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Fractional Differential Boundary Value Equation Utilizing the Convex Interpolation for Symmetry of Variables

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Abstract: In this paper, we introduce a novel form of interpolative convex contraction and develop some new theorems by utilizing the progressive method of interpolative convex contractions. We also obtain some fixed point results for a Suzuki convex contraction in orbitally S-complete F-metric spaces. The second purpose of this research is to evaluate the effectiveness of the fixed point approach in solving fractional differential equations with boundary conditions.

Keywords: interpolative convex contraction; Suzuki convex contraction; fixed point; fractional differential equation

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Citation: Hussain, A. Fractional Differential Boundary Value Equation Utilizing the Convex Interpolation for Symmetry of Variables. *Symmetry* **2023**, *15*, 1189. https://doi.org/10.3390/sym15061189

Academic Editors: Stojan Radenovic and Ghasem Soleimani Rad

Received: 21 April 2023
Revised: 24 May 2023
Accepted: 29 May 2023
Published: 2 June 2023

1. Introduction

The idea of the interpolative class of contractions was first introduced by Karapinar et al. [1], who also implemented a few fixed point results in a partial metric space. Karapinar [2] revisited Kannan's contraction principle via the notion of interpolation.

Karapinar updated Kannan's interpolative contraction in [2] and used an interpolative approach to determine the Hardy–Rogers findings in [3]. Additionally, he created a novel interpolative contraction technique in [4].

Aydi et al. [5,6] introduced interpolative and ω-interpolative Reich–Rus-type contractions and also proved some relevant fixed point findings for these mappings. Altun et al. [7] presented various proximal interpolative proximal contractions and found certain best proximity point results while taking into account the aforementioned mappings.

Hussain [8–10] recently expanded this idea of Karapinar and published a few findings pertaining to these kinds of novel contractions. Nazam et al. [11–13] introduced (Ψ, Φ)-orthogonal interpolative contractions very recently and made a few observations in the literature.

The idea of interpolation has been used by many mathematicians to obtain various analogues of classical fixed point theorems. Keeping in mind the aforementioned investigations, we develop a new concept of convex interpolative contraction and derive some results.

Let $\mathcal{F}$ represent the group of functions $g : (0, +\infty) \to \mathbb{R}$ fulfilling the following requirements:

($\mathcal{F}_1$) g is increasing, meaning that for every $u > v > 0, \Longrightarrow g(u) \leq g(v)$;
($\mathcal{F}_2$) each sequence $\{v_n\} \subset (0, +\infty)$, such that,

$$\lim_{n\to+\infty} v_n = 0 \iff \lim_{n\to+\infty} g(v_n) = -\infty.$$

Jleli and Samet [14] introduced the concept of $\mathcal{F}$-metric space as follows:

Definition 1 ([14]). *Let $Đ : X \times X \to [0, +\infty)$ be a mapping and X be a non-empty set. Suppose that there exists $(g, \mu) \in \mathcal{F} \times [0, +\infty)$ such that*

(Đ$_1$). $(\jmath, \ell) \in X \times X$, Đ $(\jmath, \ell) = 0 \iff \jmath = \ell$;
(Đ$_2$). Đ $(\jmath, \ell)$ =Đ $(\ell, \jmath)$ for all $(\jmath, \ell) \in X \times X$;
(Đ$_3$). For every $(\jmath, \ell) \in X \times X, 2 \leq N \in \mathbb{N}$, for each $(u_i)_{i \in \mathbb{N}} \subset X$ with $(u_1, u_N) = (\jmath, \ell)$, we have

$$Đ(u_1, u_N) > 0 \implies g(Đ(u_1, u_N)) \leq g(\sum_{i=1}^{N-1} d(u_i, u_{i+1})) + \mu.$$

The pair $(X, Đ)$ is called an F-metric space.

Example 1 ([14]). *Let $\mathbb{N}$ be a set and Đ be an F-metric defined by*

$$Đ(\jmath, \ell) = \begin{cases} (\jmath - \ell)^2, \text{ if } (\jmath, \ell) \in [0,3] \times [0,3] \\ |\jmath - \ell|, \text{ if } (\jmath, \ell) \notin [0,3] \times [0,3], \end{cases}$$

for all $(\jmath, \ell) \in \mathbb{N} \times \mathbb{N}$ with $f(v) = \ln(v)$ and $\mu = \ln(3)$, and Đ is not a metric, but on $\mathbb{N}$, it is an F-metric space.

Definition 2 ([14]). *Assume that $(X, Đ)$ is an F-metric space. Let $\{\jmath_n\}$ represent a sequence in X.*
(i) If $\lim_{n,m \to \infty} Đ(\jmath_n, \jmath_m) = 0$, we say that $\{\jmath_n\}$ is $\mathcal{F}$-Cauchy.
(ii) We say that $(X, Đ)$ is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a specific element in X.

Jleli and Samet [14] established an analogue of the Banach Contraction Principle as follows:

Theorem 1. *Let $h : X \to X$ be a mapping defined on an F-metric space $(X, Đ)$. Assume that the subsequent criteria are met:*
(i) $(X, Đ)$ is F-complete;
(ii) There exists $k \in (0, 1)$ such that

$$Đ(h(\jmath), h(\ell)) \leq kĐ(\jmath, \ell), \ (\jmath, \ell) \in X \times X.$$

Then h has a unique fixed point $\jmath^* \in X$. In addition, $\jmath_0 \in X$, the sequence $\{\jmath_n\} \subset X$ defined by $\jmath_{n+1} = h(\jmath_n)$, $n \in \mathbb{N}$, is $\mathcal{F}$-convergent.

In 2012, Samet et al. [15] introduced the concept of α-admissible mapping as follows:

Definition 3. *Let $S : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that S is α-admissible if for all $\jmath, \ell \in X$, with $\alpha(\jmath, \ell) \geq 1$ we have $\alpha(S\jmath, S\ell) \geq 1$.*

The idea of α-admissible mapping was then modified by Salimi et al. [16] as follows.

Definition 4 ([16]). *Let $S : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}^+$ be two functions. We say that S is α-admissible mapping with respect to η if for all $\jmath, \ell \in X$, with $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$ we have $\alpha(S\jmath, S\ell) \geq \eta(S\jmath, S\ell)$.*

Definition 5 ([17]). *Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. The mapping $S : X \to X$ is said to be α-η-continuous on (X, d) if for a sequence $\{\jmath_n\}$, we have*

$$\lim_{n \to \infty} \jmath_n = \jmath. \ \alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1}) \text{ for all } n \in \mathbb{N} \implies S\jmath_n \to S\jmath.$$

For more details, see [18,19].

If $\lim_{n \to \infty} S^n \jmath = v$ implies that $\lim_{n \to \infty} SS^n \jmath = Sv$, then a mapping $S : X \to X$ is termed as orbitally continuous at v. If S is orbitally continuous for all v, then the mapping S is orbitally continuous on v.

In Observation 1 [12], the authors proved the following inequality for $r \geq 1$, such that

$$(\text{þ} + q)^r \leq (\text{þ}q)^r, \forall\ \text{þ}, q \geq 2,$$

and remarked that the investigations in [9,10,20] did not have the correct proof.

2. Interpolative Convex Reich-Type Contraction

In this section, we offer a novel interpolative convex contraction and establish some new discoveries for interpolative convex Reich-type α-η-contraction in the context of F-complete F-metric space.

Definition 6. *Let $(X, Đ)$ be an F-metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. The mapping $S : X \to X$ is said to be an interpolative convex Reich-type α-η-contraction if there are constants $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ such that whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, we have*

$$Đ(S\jmath, S\ell)^{\text{þ}} \leq \lambda \Big[Đ(\jmath, \ell)^{\text{þ}\beta + q\alpha} \cdot Đ(\ell, S\ell)^{\text{þ}\gamma - q\gamma} \cdot Đ(\jmath, S\jmath)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \Big], \tag{1}$$

for all $\jmath, \ell \in X \backslash Fix(S)$, where $\text{þ}, q \in [1, \infty)$.

Example 2. Let $X = \{0, 1, 2, 3\}$ be endowed with F-metric space given by

$$Đ(\jmath, \ell) = \begin{cases} (\jmath - \ell)^2, \text{ if } (\jmath, \ell) \in X \times X \\ |\jmath - \ell|, \text{ if } (\jmath, \ell) \notin X \times X, \end{cases}$$

with $f(v) = \ln(v)$ and $\mu = \ln(3)$. Define $S : X \to X$ by

$$S0 = 0, S1 = 1, S2 = S3 = 0.$$

and $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(\jmath, \ell) = \begin{cases} 1, \text{ if } \jmath, \ell \in X \\ 0, \text{ otherwise} \end{cases} \quad \text{and } \eta(\jmath, \ell) = \begin{cases} \frac{1}{2}, \text{ if } \jmath, \ell \in X \\ 0, \text{ otherwise} \end{cases}.$$

If $\jmath, \ell \in X$, then clearly $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$ and so that

$$\begin{aligned} 0 &= Đ(S2, S3)^{\text{þ}} \leq \lambda \Big[Đ\ (2, 3)^{\text{þ}\beta + q\alpha} \cdot Đ(2, S2)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ\ (3, S3)^{\text{þ}\gamma - q\gamma} \Big] \\ &= \lambda \Big[(1)^{\text{þ}\beta + q\alpha} \cdot Đ\ (2, 0)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Đ(3, 0)^{\text{þ}\gamma - q\gamma} \Big] \\ &= \lambda \left[(4)^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot \left(\frac{9}{4} \right)^{\text{þ}\gamma - q\gamma} \right]. \end{aligned}$$

By taking any value of constants $\lambda \in [0, 1)$, $\alpha, \beta, \gamma \in (0, 1)$ and $\text{þ}, q \in [1, \infty)$. Clearly, (1) holds for all $\jmath, \ell \in X \backslash Fix(S)$ and S has two fixed points, 0 and 1; see for more information and examples [1].

Now, we state the key theorem of this article.

Theorem 2. *Let $(X, Đ)$ be an F-complete F-metric space and S be an interpolative convex Reich-type α-η-contraction assuring the following conditions:*
(i) S is α-admissible with respect to η;
(ii) There exists $\jmath_0 \in X$ such that $\alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0)$;
(iii) S is α-η-continuous mapping.
Then, S attains a fixed point in X.

Proof. Let $\jmath_0$ in X such that $\alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0)$. For $\jmath_0 \in X$, we construct a sequence $\{\jmath_n\}_{n=1}^{\infty}$ such that $\jmath_1 = S\jmath_0$, $\jmath_2 = S\jmath_1 = S^2\jmath_0$. Continue this approach until for every $n \in \mathbb{N}, \jmath_{n+1} = S\jmath_n = S^{n+1}\jmath_0$. Because of (i), S is α-admissible in terms of η after that $\alpha(\jmath_0, \jmath_1) = \alpha(\jmath_0, S\jmath_0) \geq \eta(\jmath_0, S\jmath_0) = \eta(\jmath_0, \jmath_1)$. By carrying out this procedure further, we have

$$\alpha(\jmath_{n-1}, \jmath_n) \geq \eta(\jmath_{n-1}, S\jmath_{n-1}) = \eta(\jmath_{n-1}, \jmath_n), \text{ for all } n \in \mathbb{N}. \tag{2}$$

On the condition that $\jmath_{n+1} = \jmath_n$ a few $n \in \mathbb{N}$, afterward, $\jmath_n = \jmath^*$ is a fixed point of S. Thus, we presume $\jmath_n \neq \jmath_{n+1}$ with

$$Ð(S\jmath_{n-1}, S\jmath_n) = Ð(\jmath_n, S\jmath_n) > 0, \text{ for all } n \in \mathbb{N}.$$

Since S is an interpolative convex Reich-type α-η-contraction, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
Ð(\jmath_n, \jmath_{n+1})^{þ} &= Ð(S\jmath_{n-1}, S\jmath_n)^{þ} \\
&\leq \lambda\Big[Ð(\jmath_{n-1}, \jmath_n)^{þ\beta+q\alpha} \cdot Ð(\jmath_{n-1}, S\jmath_{n-1})^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Ð(\jmath_n, S\jmath_n)^{þ\gamma-q\gamma}\Big], \\
&= \lambda\Big[Ð(\jmath_{n-1}, \jmath_n)^{þ\beta+q\alpha} \cdot Ð(\jmath_{n-1}, \jmath_n)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot Ð(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma}\Big], \\
&= \lambda\Big[Ð(\jmath_{n-1}, \jmath_n)^{þ(1-\gamma)+q\gamma} \cdot Ð(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma}\Big],
\end{aligned}$$

and we obtain

$$Ð(\jmath_n, \jmath_{n+1})^{þ(1-\gamma)+q\gamma} \leq \lambda Ð(\jmath_{n-1}, \jmath_n)^{þ(1-\gamma)+q\gamma}. \tag{3}$$

Afterward, we decide that $\{Ð(\jmath_{n-1}, \jmath_n)\}$ represents decreasing terms. As a result, there is a positive term ϱ s.t. $\lim_{n\to\infty} Ð(\jmath_{n-1}, \jmath_n) = \varrho$. Take note that $\varrho \geq 0$; we deduce using (3) that we have

$$Ð(\jmath_n, \jmath_{n+1}) \leq \lambda Ð(\jmath_{n-1}, \jmath_n) \leq \lambda^n Ð(\jmath_0, \jmath_1).$$

which provides

$$m > n, \ \sum_{i=n}^{m-1} Ð(\jmath_i, \jmath_{i+1}) \leq \frac{\lambda^n}{1-\lambda} Ð(\jmath_0, \jmath_1).$$

Subsequently, as we know λ belongs to $(0, 1)$, we have

$$\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} Ð(\jmath_0, \jmath_1) = 0.$$

There exists some $N, \in \mathbb{N}$ thus

$$N \leq n, \Rightarrow \ 0 < \frac{\lambda^n}{1-\lambda} Ð(\jmath_0, \jmath_1) < \delta.$$

Let $\epsilon > 0$ be fixed and $(g, \mu) \in \mathcal{F} \times \mathbb{R}^+$ satisfy $(Ð_3)$. By $(\mathcal{F}_2)$, for each $\delta > 0$, there is some N such that

$$0 < v < \delta \text{ implies that } g(v) < g(\epsilon) - a. \tag{4}$$

By (4) and $(\mathcal{F}_1)$, we obtain

$$g\left[\sum_{i=n}^{m-1} Ð(\jmath_i, \jmath_{i+1})\right] \leq g\left[\frac{\lambda^n}{1-\lambda} Ð(\jmath_0, \jmath_1)\right] < g(\epsilon) - \mu, \tag{5}$$

where $Ð(\jmath_n, \jmath_m) > 0$ and $m, n \in \mathbb{N}$ are such that $m > n \geq N$. Consequently, combining (5) and $(Ð_3)$, we have

$$g[Ð(\jmath_m, \jmath_n)] \leq g\left[\sum_{i=n}^{m-1} Ð(\jmath_i, \jmath_{i+1})\right] + \mu < g(\epsilon),$$

then, by $(\mathcal{F}_1)$, we obtain

$$Đ(\jmath_m, \jmath_n) < \epsilon, m > n \geq N.$$

This shows that $\{\jmath_n\}$ is an $\mathcal{F}$-Cauchy sequence. Thus, there exists $\jmath^* \in X$ such that $\jmath_n$ is $\mathcal{F}$-convergent to $\jmath^*$, because $(X, Đ)$ is an F-complete metric space: that is,

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0. \tag{6}$$

S is α-η-continuous and has the property $\alpha(\jmath_{n-1}, \jmath_n) \geq \eta(\jmath_{n-1}, \jmath_n)$ for every $n \in \mathbb{N}$. Now, applying a limit as n approaches infinity to $\jmath_{n+1} = S\jmath_n \to S\jmath^*$, we have $\jmath^* = S\jmath^*$. We will now demonstrate that $\jmath^*$ is a fixed point of S. We use contradiction to argue by assuming that $Đ(S\jmath^*, \jmath^*) > 0$ and (Đ3) gives us

$$g(Đ(S\jmath^*, \jmath^*)) \leq g\Big(Đ(S\jmath^*, S\jmath_n)^{þ} + Đ(S\jmath_n, \jmath^*)\Big) + \mu,\ n \in \mathbb{N}.$$

Using $(\mathcal{F}_1)$ and the contractive condition, we obtain

$$g(Đ(S\jmath^*, \jmath^*)) \leq g\left(\begin{array}{l} \lambda Đ\ (\jmath^*, \jmath_n)^{þ\beta+q\alpha} \\ \cdot Đ\ (S\jmath^*, \jmath^*)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \\ \cdot Đ(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma} + Đ(\jmath_{n+1}, \jmath^*) \end{array}\right) + \mu,$$

for every $n \in \mathbb{N}$. Using (6) information and $(\mathcal{F}_2)$, we obtain

$$\lim_{n\to\infty} g\Big(\lambda Đ(\jmath^*, \jmath_n)^{þ\beta+q\alpha} + Đ(\jmath_{n+1}, \jmath^*)\Big) + \mu = -\infty,$$

which results in a contradiction. In light of the fact that $Đ(S\jmath^*, \jmath^*) = 0$, hence, $\jmath^*$ is a fixed point of S. □

Theorem 3. *The mapping S also has a fixed point in X if we replace the hypothesis (iii) of Theorem 2 with the following:*

(iv) If $\{\jmath_n\}$ is a sequence in X such that $\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1})$ with $\lim_{n\to\infty} \jmath_n = \jmath^*$, then $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$ holds for all $n \in \mathbb{N}$.

Proof. In a manner similar to the proof of Theorem 2, we obtain $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$ for every $n \in \mathbb{N}$. By (D3), we obtain

$$g(Đ(S\jmath^*, \jmath^*)) \leq g(Đ(S\jmath^*, S\jmath_n) + Đ(\jmath_n, \jmath^*)) + \mu.$$

(1) and $(\mathcal{F}_1)$ give us

$$\begin{aligned} g(Đ(S\jmath^*, \jmath^*)) &\leq g\Big(\Big(Đ(S\jmath^*, S\jmath_n)^{þ}\Big) + Đ(S\jmath_n, \jmath^*)\Big) + \mu \\ &\leq g\left(\lambda\left[\begin{array}{l} Đ(\jmath^*, \jmath_n)^{þ\beta+q\alpha} \\ \cdot Đ(\jmath^*, S\jmath^*)^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \\ \cdot Đ(\jmath_n, \jmath_{n+1})^{þ\gamma-q\gamma} \end{array}\right] + Đ(\jmath_n, \jmath^*)\right) + \mu. \end{aligned}$$

Using the information in (6)

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0 = \lim_{n\to\infty} Đ(\jmath_{n+1}, \jmath^*),$$

we obtain

$$g(Đ(\jmath^*, S\jmath^*)) \leq g(Đ(\jmath^*, S\jmath^*)) + \mu.$$

Using $(\mathcal{F}_2)$ gives that

$$\lim_{n\to\infty} g(Đ(\jmath^*, S\jmath^*)) + \mu = -\infty,$$

which results in a contradiction. In light of the fact that $Ð(\jmath^*, S\jmath^*) = 0$, it is an established point $\jmath^*$ that possesses a fixed point of S. $\square$

Example 3. Assume $X = \mathbb{R}$ to F-metric space $Ð : X \times X \to \mathbb{R}^+$ by

$$Ð(\jmath, \ell) = \begin{cases} (\jmath - \ell)^2, \text{ if } (\jmath, \ell) \in \mathbb{N} \times \mathbb{N} \\ |\jmath - \ell|, \text{ if } (\jmath, \ell) \notin \mathbb{N} \times \mathbb{N}, \end{cases}$$

with $\mu = \ln(100)$ and $f(v) = \ln(v)$. Define $S : X \to X$ by

$$S\jmath = \begin{cases} 1 - \frac{\jmath}{2}, \text{ if } \jmath \in \mathbb{N} \\ 0, \quad \text{ if } \jmath \notin \mathbb{N} \end{cases}$$

and $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(\jmath, \ell) = \begin{cases} 2, \text{ if } \jmath, \ell \in [0, \infty) \\ 0, \text{ otherwise} \end{cases} \text{ and } \eta(\jmath, \ell) = \begin{cases} 1, \text{ if } \jmath, \ell \in [0, \infty) \\ 0, \text{ otherwise} \end{cases}.$$

Case 1: If $\jmath = \ell$. Evidently, $Ð(\jmath, \ell) = 0$.
As a result, Theorem 2's requirements are all met.
Case 2: If $\jmath, \ell$ are in $\mathbb{N}$, but $S\jmath \notin \mathbb{N}, S\ell \notin \mathbb{N}$, then

$$Ð(S\jmath, S\ell)^{\text{þ}} = Ð(1 - \frac{\jmath}{2}, 1 - \frac{\ell}{2}) = \left[\frac{1}{2}|\jmath - \ell|\right]^{\text{þ}}.$$

It is evident that S is α-admissible in terms of η for whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, which implies

$$Ð(S\jmath, S\ell)^{\text{þ}} = \left[\frac{1}{2}|\jmath - \ell|\right]^{\text{þ}} \leq \lambda\left[(\jmath - \ell)^{2\text{þ}\beta + 2q\alpha} \cdot \left|\frac{3}{2}\jmath - 1\right|^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot \left|\frac{3}{2}\ell - 1\right|^{\text{þ}\gamma - q\gamma}\right],$$

by taking constants $\lambda \in [0, 1)$, $\text{þ}, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
Although (i) neither $\jmath$ nor ℓ are in $\mathbb{N}$, which gives

$$Ð(S\jmath, S\ell)^{\text{þ}} = 0.$$

whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, it is evident that S is an α-admissible mapping with respect to η, such that

$$Ð(S\jmath, S\ell)^{\text{þ}} = 0 \leq \lambda\left[|\jmath - \ell|^{\text{þ}\beta + q\alpha} \cdot |\jmath|^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot |\ell|^{\text{þ}\gamma - q\gamma}\right],$$

where $\lambda \in [0, 1)$, $\text{þ}, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
(ii). One belongs to $\mathbb{N}$ other outside of $\mathbb{N}$

$$Ð(S\jmath, S\ell)^{\text{þ}} = Ð(1 - \frac{\jmath}{2}, 0)^{\text{þ}} = \left|1 - \frac{\jmath}{2}\right|^{\text{þ}}.$$

It is evident that S is an α-admissible mapping with respect to η whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, such that

$$Ð(S\jmath, S\ell)^{\text{þ}} = \left|1 - \frac{\jmath}{2}\right|^{\text{þ}} \leq \lambda\left[|\jmath - \ell|^{\text{þ}\beta + q\alpha} \cdot \left|\frac{3}{2}\jmath - 1\right|^{\text{þ}(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot |\ell|^{\text{þ}\gamma - q\gamma}\right],$$

by taking constants $\lambda \in [0, 1)$, $\text{þ}, q \in [1, \infty)$ and $\alpha, \beta, \gamma \in (0, 1)$, for all $\jmath, \ell \in \mathbb{N} \backslash Fix(S)$.
As a result, Theorem 2's requirements are all fulfilled. Thus, S is a convex interpolative Reich-type α-η-contraction as a result.

Definition 7. *Assume that $(X, Đ)$ is an F-metric space, S is a self-map defined on $(X, Đ)$ and $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. We say that S is α-η-complete if each $\mathcal{F}$-Cauchy sequence $\{\jmath_n\}$ satisfying*

$$\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1}) \text{ as each } n \in \mathbb{N}.$$

$\mathcal{F}$-converges in X.

Remark 1. *Theorems 2 and 3 also apply to α-η-complete F-metric space instead of F-complete F-metric space (see for more information [21]).*

3. Convex Interpolative Kannan-Type α-η-Contraction

In this stage, we develop several fixed point theorems in the context of F-complete F-metric space and provide new convex interpolative Kannan-type contractions. The following is an explanation of an interpolative convex Kannan-type α-η-contraction:

Definition 8. *Let $(X, Đ)$ is an F-metric space. Let there are two functions $\alpha, \eta : X \times X \to [0, +\infty)$ and $S : X \to X$. If there are constants $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that whenever $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$, we say that S is an convex interpolative convex Kannan-type α-η-contraction.*

$$\left[Đ(S\jmath, S\ell)^{\flat+q}\right] \leq \lambda[Đ(\jmath, S\jmath)]^{\flat(1-\beta)+q\alpha} \cdot [Đ(\ell, S\ell)]^{\flat\beta+q(1-\alpha)}, \tag{7}$$

where $\flat, q \in [1, \infty)$ for all $\jmath, \ell \in X$ with $\jmath \neq S\jmath$.

Now, we present and prove our second important theorem.

Theorem 4. *Let the mapping $S : X \to X$ satisfy the assumptions (i)–(ii) of Theorem 2 and (iii) of Theorem 3. Then, S attains a fixed point in X.*

Proof. It is carried out in a manner similar to that of Theorem 3. The inequalities (7) and (F1) give us

$$g\left(Đ(S\jmath^*, \jmath^*)^{\flat+q}\right) \leq g\left(\lambda\left(Đ(\jmath^*, S\jmath^*)^{\flat(1-\beta)+q(1-\alpha)} \cdot Đ(\jmath^*, \jmath_n)^{\flat\beta+q\alpha}\right) + Đ(S\jmath_n, \jmath^*)\right) + \mu$$

Utilizing (6) and the information

$$\lim_{n\to\infty} Đ(\jmath_n, \jmath^*) = 0 = \lim_{n\to\infty} Đ(\jmath_{n+1}, \jmath^*).$$

We achieve

$$g(Đ(\jmath^*, S\jmath^*)) \leq g(Đ(\jmath^*, S\jmath^*)) + \mu,$$

This is a contradiction. Therefore, $Đ\,(\jmath^*, S\jmath^*) = 0$; as a result, it is a fixed point of S. $\square$

Theorems 2–4 lead to the following corollaries. We can obtain the following results if we set $\eta(\jmath, \ell) = 1$. These corollaries are results that have been amended and are published in the literature.

Corollary 1. *Let $(X, Đ)$ be an F-complete F-metric space and S be an interpolative convex Reich-type α-η-contraction satisfying:*

(i) S is an α-admissible;

(ii) There is a $\jmath_0 \in X$ such that $\alpha(\jmath_0, S\jmath_0) \geq 1$;

(iii) S is continuous.

Then, S has a fixed point in X.

Corollary 2. *Let $(X, Đ)$ be an F-complete F-metric space and S be an interpolative convex Reich-type α-η-contraction satisfying:*

(i) S is an α-admissible;

(ii) There is a $\jmath_0 \in X$ such that $\alpha(\jmath_0, S\jmath_0) \geq 1$;

(iii) If $\{\jmath_n\}$ is a sequence in X such that $\alpha(\jmath_n, \jmath_{n+1}) \geq 1$ with $\lim_{n\to\infty} \jmath_n = \jmath^*$ then $\alpha(\jmath_n, \jmath^*) \geq 1$ satisfying for every $n \in \mathbb{N}$. Then, S has a fixed point.

Corollary 3. *Let $(X, Đ)$ be an F-complete F-metric space and S be a convex interpolative Kannan-type contraction satisfying:*

(i) S is an α-admissible;

(ii) There is a $\jmath_0 \in X$ such that $\alpha(\jmath_0, S\jmath_0) \geq 1$;

(iii) S is continuous.

Then, S has a fixed point in X.

Corollary 4. *Let $(X, Đ)$ be an F-complete F-metric space and S be a convex interpolative Kannan-type contraction satisfying:*

(i) S is an α-admissible;

(ii) There is a $\jmath_0 \in X$ such that $\alpha(\jmath_0, S\jmath_0) \geq 1$;

(iii) If $\{\jmath_n\}$ is a sequence in X such that $\alpha(\jmath_n, \jmath_{n+1}) \geq 1$ with $\lim_{n\to\infty} \jmath_n = \jmath^*$ implies that $\alpha(\jmath_n, \jmath^*) \geq 1$ satisfies for every $n \in \mathbb{N}$. Then, S has a fixed point in X.

4. Findings

Our findings lead to some conclusions on Suzuki contractions in orbitally S-complete and continuous maps in F-metric space.

Theorem 5. *Let S be a continuous self-map on X and $(X, Đ)$ be an F-complete F-metric space. If there exist $r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ such that*

$$Đ(\jmath, S\jmath) \leq Đ(\jmath, \ell) \implies Đ\,(S\jmath, S\ell)^{þ} \leq r[Đ(\jmath, \ell)]^{þ\beta + q\alpha} \cdot [Đ(\jmath, S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)} \cdot [Đ(\ell, S\ell)]^{þ\gamma - q\gamma},$$

where $þ, q \in [1, \infty)$, for every $\jmath, \ell \in X$.

Then, S has a fixed point in X.

Proof. Set two functions $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(\jmath, \ell) = Đ(\jmath, \ell) \text{ and } \eta(\jmath, \ell) = Đ(\jmath, \ell), \text{ for all } \jmath, \ell \in X,$$

and $\beta, \gamma \in (0,1)$, and $r \in [0,1)$. It is clear that

$$\eta(\jmath, \ell) \leq \alpha(\jmath, \ell), \text{ for all } \jmath, \ell \in X,$$

that is, Theorem 2's criteria (i) through (iii) are satisfied. Let

$$\eta(\jmath, S\jmath) \leq \alpha(\jmath, \ell) \text{ then } Đ\,(\jmath, S\jmath) \leq Đ(\jmath, \ell),$$

it suggests a contractive condition

$$Đ(S\jmath, S\ell)^{þ} \leq r[Đ(\jmath, \ell)]^{þ\beta + q\alpha}.[Đ(\jmath, S\jmath)]^{þ(1-\beta-\gamma)+q(\gamma-\alpha)}.[Đ(\ell, S\ell)]^{þ\gamma - q\gamma}.$$

As a result, Theorem 3's criteria are all satisfied. Hence, S attains a fixed point in $\jmath$. □

Theorem 6. *Suppose a continuous map S and that $(X, Đ)$ is an F-complete F-metric space. Assume $r \in [0,1)$ and $\alpha, \beta \in (0,1)$ are present and in such a way that*

$$Đ(\jmath, S\jmath) \leq Đ(\jmath, \ell) \implies Đ\,(S\jmath, S\ell)^{þ+q} \leq r[Đ(\jmath, S\jmath)]^{þ(1-\beta)+q(1-\alpha)} \cdot [Đ(\ell, S\ell)]^{þ\beta + q\alpha}$$

where $þ, q \in [1, \infty)$, for all $\jmath, \ell \in X$. Then, S attains a fixed point.

Corollary 5. *Suppose a continuous map S and let $(X, Đ)$ be an F-complete F-metric space. Assume $r \in [0,1)$ in such a way that*

$$Đ(\jmath, S\jmath) \leq Đ(\jmath, \ell) \implies Đ\,(S\jmath, S\ell) \leq rĐ(\jmath, \ell),$$

for all $\jmath, \ell \in X$. Then, S possesses a fixed point.

Theorem 7. *Suppose S is a self-map and $(X, Đ)$ is an F-metric space in X. Surmise the following claims are true:*

(i) $(X, Đ)$ is an orbitally S-complete F-metric space;

(ii) There exists $r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ such that

$$Đ(S\jmath, S\ell)^{\flat} \leq r[Đ(\jmath, \ell)]^{\flat\beta+q\alpha}.[Đ(\jmath, S\jmath)]^{\flat(1-\beta-\gamma)+q(\gamma-\alpha)}.[Đ(\ell, S\ell)]^{\flat\gamma-q\gamma},$$

where $q \in [1,\infty)$ for all $\jmath, \ell \in O(\omega)$ for some $\omega \in X$, where $O(\omega)$ is an orbit of ω, where $O(\omega)$ is an orbit of ω, and $\flat, q$ are in $[1,\infty)$ for every $\jmath, \ell \in O(\omega)$ and for some $\omega \in X$;

(iii) If $\{\jmath_n\}$ is a sequence where $\{\jmath_n\} \subseteq O(\omega)$ along $\lim_{n\to\infty} \jmath_n = \jmath^*$, then $\jmath^* \in O(\omega)$.

Then, S has a fixed point.

Proof. Set $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(\jmath, \ell) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(\jmath, \ell) = 0$ otherwise and $\eta(\jmath, \ell) = 1$ for all $\jmath, \ell \in \jmath$ (see Remark 6 [21]). Then, $(X, Đ)$ is an α-η-complete $\mathcal{F}$-metric and S is α-admissible with regard to η. Let $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$; later, $\jmath, \ell \in O(\omega)$, afterward, from (ii), give us

$$Đ(S\jmath, S\ell)^{\flat} \leq r[Đ(\jmath, \ell)]^{\flat\beta+q\alpha}.[Đ(\jmath, S\jmath)]^{\flat(1-\beta-\gamma)+q(\gamma-\alpha)}.[Đ(\ell, S\ell)]^{\flat\gamma-q\gamma}.$$

That is, S is an interpolative convex α-η-contraction of the Reich type. Let a sequence $\{\jmath_n\}$ apply, which reads $\alpha(\jmath_n, \jmath_{n+1}) \geq \eta(\jmath_n, \jmath_{n+1})$ and $\lim_{n\to\infty} \jmath_n = \jmath^*$. Therefore, $\{\jmath_n\} \subseteq O(\omega)$. The expression is taken from (iii) $\jmath^* \in O(\omega)$, $\alpha(\jmath_n, \jmath^*) \geq \eta(\jmath_n, \jmath^*)$. As a result, Theorem 3's criteria are all fulfilled. S therefore has a fixed point. □

Theorem 8. *Similar to Theorem 7's hypotheses, this satisfies*

$$Đ(S\jmath, S\ell)^{\flat+q} \leq r[Đ(\jmath, S\jmath)]^{\flat(1-\beta)+q(1-\alpha)}.[Đ(\ell, S\ell)]^{\flat\beta+q\alpha}.$$

Therefore, S attains a fixed point.

Theorem 9. *Let S be a self-map and $(X, Đ)$ be an F-complete F-metric space. Suppose the subsequent claims are true:*

(i) For all $\jmath, \ell \in O(\omega)$, there exists $r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$, $\flat, q \in [1,\infty)$ such that

$$Đ(S\jmath, S\ell)^{\flat} \leq r[Đ(\jmath, \ell)]^{\flat\beta+q\alpha}.[Đ(\jmath, S\jmath)]^{\flat(1-\beta-\gamma)+q(\gamma-\alpha)}.[Đ(\ell, S\ell)]^{\flat\gamma-q\gamma},$$

for some $\omega \in X$;

(ii) S is orbitally continuous.

Then, S possesses a fixed point.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(\jmath, \ell) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(\jmath, \ell) = 0$ otherwise and $\eta(\jmath, \ell) = 1$ (see Remark 1.1 [22]); we know S is an α-η-continuous map. Assume $\alpha(\jmath, \ell) \geq \eta(\jmath, \ell)$; afterward, $\jmath, \ell \in O(\omega)$. Therefore, $S\jmath, S\ell \in O(\omega)$: that is, $\alpha(S\jmath, S\ell) \geq \eta(S\jmath, S\ell)$. In light of this, S is therefore a mapping that is α-admissible. We have from (i)

$$Đ(S\jmath, S\ell)^{\flat} \leq r[Đ(\jmath, \ell)]^{\flat\beta+q\alpha}.[Đ(\jmath, S\jmath)]^{\flat(1-\beta-\gamma)+q(\gamma-\alpha)}.[Đ(\ell, S\ell)]^{\flat\gamma-q\gamma}.$$

That is to say, S is a Reich-type interpolative convex α-η-contraction. As a result, Theorem 2's entire premise is true. S therefore attains a fixed point. □

Theorem 10. *In Theorem 9, if the assumption (i) is replaced with*

$$Đ(S\jmath, S\ell)^{p+q} \leq r[Đ(\jmath, S\jmath)]^{p(1-\beta)+q(1-\alpha)}.[Đ(\ell, S\ell)]^{p\beta+q\alpha},$$

then S also attains a fixed point.

Corollary 6. *Let $(X, Đ)$ be an F-complete F-metric space and S be a self-map. Assume that the following conditions hold:*

(i) There exist $r \in [0,1)$ such that for every $\jmath, \ell \in O(\omega)$,

$$Đ(S\jmath, S\ell) \leq r(Đ(\jmath, \ell)$$

for some $\omega \in X$;

(ii) S is orbitally continuous.

Then, S possesses a fixed point.

5. Application

Recent research has shown that the local and nonlocal fractional differential equations are useful tools for simulating a wide range of phenomena in a variety of scientific and architectural domains. Numerous fields, including viscoelasticity, etc., make use of the fractional-order differential equations. For more information, see [23–25]. Bai [26], using the monotone iterative method, looked into whether the periodic boundary value problem for the nonlinear impulsive fractional differential equation involving the sequential fractional derivative has any solutions. Alexandru et al. [27] achieved the existence and unique solution for the system of fractional equations with sequential Caputo derivatives, two positive parameters, along with the general Riemann–Stieltjes integral nonlocal boundary conditions. Hammad et al. [28] analyzed the existence and uniqueness of solutions to a system of fractional defferential equations (FDEs) by using Riemann–Liouville (R-L) integral boundary conditions. Using the fractional generalized derivative in the sense of Riemann involving a boundary condition, we want to demonstrate the existence and uniqueness of a bounded solution.

The left Riemann–Liouville fraction of a Lebesgue integrable function g with regard to an increasing function h is provided by [29].

$${}_aI_h^\alpha\, g(v) = \frac{1}{\Gamma(\alpha)} \int_a^v (h(v) - h(\omega))^{\alpha-1}\, f(\omega)h^{'}(\omega)d\omega, \text{ where } \alpha > 0. \tag{8}$$

With regard to the identical rising function h, the related left Riemann Liouville fractional derivative of g is given by [29]

$${}_aĐ_h^\alpha g(v) = \left(\frac{1}{h^{'}(v)}\frac{d}{dv}\right)^n I^{(n-\alpha)} g(v)$$

$$= \left(\frac{1}{h^{'}(v)}\frac{d}{dv}\right)^n \frac{1}{\Gamma(\alpha)} \int_a^v (h(v) - h(\omega))^{n-\alpha-1}\, g(\omega)h^{'}(\omega)d\omega, \tag{9}$$

where α is the largest integer, $\alpha \geq 0$ and $n = \alpha + 1$. The fractional integral and fractional derivative are combined in the following theorem.

Theorem 11 ([30])**.** *Let $\alpha > 0$, $n = -[-\alpha], g \in L[c,d]$ and ${}_aI_h^\alpha g \in AC_h^n[c,d]$. Then*

$${}_aI_{ha}^\alpha Đ_h^\alpha g(v) = g(v) - \sum_{k=1}^{n} c_k (h(v) - h(a))^{\alpha-k}.$$

We are thinking about the ensuing boundary value problem

$$ {}_{c}\text{Đ}_{h}^{\alpha}\ell(v) \ + g(v,\ell(v)) = 0, \text{ with } \ \ell(c) = \ell(d) = 0, \text{ where } 1 < \alpha \leq 2. \tag{10}$$

Lemma 1. *Let $\alpha > 0$, $n = -[-\alpha]$, $g \in L[c,d]$ and ${}_{c}I_{h}^{\alpha}g \in AC_{h}^{n}[c,d]$ exist if and only if ℓ is a solution to the boundary value problem (10),*

$$\ell(v) = \int_{c}^{d} \aleph(\omega,v)\, g(\omega,\ell(\omega))h^{'}(\omega)d\omega,$$

where the Greens' function

$$\aleph(\omega,v) = \frac{1}{\Gamma(\alpha)}\begin{cases} \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1} - (h(v)-h(\omega))^{\alpha-1}, & c < \omega \leq v \\ \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}, & v \leq \omega < d \end{cases}$$

satisfies the following:

- $\aleph(\omega,v) \geq 0$.
- $$\max_{c\leq\omega,v\leq d} \aleph(\omega,v) = \frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(c)}{4}\right)^{\alpha-1}.$$

Proof. Applying the integral (8) to (10), we obtain

$${}_{c}I_{hc}^{\alpha}\text{Đ}_{h}^{\alpha}\ell(v) = -\,{}_{c}I_{h}^{\alpha}\, g(v,\ell(v)) = -\frac{1}{\Gamma(\alpha)}\int_{c}^{v}(h(v)-h(\omega))^{\alpha-1}\, g(\omega)h^{'}(\omega)d\omega,$$

using Theorem 11, we obtain

$$\ell(v) = c_1(h(v)-h(c))^{\alpha-1} + c_2(h(v)-h(c))^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_{c}^{v}(h(v)-h(\omega))^{\alpha-1}\, g(\omega)h^{'}(\omega)d\omega,$$

$$\ell(c) = 0, \text{ gives } c_2 = 0.$$

$$\begin{aligned} \ell(d) &= 0, \text{ gives} \\ c_1 &= \frac{(h(d)-h(c))^{1-\alpha}}{\Gamma(\alpha)}\int_{c}^{d}(h(d)-h(\omega))^{\alpha-1}\, g(\omega,\ell(\omega))h^{'}(\omega)d\omega. \end{aligned}$$

Therefore

$$\begin{aligned} \ell(v) &= \frac{1}{\Gamma(\alpha)}\int_{c}^{d}\frac{(h(d)-h(\omega))(h(v)-h(c))^{\alpha-1}}{(h(d)-h(c))}\, g(\omega,\ell(\omega))h^{'}(\omega)d\omega \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_{c}^{v}(h(v)-h(\omega))^{\alpha-1}\, g(\omega,\ell(\omega))h^{'}(\omega)d\omega. \end{aligned}$$

Hence

$$\ell(v) = \int_{c}^{d}\aleph(\omega,v)\, g(\omega,\ell(\omega))h^{'}(\omega)d\omega, \text{ where}$$

$$\aleph(\omega,v) = \frac{1}{\Gamma(\alpha)}\begin{cases} \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1} - (h(v)-h(\omega))^{\alpha-1}, & c < \omega \leq v \\ \left(\frac{(h(d)-h(\omega))(h(v)-h(c))}{(h(d)-h(c))}\right)^{\alpha-1}, & v \leq \omega < d \end{cases}.$$

It is obvious that $\aleph(\omega, v) \geq 0$ for $\omega \geq v$. $\square$

For $c \leq \omega < v$, it is clear that $\aleph(\omega, v) \geq 0$ when $\omega \geq v$.

$$\aleph(\omega, v) = \left(\frac{(h(v) - h(c))}{(h(d) - h(c))} \right)^{\alpha-1} \left[(h(d) - h(c))^{\alpha-1} - \left(\begin{array}{c} h(d)- \\ \left(h(c) + \frac{(h(\omega)-h(c))(h(d)-h(c))}{(h(v)-h(c))} \right)^{\alpha-1} \end{array} \right) \right].$$

Since

$$h(c) + \frac{(h(\omega) - h(c))(h(d) - h(c))}{(h(v) - h(c))} \geq h(\omega),$$

It follows that $\aleph(\omega, v) \geq 0$, where $\omega \leq v$ and for $v \leq \omega$,

$$\frac{\partial \aleph}{\partial v} = \frac{1}{\Gamma(\alpha)} \left(\frac{h(d) - h(\omega)}{(h(d) - h(c))} \right)^{\alpha-1} \cdot (\alpha - 1)(h(v) - h(c))^{\alpha-2} h'(v) \geq 0,$$

thus, $\aleph(\omega, v)$ is rising in proportion to v.

Now $\omega \leq v$

$$\frac{\partial \aleph}{\partial v} = \frac{h'(v)(\alpha - 1)}{\Gamma(\alpha)} \cdot \left[-(h(v) - h(\omega))^{\alpha-2} + \left(\frac{h(d) - h(\omega)}{h(d) - h(c)} \right)^{\alpha-1} (h(v) - h(c))^{\alpha-2} \right]$$

$$= \frac{h'(v)}{\Gamma(\alpha - 1)} \cdot \left(\frac{h(v) - h(c)}{h(d) - h(c)} \right)^{\alpha-2} \left[\left(\frac{h(d) - h(\omega)}{h(d) - h(c)} \right)^{\alpha-1} - \left(\frac{(h(d) - h(c))(h(v) - h(\omega))}{h(d) - h(c)} \right)^{\alpha-2} \right]$$

$$\begin{aligned} &\leq \frac{h'(v)}{\Gamma(\alpha - 1)} \cdot \left(\frac{h(v) - h(c)}{h(d) - h(c)} \right)^{\alpha-2} [(h(d) - h(c))^{\alpha-2} \\ &\quad - \left(h(d) - \left(h(c) + \frac{h(d) - h(c)}{h(v) - h(c)} (h(\omega) - h(c)) \right)^{\alpha-2} \right)] \\ &< 0. \end{aligned}$$

Thus, $\aleph(\omega, v)$ is decreasing when $\omega \leq v$. Therefore, at $\omega = v$, $\aleph(\omega, v)$ reaches its maximum.

$$\aleph(\omega, \omega) = \frac{1}{\Gamma(\alpha)} \frac{(h(d) - h(\omega))^{\alpha-1} (h(\omega) - h(c))^{\alpha-1}}{(h(d) - h(c))^{\alpha-1}} = \hat{G}(\omega)$$

$$\begin{aligned} \hat{G}'(\omega) &= -\frac{1}{\Gamma(\alpha)} (\alpha - 1) \frac{(h(d) - h(\omega))^{\alpha-2}}{(h(d) - h(c))^{\alpha-1}} h'(\omega) \cdot (h(\omega) - h(c))^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \frac{(h(d) - h(\omega))^{\alpha-1} (\alpha - 1)(h(\omega) - h(c))^{\alpha-2} h'(\omega)}{(h(d) - h(c))^{\alpha-1}} \\ &= 0 \end{aligned}$$

yields

$$h(\omega) = \frac{h(c) + h(d)}{2},$$

or the critical point

$$\omega^* = h^{-1} \left(\frac{h(c) + h(d)}{2} \right).$$

Therefore, the maximum value of $h(\omega, v)$ is

$$\check{N}(\omega^*) = \frac{1}{\Gamma(\alpha)} \left(\frac{h(d) - h(c)}{4} \right)^{\alpha-1},$$

$$|\aleph(\omega,v)| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{h(d)-h(c)}{4}\right)^{\alpha-1}.$$

The Riemann–Stieltjes integrable function of w with respect to ω and g is denoted as follows: a continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$. Assume that $\hat{C}_i$ is the linear space of all continuous functions defined on $I=[0,1]$ and that

$$\text{Đ}(w,v) = \|w-v\|_\infty^2 = \max_{v\in I}|w(v)-v(v)|^2 \text{ for every } w,v\in\hat{C}_i.$$

So, $(\hat{C}_i, \text{Đ})$ is a metric space that is F-complete.

We take into account the following situations:

(a) There exists $r\in[0,1)$, $\zeta:\mathbb{R}^2\to\mathbb{R}$ is a function for each $c,d\in\mathbb{R}$ with $\zeta(c,d)\geq\xi(c,d)$, such that

$$\begin{aligned}|g(\omega,w(\omega))d\omega - g(\omega,v(\omega))d\omega|^{\text{þ}} &\leq |w(\omega)-v(\omega)|^{2(\text{þ}\gamma+q\beta)}\\ &\cdot\ |w(\omega)-Sw(\omega)|^{2(\text{þ}(1-\gamma-\hat{w})+q(\hat{w}-\beta))}\\ &\cdot\ |v(\omega)-vv(\omega)|^{2(\text{þ}\hat{w}-q\hat{w})},\end{aligned}$$

where $\text{þ},q\geq[1,\infty)$, $\beta,\gamma,\hat{w}\in(0,1)$;

(b) For every $w_1\in\hat{C}_i$, there exists such that

$$\zeta\left(w_1(v),\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega\right)\geq\xi\left(w_1(v),\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega\right),$$

satisfies for each $v\in I$.

(c) There exists a $w_1,v_1\in\hat{C}_i$ for each $w,v\in\hat{C}_i$, such that

$$\begin{aligned}&\zeta(w(v),v(v))\geq\xi(w(v),v(v))\\ &\text{implies } \zeta\left(\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega,\int_c^d\aleph(v,\omega)g(\omega,v_1(\omega))h'(\omega)d\omega\right)\\ \geq\ &\xi\left(\int_c^d\aleph(v,\omega)g(\omega,w_1(\omega))h'(\omega)d\omega,\int_c^d\aleph(v,\omega)g(\omega,v_1(\omega))h'(\omega)d\omega\right),\end{aligned}$$

holds for all values of $v\in I$.

(d) Any group of points w in a sequence $\{w_n\}$ of points in $\hat{C}_i$ will have

$$\zeta(w_n,w_{n+1})\geq\xi(w_n,w_{n+1}),\ \lim_{n\to\infty}\inf\zeta(w_n,w)\geq\lim_{n\to\infty}\inf\xi(w_n,w).$$

Theorem 12. *Assume that the conditions (a)–(d) are met. So, (10) has at least one $w\in\hat{C}_i$ solution.*

Proof. We know that $w\in\hat{C}_i$ is a solution of the fractional-order integral equation if and only $w\in\hat{C}_i$ is a solution of (10),

$$w(v)=\lambda\int_a^b\aleph(v,s)g(s,w(s))h'(s)ds \text{ for all } v\in I,$$

where $0\leq\lambda<1$. Define a map $S:\hat{C}_i\to\hat{C}_i$ by

$$Sw(v)=\lambda\int_a^b\aleph(v,s)g(s,w(s))h'(s)ds \text{ for all } v\in I.$$

Then, solving problem (10) is identical to discovering $w^* \in \hat{C}_i$, which is a fixed point of S. Let $w, v \in \hat{C}_i$, be such that for all $v \in I$, $\zeta(w(v), v(v)) \geq 0$. Using (a), we obtain

$$\begin{aligned} |Sw(v) - Sv(v)|^p &= \left|\lambda \int_a^b \aleph(v,s)[g(s,w(s)) - g(s,v(s))]h'(s)ds\right|^p \\ &\leq |\lambda| \int_a^b |\aleph(v,s)| \left|g(s,w(s)) - g(s,v(s))h'(s)ds\right|^p \\ &\leq |\lambda| \int_a^b |\aleph(v,s)|h'(s)rds|w(s) - v(s)|^{2(p\gamma+q\beta)} \\ &\cdot\ |w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w})+q(\hat{w}-\beta))} \\ &\cdot\ |v(s) - vv(s)|^{2(p\hat{w}-q\hat{w})} \\ &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{h(b)-h(a)}{4}\right)^{\alpha-1}(h(b)-h(a))\|w(s) - v(s)\|_\infty^{2(p\gamma+q\beta)} \\ &\cdot\ \|w(s) - Sw(s)\|_\infty^{2(p(1-\gamma-\hat{w})+q(\hat{w}-\beta))} \cdot \|v(s) - Sv(s)\|_\infty^{2(p\hat{w}-q\hat{w})} \\ &\leq r\|w(s) - v(s)\|_\infty^{2(p\gamma+q\beta)} \\ &\cdot\ \|w(s) - Sw(s)\|_\infty^{2(p(1-\gamma-\hat{w})+q(\hat{w}-\beta))} \cdot \|v(s) - Sv(s)\|_\infty^{2(p\hat{w}-q\hat{w})}. \end{aligned}$$

Thus,

$$D(Sw,Sv)^p < \left\{ \begin{array}{l} |w(s) - v(s)|^{2(p\gamma+q\beta)} \\ \cdot|w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w})+q(\hat{w}-\beta))} \\ \cdot|v(s) - Sv(s)|^{2(p\hat{w}-q\hat{w})} \end{array} \right\}$$

holds for each $w, v \in \hat{C}_i$ such that $\zeta(w(v), v(v)) \geq \xi(w(v), v(v))$ for each $v \in I$.

We define $\alpha : \hat{C}_i \times \hat{C}_i \to [0, \infty)$ by

$$\alpha(w,v) = \left\{ \begin{array}{ll} 2, & \text{if } \zeta(w(v),v(v)) \geq 0,\ v \in I, \\ 0, & \text{otherwise} \end{array} \right\}$$

$$\text{and } \eta(w,v) = \left\{ \begin{array}{ll} \frac{1}{3}, & \text{if } \xi(w(v),v(v)) \geq 0,\ v \in I, \\ 0, & \text{otherwise} \end{array} \right\}$$

Then, for all $w, v \in \hat{C}_i$, $\alpha(w, v) \geq \eta(w, v)$, we have

$$D(Sw,Sv)^p \leq r\left\{ \begin{array}{l} |w(s) - v(s)|^{2(p\gamma+q\beta)} \\ \cdot|w(s) - Sw(s)|^{2(p(1-\gamma-\hat{w})+q(\hat{w}-\beta))} \\ \cdot|v(s) - Sv(s)|^{2(p\hat{w}-q\hat{w})} \end{array} \right\}.$$

Obviously, $\alpha(w, v) \geq \eta(w, v)$ for every $w, v \in \hat{C}_i$. If $\alpha(w, v) \geq \eta(w, v)$ for each $w, v \in \hat{C}_i$, then $\zeta(w(v), v(v)) \geq \xi(w(v), v(v))$.

From (c), we have $\zeta(Sw(v), Sv(v)) \geq \xi(Sw(v), Sv(v))$ and so $\alpha(Sw, Sv) \geq \eta(Sw, Sv)$.

Thus, S is an α-admissible map concerning η.

From (b), there subsists $w_1 \in \hat{C}_i$ parallel to $\alpha(w_1, Sw_1) = \eta(w_1, Sw_1)$.

By (d), we know that any group of points in a sequence $\{w_n\}$ of points in $\hat{C}_i$ with w will have

$$\alpha(w_n, w_{n+1}) = \eta(w_n, w_{n+1}),$$

and

$$\lim_{n\to\infty} \inf \alpha(w_n, w) = \lim_{n\to\infty} \inf \eta(w_n, w).$$

By using Theorem 2, it can be shown that S attains a fixed point in $\hat{C}_i$. Finally, w^* is a solution to the equation $Sw^* = w^*$ in $\hat{C}_i$ (10). □

6. Conclusions

In the context of F-metric space, this study focuses on a novel notion of convex interpolative contraction of the Reich and Kannan type that is more inclusive than standard metrics. Results for the Suzuki-type fixed point are driven in the F-metric space. This work expands the idea of interpolative contractions and yields a few significant theorems. This study will add fresh information to the body of knowledge. In order to demonstrate our theorems and as an application, we find a solution to the fractional differential equation problem. These new studies and uses would increase the effectiveness of the new arrangement.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Aftab Hussain expresses gratitude for the support received from King Abdulaziz University's (KAU) Deanship of Scientific project (DSR) throughout this project IFPIP: 401-130-1443. The referees and editor are thanked by the author for their insightful criticism and recommendations, which significantly raised the caliber of this research.

Conflicts of Interest: The authors declare no conflict of interest.

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