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Classical and Bayesian Inference for the Kavya–Manoharan Generalized Exponential Distribution under Generalized Progressively Hybrid Censored Data

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Abstract: This manuscript focuses on the statistical inference of the Kavya–Manoharan generalized exponential distribution under the generalized type-I progressive hybrid censoring sample (GTI-PHCS). Different classical approaches of estimation, such as maximum likelihood, the maximum product of spacing, least squares (LS), weighted LS, and percentiles under GTI-PHCS, are investigated. Based on the squared error and linear exponential loss functions, the Bayes estimates for the unknown parameters utilizing separate gamma priors under GTI-PHCS have been derived. Point and interval estimates of unknown parameters are developed. We carry out a simulation using the Monte Carlo algorithm to show the performance of the inferential procedures. Finally, real-world data collection is examined for illustration purposes.

Keywords: generalized progressive hybrid censoring; maximum likelihood; percentiles; maximum product of spacing; weighted least squares; Bayesian

1. Introduction

Censoring schemes (CSs) play a significant role in lifespan and reliability studies. According to the estimated experiment time and accompanying cost, many practical experiments that rely on the lifespan of objects may be completed before failing all of the items. In these situations, just a subset of an item's failure information is recorded, and the data collected is known as censored data.

The most popular censoring methods used in life tests are type-I and type-II CSs. Ref. [1] proposed a hybrid CS, which is a combination of type-I and type-II CSs. In many cases, it is prepared in advance to remove items prior to failure at several stages of the experiment; however, the above CS lack the flexibility to allow for items to be removed from the experiment at stages other than the trial's endpoint. To address this issue, Ref. [2] proposed a type-II progressive CS (T-IIPCS) as a generalization for the censoring systems described previously.

The following can be used to establish T-IIPCS: Assume that a life-test experiment is conducted on a random sampling of n items and that the trial's starting point is the number of reported failures $m(< n)$, which was previously progressive CS $(R_1, R_2, \ldots, R_m)$. During the time of the smallest failure, $X_{1:n}$, R_1 operating items are picked at random and left out of the experiment. At the second lowest failure time $X_{2:n}$, R_2 operational items are randomly chosen and eliminated from the experiment. The procedure is continued until the final failure time $X_{m:n}$ takes place, at which point all remaining operational items $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are removed from the experiment. The experiment is then terminated at $X_{m:n}$.

T-IIPCS has one big drawback in that if the items being studied are reliable and of excellent quality, the experiment time may be very long. This restriction was addressed in Ref. [3] with an improved system known as a type-I progressive hybrid CS (TI-PHCS), in which n, m, and $(R_1, R_2, \ldots, R_m)$, as well as the experimental duration τ, is determined beforehand. In this case, the experiment is completed at time $X^* = \min\{X_{m:n}, \tau\}$. Except for the last time point, this scheme is identical to T-IIPCS.

One of T-IIPHCS's most significant shortcomings is that the effective sample size is random and might be relatively small. As a result, statistical inference techniques may be unreliable or less efficient. A novel variation of progressive censoring called the generalized TI-PHCS (GTI-PHCS) was introduced in Ref. [4] to eliminate the problems that emerged in TI-PHCS, in which a smaller number of failures is predetermined. Using this filtering method would save time and money throughout a lifetime test trial. Additionally, the experiment having more failures improves estimates of statistical efficacy. The CS aids in ensuring that at least a constant number of observed items $w(< m < n)$ are satisfied to attain the efficiency necessary for statistical evaluation. It also controls the experiment's overall duration to be close to that time if the number of observed failures appears to be very low up until τ. In this case, the experiment is completed at the moment $X^* = \max\{X_{w:n}, \min\{X_{m:n}, \tau\}\}$ and any remaining operational items are removed from the experiment.

Numerous researchers have reported different techniques of estimation using the GTI-PHCS. In accordance with maximum product spacing (MPS), Ref. [5] introduced progressive type-II hybrid CS. For an exponential (E) model and a Weibull model, respectively Refs. [4,6] provided an accurate likelihood inference and entropy estimation methodology. Salem et al. [7] discussed a joint Type-II generalized progressive hybrid censoring scheme based on exponential distribution. By combining the generalized E (GE) and the basic step-stress accelerated life test with the competing risks model Ref. [8] investigated the statistical prediction problem of unobserved failure durations. In Refs [9,10], Bayesian and maximum likelihood (ML) estimation strategies for the E and Weibull under competing risks models were examined. When applying partially accelerated life tests to units whose lives are exponentially distributed under normal stress circumstances, [11] explored several point and interval estimates for the parameters involved, as well as the ideal stress change time. Ref. [12] examined the competing risk models under GTI-PHCS based on Chen distribution. Ref. [13] used GTI-PHCS to estimate the Weibull distribution's unknown parameters, reliability, and hazard functions with application to real data. Ref. [14] provided the ML and Bayesian estimators of the distribution's parameters, together with the reliability and hazard functions, based on GTI-PHCS data, from a GE distribution with application to numbers of million revolutions data before failure for each of the 23 ball bearings in the life test.

The GE model has been shown to be beneficial in a wide range of applications involving life testing, survival analysis, and reliability. Ref. [15] investigated this model, which is a special instance of the exponentiated Weibull model [16,17]. The followings are the cumulative distribution function (CDF) and probability density function (PDF) of the GE model, with scale parameter λ and shape parameter θ, for $x > 0$:

$$G(x;\theta,\lambda) = \left(1 - e^{-\lambda x}\right)^{\theta}, \qquad \theta,\lambda > 0, \tag{1}$$

$$g(x;\theta,\lambda) = \lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta-1}. \tag{2}$$

Several authors used the PDF (2) and CDF (1) to generate new extensions of the GE model, such as beta GE model [18], Marshall–Olkin GE [19], half-Cauchy GE [20], odd Lomax GE [21], and modified slashed GE [22]. Recently, [23] introduced the Kavya–Manoharan GE (KMGE) model as the special case of the Kavya–Manoharan exponentiated Weibull model. The KMGE distribution is a new extension where it does not require any additional parameters to the baseline distribution which absolutely is an advantage with no more parameters. The CDF, PDF, and hazard rate function (HRF) of the KMGE model are

$$F(x;\theta,\lambda) = \frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\lambda x}\right)^{\theta}}\right], \quad x > 0, \ \ \theta, \lambda > 0, \tag{3}$$

$$f(x;\theta,\lambda) = \frac{e}{e-1}\lambda\theta e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1} e^{-\left(1-e^{-\lambda x}\right)^{\theta}}, \tag{4}$$

and

$$h(x;\theta,\lambda) = \frac{e\lambda\theta e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1} e^{-\left(1-e^{-\lambda x}\right)^{\theta}}}{e^{-\left(1-e^{-\lambda x}\right)^{\theta}} - 1}.$$

The plots of these PDF and HRF are displayed in Figure 1. It can be noticed from this figure that the PDF can be uni-modal, decreasing, and right-skewed, but the HRF can be increasing, decreasing, and constant. These curves indicate that the KMGE model is very flexible in modeling different types of data.

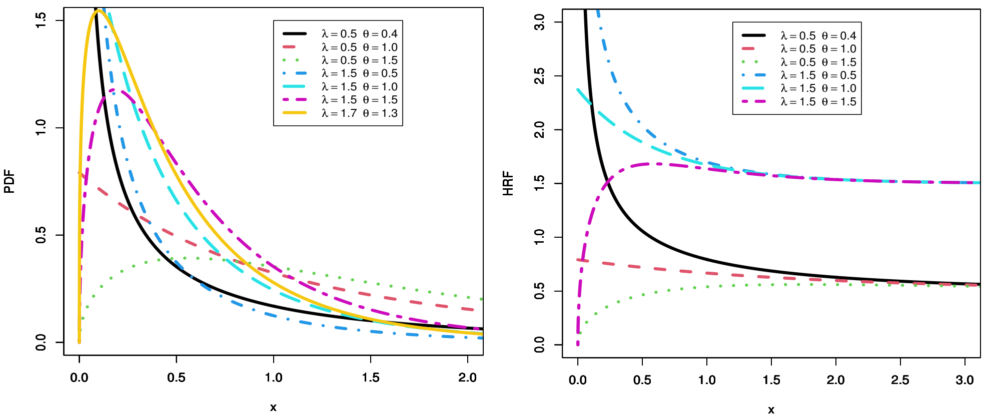

Figure 1. Plots of the PDF and HRF of the KMGE model.

As far as we are aware, there has not been any research that uses MPS, LS, and weighted LS (WLS) estimation techniques to estimate model parameters of probability distribution in the presence of the GTI-PHC data. Then according to the novelty of the KMGE distribution, we provide three important estimation methods besides the ML, percentiles (PE), and Bayesian methods. After that, a medical and engineering data application is supplied in accordance with the flexibility of the KMGE distribution (see Figure 1). In this regard, we summarized our study's objectives as follows:

- Discuss the point and interval statistical inference of the two unknown parameters θ and λ for the KMGE distribution using five classical estimation approaches such as ML, MPS, LS, WLS, and PE based on GTI-PHCS.
- Estimate the model's parameters of the KMGE distribution in view of the Bayesian estimation strategy using symmetric and asymmetric loss functions.
- Using specific metrics of accuracy, a simulation study is run to look at how different estimates behave.
- A potential application based on GTI-PHCS has been explored for data from engineering and medical sciences.

The rest of this paper is structured as follows: the model formulation of GTI-PHCS is proposed in the Section 2. Five classical estimation approaches such as ML, MPS, LS, WLS, and PE are investigated in the Section 3. Bayesian estimation with credible intervals is discussed in the Section 4. In Section 5, we evaluate the quality points and interval estimators using a Monte Carlo approach. In Section 6, we employ the theoretical study findings to real-world data. Finally, the discussion and conclusion are presented in Section 7.

2. Generalized Type-I Progressive Hybrid Censoring

The implementation steps of the GTI-PHCS are described below:

1. Assume that a random sample of n units undergoes a lifetime testing trial.
2. Suppose that $X_1, X_2, \ldots, X_n$, have the KMGE model with CDF (3) and PDF (4).
3. Assume that before starting the experiment, the integers w, m, the experimental time τ and $(R_1, R_2, \ldots, R_m)$ are assigned, so that $0 < \tau < \infty, 0 < w < m \leq n$.
4. The operational units R_1 are chosen at random and eliminated from the experiment at $X_{1:n}$, the first failure time. At the subsequent failure time $X_{2:n}$, R_2 operating units are randomly selected and eliminated from the experiment, and the procedure is repeated. Eventually, the experiment is completed when $X^* = \max\{X_{w:n}, \min\{X_{m:n}, \tau\}\}$, and any remaining operational units $R_{\mathcal{C}}$ are omitted from the experiment. Table 1 contains the values of the final censoring number $R_{\mathcal{C}}$.
5. Assume that $\mathfrak{B}$ represents the number of units that fail prior to τ. The experiment's end time X^* is, therefore, provided by

$$X^* = \begin{cases} X_{w:m:n}, & \text{if } \tau \leq X_{w:m:n} < X_{m:m:n}, \\ \tau, & \text{if } X_{w:m:n} \leq \tau < X_{m:m:n}, \\ X_{m:m:n}, & \text{if } X_{w:m:n} < X_{m:m:n} \leq \tau. \end{cases}$$

Any one of the subsequent six cases could be observed for the results:

- Case I: If the observed time τ happens to occur before the wth failure time $X_{w:n}$ and $\mathfrak{B}(< w)$ failures occur up to time τ, $\tau < X_{w:n} < X_{m:n}$. Afterward, we won't remove any operating units from the experiment until the $(\mathfrak{B}+1)$th, $\ldots, (w-1)$th failure times, after which we will remove all of the remaining operating units $R_w^* = n - w - \sum_{i=1}^{w-1} R_i$ from the experiment at the wth failure time, thereby stopping the experiment at $X^* = X_{w:n}$, where $R_{\mathfrak{B}+1} = \cdots = R_{w-1} = 0$, see Figure 2. In this case, we allow the experiment to continue after experimental time τ is reached to guarantee that at least the wth failure time $X_{w:n}$ happens. The following remarks, in this case, will be made:
$X_{1:n} < \cdots < X_{\mathfrak{B}:n} \leq \tau < X_{\mathfrak{B}+1:n} < \cdots < X_{w:n}$.
- Case II: When wth failure time $X_{w:n}$ happens before the τ, $X_{w:n} \leq \tau < X_{m:n}$, and $\mathfrak{B}(\geq w)$ failures occur up to the τ time. The experiment is terminated at $X^* = \tau$ by removing all of the remaining operational units $R_\tau^* = n - \mathfrak{B} - \sum_{i=1}^{\mathfrak{B}} R_i$, as shown in Figure 3. The following observations will be made in this situation: $X_{1:n} < \cdots < X_{w:n} < \cdots < X_{\mathfrak{B}:n} \leq \tau$.
- Case III: When the mth failure time $X_{m:n}$ happens before the time τ, $X_{w:n} < X_{m:n} \leq \tau$, then all the remaining operational units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are deleted from the experiment, terminating it at $X^* = X_{m:n}$, as shown in Figure 4. The following observations will be made in this situation: $X_{1:n} < \cdots < X_{w:n} < \cdots < X_{m:n} \leq \tau$.

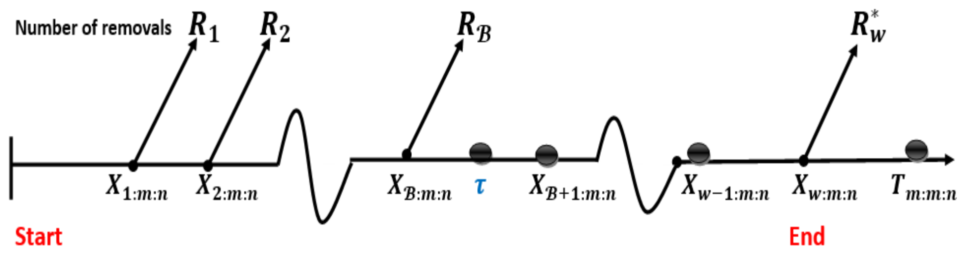

Figure 2. Case 1. The sampling process under GTI-PHCS, when $\tau < X_{w:n} < X_{m:n}$.

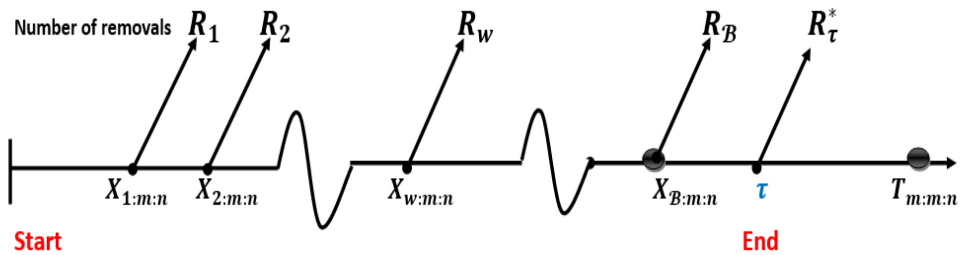

Figure 3. Case 2. The sampling process under GTI-PHCS, when $X_{w:n} \leq \tau < X_{m:n}$.

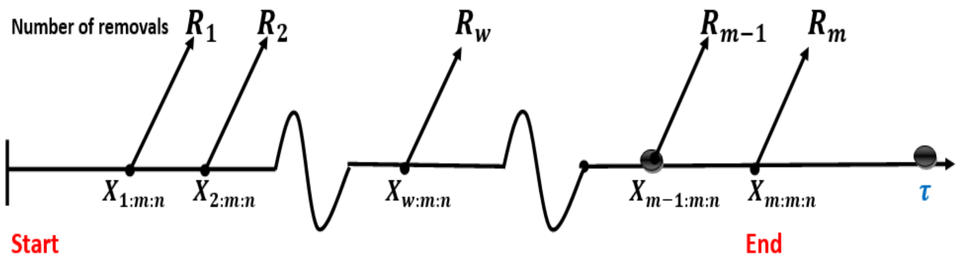

Figure 4. Case 3. The sampling process under GTI-PHCS, when $X_{w:n} < X_{m:n} \leq \tau$.

Table 1. Values of X^*, $\mathcal{K}$, ϵ, and $R_{\mathcal{C}}$, for three cases.

	Case I	Case II	Case III
X^*	$X_{w:n}$	τ	$X_{m:n}$
$\mathcal{K}$	w	$\mathfrak{B}$	m
ϵ	0	1	0
$R_{\mathcal{C}}$	$R_w = R_w^* = n - w - \sum_{i=1}^{w-1} R_i$, $\sum_{i=\mathfrak{B}+1}^{w-1} R_i = 0$	$R_\tau^* = n - \mathfrak{B} - \sum_{i=1}^{\mathfrak{B}} R_i$	$R_m = n - m - \sum_{i=1}^{m-1} R_i$

3. Different Classical Approaches of Estimation

This section discusses five classical methods for calculating ML, MPS, LS, WLS, and PE of the underlying parameters θ and λ using data gathered via GTI-PHCS.

3.1. Approach of ML Estimation

The likelihood function based on GTI-PHCS is provided via

$$\mathbf{L}(\theta, \lambda; \mathbf{x}) \propto \left[\prod_{j=1}^{\mathcal{K}} f(x_{j:n})\left[1 - F(x_{j:n})\right]^{R_j}\right]\left[1 - F(\tau)\right]^{\epsilon R_\tau^*}, \tag{5}$$

where $\mathbf{x} = (x_{1:n}, \ldots, x_{\mathcal{K}:n})$, $R_\tau^* = n - \mathcal{K} - \sum_{i=1}^{\mathcal{K}} R_i$, final censored number $R_{\mathcal{C}}$, and $X^*, \mathcal{K}, \epsilon$ be experiment end time values based on three cases are reported in Table 1. It is interesting to note that, in Case I, several values of R_i, $i = 1, 2, \ldots, m$ may be different throughout the test than those set before the test even starts.

Utilizing CDF (3) and PDF (4), the log-likelihood function takes the below formula:

$$\begin{aligned}\mathcal{L} = \log \mathbf{L}(\theta, \lambda; \mathbf{x}) \propto \mathcal{K} \log[\lambda\theta] + \sum_{j=1}^{\mathcal{K}} \Bigg\{ &-\lambda x_j + (\theta - 1) \log\left[1 - e^{-\lambda x_j}\right] - \left(1 - e^{-\lambda x_j}\right)^{\theta} \\ &+ R_j \left(\log\left[e^{1-\left(1-e^{-\lambda x_j}\right)^{\theta}} - 1 \right] \right) + \epsilon R_\tau^* \left(\log\left[e^{1-\left(1-e^{-\lambda \tau}\right)^{\theta}} - 1 \right] \right) \Bigg\}. \end{aligned} \tag{6}$$

Note that, we write x_j instead of $x_{j:n}$ for the simplified form. According to Equation (6), $(\hat{\theta}, \hat{\lambda})$ of (θ, λ), can be computed as below: The ML estimates (MLEs) of θ and λ may be determined by taking the first partial derivatives of (6) with regard to θ and λ and equating them to zero as

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \theta} = \sum_{j=1}^{\mathcal{K}} \Bigg\{ &\left(1 - \left(1 - e^{-\lambda x_j}\right)^{\theta}\right) \log\left[1 - e^{-\lambda x_j}\right] - R_j \left(\frac{\left(1 - e^{-\lambda x_j}\right)^{\theta} \log\left[1 - e^{-\lambda x_j}\right]}{1 - e^{\left(1-e^{-\lambda x_j}\right)^{\theta} - 1}} \right) \Bigg\} \\ &- \epsilon R_\tau^* \left(\frac{\left(1 - e^{-\lambda \tau}\right)^{\theta} \log\left[1 - e^{-\lambda \tau}\right]}{1 - e^{\left(1-e^{-\lambda \tau}\right)^{\theta} - 1}} \right) + \frac{\mathcal{K}}{\theta} = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{j=1}^{\mathcal{K}} \Bigg\{ &-x_j + \frac{x_j e^{-\lambda x_j}}{1 - e^{-\lambda x_j}} \left(\theta - \theta\left(1 - e^{-\lambda x_j}\right)^{\theta} - 1\right) - R_j \left(\frac{\theta x_j e^{-\lambda x_j} \left(1 - e^{-\lambda x_j}\right)^{\theta - 1}}{1 - e^{\left(1-e^{-\lambda x_j}\right)^{\theta} - 1}} \right) \Bigg\} \\ &- \epsilon R_\tau^* \left(\frac{\theta \tau e^{-\lambda \tau} \left(1 - e^{-\lambda \tau}\right)^{\theta - 1}}{1 - e^{\left(1-e^{-\lambda \tau}\right)^{\theta} - 1}} \right) + \frac{\mathcal{K}}{\lambda} = 0.\end{aligned}$$

The MLEs $\hat{\theta}$ and $\hat{\lambda}$ of θ and λ may be computed by solving the score equations, $\dfrac{\partial \mathcal{L}}{\partial \theta} = 0$ and $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0$, with regards to θ and λ and solving these equations concurrently to produce the MLEs. Because analytical solutions cannot get the roots, these equations can indeed be investigated numerically utilizing iterative procedures employing statistical software via the "maxLik" package installed through the R 4.3.0 programming language.

According to the usual asymptotic normality theory of MLEs, we may assume that $\dfrac{\hat{\theta} - \theta}{\sqrt{\mathrm{V}(\hat{\theta})}}$ and $\dfrac{\hat{\lambda} - \lambda}{\sqrt{\mathrm{V}(\hat{\lambda})}}$ can be approximated by

$$\frac{\hat{\theta} - \theta}{\sqrt{\mathrm{V}(\hat{\theta})}} \sim N(0,1) \text{ and } \frac{\hat{\lambda} - \lambda}{\sqrt{\mathrm{V}(\hat{\lambda})}} \sim N(0,1),$$

where $\mathrm{V}(\hat{\theta})$ and $\mathrm{V}(\hat{\lambda})$ are the variance of $\hat{\theta}$ and $\hat{\lambda}$ which may be founded by computing the inverse of the Fisher information matrix, i.e.,

$$\mathbf{I} = -\begin{pmatrix} \dfrac{\partial^2 \pounds}{\partial \theta^2} & \dfrac{\partial^2 \pounds}{\partial \theta \partial \lambda} \\ \dfrac{\partial^2 \pounds}{\partial \lambda \partial \theta} & \dfrac{\partial^2 \pounds}{\partial \lambda^2} \end{pmatrix}^{-1}_{\theta = \hat{\theta}, \lambda = \hat{\lambda}} = \begin{pmatrix} \mathrm{V}(\hat{\theta}) & \mathrm{Cov}(\hat{\theta}, \hat{\lambda}) \\ \mathrm{Cov}(\hat{\theta}, \hat{\lambda}) & \mathrm{V}(\hat{\lambda}) \end{pmatrix}, \tag{7}$$

where the caret ˆ denotes that the derivative is evaluated at $(\hat{\theta}, \hat{\lambda})$. It is simple to obtain the second partial derivatives of the probability function's natural logarithm for θ and λ.

Suppose that $\eta_1 = \theta$ and $\eta_2 = \lambda$, then, a $100(1-\alpha)\%$ normal approximation confidence interval (NACI) of η_i, for $i = 1, 2$, may be easily computed as below:

$$\left(\max\{0, \hat{\eta}_i - z_{\alpha/2}\sqrt{\mathrm{V}(\hat{\eta}_i)}\}, \hat{\eta}_i + z_{\alpha/2}\sqrt{\mathrm{V}(\hat{\eta}_i)}\right),$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of $N(0,1)$ distribution, and $\hat{\eta}_i$ is the MLE of η_i.

In the lower bound of NACI, as $n \to \infty$, the positive parameter can occasionally have a negative value. Ref. [24], as $n \to \infty$, recommended using a log transformation confidence interval (LTCI) in this situation. Based on the log-transformed MLE's usual estimate, $\dfrac{\log \hat{\eta}_i - \log \eta_i}{\sqrt{\mathrm{V}(\log \hat{\eta}_i)}}$, where $i = 1, 2$, as a standard normal distribution can be approximated.

$$\text{as } n \to \infty, \frac{\log \hat{\eta}_i - \log \eta_i}{\sqrt{\mathrm{V}(\log \hat{\eta}_i)}} \sim N(0,1),$$

where $\mathrm{V}(\log \hat{\eta}_i) = \dfrac{\mathrm{V}(\hat{\eta}_i)}{\hat{\eta}_i^{\,2}}$. Accordingly, a $100(1-\alpha)\%$ LTCI for η_i is characterized

$$\text{as } n \to \infty, \left(\hat{\eta}_i \exp\left[-z_{\alpha/2}\frac{\sqrt{\mathrm{V}(\hat{\eta}_i)}}{\hat{\eta}_i}\right], \hat{\eta}_i \exp\left[z_{\alpha/2}\frac{\sqrt{\mathrm{V}(\hat{\eta}_i)}}{\hat{\eta}_i}\right]\right).$$

3.2. Approach of Maximum Product of Spacing Estimation

Ref. [25] proposed an alternate technique to the ML method for estimating unknown parameters in continuous distributions. Refs. [26,27] utilized progressive type-II censoring to estimate the parameters involved in the Weibull and Kavya–Manoharan inverse length biased exponential distributions. The MPS estimates (MPSEs) are yielded by maximizing the next product of spacing for θ and λ.

$$\mathcal{P}(\theta, \lambda; \mathbf{x}) = \left(\prod_{j=1}^{\mathcal{K}+1}\left[F(x_j) - F(x_{j-1})\right]\right)\left(\prod_{j=1}^{\mathcal{K}}\left[1 - F(x_j)\right]^{R_j}\right)\left[1 - F(\tau)\right]^{\epsilon R_\tau^*}, \tag{8}$$

where $F(x_0) = 0$ and $F(x_{\mathcal{K}+1}) = 1$, and the expression about GTI-PHCS model has been discussed in Table 1. Utilizing (3) and by maximizing the product of spacing for θ and λ, then the MPSEs $\tilde{\theta}$ and $\tilde{\lambda}$ of θ and λ are provided

$$\begin{aligned}\mathcal{P}(\theta, \lambda; \mathbf{x}) = a^*&\left(1 - e^{-\left(1-e^{-\lambda x_1}\right)^\theta}\right)\left(e^{1-\left(1-e^{\lambda x_{\mathcal{K}}}\right)^\theta} - 1\right)\left[\frac{e^{1-\left(1-e^{-\lambda\tau}\right)^\theta} - 1}{e-1}\right]^{\epsilon R_\tau^*}\\ &\times\left(\prod_{j=2}^{\mathcal{K}}\left[e^{-\left(1-e^{-\lambda x_{j-1}}\right)^\theta} - e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right]\right)\left(\prod_{j=1}^{\mathcal{K}}\left[\frac{e^{1-\left(1-e^{-\lambda x_j}\right)^\theta} - 1}{e-1}\right]^{R_j}\right),\end{aligned} \tag{9}$$

where $a^* = \frac{1}{e-1}\left(\frac{e}{e-1}\right)^{\mathcal{K}}$.

To obtain the MPSEs, the nonlinear equations can also be solved simultaneously. Since an exact solution cannot obtain the roots, these equations are analytically resolved using iterative strategies and statistical software:

$$\begin{aligned}\frac{\partial \log[\mathcal{P}(\theta, \lambda; \mathbf{x})]}{\partial\theta} = \frac{A(x_1)}{1 - D(x_1)} - \frac{A(x_{\mathcal{K}})}{D(x_{\mathcal{K}}) - e^{-1}} - \epsilon R_\tau^*\frac{A(\tau)}{D(\tau) - e^{-1}} + \sum_{j=2}^{\mathcal{K}}\left[\frac{A(x_j) - A(x_{j-1})}{D(x_{j-1}) - D(x_j)}\right]\\ - \sum_{j=1}^{\mathcal{K}}\left[R_j\frac{A(x_j)}{D(x_j) - e^{-1}}\right] = 0,\end{aligned}$$

$$\frac{\partial \log[\mathcal{P}(\theta,\lambda;\mathbf{x})]}{\partial\lambda} = \frac{B(x_1)}{1-D(x_1)} - \frac{B(x_{\mathcal{K}})}{D(x_{\mathcal{K}})-e^{-1}} - \epsilon R_\tau^* \frac{B(\tau)}{D(\tau)-e^{-1}} + \sum_{j=2}^{\mathcal{K}}\left[\frac{B(x_j)-B(x_{j-1})}{D(x_{j-1})-D(x_j)}\right] - \sum_{j=1}^{\mathcal{K}}\left[R_j \frac{B(x_j)}{D(x_j)-e^{-1}}\right] = 0,$$

where

$$A(x_j) = \log\left[1-e^{-\lambda x_j}\right]\left(1-e^{-\lambda x_j}\right)^\theta e^{-\left(1-e^{-\lambda x_j}\right)^\theta}, \tag{10}$$

$$B(x_j) = x_j\theta e^{-\lambda x_j}\left(1-e^{-\lambda x_j}\right)^{\theta-1} e^{-\left(1-e^{-\lambda x_j}\right)^\theta}, \tag{11}$$

$$D(x_j) = e^{-\left(1-e^{-\lambda x_j}\right)^\theta},$$

the same forms are provided for $j=1$, or $\mathcal{K}$.

3.3. Approaches of LS and WLS

Ref. [28] established the LS and WLS techniques for estimating the parameters of the beta distribution. Refs. [29,30] employed progressive type-II censoring to estimate the parameters contained in the doubly Poisson-exponential and exponential-doubly Poisson distributions. To estimate the parameters in the Poisson-logarithmic half-logistic distribution under a progressive-stress accelerated life test, Ref. [31] proposed adaptive type-II progressive hybrid censoring.

Let $(X_1,\ldots,X_{\mathcal{K}})$ be the ordered GTI-PHCS sample from the KMGE model of size $\mathcal{K}$. The LS estimates (LSEs) of λ, θ are derived by minimizing the below formula

$$\mathcal{S}(\theta,\lambda) = \sum_{j=1}^{\mathcal{K}}\left(F(x_j) - \mathrm{E}\left[\widehat{F}(x_j)\right]\right)^2,$$

in which $\mathrm{E}\left[\widehat{F}(x_j)\right]$ denotes the empirical CDF expectation, as supplied in [32]

$$\mathrm{E}\left[\widehat{F}(x_j)\right] = 1 - \prod_{\nu=\mathcal{K}-j+1}^{\mathcal{K}}\left[\frac{\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}} R_i}{1+\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}} R_i}\right], j=1,\ldots,\mathcal{K},$$

As a result, the LSEs $\breve{\theta}$ and $\breve{\lambda}$ of θ and λ are yielded by minimizing the following formula

$$\mathcal{S}(\theta,\lambda) = \sum_{j=1}^{\mathcal{K}}\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right] - \mathrm{E}\left[\widehat{F}(x_j)\right]\right)^2.$$

These estimates can also be achieved by simultaneously solving the nonlinear equations to generate the LSEs. These equations can be resolved analytically using iterative methods and statistical tools since precise solutions cannot get the roots.

$$\frac{\partial\mathcal{S}(\theta,\lambda)}{\partial\theta} = \sum_{j=1}^{\mathcal{K}} A(x_j)\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right] - \mathrm{E}\left[\widehat{F}(x_j)\right]\right) = 0,$$

$$\frac{\partial\mathcal{S}(\theta,\lambda)}{\partial\lambda} = \sum_{j=1}^{\mathcal{K}} B(x_j)\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right] - \mathrm{E}\left[\widehat{F}(x_j)\right]\right) = 0,$$

where $A(x_j)$ and $B(x_j)$ are provided in (10) and (11), respectively.

The WLS estimates (WLSEs) of θ and λ may be generated by minimizing the below formula

$$\mathcal{S}^*(\theta,\lambda)=\sum_{j=1}^{\mathcal{K}}\mathcal{W}_j\Big(F(x_j)-\mathrm{E}\Big[\widehat{F}(x_j)\Big]\Big)^2,$$

where $\mathcal{W}_j=\dfrac{1}{\mathrm{V}[\widehat{F}(x_j)]}$ is the weight factor and $\mathrm{V}[\widehat{F}(x_j)]$ is the variance of the empirical CDF, see [32], which is provided via

$$\mathrm{V}[\widehat{F}(x_j)]=\left(\prod_{\nu=\mathcal{K}-j+1}^{\mathcal{K}}\Delta_\nu\right)\left(\prod_{\nu=\mathcal{K}-j+1}^{\mathcal{K}}\Omega_\nu-\prod_{\nu=\mathcal{K}-j+1}^{\mathcal{K}}\Delta_\nu\right),j=1,\ldots,\mathcal{K},$$

where

$$\begin{aligned}\Delta_\nu&=\frac{\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}}R_i}{1+\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}}R_i},\nu=1,\ldots,\mathcal{K},\\ \Omega_\nu&=\Delta_\nu+\frac{1}{(1+\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}}R_i)(2+\nu+\sum_{i=\mathcal{K}-\nu+1}^{\mathcal{K}}R_i)},\nu=1,\ldots,\mathcal{K}.\end{aligned}$$

Minimizing the next quantity, we get the WLSEs $\ddot{\theta}$ and $\ddot{\lambda}$ of θ and λ.

$$\mathcal{S}^*(\theta,\lambda)=\sum_{j=1}^{\mathcal{K}}\mathcal{W}_j\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right]-\mathrm{E}\Big[\widehat{F}(x_j)\Big]\right)^2.$$

To produce the WLSEs, the following nonlinear equations can be numerically solved via iterative methods and statistical tools:

$$\frac{\partial\mathcal{S}^*(\theta,\lambda)}{\partial\theta}=\sum_{j=1}^{\mathcal{K}}\mathcal{W}_jA(x_j)\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right]-\mathrm{E}\Big[\widehat{F}(x_j)\Big]\right)=0,$$

$$\frac{\partial\mathcal{S}^*(\theta,\lambda)}{\partial\lambda}=\sum_{j=1}^{\mathcal{K}}\mathcal{W}_jB(x_j)\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\lambda x_j}\right)^\theta}\right]-\mathrm{E}\Big[\widehat{F}(x_j)\Big]\right)=0,$$

where $A(x_j)$ and $B(x_j)$ are given by (10) and (11), respectively.

3.4. Approach of Percentiles Estimation

Ref. [33] suggested a percentile approach for estimating distribution. If data from a closed-form CDF were gathered, it would only make sense to estimate the unknown parameter by adjusting a straight line between the theoretical points generated by the CDF and the percentile points of the sample. In this method, the empirical CDF looks like this:

$$\widehat{F}(x_j)=1-\prod_{s=1}^{j}(1-\widehat{\mathbf{q}}_s),j=1,\ldots,\mathcal{K},$$

where $\mathcal{K}$ is defined as in Table 1 and

$$\widehat{\mathbf{q}}_s=\frac{1}{n-\left[\sum_{i=1}^{s-1}R_i\right]-s+1},s=1,\ldots,\mathcal{K}.$$

Based on GTI-PHCS, the PEs of the considered parameters can be obtained as follows: It is feasible to acquire the PEs $\check{\theta}$ and $\check{\lambda}$ of θ and λ by reducing the following quantity with regards to θ and λ.

$$\mathbf{\Psi}(\theta,\lambda;\mathbf{x}) = \sum_{i=1}^{\mathcal{K}} \left(x_i + \frac{1}{\lambda} \log\left[1 - \left(-\log\left[1 - \frac{e-1}{e}\rho_i\right]\right)^{\frac{1}{\theta}}\right]\right)^2, \tag{12}$$

where

$$\rho_i = \frac{1}{2}\left(\widehat{F}(x_{i-1}) + \widehat{F}(x_i)\right). \tag{13}$$

These estimates can also be achieved by concurrently solving the nonlinear equations and obtaining the PEs. Because an exact solution cannot yield the roots, these equations can be investigated numerically by employing iterative procedures employing statistical software:

$$\begin{aligned}\frac{\partial\mathbf{\Psi}(\theta,\lambda;\mathbf{x})}{\partial\theta} &= \sum_{i=1}^{\mathcal{K}} \left\{ \frac{1}{\lambda\theta^2} \frac{\log\left[-\log\left[1-\frac{e-1}{e}\rho_i\right]\right]}{\left(-\log\left[1-\frac{e-1}{e}\rho_i\right]\right)^{\frac{-1}{\theta}} - 1} \right. \\ &\quad \left. \times \left(x_i + \frac{1}{\lambda}\log\left[1-\left(-\log\left[1-\frac{e-1}{e}\rho_i\right]\right)^{\frac{1}{\theta}}\right]\right)\right\} = 0, \\ \frac{\partial\mathbf{\Psi}(\theta,\lambda;\mathbf{x})}{\partial\lambda} &= \sum_{i=1}^{\mathcal{K}} \left\{ \frac{1}{\lambda^2}\log\left[1-\left(-\log\left[1-\frac{e-1}{e}\rho_i\right]\right)^{\frac{1}{\theta}}\right] \right. \\ &\quad \left. \times \left(x_i + \frac{1}{\lambda}\log\left[1-\left(-\log\left[1-\frac{e-1}{e}\rho_i\right]\right)^{\frac{1}{\theta}}\right]\right)\right\} = 0.\end{aligned}$$

4. Bayesian Estimation

Here, the squared error loss (SEL) function and linear exponential loss (LINEXL) function are used to generate the Bayes estimators of θ and λ. To accomplish this, we supposed that the KMGE model parameters, θ and λ, each have independent gamma $(G(.))$ priors of the forms $G(o_1, s_1)$ and $G(o_2, s_2)$. Gamma priors should be considered for a variety of reasons, including the fact that they are (a) adjustable, (b) offer diverse shapes based on parameter values, and (c) are fairly simple and brief and might not produce a solution with a challenging estimation problem. The KMGE parameters θ and λ joint prior density is given by

$$\pi(\theta,\lambda) \propto \theta^{o_1-1}e^{-s_1\theta}\lambda^{o_2-1}e^{-s_2\lambda}, \qquad \theta,\lambda > 0, o_1, o_2 > 0, s_1, s_2 > 0, \tag{14}$$

where the hyper-parameters o_1, o_2, s_1, and s_2 are the ones that hold the previous data. Many academic authors created Bayesian estimates for their parameter models utilizing instructive gamma priors, including Refs. [34,35], and Ref. [36]. The informative priors will be used to elicit the hyper-parameters. The mean and variance using the maximum likelihood estimates of the KMGE distribution are determined. The priors (gamma priors) of the o_j and s_j mean and variance will be identical to θ and λ. We may find the means and variances of $\hat{\theta}$ and $\hat{\lambda}$ by equating them with the mean and variance of the gamma priors, as below

$$\frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{\,j} = \frac{o_1}{s_1}, \& \frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\theta}^{\,\mathrm{j}} - \frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{\,\mathrm{j}}\right)^2 = \frac{o_1}{{s_1}^2},$$

$$\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{\,j} = \frac{o_2}{s_2}, \& \frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\lambda}^{\,j} - \frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{\,j}\right)^2 = \frac{o_2}{{s_2}^2}.$$

After resolving the two equations above, the estimated hyper-parameters can now be expressed as

$$o_1 = \frac{\left(\frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{j}\right)^2}{\frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\theta}^{j}-\frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{j}\right)^2}, \& s_1 = \frac{\frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{j}}{\frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\theta}^{j}-\frac{1}{L}\sum_{j=1}^{L}\hat{\theta}^{j}\right)^2},$$

$$o_2 = \frac{\left(\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{j}\right)^2}{\frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\lambda}^{j}-\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{j}\right)^2}, \& s_2 = \frac{\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{j}}{\frac{1}{L-1}\sum_{j=1}^{L}\left(\hat{\lambda}^{j}-\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^{j}\right)^2}.$$

The likelihood function (5) and the joint prior (14) may be combined to obtain the posterior distribution, say $\Pi^*(\theta,\lambda|\, data)$, which is defined as

$$\Pi^*(\theta,\lambda \mid data) = \frac{L(\theta,\lambda \mid data)\ \pi(\theta,\lambda)}{\int_0^\infty\int_0^\infty L(\theta,\lambda \mid data)\ \pi(\theta,\lambda)\ d\theta\ d\lambda}, \tag{15}$$

the joint posterior density can be denoted in the final form:

$$\Pi^*(\theta,\lambda \mid data) \propto e^{-s_1\theta-s_2\lambda}\lambda^{\mathcal{K}+o_2-1}\theta^{\mathcal{K}+o_1-1}e^{-\lambda\sum_{j=1}^{\mathcal{K}}x_j}\left[\prod_{j=1}^{\mathcal{K}}\left(1-e^{-\lambda x_j}\right)^{\theta-1}\left[1-\left(1-e^{-\lambda x_j}\right)^{\theta}\right]^{R_j}\right] \left[1-\left(1-e^{-\lambda\tau}\right)^{\theta}\right]^{\epsilon R_\tau^*}. \tag{16}$$

The SEL function should be taken into account in a Bayesian analysis for several reasons: In addition to being a typical symmetric loss and being straightforward, obvious, and easy to understand, it also assumes that overestimation and underestimation are treated equally and directly builds the Bayes estimator by using the posterior mean. However, when considering the SEL function, the posterior expectation of (16), which is expressed as

$$\begin{aligned}\hat{g}_{SEL}(\theta,\lambda|\, data) &= E_{\theta,\lambda|\, data}(g(\theta,\lambda)) \\ &= \int_0^\infty\int_0^\infty (g(\theta,\lambda))\Pi^*(\theta,\lambda \mid data)d\theta\ d\lambda.\end{aligned} \tag{17}$$

The most widely used asymmetric loss function is the LINEXL function. In many ways, the asymmetric loss function is thought to be more complete, according to Varian [37]. The Bayes estimate (BEs) of any function $g(\theta,\lambda)$ under the LINEXL function can be determined as

$$\hat{g}_{LINEXL}(\theta,\lambda \mid\ data) = -\frac{1}{\varepsilon}\log\left[E\left(e^{-\varepsilon g(\theta,\lambda)} \mid\ data\right)\right],\ \varepsilon \neq 0, \tag{18}$$

It is clear from (16), that it is impossible to express the marginal posterior densities of θ and λ explicitly. We propose utilizing Bayesian Markov chain Monte Carlo (MCMC) methods to generate samples from (16). The conditional posterior density functions of θ and λ are, thus, obtained, respectively, from (16), as

$$\Pi^*(\theta|\lambda, data) \propto e^{-s_1\theta}\theta^{\mathcal{K}+o_1-1}\left[\prod_{j=1}^{\mathcal{K}}\left(1-e^{-\lambda x_j}\right)^{\theta-1}\left[1-\left(1-e^{-\lambda x_j}\right)^{\theta}\right]^{R_j}\right]\left[1-\left(1-e^{-\lambda\tau}\right)^{\theta}\right]^{\epsilon R_\tau^*}. \tag{19}$$

and

$$\Pi^*(\lambda|\theta\ ,\ data) \propto e^{-s_2\lambda}\lambda^{\mathcal{K}+o_2-1}e^{-\lambda\sum_{j=1}^{\mathcal{K}}x_j}\left[\prod_{j=1}^{\mathcal{K}}\left(1-e^{-\lambda x_j}\right)^{\theta-1}\left[1-\left(1-e^{-\lambda x_j}\right)^{\theta}\right]^{R_j}\right] \left[1-\left(1-e^{-\lambda\tau}\right)^{\theta}\right]^{\epsilon R_\tau^*}. \tag{20}$$

The posterior density functions of θ and λ, respectively, cannot be analytically reduced to any known distribution, as shown in (19) and (20). As a result, it is believed that the Metropolis–Hastings (M–H) method is the best approach to resolving this problem; for further information, see Refs. [38–40]. The following describes the sampling method for the M–H algorithm based on the normal proposal distribution:

1. Set the starting values $\theta^{(0)} = \hat{\theta}$ and $\lambda^{(0)} = \hat{\lambda}$.
2. Set $i = 1$.
3. Create θ^* and λ^* from $N\left(\hat{\theta}, \hat{\sigma}^2_{\hat{\theta}}\right)$ and $N\left(\hat{\lambda}, \hat{\sigma}^2_{\hat{\lambda}}\right)$, respectively.
4. Find $A_\theta = min\left\{1, \frac{\Pi^*\left(\theta^*|\lambda^{(i-1)},data\right)}{\Pi^*\left(\theta^{(i-1)}|\lambda^{(i-1)},data\right)}\right\}$, $A_\lambda = min\left\{1, \frac{\Pi^*\left(\lambda^*|\theta^{(i-1)},data\right)}{\Pi^*\left(\lambda^{(i-1)}|\theta^{(i-1)},data\right)}\right\}$.
5. Utilizing the uniform $U(0,1)$ distribution, generate samples u_1 and u_2.
6. If both u_1 and u_2 are less than A_θ and A_λ, respectively, then set $\theta^{(i)} = \theta^*$ and $\lambda^{(i)} = \lambda^*$, respectively. Otherwise, set $\theta^{(i)} = \theta^{(i-1)}$ and $\lambda^{(i)} = \lambda^{(i-1)}$, respectively.
7. Set $i = i + 1$.
8. Redo steps 3–7 H times to get $\theta^{(i)}$ and $\lambda^{(i)}$ for $i = 1, 2, \ldots, H$.

5. Results of Simulation

Because assessing the efficiency of estimating methods is conceptually challenging, a Monte Carlo simulation is employed to address this obstacle. Here, we evaluate the performance and efficacy of the estimating approaches presented in earlier parts using Monte Carlo simulation. The procedure is as follows:

1. Specify the sample size n and parameter (θ, λ) values. Moreover, specify $\mathcal{K}$, m, τ, and $(R_1, R_2, \ldots, R_m)$ values.
2. Create n observations from the Uniform (0, 1) distribution $(V_1, V_2, \ldots, V_n)$.
3. The observations $(x_1, x_2, \ldots, x_n)$ may be obtained via CDF (3).
4. As described in Section 2, employ GTI-PHCS to the random sample produced in Step 3.
5. Compute the MLEs, MPSEs, LSEs, WLSEs, PEs, NACIs, and LTCIs of (θ, λ) as mentioned in Section 3.
6. Repeat the preceding steps $\mathfrak{M}$ = 1000 times.
7. Determine the average of estimates, mean squared error (MSEr), and relative bias (RB) of $\hat{\eta}$ across $\mathfrak{M}$ samples as described in the following:

$$\overline{\hat{\eta}} = \frac{1}{\mathfrak{M}}\sum_{i=1}^{\mathfrak{M}}\hat{\eta}_i, \mathrm{RB}(\hat{\eta}) = \frac{1}{\mathfrak{M}}\sum_{i=1}^{\mathfrak{M}}\frac{|\hat{\eta}_i-\eta|}{\eta}, \mathrm{MSEr}(\hat{\eta}) = \frac{1}{\mathfrak{M}}\sum_{i=1}^{\mathfrak{M}}(\hat{\eta}_i-\eta)^2,$$

   where $\hat{\eta}$ is an estimate of η.
8. Determine the mean of the different estimates with their MSErs and RBs utilizing Step 9.
9. Compute the average of the RBs (ARB) and MSErs (AMSEr) as below:

$$\mathrm{ARB} = \frac{\mathrm{RB}(\hat{\theta})+\mathrm{RB}(\hat{\lambda})}{2}, \mathrm{AMSEr} = \frac{\mathrm{MSEr}(\hat{\theta})+\mathrm{MSEr}(\hat{\lambda})}{2}.$$

10. Calculate the average lengths (ALs) and coverage probabilities (COVPs) of the parameters (θ, λ), then their 95% NACIs and LTCIs. Calculate also the average of the ALs (AAL) as below:

$$\mathrm{AAL} = \frac{\mathrm{AL}(\hat{\theta}) + \mathrm{AL}(\hat{\lambda})}{2}.$$

The sample generation uses the following CSs:

- CS.1:

$$R_i = 2, i = 1, \ldots, \frac{n-m}{2},$$
$$R_i = 0, \text{otherwise}.$$

- CS.2:

$$R_i = n - m, \quad i = 1,$$
$$R_i = 0, \text{otherwise}.$$

- CS.3:

$$R_i = n - m, \quad i = m,$$
$$R_i = 0, \text{otherwise}.$$

- CS.4:

$$R_i = \frac{n-m}{2}, \quad i = 1, m,$$
$$R_i = 0, \text{otherwise}.$$

The calculations were performed using the true parameter values $\theta = 2.5$ and $\lambda = 1.1$. Moreover, the values $n = 40, 80$, $r_m = \frac{m}{n} = 50\%, 75\%, 100\%$ (of the sample size) ($m = nr_m$), $r_{\mathcal{K}} = \frac{\mathcal{K}}{n} = 40\%$, (of the sample size), and $\tau = 2.5, 5.0$ are used in the simulation analysis via R 4.3.0 programming software by installing the "maxLik" package to estimate MLE, along with their "coda" package in R 4.3.0 programming software, to obtain the Bayes point estimates.

The following points may be detected based on the computation results contained in Tables 2–11:

1. The MPSEs are the best estimates through the AMSErs and ARBs.
2. The MLEs are comparable to the LSEs, WLSEs, and PEs through the ARBs and AMSErs.
3. The WLSEs are comparable to the LSEs and PEs through the ARBs and AMSErs.
4. The LSEs are comparable to the PEs through the ARBs and AMSErs.
5. The NACIs are comparable to the LTCIs through the AALs.
6. For similar values of m and τ, and as n rises, the RBs, MSErs, ARBs, AMSErs, AL, and AAL decrease.
7. For the same values of n, and τ, and as m increases, the RBs, MSErs, ARBs, AMSErs, AL, and AAL decrease.
8. For similar values of n and m, by rising τ, the RBs, MSErs, ARBs, and AMSErs decrease for the MPSEs, MLEs, LSEs, and WLSEs, while the RBs, MSErs, ARBs, and AMSErs increase for the PEs.
9. As τ increases, for the same values of n and m, the AL, and AAL decrease for CS.1 and CS.2, while the AL, and AAL increase for CS.3 and CS.4.

10. As τ increases, for fixed values of n and m, the $\overline{\mathcal{K}}$ increases for CS.1 and CS.2, while the $\overline{\mathcal{K}}$ equals m for CS.3 and CS.4, where $\overline{\mathcal{K}}$ is the average number of observed failures when the experiment stops.
11. The COVPs are close to 95%, as n, m, or τ increases.

Table 2. Simulation results of MLEs and MPSEs of θ and λ with their MSErs, RBs, AMSEr, and ARB at true value $\theta = 2.5$ and $\lambda = 1.1$.

					MLE				**MPSE**				
n	$\mathcal{K}$	m	τ	**CS**	$\hat{\theta}$ $\hat{\lambda}$	**MSEr** ($\hat{\theta}$) **MSEr** ($\hat{\lambda}$)	**RB** ($\hat{\theta}$) **RB** ($\hat{\lambda}$)	**AMSEr** **ARB**	$\tilde{\theta}$ $\tilde{\lambda}$	**MSEr** ($\tilde{\theta}$) **MSEr** ($\tilde{\lambda}$)	**RB** ($\tilde{\theta}$) **RB** ($\tilde{\lambda}$)	**AMSEr** **ARB**	$\overline{\mathcal{K}}$
40	16	20	2.5	1	2.8804 1.2278	1.1209 0.1577	0.2988 0.2679	0.6393 0.2834	2.3766 1.0381	0.726 0.1323	0.2544 0.2609	0.4291 0.2577	16.732
				2	2.8653 1.2106	1.0289 0.1288	0.2923 0.2485	0.5789 0.2704	2.3734 1.0524	0.7403 0.1299	0.258 0.2474	0.4351 0.2527	16.142
				3	2.9135 1.2282	1.179 0.1385	0.3079 0.257	0.6588 0.2825	2.4917 1.0858	0.9366 0.1649	0.2588 0.2566	0.5507 0.2577	20.00
				4	2.8621 1.2128	1.0522 0.1265	0.2848 0.2416	0.5893 0.2632	2.4412 1.0688	0.721 0.1244	0.25 0.2373	0.4227 0.2436	20.00
			5.0	1	2.8546 1.228	1.0412 0.1486	0.2803 0.2581	0.5949 0.2692	2.2576 0.9843	0.5311 0.0954	0.2333 0.2294	0.3132 0.2314	19.898
				2	2.8366 1.2093	0.9224 0.1184	0.2679 0.2357	0.5204 0.2518	2.2704 1.001	0.527 0.0794	0.2308 0.2115	0.3032 0.2211	19.857
				3	2.8998 1.2326	1.0683 0.1421	0.2957 0.2575	0.6052 0.2766	2.4648 1.0674	1.1056 0.1358	0.2691 0.2431	0.6207 0.2561	20.00
				4	2.8679 1.2138	0.9591 0.1174	0.2747 0.2349	0.5383 0.2548	2.4295 1.0659	0.7659 0.1132	0.251 0.2268	0.4395 0.2389	20.00
		30	2.5	1	2.7556 1.1623	0.706 0.0786	0.2398 0.1953	0.3923 0.2175	2.3457 1.0212	0.5134 0.0712	0.2226 0.1949	0.2923 0.2088	27.14
				2	2.7441 1.1557	0.7558 0.0773	0.2509 0.1962	0.4166 0.2235	2.3415 1.034	0.4592 0.0604	0.2102 0.1797	0.2598 0.1950	27.174
				3	2.8041 1.1862	0.7642 0.0825	0.2525 0.2021	0.4234 0.2273	2.4078 1.058	0.4579 0.0671	0.2094 0.1890	0.2625 0.1992	29.999
				4	2.7505 1.1606	0.6659 0.0697	0.2341 0.1843	0.3678 0.2092	2.3988 1.0568	0.4709 0.059	0.2117 0.1743	0.2649 0.1930	29.853
			5.0	1	2.7696 1.1928	0.6404 0.0788	0.2325 0.1937	0.3596 0.2131	2.2886 1.0091	0.4231 0.0606	0.2111 0.1829	0.2419 0.1970	29.814
				2	2.737 1.1621	0.6555 0.0706	0.2328 0.1835	0.363 0.2081	2.2971 1.0095	0.4015 0.0595	0.2046 0.1819	0.2305 0.1932	29.835
				3	2.8152 1.1929	0.7046 0.0758	0.2421 0.1922	0.3902 0.2171	2.4238 1.0625	0.4827 0.0633	0.2118 0.1824	0.2730 0.1971	30.0
				4	2.8301 1.1961	0.801 0.0837	0.2506 0.1992	0.4424 0.2249	2.3606 1.0524	0.4078 0.0593	0.1998 0.1757	0.2336 0.1877	30.0
	—	40	—	—	2.7524 1.1715	0.5604 0.0544	0.2173 0.1667	0.3074 0.192	2.3033 1.0174	0.3648 0.0464	0.1921 0.1616	0.2056 0.1768	40.0

Table 3. Simulation results of MLEs and MPSEs of θ and λ with their MSErs, RBs, AMSEr, and ARB at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

n	$\mathcal{K}$	m	τ	CS	MLE $\hat{\theta}$ $\hat{\lambda}$	 **MSEr ($\hat{\theta}$)** **MSEr ($\hat{\lambda}$)**	 **RB ($\hat{\theta}$)** **RB ($\hat{\lambda}$)**	 **AMSEr** **ARB**	MPSE $\bar{\tilde{\theta}}$ $\tilde{\lambda}$	 **MSEr ($\tilde{\theta}$)** **MSEr ($\tilde{\lambda}$)**	 **RB ($\tilde{\theta}$)** **RB ($\tilde{\lambda}$)**	 **AMSEr** **ARB**	$\overline{\mathcal{K}}$
80	32	40	2.5	1	2.6641 1.1644	0.3329 0.0639	0.1772 0.1802	0.1984 0.1787	2.3738 1.0431	0.2768 0.0542	0.1677 0.1716	0.1655 0.1696	34.219
				2	2.6487 1.1478	0.3428 0.0528	0.183 0.1634	0.1978 0.1732	2.3542 1.0452	0.3088 0.0512	0.1811 0.1663	0.18 0.1737	33.95
				3	2.6412 1.1449	0.3265 0.0566	0.1763 0.1659	0.1915 0.1711	2.4423 1.0658	0.3347 0.0566	0.1826 0.1754	0.1957 0.179	40.0
				4	2.6641 1.1551	0.3125 0.0483	0.1703 0.1558	0.1804 0.1631	2.3894 1.0586	0.2778 0.0439	0.166 0.1524	0.1608 0.1592	40
			5.0	1	2.6492 1.1483	0.2893 0.0484	0.1645 0.1545	0.1689 0.1595	2.3671 1.04	0.2443 0.0463	0.1578 0.1582	0.1453 0.158	39.823
				2	2.6501 1.145	0.3251 0.0434	0.1782 0.1462	0.1842 0.1622	2.2782 1.0117	0.2662 0.0398	0.1727 0.1481	0.153 0.1604	39.755
				3	2.6837 1.1581	0.368 0.0589	0.1866 0.1721	0.2135 0.1794	2.4183 1.0648	0.2824 0.0527	0.1714 0.1675	0.1676 0.1694	40.0
				4	2.6769 1.1645	0.3382 0.0492	0.176 0.1577	0.1937 0.1669	2.3818 1.0503	0.294 0.0486	0.1745 0.1633	0.1713 0.1689	40.0
		60	2.5	1	2.6142 1.132	0.248 0.0351	0.1535 0.135	0.1415 0.1443	2.4027 1.0599	0.2352 0.0334	0.1536 0.1341	0.1343 0.1439	54.537
				2	2.5978 1.1237	0.2471 0.0306	0.1493 0.1255	0.1388 0.1374	2.3771 1.0505	0.2256 0.0309	0.153 0.1285	0.1282 0.1407	54.353
				3	2.6515 1.1445	0.2771 0.0354	0.1581 0.134	0.1563 0.146	2.4288 1.0645	0.2229 0.0315	0.1488 0.128	0.1272 0.1384	60.0
				4	2.6141 1.1394	0.2527 0.0327	0.1526 0.1267	0.1427 0.1397	2.4104 1.0677	0.2234 0.0315	0.1491 0.1273	0.1275 0.1382	59.914
			5.0	1	2.6078 1.1314	0.2489 0.0309	0.1501 0.1228	0.1399 0.1364	2.3797 1.0508	0.1885 0.0277	0.1427 0.123	0.1081 0.1329	59.65
				2	2.6113 1.1299	0.2428 0.029	0.1516 0.1236	0.1359 0.1376	2.3375 1.036	0.2147 0.0313	0.1491 0.1321	0.123 0.1406	59.632
				3	2.6123 1.1296	0.2459 0.0309	0.1487 0.1255	0.1384 0.1371	2.412 1.0644	0.2268 0.0333	0.1517 0.1348	0.1301 0.1432	60.0
				4	2.628 1.1373	0.2614 0.0333	0.1563 0.1281	0.1473 0.1422	2.4015 1.0591	0.2034 0.0291	0.1456 0.1256	0.1163 0.1356	60.0
	—	80	—	—	2.6165 1.1321	0.2002 0.0222	0.1359 0.1049	0.1112 0.1204	2.3791 1.0515	0.1664 0.0218	0.1331 0.1105	0.0941 0.1218	80.0

Table 4. Simulation results of LSEs and WLSEs of θ and λ with their MSErs, RBs, AMSEr, and ARB at the exact values of $\theta = 2.5$ and $\lambda = 1.1$.

					LSE				WLSE				
n	$\mathcal{K}$	m	τ	CS	$\bar{\breve{\theta}}$ $\bar{\breve{\lambda}}$	MSEr ($\breve{\theta}$) MSEr ($\breve{\lambda}$)	RB ($\breve{\theta}$) RB ($\breve{\lambda}$)	AMSEr ARB	$\bar{\ddot{\theta}}$ $\bar{\ddot{\lambda}}$	MSEr ($\ddot{\theta}$) MSEr ($\ddot{\lambda}$)	RB ($\ddot{\theta}$) RB ($\ddot{\lambda}$)	AMSEr ARB	$\overline{\mathcal{K}}$
40	16	20	2.5	1	2.9039 1.1533	2.3725 0.2156	0.378 0.3173	1.294 0.3477	2.8684 1.1473	2.036 0.205	0.3419 0.3026	1.1205 0.3222	16.178
				2	2.8835 1.1144	2.2141 0.1767	0.3843 0.3024	1.1954 0.3434	2.9174 1.1291	2.1381 0.1663	0.3595 0.288	1.1522 0.3238	16.036
				3	2.7419 1.1389	0.945 0.1389	0.2747 0.2588	0.5419 0.2667	2.7432 1.1389	0.8902 0.1335	0.2651 0.2549	0.5119 0.26	20.0
				4	2.7329 1.1113	1.0239 0.1274	0.2933 0.2592	0.5757 0.2763	2.7389 1.117	0.8799 0.1224	0.2741 0.2532	0.5011 0.2636	20.0
			5.0	1	2.9529 1.1687	2.3042 0.1941	0.369 0.299	1.2491 0.334	2.9396 1.1726	2.0888 0.1812	0.3277 0.274	1.135 0.3008	19.91
				2	3.0507 1.1783	2.6596 0.1781	0.39 0.2784	1.4188 0.3342	3.0943 1.1998	2.3916 0.171	0.3695 0.2639	1.2813 0.3167	19.875
				3	2.8121 1.1575	1.1968 0.1504	0.3054 0.2711	0.6736 0.2882	2.8099 1.1603	1.0327 0.1432	0.2881 0.2652	0.5879 0.2767	20.0
				4	2.7197 1.1041	0.9559 0.119	0.2835 0.2496	0.5375 0.2665	2.7304 1.1099	0.8217 0.1114	0.2683 0.2421	0.4665 0.2552	20.0
		30	2.5	1	3.6837 1.4838	3.5841 0.2596	0.4991 0.3671	1.9219 0.4331	3.8038 1.5436	3.4262 0.2871	0.5311 0.4075	1.8567 0.4693	27.303
				2	3.4009 1.4171	1.6959 0.1826	0.3906 0.3128	0.9392 0.3517	3.622 1.4975	2.0821 0.2221	0.4573 0.3653	1.1521 0.4113	27.207
				3	2.6764 1.1059	0.6708 0.0814	0.2389 0.2045	0.3761 0.2217	2.6971 1.1134	0.6359 0.0804	0.2295 0.2014	0.3581 0.2155	29.999
				4	2.7726 1.1503	0.7742 0.0812	0.2482 0.1995	0.4277 0.2238	2.8001 1.1631	0.7202 0.0807	0.242 0.1978	0.4004 0.2199	29.858
			5.0	1	2.797 1.1205	0.9503 0.0934	0.2785 0.2134	0.5227 0.2468	2.8409 1.138	0.8232 0.0874	0.2603 0.199	0.4538 0.2282	29.815
				2	3.5435 1.1271	2.5167 0.0951	0.4486 0.2187	1.3732 0.2486	3.7334 1.1477	2.8315 0.0844	0.5046 0.2042	1.5468 0.2323	27.225
				3	2.6505 1.1008	0.6931 0.0853	0.2454 0.2123	0.3892 0.2289	2.6506 1.1028	0.596 0.0781	0.2275 0.201	0.337 0.2143	30.0
				4	2.6562 1.0946	0.6956 0.0824	0.2428 0.2066	0.389 0.2247	2.6704 1.1018	0.6202 0.0777	0.2234 0.1952	0.349 0.2093	30.0
	—	40	—	—	2.6743 1.0994	0.6288 0.0644	0.2287 0.182	0.3466 0.2054	2.7101 1.1138	0.5719 0.0569	0.2149 0.1677	0.3144 0.1913	40.0

Table 5. Simulation results of LSEs and WLSEs of θ and λ with their MSErs, RBs, AMSEr, and ARB at the exact values of $\theta = 2.5$ and $\lambda = 1.1$.

					LSE				WLSE				
n	$\mathcal{K}$	m	τ	CS	$\overline{\breve{\theta}}$ $\breve{\lambda}$	MSEr ($\breve{\theta}$) MSEr ($\breve{\lambda}$)	RB ($\breve{\theta}$) RB ($\breve{\lambda}$)	AMSEr ARB	$\overline{\ddot{\theta}}$ $\ddot{\lambda}$	MSEr ($\ddot{\theta}$) MSEr ($\ddot{\lambda}$)	RB ($\ddot{\theta}$) RB ($\ddot{\lambda}$)	AMSEr ARB	$\overline{\mathcal{K}}$
80	32	40	2.5	1	2.8428 1.1972	0.8565 0.113	0.2717 0.243	0.4848 0.2573	2.7992 1.1934	0.5909 0.0954	0.2308 0.2221	0.3432 0.2265	33.355
				2	2.7256 1.1186	0.8577 0.0886	0.2636 0.2154	0.4732 0.2395	2.7318 1.1257	0.6913 0.079	0.237 0.2022	0.3852 0.2196	32.071
				3	2.6344 1.1187	0.5065 0.0749	0.2035 0.1908	0.2907 0.1972	2.6428 1.1234	0.4439 0.0702	0.1916 0.1891	0.257 0.1904	40.0
				4	2.6387 1.12	0.5235 0.0677	0.2103 0.1892	0.2956 0.1998	2.6375 1.1223	0.4217 0.0598	0.1889 0.1768	0.2408 0.1829	40.0
			5.0	1	2.6647 1.121	0.5786 0.0776	0.2233 0.2003	0.3281 0.2118	2.6718 1.1296	0.4412 0.0647	0.1928 0.1785	0.253 0.1856	39.817
				2	2.7457 1.1314	0.7031 0.0698	0.2377 0.1861	0.3865 0.2119	2.7881 1.1496	0.6164 0.0635	0.2209 0.1722	0.3399 0.1966	39.785
				3	2.6489 1.1273	0.5129 0.074	0.2083 0.1914	0.2934 0.1998	2.6529 1.1311	0.4371 0.068	0.1925 0.1847	0.2526 0.1886	40.0
				4	2.6033 1.0937	0.5044 0.0677	0.2073 0.1858	0.2861 0.1966	2.6031 1.0957	0.4149 0.0602	0.1887 0.1745	0.2376 0.1816	40.0
		60	2.5	1	3.3965 1.454	1.3908 0.1757	0.3674 0.3252	0.7832 0.3463	3.56 1.5268	1.5802 0.2151	0.4248 0.388	0.8977 0.4064	54.382
				2	3.3788 1.4422	1.3606 0.164	0.362 0.3153	0.7623 0.3387	3.6484 1.5394	1.8925 0.2271	0.4599 0.3994	1.0598 0.4296	54.252
				3	2.5861 1.1069	0.3308 0.046	0.1727 0.1549	0.1884 0.1638	2.5978 1.112	0.2921 0.0439	0.1604 0.1492	0.168 0.1548	60.0
				4	2.693 1.1429	0.4485 0.0492	0.1925 0.1556	0.2489 0.1741	2.7146 1.1524	0.4577 0.0522	0.1884 0.1552	0.255 0.1718	59.87
			5.0	1	2.6588 1.1205	0.4155 0.0428	0.1814 0.1467	0.2292 0.164	2.6804 1.1337	0.3298 0.038	0.1611 0.1333	0.1839 0.1472	59.651
				2	2.6541 1.1138	0.4281 0.0438	0.1915 0.1489	0.2359 0.1702	2.7008 1.1327	0.3911 0.0398	0.1809 0.1384	0.2155 0.1596	59.628
				3	2.6026 1.1084	0.3362 0.0451	0.1785 0.1561	0.1907 0.1673	2.6195 1.1148	0.3094 0.0431	0.1655 0.148	0.1762 0.1568	60.0
				4	2.6025 1.1014	0.3702 0.0445	0.1814 0.1531	0.2073 0.1672	2.6215 1.1086	0.3348 0.042	0.169 0.1461	0.1884 0.1576	60.0
	—	80	—	—	2.5721 1.0994	0.2718 0.0299	0.1571 0.1261	0.1509 0.1416	2.5982 1.1106	0.2592 0.0289	0.1468 0.1171	0.144 0.132	80.0

Table 6. Simulation results of PEs of θ and λ with their MSErs, RBs, AMSEr, and ARB at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

n	$\mathcal{K}$	m	τ	CS	PE				$\overline{\mathcal{K}}$
					$\bar{\breve{\theta}}$ $\breve{\lambda}$	**MSEr** ($\breve{\theta}$) **MSEr** ($\breve{\lambda}$)	**RB** ($\breve{\theta}$) **RB** ($\breve{\lambda}$)	**AMSEr** **ARB**	
40	16	20	2.5	1	3.0511 1.2304	1.4295 0.1641	0.3503 0.2834	0.7968 0.3168	16.139
				2	3.1193 1.2273	1.7078 0.1671	0.3912 0.2821	0.9375 0.3366	16.01
				3	3.0038 1.2288	1.2282 0.1533	0.3224 0.2719	0.6908 0.2972	20.0
				4	3.091 1.2351	1.3674 0.1381	0.3391 0.2583	0.7528 0.2987	20.0
			5.0	1	3.2972 1.2615	2.2332 0.168	0.4447 0.2796	1.2006 0.3621	19.916
				2	3.3258 1.2603	2.2933 0.1499	0.45 0.2631	1.2216 0.3565	19.908
				3	3.0684 1.2555	1.4316 0.1837	0.3432 0.2943	0.8077 0.3188	20.0
				4	2.997 1.1998	1.2712 0.1229	0.3219 0.246	0.6971 0.2839	20.0
		30	2.5	1	3.2021 1.2489	1.7333 0.1287	0.3804 0.2493	0.931 0.3148	27.236
				2	3.1476 1.233	1.7034 0.1262	0.3831 0.2526	0.9148 0.3178	27.123
				3	2.8463 1.1596	0.8041 0.0836	0.2631 0.2022	0.4438 0.2327	30.0
				4	2.8844 1.1688	0.9053 0.0835	0.2826 0.2042	0.4944 0.2434	29.831
			5.0	1	3.1889 1.2275	1.8176 0.1166	0.3963 0.2323	0.9671 0.3143	29.848
				2	3.2246 1.2286	1.8797 0.1062	0.4083 0.2288	0.993 0.3186	29.826
				3	2.8633 1.1628	0.8774 0.0844	0.27 0.2048	0.4809 0.2374	30.0
				4	2.8472 1.1532	0.9434 0.082	0.2839 0.202	0.5127 0.2429	30.0
	—	40	—	—	2.8255 1.1447	0.8312 0.0623	0.2887 0.1793	0.4467 0.234	40.0

Table 7. Simulation results of PEs of θ and λ with their MSErs, RBs, AMSEr, and ARB, at the exact values of $\theta = 2.5$ and $\lambda = 1.1$.

n	$\mathcal{K}$	m	τ	CS	PE $\bar{\check{\theta}}$ / $\check{\lambda}$	MSEr ($\check{\theta}$) / MSEr ($\check{\lambda}$)	RB ($\check{\theta}$) / RB ($\check{\lambda}$)	AMSEr / ARB	$\overline{\mathcal{K}}$
80	32	40	2.5	1	2.7908 1.1638	0.6698 0.0855	0.2392 0.2087	0.3776 0.2239	32.907
				2	2.8753 1.1673	0.9004 0.0792	0.2744 0.1971	0.4898 0.2358	32.827
				3	2.7379 1.1552	0.5046 0.0754	0.2059 0.1897	0.29 0.1978	40.0
				4	2.7708 1.1544	0.5796 0.0688	0.2209 0.1851	0.3242 0.203	40
			5.0	1	2.9592 1.1995	1.0991 0.0889	0.3044 0.2049	0.594 0.2547	39.81
				2	2.9931 1.1872	1.1295 0.0744	0.3173 0.1895	0.6019 0.2534	39.772
				3	2.7365 1.1514	0.4856 0.072	0.2099 0.193	0.2788 0.2014	40.0
				4	2.7742 1.1553	0.5613 0.0667	0.2182 0.1846	0.314 0.2014	40.0
		60	2.5	1	2.8425 1.1715	0.6556 0.0591	0.2381 0.1705	0.3574 0.2043	54.355
				2	2.8161 1.162	0.6145 0.0526	0.2263 0.1602	0.3335 0.1933	54.098
				3	2.6434 1.1255	0.3439 0.0402	0.1773 0.1448	0.1921 0.161	60
				4	2.6793 1.1272	0.372 0.0366	0.1842 0.1376	0.2043 0.1609	59.866
			5.0	1	2.8806 1.168	0.772 0.0493	0.259 0.1532	0.4106 0.2061	59.648
				2	2.8851 1.173	0.7899 0.0546	0.2676 0.1622	0.4223 0.2149	59.633
				3	2.6653 1.1268	0.3615 0.0396	0.1775 0.1416	0.2006 0.1596	60.0
				4	2.6953 1.1321	0.4076 0.0411	0.1901 0.145	0.2244 0.1675	60.0
	—	80	—	—	2.6047 1.1059	0.3859 0.0339	0.2029 0.1369	0.2099 0.1699	80.0

Table 8. Simulation results of ALs and COVPs (in %) of 95% CIs of θ and λ at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

n	$\mathcal{K}$	m	τ	CS	NACI CI (θ) CI (λ)	NACI AL (θ) AL (λ)	NACI COVP (θ) COVP (λ)	LTCI CI (θ) CI (λ)	LTCI AL (θ) AL (λ)	LTCI COVP (θ) COVP (λ)	$\overline{\mathcal{K}}$
40	16	20	2.5	1	{1.0681, 4.6926} {0.5125, 1.9431}	3.6245 1.4306	96.3 95.4	{1.5378, 5.4128} {0.6866, 2.2017}	3.875 1.5151	94.8 93.5	16.057
				2	{1.0596, 4.671} {0.5465, 1.8748}	3.6114 1.3283	96.8 95.2	{1.5278, 5.3886} {0.7001, 2.0975}	3.8608 1.3974	95.2 94.2	16.00
				3	{1.0868, 4.7402} {0.5557, 1.9007}	3.6534 1.345	96.6 95.5	{1.5585, 5.4616} {0.7113, 2.1265}	3.9031 1.4152	94.2 92.1	20.0
				4	{1.1385, 4.5856} {0.5896, 1.836}	3.4471 1.2465	96.9 94	{1.5692, 5.233} {0.7261, 2.0294}	3.6638 1.3033	93.8 92.1	20.0
			5.0	1	{1.2269, 4.4822} {0.6009, 1.8551}	3.2553 1.2542	96.4 94.1	{1.6163, 5.056} {0.7375, 2.0479}	3.4397 1.3104	93.1 91.0	19.909
				2	{1.1881, 4.4851} {0.6207, 1.7979}	3.2971 1.1772	96.5 95.1	{1.5882, 5.0782} {0.7437, 1.9686}	3.49 1.225	95.2 92.3	19.893
				3	{1.0867, 4.713} {0.5572, 1.908}	3.6264 1.3509	96.8 96.2	{1.5536, 5.4258} {0.7135, 2.1349}	3.8721 1.4214	95.1 92.7	20.0
				4	{1.1423, 4.5936} {0.5904, 1.8371}	3.4512 1.2468	97.8 95.8	{1.5729, 5.2401} {0.7269, 2.0303}	3.6672 1.3034	95.5 92.9	20.0
		30	2.5	1	{1.3288, 4.1823} {0.6573, 1.6673}	2.8535 1.0100	96.8 94.3	{1.643, 4.6279} {0.7532, 1.7961}	2.9848 1.0429	94.7 93.2	27.169
				2	{1.2935, 4.1947} {0.654, 1.6574}	2.9012 1.0034	95.1 93.6	{1.6185, 4.6588} {0.7493, 1.7854}	3.0403 1.0361	93.0 92.8	27.083
				3	{1.3402, 4.268} {0.6809, 1.6914}	2.9278 1.0106	96.1 95.7	{1.6648, 4.7296} {0.7751, 1.817}	3.0648 1.0419	94.2 93.0	29.997
				4	{1.3343, 4.1667} {0.6735, 1.6477}	2.8324 0.9741	96.3 95.8	{1.6446, 4.6057} {0.7632, 1.7666}	2.9611 1.0034	94.0 94.4	29.794
			5.0	1	{1.3873, 4.1518} {0.7032, 1.6824}	2.7644 0.9791	96.1 95.4	{1.6824, 4.5649} {0.7915, 1.7986}	2.8825 1.0072	95.2 93.0	29.854
				2	{1.3445, 4.1294} {0.6855, 1.6387}	2.7849 0.9532	96.9 95.2	{1.6467, 4.5553} {0.7713, 1.7518}	2.9086 0.9805	94.3 94.0	29.815
				3	{1.3452, 4.2851} {0.6853, 1.7005}	2.9399 1.0152	97.1 95.9	{1.6712, 4.7486} {0.7798, 1.8263}	3.0774 1.0466	95.0 94.2	30.0
				4	{1.3652, 4.2950} {0.6995, 1.6927}	2.9298 0.9932	96.9 94.3	{1.6878, 4.7526} {0.7900, 1.8123}	3.0647 1.0223	93.4 90.9	30.0
	—	40	—	—	{1.4824, 4.0224} {0.7518, 1.5912}	2.54 0.8394	96.0 95.0	{1.7359, 4.3682} {0.8189, 1.6765}	2.6323 0.8576	93.3 92.8	40.0

Table 9. Simulation results of ALs and COVPs (in %) of 95% CIs of θ and λ at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

n	$\mathcal{K}$	m	τ	CS	NACI CI (θ) CI (λ)	NACI AL (θ) AL (λ)	NACI COVP (θ) COVP (λ)	LTCI CI (θ) CI (λ)	LTCI AL (θ) AL (λ)	LTCI COVP (θ) COVP (λ)	$\overline{\mathcal{K}}$
80	32	40	2.5	1	{1.5301, 3.798} {0.6835, 1.6453}	2.2679 0.9618	97.2 95	{1.7411, 4.079} {0.7708, 1.7606}	2.3378 0.9898	96.5 93.6	33.133
				2	{1.484, 3.8134} {0.7017, 1.594}	2.3294 0.8923	95.9 95.6	{1.707, 4.113} {0.7784, 1.6937}	2.406 0.9152	97.3 95.4	33.24
				3	{1.5093, 3.7731} {0.6899, 1.6}	2.2638 0.9101	96.2 95.7	{1.721, 4.0553} {0.7697, 1.7043}	2.3343 0.9346	96.7 94.8	40.0
				4	{1.5465, 3.7818} {0.7266, 1.5837}	2.2353 0.8571	96.8 96.6	{1.7517, 4.0536} {0.7973, 1.6743}	2.302 0.8771	96.4 95.4	40
			5.0	1	{1.6159, 3.6824} {0.727, 1.5696}	2.0665 0.8425	97.2 95.9	{1.794, 3.9138} {0.7958, 1.6575}	2.1198 0.8617	95.3 94.3	39.828
				2	{1.5584, 3.7417} {0.7445, 1.5454}	2.1833 0.8009	96.5 95.5	{1.7558, 4.002} {0.8072, 1.6246}	2.2462 0.8174	95.6 94.2	39.765
				3	{1.5286, 3.8389} {0.6998, 1.6164}	2.3103 0.9165	96.9 96.1	{1.7455, 4.1284} {0.7799, 1.7209}	2.3829 0.941	96.0 93.8	40.0
				4	{1.5516, 3.8021} {0.7329, 1.596}	2.2505 0.8631	97.7 96.4	{1.7586, 4.0765} {0.804, 1.6872}	2.3179 0.8832	95.7 94.6	40.0
		60	2.5	1	{1.6769, 3.5516} {0.7808, 1.4833}	1.8747 0.7025	95.9 95.2	{1.8268, 3.7421} {0.8302, 1.5442}	1.9154 0.714	95.1 94.8	54.395
				2	{1.6367, 3.5589} {0.7749, 1.4725}	1.9222 0.6975	95.5 95.9	{1.7947, 3.7613} {0.824, 1.5329}	1.9666 0.7089	94.4 95.5	54.273
				3	{1.6936, 3.6094} {0.7961, 1.4929}	1.9158 0.6968	96.3 94.5	{1.8478, 3.8058} {0.8442, 1.5519}	1.958 0.7077	95.0 93.8	60.0
				4	{1.6771, 3.5511} {0.7986, 1.4801}	1.874 0.6816	95.9 95.1	{1.8269, 3.7415} {0.8449, 1.5368}	1.9146 0.6918	94.7 94.2	59.898
			5.0	1	{1.7071, 3.5085} {0.7993, 1.4635}	1.8014 0.6641	95.9 94.5	{1.8464, 3.6841} {0.8437, 1.5175}	1.8377 0.6738	94.3 93.4	59.654
				2	{1.6795, 3.543} {0.7993, 1.4604}	1.8634 0.6612	95.4 96.4	{1.8279, 3.7313} {0.8433, 1.514}	1.9035 0.6707	95.6 95.0	59.64
				3	{1.672, 3.5525} {0.7843, 1.4748}	1.8805 0.6905	95.9 95.9	{1.8229, 3.7444} {0.8322, 1.5336}	1.9216 0.7014	95.0 95.3	60.0
				4	{1.6853, 3.5707} {0.7981, 1.4766}	1.8854 0.6786	95.6 94.6	{1.8361, 3.7625} {0.8441, 1.5328}	1.9264 0.6888	94.7 93.2	60.0
	—	80	—	—	{1.7783, 3.4547} {0.8429, 1.4213}	1.6764 0.5784	96.1 95.5	{1.8995, 3.6049} {0.8769, 1.4617}	1.7054 0.5848	94.2 95.0	80.0

Table 10. Simulation results of the Bayesian θ and λ with their MSErs, RBs, AMSEr, and ARB at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

						SEL				LINEXL $\varepsilon = -0.5$				LINEXL $\varepsilon = 1.5$				
n	$\mathcal{K}$	m	τ	CS		$\hat{\theta}$ / $\hat{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\hat{\theta}$ / $\hat{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\hat{\theta}$ / $\hat{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\overline{\mathcal{K}}$
40	16	20	2.5	1	θ	2.4920	0.0032	0.0221	0.0050	2.4948	0.0021	0.0220	0.0054	2.4837	0.0065	0.0224	0.0037	18.53
					λ	1.1074	0.0067	0.0151	0.0186	1.1095	0.0087	0.0152	0.0186	1.1009	0.0008	0.0149	0.0186	
				2	θ	2.4918	0.0033	0.0217	0.0021	2.4946	0.0022	0.0216	0.0025	2.4833	0.0067	0.0220	0.0059	18.143
					λ	1.1010	0.0009	0.0150	0.0183	1.1032	0.0029	0.0150	0.0183	1.0944	0.0051	0.0150	0.0185	
				3	θ	2.4964	0.0015	0.0208	0.0041	2.4991	0.0004	0.0208	0.0026	2.4883	0.0047	0.0211	0.0088	18.2365
					λ	1.0925	0.0068	0.0162	0.0185	1.0948	0.0047	0.0162	0.0185	1.0858	0.0129	0.0162	0.0186	
				4	θ	2.4921	0.0032	0.0221	0.0017	2.4949	0.0020	0.0220	0.0019	2.4837	0.0065	0.0226	0.0064	20
					λ	1.0997	0.0003	0.0136	0.0178	1.1019	0.0017	0.0136	0.0178	1.0932	0.0062	0.0135	0.0180	
			5	1	θ	2.4976	0.0010	0.0210	0.0012	2.5004	0.0001	0.0209	0.0003	2.4891	0.0043	0.0211	0.0058	19.892
					λ	1.0984	0.0014	0.0150	0.0180	1.1006	0.0005	0.0151	0.0180	1.0920	0.0073	0.0150	0.0180	
				2	θ	2.4944	0.0022	0.0213	0.0033	2.4972	0.0011	0.0229	0.0037	2.4859	0.0056	0.0236	0.0037	19.895
					λ	1.1048	0.0044	0.0150	0.0190	1.1070	0.0064	0.0150	0.0190	1.0981	0.0017	0.0149	0.0192	
				3	θ	2.4904	0.0038	0.0205	0.0024	2.4932	0.0027	0.0210	0.0018	2.4822	0.0071	0.0213	0.0070	20
					λ	1.0989	0.0010	0.0137	0.0174	1.1010	0.0010	0.0137	0.0174	1.0925	0.0068	0.0137	0.0175	
				4	θ	2.4921	0.0032	0.0221	0.0017	2.4949	0.0020	0.0220	0.0019	2.4837	0.0065	0.0226	0.0064	20
					λ	1.0997	0.0003	0.0136	0.0178	1.1019	0.0017	0.0136	0.0178	1.0932	0.0062	0.0135	0.0180	
		30	2.5	1	θ	2.4976	0.0010	0.0078	0.0010	2.4986	0.0006	0.0078	0.0012	2.4946	0.0022	0.0079	0.0018	27.292
					λ	1.1011	0.0010	0.0065	0.0072	1.1020	0.0018	0.0065	0.0072	1.0985	0.0014	0.0064	0.0072	
				2	θ	2.4970	0.0012	0.0075	0.0007	2.4980	0.0008	0.0074	0.0009	2.4940	0.0024	0.0075	0.0023	27.105
					λ	1.1002	0.0002	0.0066	0.0070	1.1011	0.0010	0.0066	0.0070	1.0976	0.0021	0.0066	0.0070	
				3	θ	2.5007	0.0003	0.0083	0.0008	2.5018	0.0007	0.0083	0.0006	2.4975	0.0010	0.0083	0.0024	27.3564
					λ	1.0985	0.0013	0.0072	0.0077	1.0994	0.0005	0.0071	0.0077	1.0958	0.0038	0.0072	0.0077	
				4	θ	2.4956	0.0018	0.0087	0.0017	2.4966	0.0013	0.0086	0.0019	2.4925	0.0030	0.0088	0.0019	29.864
					λ	1.1017	0.0016	0.0062	0.0074	1.1026	0.0024	0.0062	0.0074	1.0991	0.0009	0.0062	0.0075	
			5	1	θ	2.5021	0.0008	0.0078	0.0016	2.5031	0.0012	0.0078	0.0022	2.4990	0.0004	0.0078	0.0003	29.831
					λ	1.1027	0.0024	0.0063	0.0071	1.1035	0.0032	0.0063	0.0071	1.1001	0.0001	0.0063	0.0071	
				2	θ	2.5008	0.0003	0.0074	0.0006	2.5019	0.0008	0.0087	0.0004	2.4976	0.0010	0.0087	0.0021	29.806
					λ	1.0990	0.0009	0.0064	0.0076	1.0999	0.0001	0.0065	0.0076	1.0964	0.0033	0.0064	0.0075	
				3	θ	2.4973	0.0011	0.0083	0.0006	2.4983	0.0007	0.0083	0.0008	2.4942	0.0023	0.0084	0.0023	30
					λ	1.1001	0.0001	0.0062	0.0072	1.1010	0.0009	0.0062	0.0072	1.0975	0.0023	0.0062	0.0073	
				4	θ	2.4958	0.0017	0.0075	0.0017	2.4968	0.0013	0.0075	0.0019	2.4927	0.0029	0.0076	0.0018	30
					λ	1.1019	0.0018	0.0062	0.0069	1.1028	0.0025	0.0063	0.0069	1.0993	0.0006	0.0062	0.0069	
	40	40	-	-	θ	2.4925	0.0030	0.0193	0.0016	2.4950	0.0020	0.0192	0.0019	2.4847	0.0061	0.0197	0.0056	39.76
					λ	1.1001	0.0001	0.0099	0.0146	1.1021	0.0019	0.0100	0.0146	1.0944	0.0051	0.0098	0.0147	

Table 11. Simulation results of the Bayesian θ and λ with their MSErs, RBs, AMSEr, and ARB at the true values of $\theta = 2.5$ and $\lambda = 1.1$.

						SEL				LINEXL $\varepsilon = -0.5$				LINEXL $\varepsilon = 1.5$				
n	$\mathcal{K}$	m	τ	CS		$\tilde{\theta}$ / $\tilde{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\tilde{\theta}$ / $\tilde{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\tilde{\theta}$ / $\tilde{\lambda}$	RB (θ) / RB (λ)	MSEr (θ) / MSEr (λ)	ARB / AMSEr	$\overline{\mathcal{K}}$
80	32	40	2.5	1	θ	2.5017	0.0007	0.0210	0.0018	2.5044	0.0018	0.0210	0.0032	2.4934	0.0026	0.0208	0.0025	37.081
					λ	1.1032	0.0029	0.0110	0.0160	1.1051	0.0047	0.0110	0.0160	1.0975	0.0023	0.0109	0.0158	
				2	θ	2.4936	0.0026	0.0229	0.0013	2.4963	0.0015	0.0229	0.0016	2.4852	0.0059	0.0232	0.0055	36.261
					λ	1.1000	0.0000	0.0102	0.0166	1.1019	0.0017	0.0103	0.0166	1.0943	0.0052	0.0101	0.0167	
				3	θ	2.4921	0.0032	0.0200	0.0037	2.4947	0.0021	0.0199	0.0040	2.4842	0.0063	0.0202	0.0034	40
					λ	1.1047	0.0043	0.0094	0.0147	1.1065	0.0059	0.0095	0.0147	1.0994	0.0006	0.0093	0.0147	
				4	θ	2.4980	0.0008	0.0188	0.0017	2.5006	0.0002	0.0188	0.0023	2.4904	0.0039	0.0188	0.0031	40
					λ	1.1029	0.0027	0.0101	0.0144	1.1048	0.0043	0.0102	0.0145	1.0975	0.0023	0.0099	0.0144	
			5	1	θ	2.4938	0.0025	0.0208	0.0017	2.4964	0.0014	0.0208	0.0020	2.4860	0.0056	0.0211	0.0048	39.828
					λ	1.1011	0.0010	0.0095	0.0152	1.1029	0.0026	0.0096	0.0152	1.0956	0.0040	0.0094	0.0152	
				2	θ	2.4954	0.0018	0.0203	0.0052	2.4980	0.0008	0.0204	0.0056	2.4875	0.0050	0.0204	0.0043	39.766
					λ	1.1095	0.0086	0.0100	0.0158	1.1114	0.0103	0.0113	0.0158	1.1039	0.0035	0.0110	0.0157	
				3	θ	2.4921	0.0032	0.0200	0.0037	2.4947	0.0021	0.0199	0.0040	2.4842	0.0063	0.0202	0.0034	40
					λ	1.1047	0.0043	0.0094	0.0147	1.1065	0.0059	0.0095	0.0147	1.0994	0.0006	0.0093	0.0147	
				4	θ	2.4980	0.0008	0.0188	0.0017	2.5006	0.0002	0.0188	0.0023	2.4904	0.0039	0.0188	0.0031	40
					λ	1.1029	0.0027	0.0101	0.0144	1.1048	0.0043	0.0102	0.0145	1.0975	0.0023	0.0099	0.0144	
		60	2.5	1	θ	2.4983	0.0007	0.0074	0.0010	2.4992	0.0003	0.0074	0.0011	2.4954	0.0018	0.0075	0.0014	54.419
					λ	1.1013	0.0012	0.0051	0.0063	1.1021	0.0019	0.0051	0.0063	1.0990	0.0009	0.0051	0.0063	
				2	θ	2.4965	0.0014	0.0074	0.0013	2.4975	0.0010	0.0074	0.0015	2.4936	0.0026	0.0074	0.0017	54.21
					λ	1.1013	0.0012	0.0051	0.0062	1.1021	0.0019	0.0051	0.0062	1.0990	0.0009	0.0051	0.0063	
				3	θ	2.5015	0.0006	0.0080	0.0006	2.5025	0.0010	0.0080	0.0011	2.4985	0.0006	0.0079	0.0010	60
					λ	1.1006	0.0006	0.0048	0.0064	1.1014	0.0013	0.0048	0.0064	1.0984	0.0015	0.0048	0.0063	
				4	θ	2.5003	0.0001	0.0076	0.0017	2.5013	0.0005	0.0076	0.0023	2.4974	0.0011	0.0076	0.0011	59.853
					λ	1.1037	0.0033	0.0051	0.0063	1.1045	0.0041	0.0051	0.0063	1.1013	0.0011	0.0051	0.0063	
			5	1	θ	2.4997	0.0001	0.0081	0.0001	2.5007	0.0003	0.0081	0.0005	2.4967	0.0013	0.0081	0.0016	59.667
					λ	1.1002	0.0001	0.0051	0.0066	1.1009	0.0008	0.0051	0.0066	1.0979	0.0019	0.0051	0.0066	
				2	θ	2.4929	0.0028	0.0078	0.0018	2.4939	0.0024	0.0078	0.0020	2.4898	0.0041	0.0079	0.0027	59.589
					λ	1.1009	0.0008	0.0051	0.0064	1.1016	0.0015	0.0051	0.0064	1.0985	0.0013	0.0051	0.0065	
				3	θ	2.5015	0.0006	0.0080	0.0006	2.5025	0.0010	0.0080	0.0011	2.4985	0.0006	0.0079	0.0010	60
					λ	1.1006	0.0006	0.0048	0.0064	1.1014	0.0013	0.0048	0.0064	1.0984	0.0015	0.0048	0.0063	
				4	θ	2.5022	0.0009	0.0078	0.0015	2.5032	0.0013	0.0078	0.0021	2.4993	0.0003	0.0078	0.0002	60
					λ	1.1024	0.0022	0.0050	0.0064	1.1032	0.0029	0.0050	0.0064	1.1001	0.0001	0.0050	0.0064	
	80	80	-	-	θ	2.4980	0.0008	0.0182	0.0037	2.5006	0.0003	0.0182	0.0041	2.4903	0.0039	0.0182	0.0032	79.534
					λ	1.1072	0.0065	0.0065	0.0124	1.1086	0.0079	0.0066	0.0124	1.1029	0.0026	0.0064	0.0123	

6. Applications

The significance and applicability of the suggested KMGE model are illustrated using two real data sets from engineering and medical science. We use the "maxLik" program in the R package to compute likelihood estimates using the Newton–Raphson (NR) algorithms; for further information, see [41].

The first data set contains the minutes that 100 bank customers had to wait before receiving the service. It was first employed by [42]. "18.4, 18.9, 19, 27, 21.3, 21.4, 21.9, 23.0, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 7.1, 7.1, 7.4, 3.6, 4.0, 0.8, 0.8, 19.9, 20.6, 4.1, 1.9, 4.8, 4.9, 4.9, 5, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 6.3, 6.7, 6.9, 7.1, 7.1, 7.6, 7.7, 8, 8.2, 8.6, 13.3, 13.6, 13.7, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 4.7, 4.7, 1.3, 1.5, 1.8, 1.9, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 31.6, 33.1, 38.5".

The second dataset from [43] includes the number of hours (in thousands) between failures of secondary reactor pumps: "0.954, 0.491, 6.560, 4.992, 0.347, 0.070, 0.062, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 2.160, 0.746, 0.402, 1.921, 4.082, 0.199, 0.605, 0.273, 5.320".

For data I: From Table 12, the MLEs (with their standard errors (SE)) of θ and λ, meanwhile the Kolmogorov–Smirnov test (KS) (p-value) was 0.0366 (0.9993). Figure 5 illustrates data I: the estimated and empirical CDF of KMGE in the left, the estimated and histogram of KMGE density in the center, and the generate and quantile (Q-Q) of the KMGE in the right for the waiting time before receiving the banking service data using a graphic visualization. In the results in Table 12, we can confirm that the data I have fitted the KMGE distribution.

Table 12. MLE of complete sample for parameters of KMGE distribution.

Data		Estimates	SE	KS	p-Value
I	θ	2.3358	0.3300	0.0366	0.9993
	λ	0.1357	0.0162		
II	θ	0.8571	0.2030	0.1198	0.8579
	λ	0.4447	0.1460		

For data II: From Table 12, the MLEs (with their SE) of θ and λ, meanwhile the KS (p-value) was 0.1198 (0.8579). It means that the KMGE lifetime model fits the number of hours (in thousands) between failures of secondary reactor pumps data well. Figure 6 illustrates the estimated and empirical CDF of KMGE on the left, the estimated and histogram of KMGE density in the center, and the Q-Q of KMGE on the right for the number of hours (in thousands) between the failures of the secondary reactor pump data using a graphic visualization. From the results in Table 12, we can confirm that data II fitted the KMGE distribution.

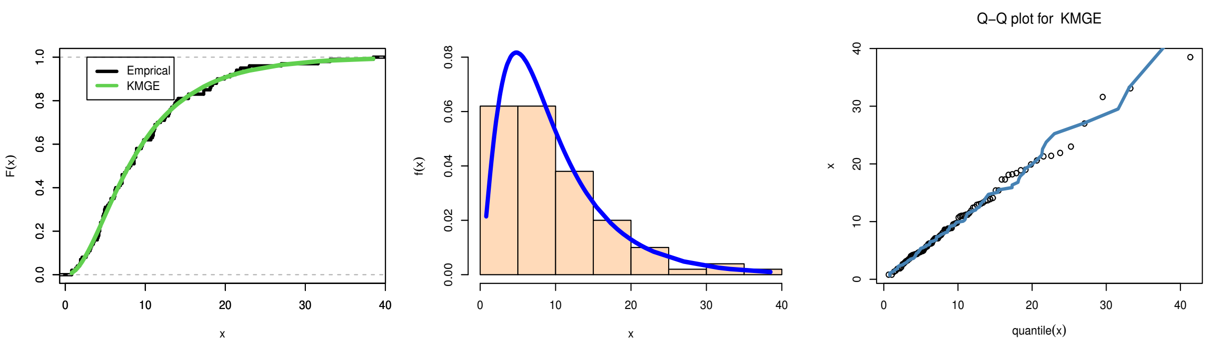

Figure 5. Estimated PDF and CDF with lines plots and Q-Q plot: data I.

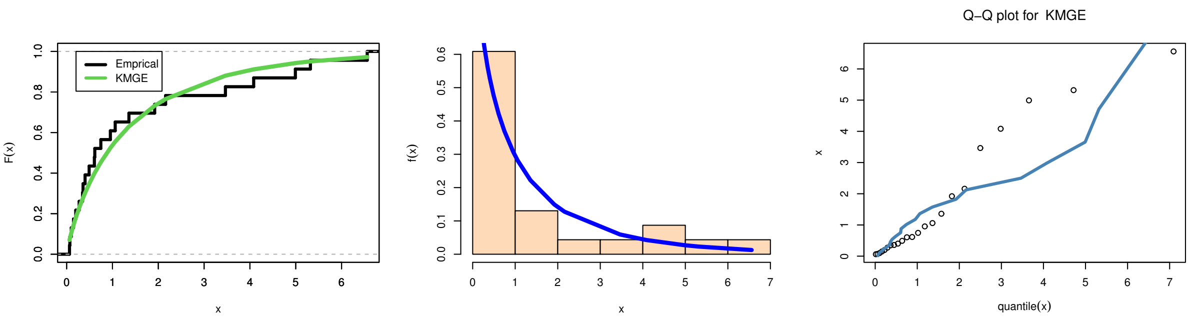

Figure 6. Estimated PDF and CDF with lines plots and Q-Q plot: data II.

For the GTI-PHCS of data I: Different GTI-PHCS samples (where m = 80, $\mathcal{K}$ = 70, and τ = 12) based on various CS selections were obtained from the waiting period before getting the banking service data and are shown in Table 13 to evaluate our acquired estimators. Also, MLE and Bayesian estimates based on GTI-PHCS for data I have been obtained in Table 14.

Table 13. Sample observation based on GTI-PHCS: data I.

CS	Observation	$\overline{\mathcal{K}}$
1	0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 4.1 4.2 4.3 4.4 4.4 4.6 4.7 4.7 4.8 4.9 4.9 5.5 5.7 5.7 6.1 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.1 7.1 7.6 7.7 8.2 8.6 8.6 8.8 8.8 9.5 9.6 9.7 10.7 10.9 11.0 11.1 11.2 11.2 11.5	55
2	0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.6 2.9 3.1 3.2 3.3 3.6 4.0 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.8 4.9 4.9 5.0 5.5 5.7 5.7 6.1 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.4 7.6 7.7 8.0 8.6 8.6 8.8 8.8 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11.0 11.0 11.1 11.2 11.9	57
3	0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 3.5 3.6 4.0 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.7 4.7 4.8 4.9 4.9 5.0 5.3 5.5 5.7 5.7 6.1 6.2 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.1 7.1 7.4 7.6 7.7 8.0 8.2 8.6 8.6 8.6 8.8 8.8 8.9 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11.0 11.0 11.1 11.2 11.2 11.5 11.9	71
4	0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 3.6 4.0 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.7 4.8 4.9 4.9 5.0 5.5 5.7 5.7 6.1 6.2 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.4 7.6 7.7 8.0 8.6 8.6 8.8 8.8 8.9 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11.0 11.0 11.1 11.2 11.5 11.9	63

For the GTI-PHCS of data II: Different GTI-PHCS samples (where m = 20, $\mathcal{K}$ = 15, and τ = 2) based on various CS as
CS.1: $R_1 = 2$, $R_2 = 1$ and the $R_i = 0$ where $i = 3, \ldots, m$.
CS.2: $R_1 = 3$, and the $R_i = 0$ where $i = 2, \ldots, m$.
CS.3: $R_m = 3$, and the $R_i = 0$ where $i = 1, \ldots, m-1$.
CS.4: $R_1 = 2$, $R_2 = 1$ and the $R_i = 0$ where $i = 2, \ldots, m-1$.
selections were obtained from the number of hours (in thousands) between failures of secondary reactor pumps data and are shown in Table 15 to evaluate our acquired estimators.

Figures 7 and 8 discussed the contour of the log-likelihood function in relation to different θ and λ values. It confirmed the results of the KS test and demonstrated the existence, uniqueness, and originality of the MLE. The Bayes estimates (along with their SE) were assessed using gamma priors which are also supplied in Tables 14 and 16 because there was no prior knowledge about the unknown KMGE parameters θ and λ from the available data set. Figures 9 and 10 show the trace plots of each generated sample to show how the MCMC iterations have converged. Figures 11 and 12 discussed MCMC posterior density and scatter plot for parameters based on CS.4 for data I, data II, respectively.

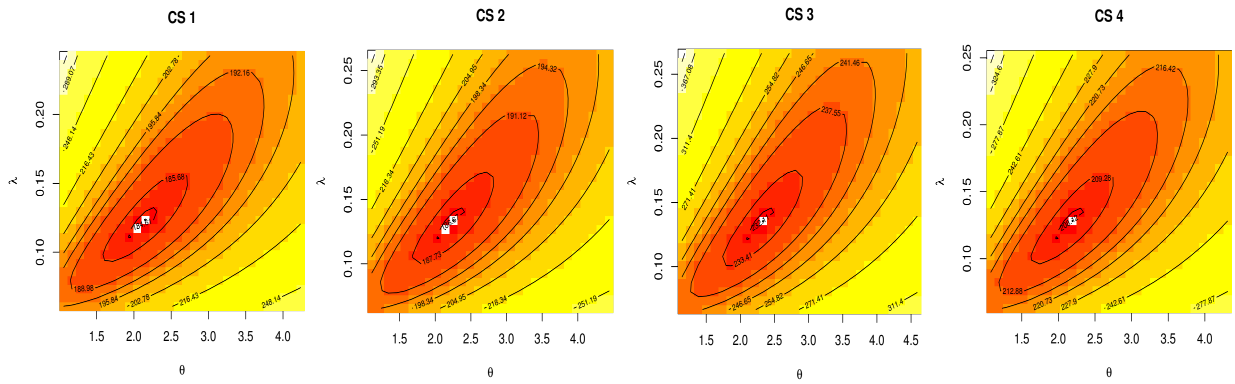

Figure 7. Contour plots: data I.

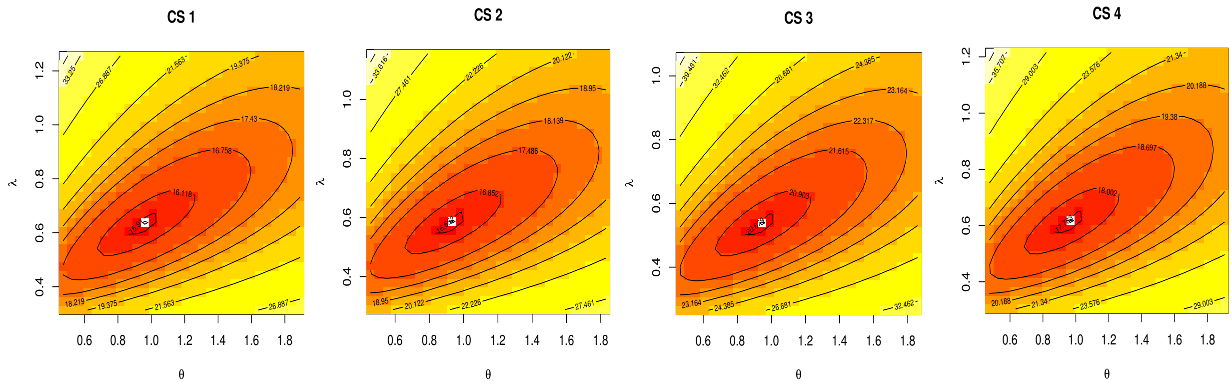

Figure 8. Contour plots: data II.

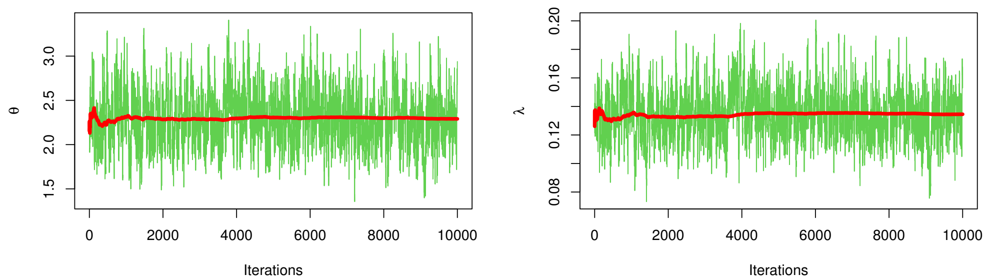

Figure 9. MCMC trace plot with convergence line: data I, CS.4.

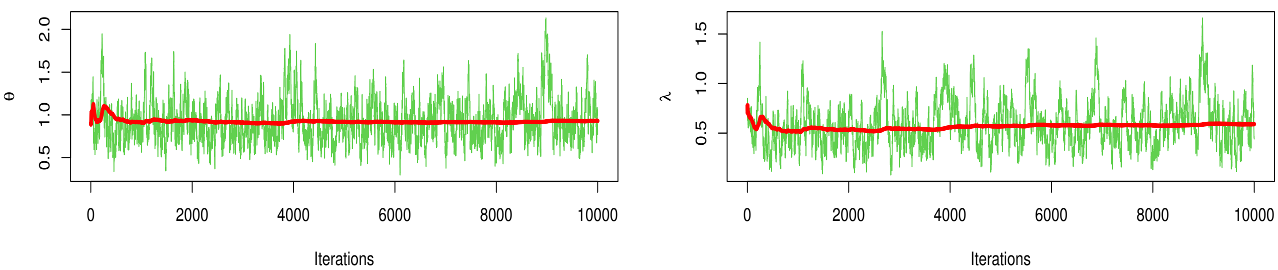

Figure 10. MCMC trace with convergence line: data II, CS.4.

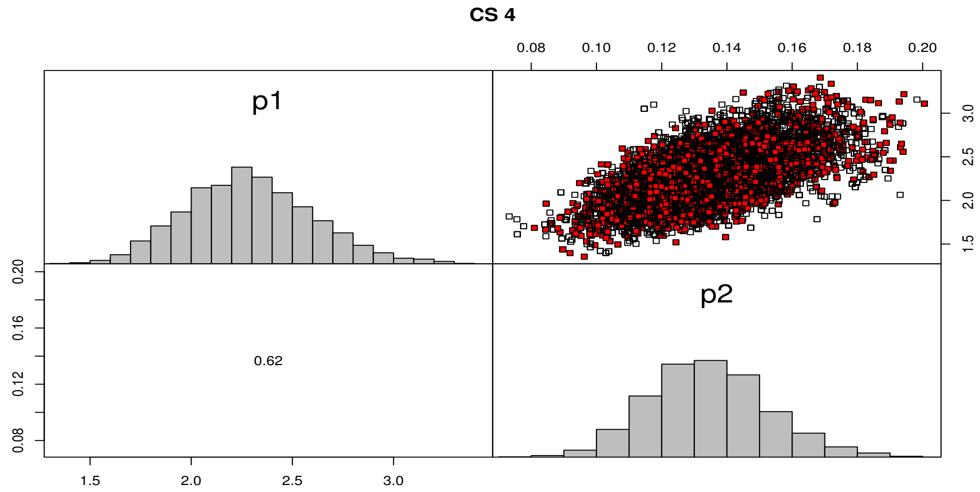

Figure 11. MCMC posterior density and scatter plot: data I, CS.4.

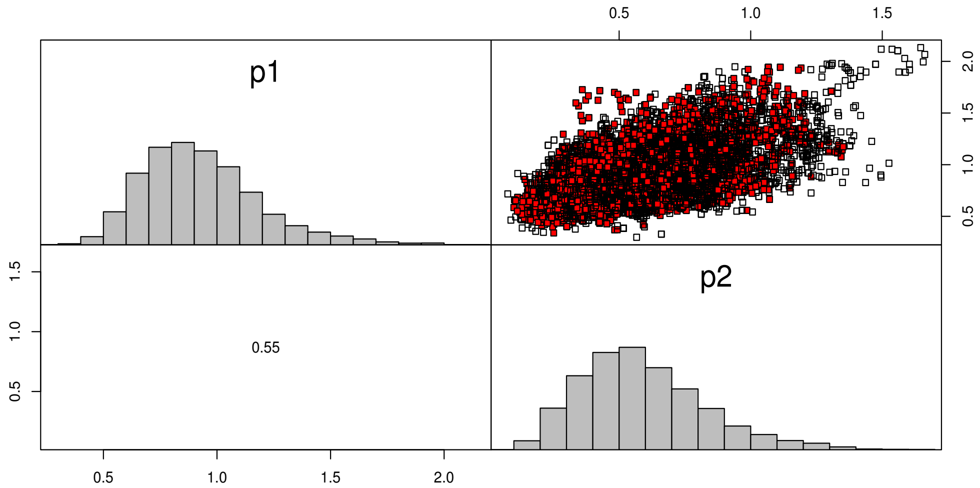

Figure 12. MCMC posterior density and scatter plot: data II, CS.4.

Table 14. MLE and Bayesian Estimate based on GTI-PHCS: data I.

		MLE		Bayesian	
CS		**Estimates**	**SE**	**Bayes**	**SEBayes**
1	θ	2.1173	0.3615	2.3120	0.3444
	λ	0.1213	0.0210	0.1322	0.0191
2	θ	2.2197	0.3847	2.3149	0.3347
	λ	0.1312	0.0218	0.1366	0.0184
3	θ	2.2943	0.3722	2.3563	0.3424
	λ	0.1331	0.0199	0.1366	0.0181
4	θ	2.1549	0.3594	2.2909	0.3154
	λ	0.1261	0.0203	0.1345	0.0181

Table 15. Sample observation based on GTI-PHCS: data II.

CS	Observation	$\overline{\mathcal{K}}$
1	0.062 0.070 0.101 0.150 0.199 0.273 0.347 0.358 0.402 0.491 0.605 0.614 0.746 0.954 1.060	15
2	0.062 0.070 0.101 0.150 0.199 0.273 0.347 0.358 0.402 0.491 0.605 0.614 0.746 0.954 1.921	15
3	0.062 0.070 0.101 0.150 0.199 0.273 0.347 0.358 0.402 0.491 0.605 0.614 0.746 0.954 1.060 1.359 1.921	17
4	0.062 0.070 0.101 0.150 0.199 0.273 0.347 0.358 0.402 0.491 0.491 0.605 0.614 0.746 0.954 1.060 1.921	16

Table 16. MLE and Bayesian Estimate based on GTI-PHCS: data II.

		MLE		Bayesian	
CS		Estimates	SE	Bayes	SEBayes
1	θ	0.9451	0.2726	0.9307	0.2508
	λ	0.6276	0.2843	0.6267	0.2656
2	θ	0.9151	0.2605	0.8824	0.2362
	λ	0.5770	0.2640	0.5357	0.2342
3	θ	0.9283	0.2576	0.9225	0.2466
	λ	0.5299	0.2265	0.5372	0.2047
4	θ	0.9514	0.2679	0.9306	0.2599
	λ	0.6075	0.2623	0.5903	0.2378

7. Concluding Remarks

In this paper, we considered the problems of parameter estimation of the KMGE distribution under the generalized progressively hybrid censored samples. For point estimation, five classical approaches of estimation such as ML, MPS, LS, WLS, and PE are discussed. Moreover, the Bayesian approach is studied. For interval estimation, we use the ML method of estimation by using the normal approximation confidence interval and the normal approximation of log-transformed MLE. The considered five classical estimation methods were then compared in terms of RB, MSEr, ARB, AMSEr, and AL of CI via Monte Carlo simulations. The MPS method shows better performance than the other four classical estimation methods for most of the considered cases. Bayesian estimation of the unknown parameters is presented under informative prior using two different loss functions. The results in the illustrative example show that the proposed ML and Bayesian work well again. In summary, the improved estimation methods of the ML, MPS, LS, WLS, PE, and Bayesian approaches contribute to more accurate parameter estimation for the KMGE distribution. These methods have practical utility in industries such as finance, insurance, engineering, healthcare, environmental modeling, social sciences, and more. By utilizing these estimation methods, practitioners can obtain reliable parameter estimates, leading to improved decision-making and predictive modeling in real-world applications. Future studies will take into account the estimation problem based on a broad framework called unified hybrid censoring.

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