4^7p_5^3p_6^4p_3^{10} + 1600830p_1^4p_2^7p_4^6p_5^3p_6^4p_3^{10}$$
$$+ 13222440p_1^3p_2^6p_4^7p_5^4p_6^5p_3^9 + 8489250p_1^3p_2^7p_4^6p_5^4p_6^5p_3^9 + 2546775p_1^4p_2^6p_4^6p_5^4p_6^5p_3^9 + 23654400p_1^3p_2^7p_4^7p_5^3p_6^5p_3^9 + 8489250p_1^4p_2^6p_4^7p_5^3p_6^5p_3^9$$
$$+ 13222440p_1^4p_2^7p_4^6p_5^3p_6^5p_3^9 + 6225450p_1^3p_2^7p_4^7p_5^4p_6^4p_3^9 + 1867635p_1^4p_2^6p_4^7p_5^4p_6^4p_3^9 + 1867635p_1^4p_2^7p_4^6p_5^4p_6^4p_3^9 + 6225450p_1^4p_2^7p_4^7p_5^3p_6^4p_3^9)z^{34}$$
$$+ (2069760p_1^2p_2^6p_4^7p_5^3p_6^5p_3^{10} + 24147200p_1^3p_2^6p_4^6p_5^3p_6^5p_3^{10} + 2069760p_1^3p_2^7p_4^6p_5^2p_6^5p_3^{10} + 11383680p_1^3p_2^6p_4^7p_5^3p_6^4p_3^{10}$$
$$+ 11383680p_1^3p_2^7p_4^6p_5^3p_6^4p_3^{10} + 205800p_1^3p_2^6p_4^6p_5^3p_6^6p_3^9 + 3773000p_1^2p_2^6p_4^7p_5^4p_6^5p_3^9 + 9240000p_1^3p_2^5p_4^7p_5^4p_6^5p_3^9 + 37560600p_1^3p_2^6p_4^6p_5^4p_6^5p_3^9$$
$$+ 82222140p_1^3p_2^6p_4^7p_5^3p_6^5p_3^9 + 82222140p_1^3p_2^7p_4^6p_5^3p_6^5p_3^9 + 37560600p_1^4p_2^6p_4^6p_5^3p_6^5p_3^9 + 9240000p_1^4p_2^7p_4^5p_5^3p_6^5p_3^9 + 3773000p_1^4p_2^7p_4^6p_5^2p_6^5p_3^9$$
$$+ 27544440p_1^3p_2^6p_4^7p_5^4p_6^4p_3^9 + 13280960p_1^3p_2^7p_4^6p_5^4p_6^4p_3^9 + 6225450p_1^4p_2^6p_4^6p_5^4p_6^4p_3^9 + 41164200p_1^3p_2^7p_4^7p_5^3p_6^4p_3^9 + 13280960p_1^4p_2^6p_4^7p_5^3p_6^4p_3^9$$
$$+ 27544440p_1^4p_2^7p_4^6p_5^3p_6^4p_3^9)z^{33} + (5588352p_1^2p_2^6p_4^6p_5^3p_6^5p_3^{10} + 5588352p_1^3p_2^6p_4^6p_5^2p_6^5p_3^{10} + 30735936p_1^3p_2^6p_4^6p_5^3p_6^4p_3^{10}$$
$$+ 5197500p_1^2p_2^5p_4^7p_5^4p_6^5p_3^9 + 10187100p_1^2p_2^6p_4^6p_5^4p_6^5p_3^9 + 38981250p_1^3p_2^5p_4^6p_5^4p_6^5p_3^9 + 28385280p_1^2p_2^6p_4^7p_5^3p_6^5p_3^9 + 63669375p_1^3p_2^5p_4^7p_5^3p_6^5p_3^9$$
$$+ 440527626p_1^3p_2^6p_4^6p_5^3p_6^5p_3^9 + 63669375p_1^3p_2^7p_4^5p_5^3p_6^5p_3^9 + 38981250p_1^4p_2^6p_4^5p_5^3p_6^5p_3^9 + 28385280p_1^3p_2^7p_4^6p_5^2p_6^5p_3^9 + 10187100p_1^4p_2^6p_4^6p_5^2p_6^5p_3^9$$
$$+ 5197500p_1^4p_2^7p_4^5p_5^2p_6^5p_3^9 + 7470540p_1^2p_2^6p_4^7p_5^4p_6^4p_3^9 + 28586250p_1^3p_2^5p_4^7p_5^4p_6^4p_3^9 + 87268104p_1^3p_2^6p_4^6p_5^4p_6^4p_3^9 + 202848030p_1^3p_2^6p_4^7p_5^3p_6^4p_3^9$$
$$+ 202848030p_1^3p_2^7p_4^6p_5^3p_6^4p_3^9 + 87268104p_1^4p_2^6p_4^6p_5^3p_6^4p_3^9 + 28586250p_1^4p_2^7p_4^5p_5^3p_6^4p_3^9 + 7470540p_1^4p_2^7p_4^6p_5^2p_6^4p_3^9$$
$$+ 11884950p_1^3p_2^6p_4^6p_5^4p_6^5p_3^8 + 11884950p_1^4p_2^6p_4^6p_5^3p_6^5p_3^8 + 8715630p_1^3p_2^6p_4^7p_5^4p_6^4p_3^8 + 2614689p_1^4p_2^6p_4^6p_5^4p_6^4p_3^8 + 8715630p_1^4p_2^7p_4^6p_5^3p_6^4p_3^8)z^{32}$$
$$+ (24948000p_1^2p_2^5p_4^6p_5^4p_6^5p_3^9 + 40748400p_1^2p_2^5p_4^7p_5^3p_6^5p_3^9 + 177031008p_1^2p_2^6p_4^6p_5^3p_6^5p_3^9 + 467082000p_1^3p_2^5p_4^6p_5^3p_6^5p_3^9$$
$$+ 467082000p_1^3p_2^6p_4^5p_5^3p_6^5p_3^9 + 177031008p_1^3p_2^6p_4^6p_5^2p_6^5p_3^9 + 40748400p_1^3p_2^7p_4^5p_5^2p_6^5p_3^9 + 24948000p_1^4p_2^6p_4^5p_5^2p_6^5p_3^9$$
$$+ 18295200p_1^2p_2^5p_4^7p_5^4p_6^4p_3^9 + 35858592p_1^2p_2^6p_4^6p_5^4p_6^4p_3^9 + 137214000p_1^3p_2^5p_4^6p_5^4p_6^4p_3^9 + 60984000p_1^2p_2^6p_4^7p_5^3p_6^4p_3^9$$
$$+ 224116200p_1^3p_2^5p_4^7p_5^3p_6^4p_3^9 + 1339753968p_1^3p_2^6p_4^6p_5^3p_6^4p_3^9 + 224116200p_1^3p_2^7p_4^5p_5^3p_6^4p_3^9 + 137214000p_1^4p_2^6p_4^5p_5^3p_6^4p_3^9$$
$$+ 60984000p_1^3p_2^7p_4^6p_5^2p_6^4p_3^9 + 35858592p_1^4p_2^6p_4^6p_5^2p_6^4p_3^9 + 18295200p_1^4p_2^7p_4^5p_5^2p_6^4p_3^9 + 29106000p_1^3p_2^5p_4^6p_5^4p_6^5p_3^8$$
$$+ 191866752p_1^3p_2^6p_4^6p_5^3p_6^5p_3^8 + 29106000p_1^4p_2^6p_4^5p_5^3p_6^5p_3^8 + 21344400p_1^3p_2^5p_4^7p_5^4p_6^4p_3^8 + 66594528p_1^3p_2^6p_4^6p_5^4p_6^4p_3^8 + 71148000p_1^3p_2^6p_4^7p_5^3p_6^4p_3^8$$
$$+ 71148000p_1^3p_2^7p_4^6p_5^3p_6^4p_3^8 + 66594528p_1^4p_2^6p_4^6p_5^3p_6^4p_3^8 + 21344400p_1^4p_2^7p_4^5p_5^3p_6^4p_3^8)z^{31} + (353089660p_1^2p_2^5p_4^6p_5^3p_6^5p_3^9$$
$$+ 247546530p_1^2p_2^6p_4^5p_5^3p_6^5p_3^9 + 707437500p_1^3p_2^5p_4^5p_5^3p_6^5p_3^9 + 60555264p_1^2p_2^6p_4^6p_5^2p_6^5p_3^9 + 247546530p_1^3p_2^5p_4^6p_5^2p_6^5p_3^9$$
$$+ 353089660p_1^3p_2^6p_4^5p_5^2p_6^5p_3^9 + 111143340p_1^2p_2^5p_4^6p_5^4p_6^4p_3^9 + 155636250p_1^2p_2^5p_4^7p_5^3p_6^4p_3^9 + 542666124p_1^2p_2^6p_4^6p_5^3p_6^4p_3^9$$
$$+ 1941993130p_1^3p_2^5p_4^6p_5^3p_6^4p_3^9 + 1941993130p_1^3p_2^6p_4^5p_5^3p_6^4p_3^9 + 542666124p_1^3p_2^6p_4^6p_5^2p_6^4p_3^9 + 155636250p_1^3p_2^7p_4^5p_5^2p_6^4p_3^9$$
$$+ 111143340p_1^4p_2^6p_4^5p_5^2p_6^4p_3^9 + 5478396p_1^3p_2^6p_4^6p_5^3p_6^3p_3^9 + 20212500p_1^2p_2^5p_4^6p_5^4p_6^5p_3^8 + 110387200p_1^2p_2^6p_4^6p_5^3p_6^5p_3^8$$
$$+ 388031490p_1^3p_2^5p_4^6p_5^3p_6^5p_3^8 + 388031490p_1^3p_2^6p_4^5p_5^3p_6^5p_3^8 + 110387200p_1^3p_2^6p_4^6p_5^2p_6^5p_3^8 + 20212500p_1^4p_2^6p_4^5p_5^2p_6^5p_3^8$$
$$+ 14822500p_1^2p_2^5p_4^7p_5^4p_6^4p_3^8 + 29052100p_1^2p_2^6p_4^6p_5^4p_6^4p_3^8 + 253998360p_1^3p_2^5p_4^6p_5^4p_6^4p_3^8 + 8715630p_1^3p_2^6p_4^5p_5^4p_6^4p_3^8 + 181575625p_1^3p_2^5p_4^7p_5^3p_6^4p_3^8$$
$$+ 1554121926p_1^3p_2^6p_4^6p_5^3p_6^4p_3^8 + 8715630p_1^4p_2^5p_4^6p_5^3p_6^4p_3^8 + 181575625p_1^3p_2^7p_4^5p_5^3p_6^4p_3^8 + 253998360p_1^4p_2^6p_4^5p_5^3p_6^4p_3^8$$
$$+ 29052100p_1^4p_2^6p_4^6p_5^2p_6^4p_3^8 + 14822500p_1^4p_2^7p_4^5p_5^2p_6^4p_3^8 + 6391462p_1^3p_2^6p_4^6p_5^4p_6^3p_3^8 + 6391462p_1^4p_2^6p_4^6p_5^3p_6^3p_3^8)z^{30}$$
$$+ (651974400p_1^2p_2^5p_4^5p_5^3p_6^5p_3^9 + 195592320p_1^2p_2^5p_4^6p_5^2p_6^5p_3^9 + 195592320p_1^2p_2^6p_4^5p_5^2p_6^5p_3^9 + 651974400p_1^3p_2^5p_4^5p_5^2p_6^5p_3^9$$
$$+ 1644128640p_1^2p_2^5p_4^6p_5^3p_6^4p_3^9 + 1075757760p_1^2p_2^6p_4^5p_5^3p_6^4p_3^9 + 3585859200p_1^3p_2^5p_4^5p_5^3p_6^4p_3^9 + 292723200p_1^2p_2^6p_4^6p_5^2p_6^4p_3^9$$
$$+ 1075757760p_1^3p_2^5p_4^6p_5^2p_6^4p_3^9 + 1644128640p_1^3p_2^6p_4^5p_5^2p_6^4p_3^9 + 413887320p_1^2p_2^5p_4^6p_5^3p_6^5p_3^8 + 356548500p_1^2p_2^6p_4^5p_5^3p_6^5p_3^8$$
$$+ 1189465200p_1^3p_2^5p_4^5p_5^3p_6^5p_3^8 + 356548500p_1^3p_2^5p_4^6p_5^2p_6^5p_3^8 + 413887320p_1^3p_2^6p_4^5p_5^2p_6^5p_3^8 + 268939440p_1^2p_2^5p_4^6p_5^4p_6^4p_3^8$$
$$+ 48024900p_1^3p_2^5p_4^5p_5^4p_6^4p_3^8 + 133402500p_1^2p_2^5p_4^7p_5^3p_6^4p_3^8 + 672348600p_1^2p_2^6p_4^6p_5^3p_6^4p_3^8 + 4320547560p_1^3p_2^5p_4^6p_5^3p_6^4p_3^8$$
$$+ 4320547560p_1^3p_2^6p_4^5p_5^3p_6^4p_3^8 + 48024900p_1^4p_2^5p_4^5p_5^3p_6^4p_3^8 + 672348600p_1^3p_2^6p_4^6p_5^2p_6^4p_3^8 + 133402500p_1^3p_2^7p_4^5p_5^2p_6^4p_3^8$$
$$+ 268939440p_1^4p_2^6p_4^5p_5^2p_6^4p_3^8 + 35218260p_1^3p_2^5p_4^6p_5^4p_6^3p_3^8 + 180457200p_1^3p_2^6p_4^6p_5^3p_6^3p_3^8 + 35218260p_1^4p_2^6p_4^5p_5^3p_6^3p_3^8$$

$+ 119528640p_1^3p_2^5p_4^6p_5^4p_6^4p_3^7 + 398428800p_1^3p_2^6p_4^6p_5^3p_6^4p_3^7 + 119528640p_1^4p_2^6p_4^5p_5^3p_6^4p_3^7)z^{29} + (651974400p_1^2p_2^5p_4^5p_5^2p_6^5p_3^9$
$+ 3585859200p_1^2p_2^5p_4^5p_5^3p_6^4p_3^9 + 1075757760p_1^2p_2^5p_4^6p_5^2p_6^4p_3^9 + 1075757760p_1^2p_2^6p_4^5p_5^2p_6^4p_3^9 + 3585859200p_1^3p_2^5p_4^5p_5^2p_6^4p_3^9$
$+ 54573750p_1^2p_2^4p_4^6p_5^3p_6^5p_3^8 + 1671169500p_1^2p_2^5p_4^5p_5^3p_6^5p_3^8 + 298045440p_1^2p_2^5p_4^6p_5^2p_6^5p_3^8 + 298045440p_1^2p_2^6p_4^5p_5^2p_6^5p_3^8$
$+ 1671169500p_1^3p_2^5p_4^5p_5^2p_6^5p_3^8 + 54573750p_1^3p_2^6p_4^4p_5^2p_6^5p_3^8 + 24502500p_1^2p_2^4p_4^6p_5^4p_6^4p_3^8 + 48024900p_1^2p_2^5p_4^5p_5^4p_6^4p_3^8$
$+ 4609356210p_1^2p_2^5p_4^6p_5^3p_6^4p_3^8 + 300155625p_1^3p_2^4p_4^6p_5^3p_6^4p_3^8 + 3054383640p_1^2p_2^6p_4^5p_5^3p_6^4p_3^8 + 14283282150p_1^3p_2^5p_4^5p_5^3p_6^4p_3^8$
$+ 300155625p_1^3p_2^6p_4^4p_5^3p_6^4p_3^8 + 446054400p_1^2p_2^6p_4^6p_5^2p_6^4p_3^8 + 3054383640p_1^3p_2^5p_4^6p_5^2p_6^4p_3^8 + 4609356210p_1^3p_2^6p_4^5p_5^2p_6^4p_3^8$
$+ 48024900p_1^4p_2^5p_4^5p_5^2p_6^4p_3^8 + 24502500p_1^4p_2^6p_4^4p_5^2p_6^4p_3^8 + 35218260p_1^2p_2^5p_4^6p_5^4p_6^3p_3^8 + 117394200p_1^2p_2^6p_4^6p_5^3p_6^3p_3^8$
$+ 607296690p_1^3p_2^5p_4^6p_5^3p_6^3p_3^8 + 607296690p_1^3p_2^6p_4^5p_5^3p_6^3p_3^8 + 117394200p_1^3p_2^6p_4^6p_5^2p_6^3p_3^8 + 35218260p_1^4p_2^6p_4^5p_5^2p_6^3p_3^8$
$+ 499167900p_1^3p_2^5p_4^5p_5^3p_6^5p_3^7 + 186763500p_1^2p_2^5p_4^6p_5^4p_6^4p_3^7 + 56029050p_1^3p_2^5p_4^5p_5^4p_6^4p_3^7 + 2655776970p_1^3p_2^5p_4^6p_5^3p_6^4p_3^7$
$+ 2655776970p_1^3p_2^6p_4^5p_5^3p_6^4p_3^7 + 56029050p_1^4p_2^5p_4^5p_5^3p_6^4p_3^7 + 186763500p_1^4p_2^6p_4^5p_5^2p_6^4p_3^7 + 41087970p_1^3p_2^5p_4^6p_5^4p_6^3p_3^7$
$+ 136959900p_1^3p_2^6p_4^6p_5^3p_6^3p_3^7 + 41087970p_1^4p_2^6p_4^5p_5^3p_6^3p_3^7)z^{28} + (4079910912p_1^2p_2^5p_4^5p_5^2p_6^4p_3^9 + 323400000p_1^2p_2^4p_4^5p_5^3p_6^5p_3^8$
$+ 2253071744p_1^2p_2^5p_4^5p_5^2p_6^5p_3^8 + 323400000p_1^3p_2^5p_4^4p_5^2p_6^5p_3^8 + 11383680p_1p_2^5p_4^6p_5^3p_6^4p_3^8 + 830060000p_1^2p_2^4p_4^6p_5^3p_6^4p_3^8$
$+ 18476731056p_1^2p_2^5p_4^5p_5^3p_6^4p_3^8 + 1778700000p_1^3p_2^4p_4^5p_5^3p_6^4p_3^8 + 1778700000p_1^3p_2^5p_4^4p_5^3p_6^4p_3^8 + 4063973760p_1^2p_2^5p_4^6p_5^2p_6^4p_3^8$
$+ 4063973760p_1^2p_2^6p_4^5p_5^2p_6^4p_3^8 + 18476731056p_1^3p_2^5p_4^5p_5^2p_6^4p_3^8 + 830060000p_1^3p_2^6p_4^4p_5^2p_6^4p_3^8 + 11383680p_1^3p_2^6p_4^5p_5p_6^4p_3^8$
$+ 871627680p_1^2p_2^5p_4^6p_5^3p_6^3p_3^8 + 766975440p_1^2p_2^6p_4^5p_5^3p_6^3p_3^8 + 2414513024p_1^3p_2^5p_4^5p_5^3p_6^3p_3^8 + 766975440p_1^3p_2^5p_4^6p_5^2p_6^3p_3^8$
$+ 871627680p_1^3p_2^6p_4^5p_5^2p_6^3p_3^8 + 887409600p_1^2p_2^5p_4^5p_5^3p_6^5p_3^7 + 887409600p_1^3p_2^5p_4^5p_5^2p_6^5p_3^7 + 50820000p_1^2p_2^4p_4^6p_5^4p_6^4p_3^7$
$+ 99607200p_1^2p_2^5p_4^5p_5^4p_6^4p_3^7 + 3252073440p_1^2p_2^5p_4^6p_5^3p_6^4p_3^7 + 622545000p_1^3p_2^4p_4^6p_5^3p_6^4p_3^7 + 2075150000p_1^2p_2^6p_4^5p_5^3p_6^4p_3^7$
$+ 19480302576p_1^3p_2^5p_4^5p_5^3p_6^4p_3^7 + 622545000p_1^3p_2^6p_4^4p_5^3p_6^4p_3^7 + 2075150000p_1^3p_2^5p_4^6p_5^2p_6^4p_3^7 + 3252073440p_1^3p_2^6p_4^5p_5^2p_6^4p_3^7$
$+ 99607200p_1^4p_2^5p_4^5p_5^2p_6^4p_3^7 + 50820000p_1^4p_2^6p_4^4p_5^2p_6^4p_3^7 + 73045280p_1^2p_2^5p_4^6p_5^4p_6^3p_3^7 + 21913584p_1^3p_2^5p_4^5p_5^4p_6^3p_3^7$
$+ 1016898960p_1^3p_2^5p_4^6p_5^3p_6^3p_3^7 + 1016898960p_1^3p_2^6p_4^5p_5^3p_6^3p_3^7 + 21913584p_1^4p_2^5p_4^5p_5^3p_6^3p_3^7 + 73045280p_1^4p_2^6p_4^5p_5^2p_6^3p_3^7)z^{27}$
$+ (356548500p_1^2p_2^4p_4^5p_5^2p_6^5p_3^8 + 356548500p_1^2p_2^5p_4^4p_5^2p_6^5p_3^8 + 48024900p_1p_2^4p_4^6p_5^3p_6^4p_3^8 + 94128804p_1p_2^5p_4^5p_5^3p_6^4p_3^8$
$+ 5042614500p_1^2p_2^4p_4^5p_5^3p_6^4p_3^8 + 1961016750p_1^2p_2^5p_4^4p_5^3p_6^4p_3^8 + 588305025p_1^2p_2^4p_4^6p_5^2p_6^4p_3^8 + 27835512516p_1^2p_2^5p_4^5p_5^2p_6^4p_3^8$
$+ 1961016750p_1^3p_2^4p_4^5p_5^2p_6^4p_3^8 + 588305025p_1^2p_2^6p_4^4p_5^2p_6^4p_3^8 + 5042614500p_1^3p_2^5p_4^4p_5^2p_6^4p_3^8 + 94128804p_1^3p_2^5p_4^5p_5p_6^4p_3^8$
$+ 48024900p_1^3p_2^6p_4^4p_5p_6^4p_3^8 + 264136950p_1^2p_2^4p_4^6p_5^3p_6^3p_3^8 + 4790284884p_1^2p_2^5p_4^5p_5^3p_6^3p_3^8 + 880456500p_1^2p_2^5p_4^6p_5^2p_6^3p_3^8$
$+ 880456500p_1^2p_2^6p_4^5p_5^2p_6^3p_3^8 + 4790284884p_1^3p_2^5p_4^5p_5^2p_6^3p_3^8 + 264136950p_1^3p_2^6p_4^4p_5^2p_6^3p_3^8 + 305613000p_1^2p_2^4p_4^5p_5^3p_6^5p_3^7$
$+ 1669054464p_1^2p_2^5p_4^5p_5^2p_6^5p_3^7 + 305613000p_1^3p_2^5p_4^4p_5^2p_6^5p_3^7 + 34303500p_1^2p_2^4p_4^5p_5^4p_6^4p_3^7 + 1413304200p_1^2p_2^4p_4^6p_5^3p_6^4p_3^7$
$+ 30824064054p_1^2p_2^5p_4^5p_5^3p_6^4p_3^7 + 6565308750p_1^3p_2^4p_4^5p_5^3p_6^4p_3^7 + 6565308750p_1^3p_2^5p_4^4p_5^3p_6^4p_3^7 + 3902976000p_1^2p_2^5p_4^6p_5^2p_6^4p_3^7$
$+ 3902976000p_1^2p_2^6p_4^5p_5^2p_6^4p_3^7 + 30824064054p_1^3p_2^5p_4^5p_5^2p_6^4p_3^7 + 1413304200p_1^3p_2^6p_4^4p_5^2p_6^4p_3^7 + 34303500p_1^4p_2^5p_4^4p_5^2p_6^4p_3^7$
$+ 25155900p_1^2p_2^4p_4^6p_5^4p_6^3p_3^7 + 49305564p_1^2p_2^5p_4^5p_5^4p_6^3p_3^7 + 1445944500p_1^2p_2^5p_4^6p_5^3p_6^3p_3^7 + 308159775p_1^3p_2^4p_4^6p_5^3p_6^3p_3^7$
$+ 1027199250p_1^2p_2^6p_4^5p_5^3p_6^3p_3^7 + 8951787306p_1^3p_2^5p_4^5p_5^3p_6^3p_3^7 + 308159775p_1^3p_2^6p_4^4p_5^3p_6^3p_3^7 + 1027199250p_1^3p_2^5p_4^6p_5^2p_6^3p_3^7$
$+ 1445944500p_1^3p_2^6p_4^5p_5^2p_6^3p_3^7 + 49305564p_1^4p_2^5p_4^5p_5^2p_6^3p_3^7 + 25155900p_1^4p_2^6p_4^4p_5^2p_6^3p_3^7 + 8199664704p_1^3p_2^5p_4^5p_5^3p_6^4p_3^6)z^{26}$
$+ (409812480p_1p_2^4p_4^5p_5^3p_6^4p_3^8 + 122943744p_1p_2^5p_4^5p_5^2p_6^4p_3^8 + 7991343360p_1^2p_2^4p_4^5p_5^2p_6^4p_3^8 + 7991343360p_1^2p_2^5p_4^4p_5^2p_6^4p_3^8$
$+ 122943744p_1^2p_2^5p_4^5p_5p_6^4p_3^8 + 409812480p_1^3p_2^5p_4^4p_5p_6^4p_3^8 + 2253968640p_1^2p_2^4p_4^5p_5^3p_6^3p_3^8 + 8611029504p_1^2p_2^5p_4^5p_5^2p_6^3p_3^8$
$+ 2253968640p_1^3p_2^5p_4^4p_5^2p_6^3p_3^8 + 599001480p_1^2p_2^4p_4^5p_5^2p_6^5p_3^7 + 599001480p_1^2p_2^5p_4^4p_5^2p_6^5p_3^7 + 114345000p_1p_2^4p_4^6p_5^3p_6^4p_3^7$
$+ 224116200p_1p_2^5p_4^5p_5^3p_6^4p_3^7 + 19609868880p_1^2p_2^4p_4^5p_5^3p_6^4p_3^7 + 9158882040p_1^2p_2^5p_4^4p_5^3p_6^4p_3^7 + 2858625000p_1^3p_2^4p_4^4p_5^3p_6^4p_3^7$
$+ 1400726250p_1^2p_2^4p_4^6p_5^2p_6^4p_3^7 + 59798117736p_1^2p_2^5p_4^5p_5^2p_6^4p_3^7 + 9158882040p_1^3p_2^4p_4^5p_5^2p_6^4p_3^7 + 1400726250p_1^2p_2^6p_4^4p_5^2p_6^4p_3^7$
$+ 19609868880p_1^3p_2^5p_4^4p_5^2p_6^4p_3^7 + 224116200p_1^3p_2^5p_4^5p_5p_6^4p_3^7 + 114345000p_1^3p_2^6p_4^4p_5p_6^4p_3^7 + 30187080p_1^2p_2^4p_4^5p_5^4p_6^3p_3^7$
$+ 966735000p_1^2p_2^4p_4^6p_5^3p_6^3p_3^7 + 17946116496p_1^2p_2^5p_4^5p_5^3p_6^3p_3^7 + 3421632060p_1^3p_2^4p_4^5p_5^3p_6^3p_3^7 + 3421632060p_1^3p_2^5p_4^4p_5^3p_6^3p_3^7$
$+ 2096325000p_1^2p_2^5p_4^6p_5^2p_6^3p_3^7 + 2096325000p_1^2p_2^6p_4^5p_5^2p_6^3p_3^7 + 17946116496p_1^3p_2^5p_4^5p_5^2p_6^3p_3^7 + 966735000p_1^3p_2^6p_4^4p_5^2p_6^3p_3^7$
$+ 30187080p_1^4p_2^5p_4^4p_5^2p_6^3p_3^7 + 439267752p_1^3p_2^5p_4^5p_5^3p_6^2p_3^7 + 16734009600p_1^2p_2^5p_4^5p_5^3p_6^4p_3^6 + 5020202880p_1^3p_2^4p_4^5p_5^3p_6^4p_3^6$
$+ 5020202880p_1^3p_2^5p_4^4p_5^3p_6^4p_3^6 + 16734009600p_1^3p_2^5p_4^5p_5^2p_6^4p_3^6 + 6754454784p_1^3p_2^5p_4^5p_5^3p_6^3p_3^6)z^{25} + (557800320p_1p_2^4p_4^5p_5^2p_6^4p_3^8$
$+ 1859334400p_1^2p_2^4p_4^4p_5^2p_6^4p_3^8 + 557800320p_1^2p_2^5p_4^4p_5p_6^4p_3^8 + 3067901760p_1^2p_2^4p_4^5p_5^2p_6^3p_3^8 + 3067901760p_1^2p_2^5p_4^4p_5^2p_6^3p_3^8$
$+ 221852400p_1^2p_2^4p_4^4p_5^2p_6^5p_3^7 + 2212838320p_1p_2^4p_4^5p_5^3p_6^4p_3^7 + 1985156250p_1^2p_2^3p_4^5p_5^3p_6^4p_3^7 + 6097637700p_1^2p_2^4p_4^4p_5^3p_6^4p_3^7$
$+ 520396800p_1p_2^5p_4^5p_5^2p_6^4p_3^7 + 40830215370p_1^2p_2^4p_4^5p_5^2p_6^4p_3^7 + 40830215370p_1^2p_2^5p_4^4p_5^2p_6^4p_3^7 + 6097637700p_1^3p_2^4p_4^4p_5^2p_6^4p_3^7$
$+ 1985156250p_1^3p_2^5p_4^3p_5^2p_6^4p_3^7 + 520396800p_1^2p_2^5p_4^5p_5p_6^4p_3^7 + 2212838320p_1^3p_2^5p_4^4p_5p_6^4p_3^7 + 69877500p_1p_2^4p_4^6p_5^3p_6^3p_3^7$
$+ 136959900p_1p_2^5p_4^5p_5^3p_6^3p_3^7 + 16980581310p_1^2p_2^4p_4^5p_5^3p_6^3p_3^7 + 5281747240p_1^2p_2^5p_4^4p_5^3p_6^3p_3^7 + 2862182400p_1^3p_2^4p_4^4p_5^3p_6^3p_3^7$
$+ 855999375p_1^2p_2^4p_4^6p_5^2p_6^3p_3^7 + 41763159900p_1^2p_2^5p_4^5p_5^2p_6^3p_3^7 + 5281747240p_1^3p_2^4p_4^5p_5^2p_6^3p_3^7 + 855999375p_1^2p_2^6p_4^4p_5^2p_6^3p_3^7$
$+ 16980581310p_1^3p_2^5p_4^4p_5^2p_6^3p_3^7 + 136959900p_1^3p_2^5p_4^5p_5p_6^3p_3^7 + 69877500p_1^3p_2^6p_4^4p_5p_6^3p_3^7 + 1220188200p_1^2p_2^5p_4^5p_5^3p_6^2p_3^7$
$+ 1220188200p_1^3p_2^5p_4^5p_5^2p_6^2p_3^7 + 18993314340p_1^2p_2^4p_4^5p_5^3p_6^4p_3^6 + 10676646750p_1^2p_2^5p_4^4p_5^3p_6^4p_3^6 + 3890016900p_1^3p_2^4p_4^4p_5^3p_6^4p_3^6$
$+ 38856294400p_1^2p_2^5p_4^5p_5^2p_6^4p_3^6 + 10676646750p_1^3p_2^4p_4^5p_5^2p_6^4p_3^6 + 18993314340p_1^3p_2^5p_4^4p_5^2p_6^4p_3^6 + 3913140p_1^2p_2^4p_4^5p_5^4p_6^3p_3^6$
$+ 81523750p_1^2p_2^4p_4^6p_5^3p_6^3p_3^6 + 16798204500p_1^2p_2^5p_4^5p_5^3p_6^3p_3^6 + 5039461350p_1^3p_2^4p_4^5p_5^3p_6^3p_3^6 + 5039461350p_1^3p_2^5p_4^4p_5^3p_6^3p_3^6$
$+ 16798204500p_1^3p_2^5p_4^5p_5^2p_6^3p_3^6 + 81523750p_1^3p_2^6p_4^4p_5^2p_6^3p_3^6 + 3913140p_1^4p_2^5p_4^4p_5^2p_6^3p_3^6 + 1423552900p_1^3p_2^5p_4^5p_5^3p_6^2p_3^6)z^{24}$
$+ (457380000p_1p_2^3p_4^5p_5^3p_6^4p_3^7 + 896464800p_1p_2^4p_4^4p_5^3p_6^4p_3^7 + 4201797600p_1p_2^4p_4^5p_5^2p_6^4p_3^7 + 5602905000p_1^2p_2^3p_4^5p_5^2p_6^4p_3^7$
$+ 268939440p_1p_2^5p_4^4p_5^2p_6^4p_3^7 + 32647658400p_1^2p_2^4p_4^4p_5^2p_6^4p_3^7 + 5602905000p_1^2p_2^5p_4^3p_5^2p_6^4p_3^7 + 268939440p_1^2p_2^4p_4^5p_5p_6^4p_3^7$
$+ 4201797600p_1^2p_2^5p_4^4p_5p_6^4p_3^7 + 896464800p_1^3p_2^4p_4^4p_5p_6^4p_3^7 + 457380000p_1^3p_2^5p_4^3p_5p_6^4p_3^7 + 1537683840p_1p_2^4p_4^5p_5^3p_6^3p_3^7$
$+ 2515590000p_1^2p_2^3p_4^5p_5^3p_6^3p_3^7 + 6940533600p_1^2p_2^4p_4^4p_5^3p_6^3p_3^7 + 402494400p_1p_2^5p_4^5p_5^2p_6^3p_3^7 + 38328942960p_1^2p_2^4p_4^5p_5^2p_6^3p_3^7$
$+ 38328942960p_1^2p_2^5p_4^4p_5^2p_6^3p_3^7 + 6940533600p_1^3p_2^4p_4^4p_5^2p_6^3p_3^7 + 2515590000p_1^3p_2^5p_4^3p_5^2p_6^3p_3^7 + 402494400p_1^2p_2^5p_4^5p_5p_6^3p_3^7$
$+ 1537683840p_1^3p_2^5p_4^4p_5p_6^3p_3^7 + 1075757760p_1^2p_2^4p_4^5p_5^3p_6^2p_3^7 + 3585859200p_1^2p_2^5p_4^5p_5^2p_6^2p_3^7 + 1075757760p_1^3p_2^5p_4^4p_5^2p_6^2p_3^7$
$+ 2561328000p_1p_2^4p_4^5p_5^3p_6^4p_3^6 + 4802490000p_1^2p_2^3p_4^5p_5^3p_6^4p_3^6 + 14386125600p_1^2p_2^4p_4^4p_5^3p_6^4p_3^6 + 48870138240p_1^2p_2^4p_4^5p_5^2p_6^4p_3^6$

$$
\begin{aligned}
&+ 48870138240p_1^2p_2^5p_4^4p_5^2p_6^4p_3^6 + 14386125600p_1^3p_2^4p_4^4p_5^2p_6^4p_3^6 + 4802490000p_1^3p_2^5p_4^3p_5^2p_6^4p_3^6 + 2561328000p_1^3p_2^5p_4^4p_5p_6^4p_3^6 \\
&+ 28131531120p_1^2p_2^4p_4^5p_5^3p_6^3p_3^6 + 12664602720p_1^2p_2^5p_4^4p_5^3p_6^3p_3^6 + 6411081600p_1^3p_2^4p_4^4p_5^3p_6^3p_3^6 + 45964195200p_1^2p_2^5p_4^5p_5^2p_6^3p_3^6 \\
&+ 12664602720p_1^3p_2^4p_4^5p_5^2p_6^3p_3^6 + 28131531120p_1^3p_2^5p_4^4p_5^2p_6^3p_3^6 + 4183502400p_1^2p_2^5p_4^5p_5^3p_6^2p_3^6 + 1255050720p_1^3p_2^4p_4^5p_5^3p_6^2p_3^6 \\
&+ 1255050720p_1^3p_2^5p_4^4p_5^3p_6^2p_3^6 + 4183502400p_1^3p_2^5p_4^5p_5^2p_6^2p_3^6)z^{23} + (1400726250p_1p_2^3p_4^5p_5^2p_6^4p_3^7 + 3186932364p_1p_2^4p_4^4p_5^2p_6^4p_3^7 \\
&+ 4669087500p_1^2p_2^3p_4^4p_5^2p_6^4p_3^7 + 4669087500p_1^2p_2^4p_4^3p_5^2p_6^4p_3^7 + 3186932364p_1^2p_2^4p_4^4p_5p_6^4p_3^7 + 1400726250p_1^2p_2^5p_4^3p_5p_6^4p_3^7 \\
&+ 628897500p_1p_2^3p_4^5p_5^3p_6^3p_3^7 + 1232639100p_1p_2^4p_4^4p_5^3p_6^3p_3^7 + 3421632060p_1p_2^4p_4^5p_5^2p_6^3p_3^7 + 7703994375p_1^2p_2^3p_4^5p_5^2p_6^3p_3^7 \\
&+ 369791730p_1p_2^5p_4^4p_5^2p_6^3p_3^7 + 38630489436p_1^2p_2^4p_4^4p_5^2p_6^3p_3^7 + 7703994375p_1^2p_2^5p_4^3p_5^2p_6^3p_3^7 + 369791730p_1^2p_2^4p_4^5p_5p_6^3p_3^7 \\
&+ 3421632060p_1^2p_2^5p_4^4p_5p_6^3p_3^7 + 1232639100p_1^3p_2^4p_4^4p_5p_6^3p_3^7 + 628897500p_1^3p_2^5p_4^3p_5p_6^3p_3^7 + 3294508140p_1^2p_2^4p_4^5p_5^2p_6^2p_3^7 \\
&+ 3294508140p_1^2p_2^5p_4^4p_5^2p_6^2p_3^7 + 1200622500p_1p_2^3p_4^5p_5^3p_6^4p_3^6 + 2353220100p_1p_2^4p_4^4p_5^3p_6^4p_3^6 + 4002075000p_1^2p_2^3p_4^4p_5^3p_6^4p_3^6 \\
&+ 6556999680p_1p_2^4p_4^5p_5^2p_6^4p_3^6 + 14707625625p_1^2p_2^3p_4^5p_5^2p_6^4p_3^6 + 76881768516p_1^2p_2^4p_4^4p_5^2p_6^4p_3^6 + 14707625625p_1^2p_2^5p_4^3p_5^2p_6^4p_3^6 \\
&+ 4002075000p_1^3p_2^4p_4^3p_5^2p_6^4p_3^6 + 6556999680p_1^2p_2^5p_4^4p_5p_6^4p_3^6 + 2353220100p_1^3p_2^4p_4^4p_5p_6^4p_3^6 + 1200622500p_1^3p_2^5p_4^3p_5p_6^4p_3^6 \\
&+ 3447314640p_1p_2^4p_4^5p_5^3p_6^3p_3^6 + 10017315000p_1^2p_2^3p_4^5p_5^3p_6^3p_3^6 + 27874845306p_1^2p_2^4p_4^4p_5^3p_6^3p_3^6 + 80030488230p_1^2p_2^4p_4^5p_5^2p_6^3p_3^6 \\
&+ 80030488230p_1^2p_2^5p_4^4p_5^2p_6^3p_3^6 + 27874845306p_1^3p_2^4p_4^4p_5^2p_6^3p_3^6 + 10017315000p_1^3p_2^5p_4^3p_5^2p_6^3p_3^6 + 3447314640p_1^3p_2^5p_4^4p_5p_6^3p_3^6 \\
&+ 8519617260p_1^2p_2^4p_4^5p_5^3p_6^2p_3^6 + 3843592830p_1^2p_2^5p_4^4p_5^3p_6^2p_3^6 + 1967099904p_1^3p_2^4p_4^4p_5^3p_6^2p_3^6 + 13340250000p_1^2p_2^5p_4^5p_5^2p_6^2p_3^6 \\
&+ 3843592830p_1^3p_2^4p_4^5p_5^2p_6^2p_3^6 + 8519617260p_1^3p_2^5p_4^4p_5^2p_6^2p_3^6 + 2629630080p_1^2p_2^4p_4^5p_5^3p_6^3p_3^5 + 788889024p_1^3p_2^4p_4^4p_5^3p_6^3p_3^5 \\
&+ 2629630080p_1^3p_2^5p_4^4p_5^2p_6^3p_3^5)z^{22} + (1328096000p_1p_2^3p_4^4p_5^2p_6^4p_3^7 + 398428800p_1p_2^4p_4^4p_5p_6^4p_3^7 + 1328096000p_1^2p_2^4p_4^3p_5p_6^4p_3^7 \\
&+ 2191358400p_1p_2^3p_4^5p_5^2p_6^3p_3^7 + 4881115008p_1p_2^4p_4^4p_5^2p_6^3p_3^7 + 7304528000p_1^2p_2^3p_4^4p_5^2p_6^3p_3^7 + 7304528000p_1^2p_2^4p_4^3p_5^2p_6^3p_3^7 \\
&+ 4881115008p_1^2p_2^4p_4^4p_5p_6^3p_3^7 + 2191358400p_1^2p_2^5p_4^3p_5p_6^3p_3^7 + 3123681792p_1^2p_2^4p_4^4p_5^2p_6^2p_3^7 + 1138368000p_1p_2^3p_4^4p_5^3p_6^4p_3^6 \\
&+ 3890016900p_1p_2^3p_4^5p_5^2p_6^4p_3^6 + 10875161528p_1p_2^4p_4^4p_5^2p_6^4p_3^6 + 28921662000p_1^2p_2^3p_4^4p_5^2p_6^4p_3^6 + 28921662000p_1^2p_2^4p_4^3p_5^2p_6^4p_3^6 \\
&+ 10875161528p_1^2p_2^4p_4^4p_5p_6^4p_3^6 + 3890016900p_1^2p_2^5p_4^3p_5p_6^4p_3^6 + 1138368000p_1^3p_2^4p_4^3p_5p_6^4p_3^6 + 2752521200p_1p_2^3p_4^5p_5^3p_6^3p_3^6 \\
&+ 5394941552p_1p_2^4p_4^4p_5^3p_6^3p_3^6 + 10284615000p_1^2p_2^3p_4^4p_5^3p_6^3p_3^6 + 97828500p_1^2p_2^4p_4^3p_5^3p_6^3p_3^6 + 10284615000p_1p_2^4p_4^5p_5^2p_6^3p_3^6 \\
&+ 33718384700p_1^2p_2^3p_4^5p_5^2p_6^3p_3^6 + 97828500p_1p_2^5p_4^4p_5^2p_6^3p_3^6 + 163830961008p_1^2p_2^4p_4^4p_5^2p_6^3p_3^6 + 97828500p_1^3p_2^3p_4^4p_5^2p_6^3p_3^6 \\
&+ 33718384700p_1^2p_2^5p_4^3p_5^2p_6^3p_3^6 + 10284615000p_1^3p_2^4p_4^3p_5^2p_6^3p_3^6 + 97828500p_1^2p_2^4p_4^5p_5p_6^3p_3^6 + 10284615000p_1^2p_2^5p_4^4p_5p_6^3p_3^6 \\
&+ 5394941552p_1^3p_2^4p_4^4p_5p_6^3p_3^6 + 2752521200p_1^3p_2^5p_4^3p_5p_6^3p_3^6 + 1138368000p_1p_2^4p_4^5p_5^3p_6^2p_3^6 + 3890016900p_1^2p_2^3p_4^5p_5^3p_6^2p_3^6 \\
&+ 10875161528p_1^2p_2^4p_4^4p_5^3p_6^2p_3^6 + 28921662000p_1^2p_2^4p_4^5p_5^2p_6^2p_3^6 + 28921662000p_1^2p_2^5p_4^4p_5^2p_6^2p_3^6 + 10875161528p_1^3p_2^4p_4^4p_5^2p_6^2p_3^6 \\
&+ 3890016900p_1^3p_2^5p_4^3p_5^2p_6^2p_3^6 + 1138368000p_1^3p_2^5p_4^4p_5p_6^2p_3^6 + 3123681792p_1^2p_2^4p_4^4p_5^2p_6^4p_3^5 + 2191358400p_1^2p_2^3p_4^5p_5^3p_6^3p_3^5 \\
&+ 4881115008p_1^2p_2^4p_4^4p_5^3p_6^3p_3^5 + 7304528000p_1^2p_2^4p_4^5p_5^2p_6^3p_3^5 + 7304528000p_1^2p_2^5p_4^4p_5^2p_6^3p_3^5 + 4881115008p_1^3p_2^4p_4^4p_5^2p_6^3p_3^5 \\
&+ 2191358400p_1^3p_2^5p_4^3p_5^2p_6^3p_3^5 + 1328096000p_1^2p_2^4p_4^5p_5^3p_6^2p_3^5 + 398428800p_1^3p_2^4p_4^4p_5^3p_6^2p_3^5 + 1328096000p_1^3p_2^5p_4^4p_5^2p_6^2p_3^5)z^{21} \\
&+ (2629630080p_1p_2^3p_4^4p_5^2p_6^3p_3^7 + 788889024p_1p_2^4p_4^4p_5p_6^3p_3^7 + 2629630080p_1^2p_2^4p_4^3p_5p_6^3p_3^7 + 8519617260p_1p_2^3p_4^4p_5^2p_6^4p_3^6 \\
&+ 3843592830p_1p_2^4p_4^3p_5^2p_6^4p_3^6 + 13340250000p_1^2p_2^3p_4^3p_5^2p_6^4p_3^6 + 1967099904p_1p_2^4p_4^4p_5p_6^4p_3^6 + 3843592830p_1^2p_2^3p_4^4p_5p_6^4p_3^6 \\
&+ 8519617260p_1^2p_2^4p_4^3p_5p_6^4p_3^6 + 3447314640p_1p_2^3p_4^4p_5^3p_6^3p_3^6 + 10017315000p_1p_2^3p_4^5p_5^2p_6^3p_3^6 + 27874845306p_1p_2^4p_4^4p_5^2p_6^3p_3^6 \\
&+ 80030488230p_1^2p_2^3p_4^4p_5^2p_6^3p_3^6 + 80030488230p_1^2p_2^4p_4^3p_5^2p_6^3p_3^6 + 27874845306p_1^2p_2^4p_4^4p_5p_6^3p_3^6 + 10017315000p_1^2p_2^5p_4^3p_5p_6^3p_3^6 \\
&+ 3447314640p_1^3p_2^4p_4^3p_5p_6^3p_3^6 + 1200622500p_1p_2^3p_4^5p_5^3p_6^2p_3^6 + 2353220100p_1p_2^4p_4^4p_5^3p_6^2p_3^6 + 6556999680p_1^2p_2^3p_4^4p_5^3p_6^2p_3^6 \\
&+ 4002075000p_1p_2^4p_4^5p_5^2p_6^2p_3^6 + 14707625625p_1^2p_2^3p_4^5p_5^2p_6^2p_3^6 + 76881768516p_1^2p_2^4p_4^4p_5^2p_6^2p_3^6 + 14707625625p_1^2p_2^5p_4^3p_5^2p_6^2p_3^6 \\
&+ 6556999680p_1^3p_2^4p_4^3p_5^2p_6^2p_3^6 + 4002075000p_1^2p_2^5p_4^4p_5p_6^2p_3^6 + 2353220100p_1^3p_2^4p_4^4p_5p_6^2p_3^6 + 1200622500p_1^3p_2^5p_4^3p_5p_6^2p_3^6 \\
&+ 3294508140p_1^2p_2^3p_4^4p_5^2p_6^4p_3^5 + 3294508140p_1^2p_2^4p_4^3p_5^2p_6^4p_3^5 + 628897500p_1p_2^3p_4^5p_5^3p_6^3p_3^5 + 1232639100p_1p_2^4p_4^4p_5^3p_6^3p_3^5 \\
&+ 3421632060p_1^2p_2^3p_4^4p_5^3p_6^3p_3^5 + 369791730p_1^2p_2^4p_4^3p_5^3p_6^3p_3^5 + 7703994375p_1^2p_2^3p_4^5p_5^2p_6^3p_3^5 + 38630489436p_1^2p_2^4p_4^4p_5^2p_6^3p_3^5 \\
&+ 369791730p_1^3p_2^3p_4^4p_5^2p_6^3p_3^5 + 7703994375p_1^2p_2^5p_4^3p_5^2p_6^3p_3^5 + 3421632060p_1^3p_2^4p_4^3p_5^2p_6^3p_3^5 + 1232639100p_1^3p_2^4p_4^4p_5p_6^3p_3^5 \\
&+ 628897500p_1^3p_2^5p_4^3p_5p_6^3p_3^5 + 1400726250p_1^2p_2^3p_4^5p_5^3p_6^2p_3^5 + 3186932364p_1^2p_2^4p_4^4p_5^3p_6^2p_3^5 + 4669087500p_1^2p_2^4p_4^5p_5^2p_6^2p_3^5 \\
&+ 4669087500p_1^2p_2^5p_4^4p_5^2p_6^2p_3^5 + 3186932364p_1^3p_2^4p_4^4p_5^2p_6^2p_3^5 + 1400726250p_1^3p_2^5p_4^3p_5^2p_6^2p_3^5)z^{20} + (4183502400p_1p_2^3p_4^3p_5^2p_6^4p_3^6 \\
&+ 1255050720p_1p_2^3p_4^4p_5p_6^4p_3^6 + 1255050720p_1p_2^4p_4^3p_5p_6^4p_3^6 + 4183502400p_1^2p_2^3p_4^3p_5p_6^4p_3^6 + 28131531120p_1p_2^3p_4^4p_5^2p_6^3p_3^6 \\
&+ 12664602720p_1p_2^4p_4^3p_5^2p_6^3p_3^6 + 45964195200p_1^2p_2^3p_4^3p_5^2p_6^3p_3^6 + 6411081600p_1p_2^4p_4^4p_5p_6^3p_3^6 + 12664602720p_1^2p_2^3p_4^4p_5p_6^3p_3^6 \\
&+ 28131531120p_1^2p_2^4p_4^3p_5p_6^3p_3^6 + 2561328000p_1p_2^3p_4^4p_5^3p_6^2p_3^6 + 4802490000p_1p_2^3p_4^5p_5^2p_6^2p_3^6 + 14386125600p_1p_2^4p_4^4p_5^2p_6^2p_3^6 \\
&+ 48870138240p_1^2p_2^3p_4^4p_5^2p_6^2p_3^6 + 48870138240p_1^2p_2^4p_4^3p_5^2p_6^2p_3^6 + 14386125600p_1^2p_2^4p_4^4p_5p_6^2p_3^6 + 4802490000p_1^2p_2^5p_4^3p_5p_6^2p_3^6 \\
&+ 2561328000p_1^3p_2^4p_4^3p_5p_6^2p_3^6 + 1075757760p_1p_2^3p_4^4p_5^2p_6^4p_3^5 + 3585859200p_1^2p_2^3p_4^3p_5^2p_6^4p_3^5 + 1075757760p_1^2p_2^4p_4^3p_5p_6^4p_3^5 \\
&+ 1537683840p_1p_2^3p_4^4p_5^3p_6^3p_3^5 + 402494400p_1^2p_2^3p_4^3p_5^3p_6^3p_3^5 + 2515590000p_1p_2^3p_4^5p_5^2p_6^3p_3^5 + 6940533600p_1p_2^4p_4^4p_5^2p_6^3p_3^5 \\
&+ 38328942960p_1^2p_2^3p_4^4p_5^2p_6^3p_3^5 + 38328942960p_1^2p_2^4p_4^3p_5^2p_6^3p_3^5 + 402494400p_1^3p_2^3p_4^3p_5^2p_6^3p_3^5 + 6940533600p_1^2p_2^4p_4^4p_5p_6^3p_3^5 \\
&+ 2515590000p_1^2p_2^5p_4^3p_5p_6^3p_3^5 + 1537683840p_1^3p_2^4p_4^3p_5p_6^3p_3^5 + 457380000p_1p_2^3p_4^5p_5^3p_6^2p_3^5 + 896464800p_1p_2^4p_4^4p_5^3p_6^2p_3^5 \\
&+ 4201797600p_1^2p_2^3p_4^4p_5^3p_6^2p_3^5 + 268939440p_1^2p_2^4p_4^3p_5^3p_6^2p_3^5 + 5602905000p_1^2p_2^3p_4^5p_5^2p_6^2p_3^5 + 32647658400p_1^2p_2^4p_4^4p_5^2p_6^2p_3^5 \\
&+ 268939440p_1^3p_2^3p_4^4p_5^2p_6^2p_3^5 + 5602905000p_1^2p_2^5p_4^3p_5^2p_6^2p_3^5 + 4201797600p_1^3p_2^4p_4^3p_5^2p_6^2p_3^5 + 896464800p_1^3p_2^4p_4^4p_5p_6^2p_3^5 \\
&+ 457380000p_1^3p_2^5p_4^3p_5p_6^2p_3^5)z^{19} + (1423552900p_1p_2^3p_4^3p_5p_6^4p_3^6 + 3913140p_1^2p_2^4p_4^3p_6^3p_3^6 + 3913140p_2^3p_4^4p_5^2p_6^3p_3^6 \\
&+ 81523750p_1p_2^2p_4^4p_5^2p_6^3p_3^6 + 16798204500p_1p_2^3p_4^3p_5^2p_6^3p_3^6 + 5039461350p_1p_2^3p_4^4p_5p_6^3p_3^6 + 5039461350p_1p_2^4p_4^3p_5p_6^3p_3^6 \\
&+ 16798204500p_1^2p_2^3p_4^3p_5p_6^3p_3^6 + 81523750p_1^2p_2^4p_4^2p_5p_6^3p_3^6 + 18993314340p_1p_2^3p_4^4p_5^2p_6^2p_3^6 + 10676646750p_1p_2^4p_4^3p_5^2p_6^2p_3^6 \\
&+ 38856294400p_1^2p_2^3p_4^3p_5^2p_6^2p_3^6 + 3890016900p_1p_2^4p_4^4p_5p_6^2p_3^6 + 10676646750p_1^2p_2^3p_4^4p_5p_6^2p_3^6 + 18993314340p_1^2p_2^4p_4^3p_5p_6^2p_3^6 \\
&+ 1220188200p_1p_2^3p_4^3p_5^2p_6^4p_3^5 + 1220188200p_1^2p_2^3p_4^3p_5p_6^4p_3^5 + 69877500p_1p_2^2p_4^4p_5^3p_6^3p_3^5 + 136959900p_1p_2^3p_4^3p_5^3p_6^3p_3^5 \\
&+ 16980581310p_1p_2^3p_4^4p_5^2p_6^3p_3^5 + 855999375p_1^2p_2^2p_4^4p_5^2p_6^3p_3^5 + 5281747240p_1p_2^4p_4^3p_5^2p_6^3p_3^5 + 41763159900p_1^2p_2^3p_4^3p_5^2p_6^3p_3^5 \\
&+ 855999375p_1^2p_2^4p_4^2p_5^2p_6^3p_3^5 + 2862182400p_1p_2^4p_4^4p_5p_6^3p_3^5 + 5281747240p_1^2p_2^3p_4^4p_5p_6^3p_3^5 + 16980581310p_1^2p_2^4p_4^3p_5p_6^3p_3^5 \\
&+ 136959900p_1^3p_2^3p_4^3p_5p_6^3p_3^5 + 69877500p_1^3p_2^4p_4^2p_5p_6^3p_3^5 + 2212838320p_1p_2^3p_4^4p_5^3p_6^2p_3^5 + 520396800p_1^2p_2^3p_4^3p_5^3p_6^2p_3^5 \\
&+ 1985156250p_1p_2^3p_4^5p_5^2p_6^2p_3^5 + 6097637700p_1p_2^4p_4^4p_5^2p_6^2p_3^5 + 40830215370p_1^2p_2^3p_4^4p_5^2p_6^2p_3^5 + 40830215370p_1^2p_2^4p_4^3p_5^2p_6^2p_3^5
\end{aligned}
$$

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\begin{aligned}
&+ 520396800p_1^3p_2^3p_4^3p_5^2p_6^2p_3^5 + 6097637700p_1^2p_2^4p_4^4p_5p_6^2p_3^5 + 1985156250p_1^2p_2^5p_4^3p_5p_6^2p_3^5 + 2212838320p_1^3p_2^4p_4^3p_5p_6^2p_3^5 \\
&+ 221852400p_1^2p_2^4p_4^4p_5^2p_6p_3^5 + 3067901760p_1^2p_2^3p_4^4p_5^2p_6^3p_3^4 + 3067901760p_1^2p_2^4p_4^3p_5^2p_6^3p_3^4 + 557800320p_1^2p_2^3p_4^4p_5^3p_6^2p_3^4 \\
&+ 1859334400p_1^2p_2^4p_4^4p_5^2p_6^2p_3^4 + 557800320p_1^3p_2^4p_4^3p_5^2p_6^2p_3^4)z^{18} + (6754454784p_1p_2^3p_4^3p_5p_6^3p_3^6 + 16734009600p_1p_2^3p_4^3p_5^2p_6^2p_3^6 \\
&+ 5020202880p_1p_2^3p_4^4p_5p_6^2p_3^6 + 5020202880p_1p_2^4p_4^3p_5p_6^2p_3^6 + 16734009600p_1^2p_2^3p_4^3p_5p_6^2p_3^6 + 439267752p_1p_2^3p_4^3p_5p_6^4p_3^5 \\
&+ 30187080p_1^2p_2^4p_4^3p_6^3p_3^5 + 30187080p_2^3p_4^4p_5^2p_6^3p_3^5 + 966735000p_1p_2^2p_4^4p_5^2p_6^3p_3^5 + 17946116496p_1p_2^3p_4^3p_5^2p_6^3p_3^5 \\
&+ 2096325000p_1^2p_2^2p_4^3p_5^2p_6^3p_3^5 + 2096325000p_1^2p_2^3p_4^2p_5^2p_6^3p_3^5 + 3421632060p_1p_2^3p_4^4p_5p_6^3p_3^5 + 3421632060p_1p_2^4p_4^3p_5p_6^3p_3^5 \\
&+ 17946116496p_1^2p_2^3p_4^3p_5p_6^3p_3^5 + 966735000p_1^2p_2^4p_4^2p_5p_6^3p_3^5 + 114345000p_1p_2^2p_4^4p_5^3p_6^2p_3^5 + 224116200p_1p_2^3p_4^3p_5^3p_6^2p_3^5 \\
&+ 19609868880p_1p_2^3p_4^4p_5^2p_6^2p_3^5 + 1400726250p_1^2p_2^2p_4^4p_5^2p_6^2p_3^5 + 9158882040p_1p_2^4p_4^3p_5^2p_6^2p_3^5 + 59798117736p_1^2p_2^3p_4^3p_5^2p_6^2p_3^5 \\
&+ 1400726250p_1^2p_2^4p_4^2p_5^2p_6^2p_3^5 + 2858625000p_1p_2^4p_4^4p_5p_6^2p_3^5 + 9158882040p_1^2p_2^3p_4^4p_5p_6^2p_3^5 + 19609868880p_1^2p_2^4p_4^3p_5p_6^2p_3^5 \\
&+ 224116200p_1^3p_2^3p_4^3p_5p_6^2p_3^5 + 114345000p_1^3p_2^4p_4^2p_5p_6^2p_3^5 + 599001480p_1^2p_2^3p_4^4p_5^2p_6p_3^5 + 599001480p_1^2p_2^4p_4^3p_5^2p_6p_3^5 \\
&+ 2253968640p_1p_2^3p_4^4p_5^2p_6^3p_3^4 + 8611029504p_1^2p_2^3p_4^3p_5^2p_6^3p_3^4 + 2253968640p_1^2p_2^4p_4^3p_5p_6^3p_3^4 + 409812480p_1p_2^3p_4^4p_5^3p_6^2p_3^4 \\
&+ 122943744p_1^2p_2^3p_4^3p_5^3p_6^2p_3^4 + 7991343360p_1^2p_2^3p_4^4p_5^2p_6^2p_3^4 + 7991343360p_1^2p_2^4p_4^3p_5^2p_6^2p_3^4 + 122943744p_1^3p_2^3p_4^3p_5^2p_6^2p_3^4 \\
&+ 409812480p_1^3p_2^4p_4^3p_5p_6^2p_3^4)z^{17} + (8199664704p_1p_2^3p_4^3p_5p_6^2p_3^6 + 49305564p_1^2p_2^3p_4^3p_6^3p_3^5 + 25155900p_1^2p_2^4p_4^2p_6^3p_3^5 + 25155900p_2^2p_4^4p_5^2p_6^3p_3^5 \\
&+ 49305564p_2^3p_4^3p_5^2p_6^3p_3^5 + 1445944500p_1p_2^2p_4^3p_5^2p_6^3p_3^5 + 1027199250p_1p_2^3p_4^2p_5^2p_6^3p_3^5 + 308159775p_1p_2^2p_4^4p_5p_6^3p_3^5 \\
&+ 8951787306p_1p_2^3p_4^3p_5p_6^3p_3^5 + 1027199250p_1^2p_2^2p_4^3p_5p_6^3p_3^5 + 308159775p_1p_2^4p_4^2p_5p_6^3p_3^5 + 1445944500p_1^2p_2^3p_4^2p_5p_6^3p_3^5 \\
&+ 34303500p_1^2p_2^4p_4^3p_6^2p_3^5 + 34303500p_2^3p_4^4p_5^2p_6^2p_3^5 + 1413304200p_1p_2^2p_4^4p_5^2p_6^2p_3^5 + 30824064054p_1p_2^3p_4^3p_5^2p_6^2p_3^5 \\
&+ 3902976000p_1^2p_2^2p_4^3p_5^2p_6^2p_3^5 + 3902976000p_1^2p_2^3p_4^2p_5^2p_6^2p_3^5 + 6565308750p_1p_2^3p_4^4p_5p_6^2p_3^5 + 6565308750p_1p_2^4p_4^3p_5p_6^2p_3^5 \\
&+ 30824064054p_1^2p_2^3p_4^3p_5p_6^2p_3^5 + 1413304200p_1^2p_2^4p_4^2p_5p_6^2p_3^5 + 305613000p_1p_2^3p_4^4p_5^2p_6p_3^5 + 1669054464p_1^2p_2^3p_4^3p_5^2p_6p_3^5 \\
&+ 305613000p_1^2p_2^4p_4^3p_5p_6p_3^5 + 264136950p_1p_2^2p_4^4p_5^2p_6^3p_3^4 + 4790284884p_1p_2^3p_4^3p_5^2p_6^3p_3^4 + 880456500p_1^2p_2^2p_4^3p_5^2p_6^3p_3^4 \\
&+ 880456500p_1^2p_2^3p_4^2p_5^2p_6^3p_3^4 + 4790284884p_1^2p_2^3p_4^3p_5p_6^3p_3^4 + 264136950p_1^2p_2^4p_4^2p_5p_6^3p_3^4 + 48024900p_1p_2^2p_4^4p_5^3p_6^2p_3^4 \\
&+ 94128804p_1p_2^3p_4^3p_5^3p_6^2p_3^4 + 5042614500p_1p_2^3p_4^4p_5^2p_6^2p_3^4 + 588305025p_1^2p_2^2p_4^4p_5^2p_6^2p_3^4 + 1961016750p_1p_2^4p_4^3p_5^2p_6^2p_3^4 \\
&+ 27835512516p_1^2p_2^3p_4^3p_5^2p_6^2p_3^4 + 588305025p_1^2p_2^4p_4^2p_5^2p_6^2p_3^4 + 1961016750p_1^2p_2^3p_4^4p_5p_6^2p_3^4 + 5042614500p_1^2p_2^4p_4^3p_5p_6^2p_3^4 \\
&+ 94128804p_1^3p_2^3p_4^3p_5p_6^2p_3^4 + 48024900p_1^3p_2^4p_4^2p_5p_6^2p_3^4 + 356548500p_1^2p_2^3p_4^4p_5^2p_6p_3^4 + 356548500p_1^2p_2^4p_4^3p_5^2p_6p_3^4)z^{16} \\
&+ (21913584p_1p_2^3p_4^3p_6^3p_3^5 + 73045280p_1^2p_2^3p_4^2p_6^3p_3^5 + 73045280p_2^2p_4^3p_5^2p_6^3p_3^5 + 21913584p_2^3p_4^3p_5p_6^3p_3^5 \\
&+ 1016898960p_1p_2^2p_4^3p_5p_6^3p_3^5 + 1016898960p_1p_2^3p_4^2p_5p_6^3p_3^5 + 99607200p_1^2p_2^3p_4^3p_6^2p_3^5 + 50820000p_1^2p_2^4p_4^2p_6^2p_3^5 \\
&+ 50820000p_2^2p_4^4p_5^2p_6^2p_3^5 + 99607200p_2^3p_4^3p_5^2p_6^2p_3^5 + 3252073440p_1p_2^2p_4^3p_5^2p_6^2p_3^5 + 2075150000p_1p_2^3p_4^2p_5^2p_6^2p_3^5 + 622545000p_1p_2^2p_4^4p_5p_6^2p_3^5 \\
&+ 19480302576p_1p_2^3p_4^3p_5p_6^2p_3^5 + 2075150000p_1^2p_2^2p_4^3p_5p_6^2p_3^5 + 622545000p_1p_2^4p_4^2p_5p_6^2p_3^5 + 3252073440p_1^2p_2^3p_4^2p_5p_6^2p_3^5 \\
&+ 887409600p_1p_2^3p_4^3p_5^2p_6p_3^5 + 887409600p_1^2p_2^3p_4^3p_5p_6p_3^5 + 871627680p_1p_2^2p_4^3p_5^2p_6^3p_3^4 + 766975440p_1p_2^3p_4^2p_5^2p_6^3p_3^4 \\
&+ 2414513024p_1p_2^3p_4^3p_5p_6^3p_3^4 + 766975440p_1^2p_2^2p_4^3p_5p_6^3p_3^4 + 871627680p_1^2p_2^3p_4^2p_5p_6^3p_3^4 + 11383680p_1p_2^2p_4^3p_5^3p_6^2p_3^4 \\
&+ 830060000p_1p_2^2p_4^4p_5^2p_6^2p_3^4 + 18476731056p_1p_2^3p_4^3p_5^2p_6^2p_3^4 + 4063973760p_1^2p_2^2p_4^3p_5^2p_6^2p_3^4 + 4063973760p_1^2p_2^3p_4^2p_5^2p_6^2p_3^4 \\
&+ 1778700000p_1p_2^3p_4^4p_5p_6^2p_3^4 + 1778700000p_1p_2^4p_4^3p_5p_6^2p_3^4 + 18476731056p_1^2p_2^3p_4^3p_5p_6^2p_3^4 + 830060000p_1^2p_2^4p_4^2p_5p_6^2p_3^4 \\
&+ 11383680p_1^3p_2^3p_4^2p_5p_6^2p_3^4 + 323400000p_1p_2^3p_4^4p_5^2p_6p_3^4 + 2253071744p_1^2p_2^3p_4^3p_5^2p_6p_3^4 + 323400000p_1^2p_2^4p_4^3p_5p_6p_3^4 \\
&+ 4079910912p_1^2p_2^3p_4^3p_5^2p_6^2p_3^3)z^{15} + (41087970p_1p_2^3p_4^2p_6^3p_3^5 + 41087970p_2^2p_4^3p_5p_6^3p_3^5 + 136959900p_1p_2^2p_4^2p_5p_6^3p_3^5 + 56029050p_1p_2^3p_4^3p_6^2p_3^5 \\
&+ 186763500p_1^2p_2^3p_4^2p_6^2p_3^5 + 186763500p_2^2p_4^3p_5^2p_6^2p_3^5 + 56029050p_2^3p_4^3p_5p_6^2p_3^5 + 2655776970p_1p_2^2p_4^3p_5p_6^2p_3^5 + 2655776970p_1p_2^3p_4^2p_5p_6^2p_3^5 \\
&+ 499167900p_1p_2^3p_4^3p_5p_6p_3^5 + 35218260p_1^2p_2^3p_4^2p_6^3p_3^4 + 35218260p_2^2p_4^3p_5^2p_6^3p_3^4 + 117394200p_1p_2^2p_4^2p_5^2p_6^3p_3^4 \\
&+ 607296690p_1p_2^2p_4^3p_5p_6^3p_3^4 + 607296690p_1p_2^3p_4^2p_5p_6^3p_3^4 + 117394200p_1^2p_2^2p_4^2p_5p_6^3p_3^4 + 48024900p_1^2p_2^3p_4^3p_6^2p_3^4 \\
&+ 24502500p_1^2p_2^4p_4^2p_6^2p_3^4 + 24502500p_2^2p_4^4p_5^2p_6^2p_3^4 + 48024900p_2^3p_4^3p_5^2p_6^2p_3^4 + 4609356210p_1p_2^2p_4^3p_5^2p_6^2p_3^4 + 3054383640p_1p_2^3p_4^2p_5^2p_6^2p_3^4 \\
&+ 446054400p_1^2p_2^2p_4^2p_5^2p_6^2p_3^4 + 300155625p_1p_2^2p_4^4p_5p_6^2p_3^4 + 14283282150p_1p_2^3p_4^3p_5p_6^2p_3^4 + 3054383640p_1^2p_2^2p_4^3p_5p_6^2p_3^4 \\
&+ 300155625p_1p_2^4p_4^2p_5p_6^2p_3^4 + 4609356210p_1^2p_2^3p_4^2p_5p_6^2p_3^4 + 54573750p_1p_2^2p_4^4p_5^2p_6p_3^4 + 1671169500p_1p_2^3p_4^3p_5^2p_6p_3^4 \\
&+ 298045440p_1^2p_2^2p_4^3p_5^2p_6p_3^4 + 298045440p_1^2p_2^3p_4^2p_5^2p_6p_3^4 + 1671169500p_1^2p_2^3p_4^3p_5p_6p_3^4 + 54573750p_1^2p_2^4p_4^2p_5p_6p_3^4 \\
&+ 3585859200p_1p_2^3p_4^3p_5^2p_6^2p_3^3 + 1075757760p_1^2p_2^2p_4^3p_5^2p_6^2p_3^3 + 1075757760p_1^2p_2^3p_4^2p_5^2p_6^2p_3^3 + 3585859200p_1^2p_2^3p_4^3p_5p_6^2p_3^3 \\
&+ 651974400p_1^2p_2^3p_4^3p_5^2p_6p_3^3)z^{14} + (119528640p_1p_2^3p_4^2p_6^2p_3^5 + 119528640p_2^2p_4^3p_5p_6^2p_3^5 + 398428800p_1p_2^2p_4^2p_5p_6^2p_3^5 \\
&+ 35218260p_1p_2^3p_4^2p_6^3p_3^4 + 35218260p_2^2p_4^3p_5p_6^3p_3^4 + 180457200p_1p_2^2p_4^2p_5p_6^3p_3^4 + 48024900p_1p_2^3p_4^3p_6^2p_3^4 \\
&+ 268939440p_1^2p_2^3p_4^2p_6^2p_3^4 + 268939440p_2^2p_4^3p_5^2p_6^2p_3^4 + 133402500p_1p_2p_4^3p_5^2p_6^2p_3^4 + 672348600p_1p_2^2p_4^2p_5^2p_6^2p_3^4 + 48024900p_2^3p_4^3p_5p_6^2p_3^4 \\
&+ 4320547560p_1p_2^2p_4^3p_5p_6^2p_3^4 + 4320547560p_1p_2^3p_4^2p_5p_6^2p_3^4 + 672348600p_1^2p_2^2p_4^2p_5p_6^2p_3^4 + 133402500p_1^2p_2^3p_4p_5p_6^2p_3^4 \\
&+ 413887320p_1p_2^2p_4^3p_5^2p_6p_3^4 + 356548500p_1p_2^3p_4^2p_5^2p_6p_3^4 + 1189465200p_1p_2^3p_4^3p_5p_6p_3^4 + 356548500p_1^2p_2^2p_4^3p_5p_6p_3^4 \\
&+ 413887320p_1^2p_2^3p_4^2p_5p_6p_3^4 + 1644128640p_1p_2^2p_4^3p_5^2p_6^2p_3^3 + 1075757760p_1p_2^3p_4^2p_5^2p_6^2p_3^3 + 292723200p_1^2p_2^2p_4^2p_5^2p_6^2p_3^3 \\
&+ 3585859200p_1p_2^3p_4^3p_5p_6^2p_3^3 + 1075757760p_1^2p_2^2p_4^3p_5p_6^2p_3^3 + 1644128640p_1^2p_2^3p_4^2p_5p_6^2p_3^3 + 651974400p_1p_2^3p_4^3p_5^2p_6p_3^3 \\
&+ 195592320p_1^2p_2^2p_4^3p_5^2p_6p_3^3 + 195592320p_1^2p_2^3p_4^2p_5^2p_6p_3^3 + 651974400p_1^2p_2^3p_4^3p_5p_6p_3^3)z^{13} + (6391462p_1p_2^2p_4^2p_6^3p_3^4 \\
&+ 6391462p_2^2p_4^2p_5p_6^3p_3^4 + 8715630p_1p_2^2p_4^3p_6^2p_3^4 + 253998360p_1p_2^3p_4^2p_6^2p_3^4 + 29052100p_1^2p_2^2p_4^2p_6^2p_3^4 \\
&+ 14822500p_2p_4^3p_5^2p_6^2p_3^4 + 29052100p_2^2p_4^2p_5^2p_6^2p_3^4 + 14822500p_1^2p_2^3p_4p_6^2p_3^4 + 253998360p_2^2p_4^3p_5p_6^2p_3^4 + 181575625p_1p_2p_4^3p_5p_6^2p_3^4 \\
&+ 8715630p_2^3p_4^2p_5p_6^2p_3^4 + 1554121926p_1p_2^2p_4^2p_5p_6^2p_3^4 + 181575625p_1p_2^3p_4p_5p_6^2p_3^4 + 20212500p_1^2p_2^3p_4^2p_6p_3^4 + 20212500p_2^2p_4^3p_5^2p_6p_3^4 \\
&+ 110387200p_1p_2^2p_4^2p_5^2p_6p_3^4 + 388031490p_1p_2^2p_4^3p_5p_6p_3^4 + 388031490p_1p_2^3p_4^2p_5p_6p_3^4 + 110387200p_1^2p_2^2p_4^2p_5p_6p_3^4 \\
&+ 5478396p_1p_2^2p_4^2p_5p_6^3p_3^3 + 111143340p_1^2p_2^3p_4^2p_6^2p_3^3 + 111143340p_2^2p_4^3p_5^2p_6^2p_3^3 + 155636250p_1p_2p_4^3p_5^2p_6^2p_3^3 \\
&+ 542666124p_1p_2^2p_4^2p_5^2p_6^2p_3^3 + 1941993130p_1p_2^2p_4^3p_5p_6^2p_3^3 + 1941993130p_1p_2^3p_4^2p_5p_6^2p_3^3 + 542666124p_1^2p_2^2p_4^2p_5p_6^2p_3^3 \\
&+ 155636250p_1^2p_2^3p_4p_5p_6^2p_3^3 + 353089660p_1p_2^2p_4^3p_5^2p_6p_3^3 + 247546530p_1p_2^3p_4^2p_5^2p_6p_3^3 + 60555264p_1^2p_2^2p_4^2p_5^2p_6p_3^3 \\
&+ 707437500p_1p_2^3p_4^3p_5p_6p_3^3 + 247546530p_1^2p_2^2p_4^3p_5p_6p_3^3 + 353089660p_1^2p_2^3p_4^2p_5p_6p_3^3)z^{12} + (66594528p_1p_2^2p_4^2p_6^2p_3^4 \\
&+ 21344400p_1p_2^3p_4p_6^2p_3^4 + 21344400p_2p_4^3p_5p_6^2p_3^4 + 66594528p_2^2p_4^2p_5p_6^2p_3^4 + 71148000p_1p_2p_4^2p_5p_6^2p_3^4 + 71148000p_1p_2^2p_4p_5p_6^2p_3^4 \\
&+ 29106000p_1p_2^3p_4^2p_6p_3^4 + 29106000p_2^2p_4^3p_5p_6p_3^4 + 191866752p_1p_2^2p_4^2p_5p_6p_3^4 + 137214000p_1p_2^3p_4^2p_6^2p_3^3 + 35858592p_1^2p_2^2p_4^2p_6^2p_3^3
\end{aligned}
$$

$$+ 18295200p_2p_4^3p_5^2p_6^2p_3^3 + 35858592p_2^2p_4^2p_5^2p_6^2p_3^3 + 60984000p_1p_2p_4^2p_5^2p_6^2p_3^3 + 18295200p_1^2p_2^3p_4p_6^2p_3^3 + 137214000p_2^2p_4^3p_5p_6^2p_3^3$$
$$+ 224116200p_1p_2p_4^3p_5p_6^2p_3^3 + 1339753968p_1p_2^2p_4^2p_5p_6^2p_3^3 + 224116200p_1p_2^3p_4p_5p_6^2p_3^3 + 60984000p_1^2p_2^2p_4p_5p_6^2p_3^3$$
$$+ 24948000p_1^2p_2^3p_4^2p_6p_3^3 + 24948000p_2^2p_4^3p_5^2p_6p_3^3 + 40748400p_1p_2p_4^3p_5^2p_6p_3^3 + 177031008p_1p_2^2p_4^2p_5^2p_6p_3^3 + 467082000p_1p_2^2p_4^3p_5p_6p_3^3$$
$$+ 467082000p_1p_2^3p_4^2p_5p_6p_3^3 + 177031008p_1^2p_2^2p_4^2p_5p_6p_3^3 + 40748400p_1^2p_2^3p_4p_5p_6p_3^3)z^{11} + (2614689p_2^2p_4^2p_6^2p_3^4$$
$$+ 8715630p_1p_2^2p_4p_6^2p_3^4 + 8715630p_2p_4^2p_5p_6^2p_3^4 + 11884950p_1p_2^2p_4^2p_6p_3^4 + 11884950p_2^2p_4^2p_5p_6p_3^4 + 87268104p_1p_2^2p_4^2p_6^2p_3^3$$
$$+ 7470540p_2p_4^2p_5^2p_6^2p_3^3 + 28586250p_1p_2^3p_4p_6^2p_3^3 + 7470540p_1^2p_2^2p_4p_6^2p_3^3 + 28586250p_2p_4^3p_5p_6^2p_3^3 + 87268104p_2^2p_4^2p_5p_6^2p_3^3$$
$$+ 202848030p_1p_2p_4^2p_5p_6^2p_3^3 + 202848030p_1p_2^2p_4p_5p_6^2p_3^3 + 38981250p_1p_2^3p_4^2p_6p_3^3 + 10187100p_1^2p_2^2p_4^2p_6p_3^3 + 5197500p_2p_4^3p_5^2p_6p_3^3$$
$$+ 10187100p_2^2p_4^2p_5^2p_6p_3^3 + 28385280p_1p_2p_4^2p_5^2p_6p_3^3 + 5197500p_1^2p_2^3p_4p_6p_3^3 + 38981250p_2^2p_4^3p_5p_6p_3^3 + 63669375p_1p_2p_4^3p_5p_6p_3^3$$
$$+ 440527626p_1p_2^2p_4^2p_5p_6p_3^3 + 63669375p_1p_2^3p_4p_5p_6p_3^3 + 28385280p_1^2p_2^2p_4p_5p_6p_3^3 + 30735936p_1p_2^2p_4^2p_5p_6^2p_3^2 + 5588352p_1p_2^2p_4^2p_5^2p_6p_3^2$$
$$+ 5588352p_1^2p_2^2p_4^2p_5p_6p_3^2)z^{10} + (6225450p_2^2p_4^2p_6^2p_3^3 + 13280960p_1p_2p_4^2p_6^2p_3^3 + 27544440p_1p_2^2p_4p_6^2p_3^3 + 27544440p_2p_4^2p_5p_6^2p_3^3$$
$$+ 13280960p_2^2p_4p_5p_6^2p_3^3 + 41164200p_1p_2p_4p_5p_6^2p_3^3 + 205800p_1p_2^2p_4^2p_5p_3^3 + 37560600p_1p_2^2p_4^2p_6p_3^3 + 3773000p_2p_4^2p_5^2p_6p_3^3$$
$$+ 9240000p_1p_2^3p_4p_6p_3^3 + 3773000p_1^2p_2^2p_4p_6p_3^3 + 9240000p_2p_4^3p_5p_6p_3^3 + 37560600p_2^2p_4^2p_5p_6p_3^3$$
$$+ 82222140p_1p_2p_4^2p_5p_6p_3^3 + 82222140p_1p_2^2p_4p_5p_6p_3^3 + 11383680p_1p_2p_4^2p_5p_6^2p_3^2 + 11383680p_1p_2^2p_4p_5p_6^2p_3^2 + 2069760p_1p_2p_4^2p_5^2p_6p_3^2$$
$$+ 24147200p_1p_2^2p_4^2p_5p_6p_3^2 + 2069760p_1^2p_2^2p_4p_5p_6p_3^2)z^9 + (1867635p_2p_4^2p_6^2p_3^3 + 1867635p_2^2p_4p_6^2p_3^3 + 6225450p_1p_2p_4p_6^2p_3^3$$
$$+ 6225450p_2p_4p_5p_6^2p_3^3 + 2546775p_2^2p_4^2p_6p_3^3 + 8489250p_1p_2p_4^2p_6p_3^3 + 13222440p_1p_2^2p_4p_6p_3^3 + 13222440p_2p_4^2p_5p_6p_3^3$$
$$+ 8489250p_2^2p_4p_5p_6p_3^3 + 23654400p_1p_2p_4p_5p_6p_3^3 + 1600830p_1p_2^2p_4p_6^2p_3^2 + 1600830p_2p_4^2p_5p_6^2p_3^2 + 5336100p_1p_2p_4p_5p_6^2p_3^2$$
$$+ 396900p_1p_2^2p_4^2p_5p_3^2 + 2182950p_1p_2^2p_4^2p_6p_3^2 + 291060p_2p_4^2p_5^2p_6p_3^2 + 291060p_1^2p_2^2p_4p_6p_3^2 + 2182950p_2^2p_4^2p_5p_6p_3^2$$
$$+ 9168390p_1p_2p_4^2p_5p_6p_3^2 + 9168390p_1p_2^2p_4p_5p_6p_3^2)z^8 + (996072p_2p_4p_6^2p_3^3 + 1358280p_2p_4^2p_6p_3^3 + 1358280p_2^2p_4p_6p_3^3$$
$$+ 4527600p_1p_2p_4p_6p_3^3 + 4527600p_2p_4p_5p_6p_3^3 + 853776p_1p_2p_4p_6^2p_3^2 + 853776p_2p_4p_5p_6^2p_3^2 + 211680p_1p_2p_4^2p_5p_3^2$$
$$+ 211680p_1p_2^2p_4p_5p_3^2 + 1164240p_1p_2p_4^2p_6p_3^2 + 1853280p_1p_2^2p_4p_6p_3^2 + 1853280p_2p_4^2p_5p_6p_3^2 + 1164240p_2^2p_4p_5p_6p_3^2$$
$$+ 6044544p_1p_2p_4p_5p_6p_3^2)z^7 + (916839p_2p_4p_6p_3^3 + 148225p_2p_4p_6^2p_3^2 + 36750p_1p_2^2p_4p_3^2 + 36750p_2p_4^2p_5p_3^2$$
$$+ 200704p_1p_2p_4p_5p_3^2 + 26950p_1p_2^2p_6p_3^2 + 202125p_2p_4^2p_6p_3^2 + 202125p_2^2p_4p_6p_3^2$$
$$+ 1539384p_1p_2p_4p_6p_3^2 + 26950p_4^2p_5p_6p_3^2 + 1539384p_2p_4p_5p_6p_3^2 + 369600p_1p_2p_4p_5p_6p_3)z^6 + (44100p_1p_2p_4p_3^2 + 44100p_2p_4p_5p_3^2$$
$$+ 32340p_1p_2p_6p_3^2 + 407484p_2p_4p_6p_3^2 + 32340p_4p_5p_6p_3^2 + 24192p_1p_2p_4p_5p_3 + 133056p_1p_2p_4p_6p_3 + 133056p_2p_4p_5p_6p_3)z^5$$
$$+ (11025p_2p_4p_3^2 + 8085p_2p_6p_3^2 + 8085p_4p_6p_3^2 + 9450p_1p_2p_4p_3 + 9450p_2p_4p_5p_3 + 6930p_1p_2p_6p_3$$
$$+ 51975p_2p_4p_6p_3 + 6930p_4p_5p_6p_3)z^4 + (560p_1p_2p_3 + 4200p_2p_4p_3 + 560p_4p_5p_3 + 3080p_2p_6p_3 + 3080p_4p_6p_3)z^3$$
$$+ (315p_2p_3 + 315p_4p_3 + 231p_6p_3)z^2 + 42p_3z + 1$$

$$H_4 = p_1^3p_2^6p_3^8p_4^6p_5^3p_6^4z^{30} + 30p_1^3p_2^5p_3^8p_4^6p_5^3p_6^4z^{29} + (120p_1^2p_2^5p_4^6p_5^3p_6^4p_3^8 + 315p_1^3p_2^5p_4^6p_5^3p_6^4p_3^7)z^{28} + (2240p_1^2p_2^5p_4^6p_5^3p_6^4p_3^7 + 1050p_1^3p_2^5p_4^5p_5^3p_6^4p_3^7$$
$$+ 770p_1^3p_2^5p_4^6p_5^3p_6^3p_3^7)z^{27} + (4200p_1^2p_2^4p_4^6p_5^3p_6^4p_3^7 + 9450p_1^2p_2^5p_4^5p_5^3p_6^4p_3^7 + 1050p_1^3p_2^5p_4^5p_5^2p_6^4p_3^7 + 6930p_1^2p_2^5p_4^6p_5^3p_6^3p_3^7$$
$$+ 5775p_1^3p_2^5p_4^5p_5^3p_6^3p_3^7)z^{26} + (31500p_1^2p_2^4p_4^5p_5^3p_6^4p_3^7 + 10752p_1^2p_2^5p_4^5p_5^2p_6^4p_3^7 + 23100p_1^2p_2^4p_4^6p_5^3p_6^3p_3^7 + 59136p_1^2p_2^5p_4^5p_5^3p_6^3p_3^7$$
$$+ 8316p_1^3p_2^5p_4^5p_5^2p_6^3p_3^7 + 9702p_1^3p_2^5p_4^5p_5^3p_6^3p_3^6)z^{25} + (45360p_1^2p_2^4p_4^5p_5^2p_6^4p_3^7 + 249480p_1^2p_2^4p_4^5p_5^3p_6^3p_3^7 + 92400p_1^2p_2^5p_4^5p_5^2p_6^3p_3^7$$
$$+ 36750p_1^2p_2^4p_4^5p_5^3p_6^4p_3^6 + 26950p_1^2p_2^4p_4^6p_5^3p_6^3p_3^6 + 107800p_1^2p_2^5p_4^5p_5^3p_6^3p_3^6 + 8085p_1^3p_2^4p_4^5p_5^3p_6^3p_3^6 + 26950p_1^3p_2^5p_4^5p_5^2p_6^3p_3^6)z^{24}$$
$$+ (443520p_1^2p_2^4p_4^5p_5^2p_6^3p_3^7 + 94080p_1^2p_2^4p_4^5p_5^2p_6^4p_3^6 + 1132560p_1^2p_2^4p_4^5p_5^3p_6^3p_3^6 + 316800p_1^2p_2^5p_4^5p_5^2p_6^3p_3^6 + 32340p_1^3p_2^4p_4^5p_5^2p_6^3p_3^6$$
$$+ 16500p_1^3p_2^5p_4^4p_5^2p_6^3p_3^6)z^{23} + (44100p_1^2p_2^4p_4^4p_5^2p_6^4p_3^6 + 32340p_1p_2^4p_4^5p_5^3p_6^3p_3^6 + 495000p_1^2p_2^3p_4^5p_5^3p_6^3p_3^6 + 242550p_1^2p_2^4p_4^4p_5^3p_6^3p_3^6$$
$$+ 3256110p_1^2p_2^4p_4^5p_5^2p_6^3p_3^6 + 202125p_1^2p_2^5p_4^4p_5^2p_6^3p_3^6 + 44550p_1^3p_2^4p_4^4p_5^2p_6^3p_3^6 + 177870p_1^2p_2^4p_4^5p_5^3p_6^2p_3^6 + 1358280p_1^2p_2^4p_4^5p_5^3p_6^3p_3^5)z^{22}$$
$$+ (178200p_1p_2^3p_4^5p_5^3p_6^3p_3^6 + 168960p_1p_2^4p_4^5p_5^2p_6^3p_3^6 + 2182950p_1^2p_2^3p_4^5p_5^2p_6^3p_3^6 + 2674100p_1^2p_2^4p_4^4p_5^2p_6^3p_3^6 + 711480p_1^2p_2^4p_4^5p_5^2p_6^2p_3^6$$
$$+ 1478400p_1^2p_2^3p_4^5p_5^3p_6^3p_3^5 + 1131900p_1^2p_2^4p_4^4p_5^3p_6^3p_3^5 + 4928000p_1^2p_2^4p_4^5p_5^2p_6^3p_3^5 + 23100p_1^3p_2^4p_4^4p_5^2p_6^3p_3^5 + 830060p_1^2p_2^4p_4^5p_5^3p_6^2p_3^5)z^{21}$$
$$+ (970200p_1p_2^3p_4^5p_5^2p_6^3p_3^6 + 349272p_1p_2^4p_4^4p_5^2p_6^3p_3^6 + 3234000p_1^2p_2^3p_4^4p_5^2p_6^3p_3^6 + 155232p_1^2p_2^4p_4^4p_5p_6^3p_3^6 + 853776p_1^2p_2^4p_4^4p_5^2p_6^2p_3^6$$
$$+ 577500p_1p_2^3p_4^5p_5^3p_6^3p_3^5 + 1559250p_1^2p_2^3p_4^4p_5^3p_6^3p_3^5 + 7074375p_1^2p_2^3p_4^5p_5^2p_6^3p_3^5 + 9315306p_1^2p_2^4p_4^4p_5^2p_6^3p_3^5 + 1143450p_1^2p_2^3p_4^5p_5^3p_6^2p_3^5$$
$$+ 996072p_1^2p_2^4p_4^4p_5^3p_6^2p_3^5 + 3811500p_1^2p_2^4p_4^5p_5^2p_6^2p_3^5 + 5082p_1^3p_2^4p_4^4p_5^2p_6^2p_3^5)z^{20} + (2069760p_1p_2^3p_4^4p_5^2p_6^3p_3^6 + 693000p_1p_2^3p_4^4p_5^3p_6^3p_3^5$$
$$+ 3326400p_1p_2^3p_4^5p_5^2p_6^3p_3^5 + 369600p_1p_2^4p_4^4p_5^2p_6^3p_3^5 + 21801780p_1^2p_2^3p_4^4p_5^2p_6^3p_3^5 + 4331250p_1^2p_2^4p_4^3p_5^2p_6^3p_3^5 + 1478400p_1^2p_2^4p_4^4p_5p_6^3p_3^5$$
$$+ 508200p_1p_2^3p_4^5p_5^3p_6^2p_3^5 + 2439360p_1^2p_2^3p_4^4p_5^3p_6^2p_3^5 + 6225450p_1^2p_2^3p_4^5p_5^2p_6^2p_3^5 + 11384100p_1^2p_2^4p_4^4p_5^2p_6^2p_3^5)z^{19} + (14314300p_1p_2^3p_4^4p_5^2p_6^3p_3^5$$
$$+ 14437500p_1^2p_2^3p_4^3p_5^2p_6^3p_3^5 + 3056130p_1^2p_2^3p_4^4p_5p_6^3p_3^5 + 1559250p_1^2p_2^4p_4^3p_5p_6^3p_3^5 + 1372140p_1p_2^3p_4^4p_5^3p_6^2p_3^5$$
$$+ 3176250p_1p_2^3p_4^5p_5^2p_6^2p_3^5 + 127050p_1p_2^4p_4^4p_5^2p_6^2p_3^5 + 28420210p_1^2p_2^3p_4^4p_5^2p_6^2p_3^5 + 8575875p_1^2p_2^4p_4^3p_5^2p_6^2p_3^5$$
$$+ 2032800p_1^2p_2^4p_4^4p_5p_6^2p_3^5 + 6338640p_1^2p_2^3p_4^4p_5^2p_6^3p_3^4 + 711480p_1^2p_2^3p_4^4p_5^3p_6^2p_3^4 + 2371600p_1^2p_2^4p_4^4p_5^2p_6^2p_3^4)z^{18}$$
$$+ (577500p_1p_2^2p_4^4p_5^2p_6^3p_3^5 + 10187100p_1p_2^3p_4^3p_5^2p_6^3p_3^5 + 1774080p_1p_2^3p_4^4p_5p_6^3p_3^5 + 5913600p_1^2p_2^3p_4^3p_5p_6^3p_3^5 + 18705960p_1p_2^3p_4^4p_5^2p_6^2p_3^5$$
$$+ 32524800p_1^2p_2^3p_4^3p_5^2p_6^2p_3^5 + 7470540p_1^2p_2^3p_4^4p_5p_6^2p_3^5 + 3811500p_1^2p_2^4p_4^3p_5p_6^2p_3^5 + 8279040p_1p_2^3p_4^4p_5^2p_6^3p_3^4 + 8731800p_1^2p_2^3p_4^3p_5^2p_6^3p_3^4$$
$$+ 711480p_1p_2^3p_4^4p_5^3p_6^2p_3^4 + 16625700p_1^2p_2^3p_4^4p_5^2p_6^2p_3^4 + 4446750p_1^2p_2^4p_4^3p_5^2p_6^2p_3^4)z^{17} + (4527600p_1p_2^3p_4^3p_5p_6^3p_3^5 + 508200p_1p_2^2p_4^4p_5^2p_6^2p_3^5$$
$$+ 24901800p_1p_2^3p_4^3p_5^2p_6^2p_3^5 + 5488560p_1p_2^3p_4^4p_5p_6^2p_3^5 + 18295200p_1^2p_2^3p_4^3p_5p_6^2p_3^5 + 2182950p_1p_2^2p_4^4p_5^2p_6^3p_3^4 + 11884950p_1p_2^3p_4^3p_5^2p_6^3p_3^4$$
$$+ 3880800p_1^2p_2^3p_4^3p_5p_6^3p_3^4 + 108900p_1p_2^2p_4^4p_5^3p_6^2p_3^4 + 18478980p_1p_2^3p_4^4p_5^2p_6^2p_3^4 + 1334025p_1^2p_2^2p_4^4p_5^2p_6^2p_3^4 + 45530550p_1^2p_2^3p_4^3p_5^2p_6^2p_3^4$$
$$+ 4446750p_1^2p_2^3p_4^4p_5p_6^2p_3^4 + 2268750p_1^2p_2^4p_4^3p_5p_6^2p_3^4 + 1584660p_1^2p_2^3p_4^4p_5^2p_6p_3^4)z^{16} + (15937152p_1p_2^3p_4^3p_5p_6^2p_3^5 + 3234000p_1p_2^2p_4^3p_5^2p_6^3p_3^4$$
$$+ 5588352p_1p_2^3p_4^3p_5p_6^3p_3^4 + 6203600p_1p_2^2p_4^4p_5^2p_6^2p_3^4 + 53742416p_1p_2^3p_4^3p_5^2p_6^2p_3^4 + 5808000p_1^2p_2^2p_4^3p_5^2p_6^2p_3^4 + 5808000p_1p_2^3p_4^4p_5p_6^2p_3^4$$
$$+ 34036496p_1^2p_2^3p_4^3p_5p_6^2p_3^4 + 3234000p_1p_2^3p_4^4p_5^2p_6p_3^4 + 5588352p_1^2p_2^3p_4^3p_5^2p_6p_3^4 + 15937152p_1^2p_2^3p_4^3p_5^2p_6^2p_3^3)z^{15} + (1584660p_1p_2^2p_4^3p_5p_6^3p_3^4$$
$$+ 108900p_2^2p_4^4p_5^2p_6^2p_3^4 + 18478980p_1p_2^2p_4^3p_5^2p_6^2p_3^4 + 1334025p_1p_2^2p_4^4p_5p_6^2p_3^4 + 45530550p_1p_2^3p_4^3p_5p_6^2p_3^4 + 4446750p_1^2p_2^2p_4^3p_5p_6^2p_3^4$$
$$+ 2268750p_1^2p_2^3p_4^2p_5p_6^2p_3^4 + 2182950p_1p_2^2p_4^4p_5^2p_6p_3^4 + 11884950p_1p_2^3p_4^3p_5^2p_6p_3^4 + 3880800p_1^2p_2^3p_4^3p_5p_6p_3^4 + 508200p_1p_2^2p_4^4p_5^2p_6^2p_3^3$$
$$+ 24901800p_1p_2^3p_4^3p_5^2p_6^2p_3^3 + 5488560p_1^2p_2^2p_4^3p_5^2p_6^2p_3^3 + 18295200p_1^2p_2^3p_4^3p_5p_6^2p_3^3 + 4527600p_1^2p_2^3p_4^3p_5^2p_6p_3^3)z^{14} + (711480p_2^2p_4^3p_5^2p_6^2p_3^4$$
$$+ 16625700p_1p_2^2p_4^3p_5p_6^2p_3^4 + 4446750p_1p_2^3p_4^2p_5p_6^2p_3^4 + 8279040p_1p_2^2p_4^3p_5^2p_6p_3^4 + 8731800p_1p_2^3p_4^3p_5p_6p_3^4 + 18705960p_1p_2^2p_4^3p_5^2p_6^2p_3^3$$
$$+ 32524800p_1p_2^3p_4^3p_5p_6^2p_3^3 + 7470540p_1^2p_2^2p_4^3p_5p_6^2p_3^3 + 3811500p_1^2p_2^3p_4^2p_5p_6^2p_3^3 + 577500p_1p_2^2p_4^4p_5^2p_6p_3^3 + 10187100p_1p_2^3p_4^3p_5^2p_6p_3^3$$

$$
\begin{aligned}
&+1774080p_1^2p_2^2p_4^3p_5^2p_6p_3^3+5913600p_1^2p_2^3p_4^3p_5p_6p_3^3)z^{13}+(711480p_2^2p_4^3p_5p_6^2p_3^4+2371600p_1p_2^2p_4^2p_5p_6^2p_3^4+6338640p_1p_2^2p_4^3p_5p_6p_3^4\\
&+1372140p_2^2p_4^3p_5^2p_6^2p_3^3+3176250p_1p_2p_4^3p_5^2p_6^2p_3^3+127050p_1p_2^2p_4^2p_5^2p_6^2p_3^3+28420210p_1p_2^2p_4^3p_5p_6^2p_3^3+8575875p_1p_2^3p_4^2p_5p_6^2p_3^3\\
&+2032800p_1^2p_2^2p_4^2p_5p_6^2p_3^3+14314300p_1p_2^2p_4^3p_5^2p_6p_3^3+14437500p_1p_2^3p_4^3p_5p_6p_3^3+3056130p_1^2p_2^2p_4^3p_5p_6p_3^3+1559250p_1^2p_2^3p_4^2p_5p_6p_3^3)z^{12}\\
&+(508200p_2p_4^3p_5^2p_6^2p_3^3+2439360p_2^2p_4^3p_5p_6^2p_3^3+6225450p_1p_2p_4^3p_5p_6^2p_3^3+11384100p_1p_2^2p_4^2p_5p_6^2p_3^3+693000p_2^2p_4^3p_5^2p_6p_3^3\\
&+3326400p_1p_2p_4^3p_5^2p_6p_3^3+369600p_1p_2^2p_4^2p_5^2p_6p_3^3+21801780p_1p_2^2p_4^3p_5p_6p_3^3+4331250p_1p_2^3p_4^2p_5p_6p_3^3+1478400p_1^2p_2^2p_4^2p_5p_6p_3^3\\
&+2069760p_1p_2^2p_4^3p_5^2p_6p_3^2)z^{11}+(5082p_1p_2^2p_4^2p_6^2p_3^3+1143450p_2p_4^3p_5p_6^2p_3^3+996072p_2^2p_4^2p_5p_6^2p_3^3+3811500p_1p_2p_4^2p_5p_6^2p_3^3\\
&+577500p_2p_4^3p_5^2p_6p_3^3+1559250p_2^2p_4^3p_5p_6p_3^3+7074375p_1p_2p_4^3p_5p_6p_3^3+9315306p_1p_2^2p_4^2p_5p_6p_3^3+853776p_1p_2^2p_4^2p_5p_6^2p_3^2\\
&+970200p_1p_2p_4^3p_5^2p_6p_3^2+349272p_1p_2^2p_4^2p_5^2p_6p_3^2+3234000p_1p_2^2p_4^3p_5p_6p_3^2+155232p_1^2p_2^2p_4^2p_5p_6p_3^2)z^{10}+(830060p_2p_4^2p_5p_6^2p_3^3\\
&+23100p_1p_2^2p_4^2p_6p_3^3+1478400p_2p_4^3p_5p_6p_3^3+1131900p_2^2p_4^2p_5p_6p_3^3+4928000p_1p_2p_4^2p_5p_6p_3^3+711480p_1p_2p_4^2p_5p_6^2p_3^2\\
&+178200p_2p_4^3p_5^2p_6p_3^2+168960p_1p_2p_4^2p_5^2p_6p_3^2+2182950p_1p_2p_4^3p_5p_6p_3^2+2674100p_1p_2^2p_4^2p_5p_6p_3^2)z^9+(1358280p_2p_4^2p_5p_6p_3^3\\
&+177870p_2p_4^2p_5p_6^2p_3^2+44100p_1p_2^2p_4^2p_5p_3^2+44550p_1p_2^2p_4^2p_6p_3^2+32340p_2p_4^2p_5^2p_6p_3^2+495000p_2p_4^3p_5p_6p_3^2\\
&+242550p_2^2p_4^2p_5p_6p_3^2+3256110p_1p_2p_4^2p_5p_6p_3^2+202125p_1p_2^2p_4p_5p_6p_3^2)z^8+(94080p_1p_2p_4^2p_5p_3^2+32340p_1p_2p_4^2p_6p_3^2\\
&+16500p_1p_2^2p_4p_6p_3^2+1132560p_2p_4^2p_5p_6p_3^2+316800p_1p_2p_4p_5p_6p_3^2+443520p_1p_2p_4^2p_5p_6p_3)z^7+(36750p_2p_4^2p_5p_3^2+8085p_2p_4^2p_6p_3^2\\
&+26950p_1p_2p_4p_6p_3^2+26950p_4^2p_5p_6p_3^2+107800p_2p_4p_5p_6p_3^2+45360p_1p_2p_4^2p_5p_3+249480p_2p_4^2p_5p_6p_3\\
&+92400p_1p_2p_4p_5p_6p_3)z^6+(9702p_2p_4p_6p_3^2+31500p_2p_4^2p_5p_3+10752p_1p_2p_4p_5p_3+8316p_1p_2p_4p_6p_3+23100p_4^2p_5p_6p_3\\
&+59136p_2p_4p_5p_6p_3)z^5+(4200p_3p_5p_4^2+1050p_1p_2p_3p_4+9450p_2p_3p_5p_4+5775p_2p_3p_6p_4+6930p_3p_5p_6p_4)z^4+(1050p_2p_3p_4\\
&+2240p_3p_5p_4+770p_3p_6p_4)z^3+(315p_3p_4+120p_5p_4)z^2+30p_4z+1
\end{aligned}
$$

$$
\begin{aligned}
H_5={}&p_1^2p_2^3p_3^4p_4^3p_5^2p_6^2z^{16}+16p_1p_2^3p_3^4p_4^3p_5^2p_6^2z^{15}+120p_1p_2^2p_3^4p_4^3p_5^2p_6^2z^{14}+560p_1p_2^2p_3^3p_4^3p_5^2p_6^2z^{13}+(1050p_1p_2^2p_4^2p_5^2p_6^2p_3^3\\
&+770p_1p_2^2p_4^3p_5^2p_6p_3^3)z^{12}+(672p_1p_2^2p_4^2p_5p_6^2p_3^3+3696p_1p_2^2p_4^2p_5^2p_6p_3^3)z^{11}+(3696p_1p_2^2p_4^2p_5p_6p_3^3+4312p_1p_2^2p_4^2p_5^2p_6p_3^2)z^{10}\\
&+(2640p_1p_2p_3^2p_5^2p_6p_4^2+8800p_1p_2^2p_3^2p_5p_6p_4^2)z^9+(660p_2p_4^2p_5^2p_6p_3^2+8085p_1p_2p_4^2p_5p_6p_3^2+4125p_1p_2^2p_4p_5p_6p_3^2)z^8\\
&+(2640p_2p_4^2p_5p_6p_3^2+8800p_1p_2p_4p_5p_6p_3^2)z^7+(4312p_2p_4p_5p_6p_3^2+3696p_1p_2p_4p_5p_6p_3)z^6+(672p_1p_2p_3p_4p_5\\
&+3696p_2p_3p_4p_6p_5)z^5+(1050p_2p_3p_4p_5+770p_3p_4p_6p_5)z^4+560p_3p_4p_5z^3+120p_4p_5z^2+16p_5z+1
\end{aligned}
$$

$$
\begin{aligned}
H_6={}&p_1^2p_2^4p_3^6p_4^4p_5^2p_6^4z^{22}+22p_1^2p_2^4p_3^6p_4^4p_5^2p_6^3z^{21}+231p_1^2p_2^4p_3^5p_4^4p_5^2p_6^3z^{20}+(770p_1^2p_2^3p_4^4p_5^2p_6^3p_3^5+770p_1^2p_2^4p_4^3p_5^2p_6^3p_3^5)z^{19}\\
&+(770p_1p_2^3p_4^4p_5^2p_6^3p_3^5+5775p_1^2p_2^3p_4^3p_5^2p_6^3p_3^5+770p_1^2p_2^4p_4^3p_5p_6^3p_3^5)z^{18}+(8316p_1p_2^3p_4^3p_5^2p_6^3p_3^5+8316p_1^2p_2^3p_4^3p_5p_6^3p_3^5\\
&+9702p_1^2p_2^3p_4^3p_5^2p_6^3p_3^4)z^{17}+(14784p_1p_2^3p_4^3p_5p_6^3p_3^5+26950p_1p_2^3p_4^3p_5^2p_6^3p_3^4+26950p_1^2p_2^3p_4^3p_5p_6^3p_3^4+5929p_1^2p_2^3p_4^3p_5^2p_6^2p_3^4)z^{16}\\
&+(16500p_1p_2^2p_4^3p_5^2p_6^3p_3^4+90112p_1p_2^3p_4^3p_5p_6^3p_3^4+16500p_1^2p_2^3p_4^2p_5p_6^3p_3^4+23716p_1p_2^3p_4^3p_5^2p_6^2p_3^4+23716p_1^2p_2^3p_4^3p_5p_6^2p_3^4)z^{15}\\
&+(72765p_1p_2^2p_4^3p_5p_6^3p_3^4+72765p_1p_2^3p_4^2p_5p_6^3p_3^4+32670p_1p_2^2p_4^3p_5^2p_6^2p_3^4+108900p_1p_2^3p_4^3p_5p_6^2p_3^4+32670p_1^2p_2^3p_4^2p_5p_6^2p_3^4)z^{14}\\
&+(107800p_1p_2^2p_4^2p_5p_6^3p_3^4+177870p_1p_2^2p_4^3p_5p_6^2p_3^4+177870p_1p_2^3p_4^2p_5p_6^2p_3^4+16940p_1p_2^2p_4^3p_5^2p_6^2p_3^3+16940p_1^2p_2^3p_4^2p_5p_6^2p_3^3)z^{13}\\
&+(379456p_1p_2^2p_4^2p_5p_6^2p_3^4+45276p_1p_2^2p_4^2p_5p_6^3p_3^3+5082p_1p_2^2p_4^2p_5^2p_6^2p_3^3+105875p_1p_2^2p_4^3p_5p_6^2p_3^3+105875p_1p_2^3p_4^2p_5p_6^2p_3^3\\
&+5082p_1^2p_2^2p_4^2p_5p_6^2p_3^3)z^{12}+705432p_1p_2^2p_3^3p_4^2p_5p_6^2z^{11}+(5082p_1p_2^2p_4^2p_6^2p_3^3+5082p_2^2p_4^2p_5p_6^2p_3^3\\
&+105875p_1p_2p_4^2p_5p_6^2p_3^3+105875p_1p_2^2p_4p_5p_6^2p_3^3+45276p_1p_2^2p_4^2p_5p_6p_3^3+379456p_1p_2^2p_4^2p_5p_6^2p_3^2)z^{10}+(16940p_1p_2^2p_4p_6^2p_3^3\\
&+16940p_2p_4^2p_5p_6^2p_3^3+177870p_1p_2p_4^2p_5p_6^2p_3^2+177870p_1p_2^2p_4p_5p_6^2p_3^2+107800p_1p_2^2p_4^2p_5p_6p_3^2)z^9+(32670p_1p_2^2p_4p_6^2p_3^2\\
&+32670p_2p_4^2p_5p_6^2p_3^2+108900p_1p_2p_4p_5p_6^2p_3^2+72765p_1p_2p_4^2p_5p_6p_3^2+72765p_1p_2^2p_4p_5p_6p_3^2)z^8+(23716p_1p_2p_4p_6^2p_3^2\\
&+23716p_2p_4p_5p_6^2p_3^2+16500p_1p_2^2p_4p_6p_3^2+16500p_2p_4^2p_5p_6p_3^2+90112p_1p_2p_4p_5p_6p_3^2)z^7+(5929p_2p_4p_6^2p_3^2\\
&+26950p_1p_2p_4p_6p_3^2+26950p_2p_4p_5p_6p_3^2+14784p_1p_2p_4p_5p_6p_3)z^6+(9702p_2p_4p_6p_3^2+8316p_1p_2p_4p_6p_3+8316p_2p_4p_5p_6p_3)z^5\\
&+(770p_1p_2p_3p_6+5775p_2p_3p_4p_6+770p_3p_4p_5p_6)z^4+(770p_2p_3p_6+770p_3p_4p_6)z^3+231p_3p_6z^2+22p_6z+1
\end{aligned}
$$

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