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On the Asymptotic Behavior of Class of Third-Order Neutral Differential Equations with Symmetrical and Advanced Argument

Munirah Aldiaiji [1], Belgees Qaraad [1,*], Loredana Florentina Iambor [2,*] and Elmetwally M. Elabbasy [1]

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; dr.moneerah.aldaiji@gmail.com (M.A.); emelabbasy@mans.edu.eg (E.M.E.)
[2] Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania
* Correspondence: belgeesmath2016@gmail.com (B.Q.); iambor.loredana@gmail.com (L.F.I.)

Abstract: In this paper, we aimed to study some asymptotic properties of a class of third-order neutral differential equations with advanced argument in canonical form. We provide new and simplified oscillation criteria that improve and complement a number of existing results. We also show some examples to illustrate the importance of our results.

Keywords: third-order differential equations; asymptotic behavior; advanced argument; neutral differential equations

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1. Introduction

The study of functional differential equations (FDEs) and their symmetric properties is one of the most important studies, as it has been, and still is, the focus of attention of researchers for its effective role in the understanding and interpreting of real-world phenomena. As it is not easy to find solutions in their closed forms, studying the properties of solutions is one of the best ways to understand and analyze the phenomena; see [1–3].

In particular, delay differential equations (DDEs) are defined as giving the derivative of an unknown function at a certain time in terms of the values of the function at previous times, and are referred to in the literature as delay systems or systems with delay arguments. The mathematical modeling that includes DEs with a delay has been extensively studied in different fields of life sciences, such as, for example, immunology, population dynamics, epidemiology, neural networks, and physiology. For more details, see [4–8] and the references therein. This delay may be related to a certain period of hidden processes, as can be seen in the time interval between the infection of cells and the production of new viruses, the duration of the immunity period, and the duration of the infection period in the stages of the life cycle.

Recently, advanced differential equations (ADEs) have been used to model some phenomena whose development depends not only on the present, but also on the future. Whereas delays in DDEs are retrospective, in relation to the past, developments in ADEs are prospective in the future (i.e., taking into account the effect on any possible future actions that are currently available). For instance, it is believed that economic problems, population dynamics, or mechanical control engineering are among the phenomena in which such a phenomenon may occur (see [9,10] for details).

The importance of oscillation theory has evolved into a widely used numerical mathematical method in many disciplines and fields of technology. Finding better conditions to ensure the oscillation of the solutions of any DE is one of the most important and prominent goals of this theory, as attested to by its many studies that have appeared over the past decades. See the following references: [11–15].

The study of second-order ADEs has received relatively more attention compared to higher-order ADEs. For example, the following linear advanced differential equation of the second order

$$\varpi''(\top)+\Theta(\top)\varpi(\Omega(\top))=0$$

has been discussed in [16,17].

Furthermore, Dzurina [18] studied the oscillatory behavior of the ADE

$$(r(\top)\varpi'(\top))'+\Theta(\top)\varpi(\Omega(\top))=0$$

in canonical form and presented some new oscillation criteria.

We also found a number of similar studies and results that we refer the reader to [19–22].

There is a very limited amount of literature that studies the oscillatory behavior of third-order ADEs. Yao et al. in [23], discussed some results of the oscillation of the equation

$$\left(r_2(\top)\left(\left(r_1(\top)(\varpi'(\top))^{\ell}\right)'\right)^{\beta}\right)'+\Theta(\top)\varpi(\Omega(\top))=0,\ \ \top\geq\top_0>0, \tag{1}$$

They offered some conditions that ensure that the solutions of Equation (1) are either oscillatory or converge to zero, where

$$\int^{\top} r_1^{-1/\ell}(s)\mathrm{d}s<\infty \text{ and } \int^{\top} r_2^{-1/\ell}(s)\mathrm{d}s<\infty.$$

Furthermore, Dzurina and Baculikova [24] presented some results, in canonical form, that complement the previous oscillation results for equation

$$\left(r(\top)(\varpi'(\top))^{\ell}\right)''+\Theta(\top)\varpi(\Omega(\top))=0,\ \ \top\geq\top_0>0$$

In this work, we study some properties of third-order nonlinear DEs with an advanced argument of the form

$$\left(r(\top)(\varpi''(\top))^{\ell}\right)'+\Theta(\top)f(\varkappa(\Omega(\top)))=0, \tag{2}$$

where ℓ is a quotient of odd positive integers. Furthermore, we applied our results to the following ADEs:

$$\left(r(\top)(\varpi''(\top))^{\ell-1}\varpi''(\top)\right)'+\Theta(\top)[\varkappa(\Omega(\top))]^{\ell-1}\varkappa(\Omega(\top))=0, \text{ where } \ell>0, \tag{3}$$

and

$$\varpi(\top)=\varkappa(\top)+p(\top)\varkappa(\varsigma(\top)).$$

We assumed the following conditions:

(H_1) Θ, Ω, ς, r, $p\in C([\top_0,\infty),(0,\infty))$, $\Omega(\top)\geq\top$, $\varsigma(\top)\geq\top$, $\varsigma'(\top)\geq\varsigma_0>0$, $\Omega'(\top)\geq\Omega_0>0$, $p(\top)\leq p_0<\infty$, $\lim_{\top\to\infty}\varsigma(\top)=\infty$, Θ does not vanish identically and

$$R(\top)=\int_{\top_0}^{\top}\frac{1}{r^{1/\ell}(s)}\mathrm{d}s=\infty; \tag{4}$$

(H_2) $f\in(\mathbb{R},\mathbb{R})$ such that $\varkappa f(\varkappa)>0$, $f(\varkappa)/\kappa>\varkappa^{\ell}$ $\forall$ $\varkappa\neq0$, where $\kappa>0$.

Definition 1 ([11]). *A solution of (2) means $\varkappa \in C^2([\top_\varkappa,\infty),[0,\infty))$, $\top_\varkappa > \top_0$, which satisfies the property $r(\varkappa'')^\ell \in C^1([\top_\varkappa,\infty),[0,\infty))$ and (2) on $[\top_\varkappa,\infty)$. We consider the solutions of (2) existing on some half-line $[\top_\varkappa,\infty)$ and satisfy*

$$\sup\{|\varkappa(\top)| : T_* \leq \top < \infty\} > 0 \text{ for any } T_* \geq \top_\varkappa.$$

Such a solution is called oscillatory if it has arbitrarily large zeros on $[\top_\varkappa,\infty)$; otherwise, it is said to be nonoscillatory. Equation (2) itself is said to be oscillatory if all its solutions are oscillatory.

In this paper, we aimed to study some asymptotic properties of a class of third-order neutral DEs with advanced argument in canonical form. First, we classified the derivatives of the nonoscillatory (positive) solutions of the Equation (2) and presented some new monotonic properties. Next, by means of these properties, we were able to obtain relationships between the solution and the corresponding function of Equation (2). We used these new relationships to exclude positive increasing solutions. The results of this paper are an improvement of, and complement to, a number of existing results. We also show some examples to illustrate the importance of our results.

The paper is organized as follows. In Section 1, we present the importance of oscillation theory in many disciplines and fields of technology, which is the starting point of this paper. After that, in Section 2, we give some conditions and auxiliary results that are used in Section 3, where there are some new oscillation results of the studied equation in canonical form. We also present some examples and their discussions to illustrate the significance of our findings in Section 4. We end with Section 5, addressing the conclusions and future works, and posing an interesting open question.

2. Auxiliary Lemmas

We show here some auxiliary results that are used in the theorems that follow. For ease, we use the following notation:

$$\begin{aligned} L_1\varpi(\top) &= \frac{1}{\Omega_0} r\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)^\ell + \frac{p_0^\ell}{\Omega_0 \varsigma_0} r\left(\Omega^{-1}(\varsigma(\top))\right)\left(\varpi''\left(\Omega^{-1}(\varsigma(\top))\right)\right)^\ell, \\ L_2\varpi(\top) &= r(\top)(\varpi''(\top))^\ell + \frac{p_0^\ell}{\varsigma_0} r(\varsigma(\top))(\varpi''((\varsigma(\top))))^\ell \end{aligned}$$

and

$$\widetilde{\Theta}_1(\top) = \min\left\{\Theta\left(\Omega^{-1}(\top)\right), \Theta\left(\Omega^{-1}(\varsigma(\top))\right)\right\},\ \widetilde{\Theta}_2(\top) = \min\{\Theta(\top), \Theta(\varsigma(\top))\}.$$

Lemma 1. *Let $\varkappa > 0$ be a solution of (2). Then, $\varpi(\top)\left(r(\top)(\varpi''(\top))^\ell\right)' < 0$, $\varpi(\top)\varpi''(\top) > 0$ and either*

$$\varpi(\top)\varpi'(\top) < 0, \tag{5}$$

or

$$\varpi(\top)\varpi'(\top) > 0. \tag{6}$$

Proof. Let $\varkappa > 0$ be a solution of (2), for $\top \geq \top_0$. By (2), we see that

$$\left(r(\top)(\varpi''(\top))^\ell\right)' < 0.$$

This means that $r(\varpi'')^\ell$ is decreasing and has a fixed sign. If $r(\top)(\varpi''(\top))^\ell < 0$, then is $\varpi(\top)$ is decreasing and negative. This contradiction is

$$r(\top)(\varpi''(\top))^\ell > 0, \text{ eventually.}$$

Thus, $\varpi(\top)$ has a fixed sign for all $\top$ that are large enough. Consequently, one of two cases, Case (5) or Case (6), is satisfied. The proof is complete. □

Definition 2. *When we say that property N holds, we mean that all nonoscillatory solutions to (2) satisfy only the Case (5).*

Lemma 2. *Assume that $\varkappa > 0$ is a solution of (2). Then*

$$L_1\varpi(\top) \leq -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top)\varpi^{\ell}(\top). \tag{7}$$

Moreover, assume that $\varsigma \circ \Omega = \Omega \circ \varsigma$. Then

$$L_2\varpi(\top) \leq -\frac{\kappa}{\mu}\widetilde{\Theta}_2(\top)\varpi^{\ell}(\Omega(\top)). \tag{8}$$

Proof. Let $\varkappa > 0$ be a solution of (2). By [25], we note that

$$\frac{(\varkappa(\top) + p_0\varkappa(\varsigma(\top)))^{\ell}}{\varkappa(\top)^{\ell} + p_0^{\ell}\varkappa(\varsigma(\top))^{\ell}} \leq \mu := \begin{cases} 2^{\ell-1} & \text{if } \ell > 1 \\ 1 & \text{if } \ell \leq 1 \end{cases}.$$

That is

$$\frac{1}{\mu}\varpi^{\ell}(\top) \leq \varkappa(\top)^{\ell} + p_0^{\ell}\varkappa(\varsigma(\top))^{\ell} \tag{9}$$

and

$$\frac{1}{\mu}\varpi^{\ell}(\Omega(\top)) \leq \varkappa(\Omega(\top))^{\ell} + p_0^{\ell}\varkappa(\varsigma(\Omega(\top)))^{\ell}. \tag{10}$$

Using (H2) in (2), we have

$$\left(r(\top)(\varpi''(\top))^{\ell}\right)' + \kappa\Theta(\top)\varkappa^{\ell}(\Omega(\top)) \leq 0. \tag{11}$$

Furthermore,

$$\frac{1}{\Omega_0}\left(r\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)^{\ell}\right)' + \kappa\Theta\left(\Omega^{-1}(\top)\right)\varkappa^{\ell}(\top) \leq 0.$$

Since $\Omega'(\top) \geq \Omega_0 > 0$ and $\varsigma'(\top) \geq \varsigma_0 > 0$, it follows that

$$\frac{p_0^{\ell}}{\Omega_0\varsigma_0}\left(r\left(\Omega^{-1}(\varsigma(\top))\right)\left(\varpi''\left(\Omega^{-1}(\varsigma(\top))\right)\right)^{\ell}\right)' + \kappa\Theta\left(\Omega^{-1}(\varsigma(\top))\right)\varkappa^{\ell}(\varsigma(\top)) \leq 0. \tag{12}$$

Combining (11) and (12), and using (9), we obtain

$$\left(\tfrac{1}{\Omega_0}r(\Omega^{-1}(\top))(\varpi''(\Omega^{-1}(\top)))^{\ell} + \tfrac{p_0^{\ell}}{\Omega_0\varsigma_0}r(\Omega^{-1}(\varsigma(\top)))(\varpi''(\Omega^{-1}(\varsigma(\top))))^{\ell}\right)' + \frac{\kappa}{\mu}\widetilde{\Theta}_1(\top)\varpi^{\ell}(\top) \leq 0. \tag{13}$$

On the other hand, since $\varsigma'(\top) \geq \varsigma_0 > 0$ and $\varsigma \circ \Omega = \Omega \circ \varsigma$, we obtain

$$\frac{p_0^{\ell}}{\varsigma_0}\left(r(\varsigma(\top))(\varpi''(\varsigma(\top)))^{\ell}\right)' + \kappa\Theta(\varsigma(\top))\varkappa^{\ell}(\varsigma(\Omega(\top))) \leq 0. \tag{14}$$

Now, combining (11) and (14), then, using (10), we obtain

$$\left(r(\top)(\varpi''(\top))^{\ell} + \frac{p_0^{\ell}}{\varsigma_0}r(\varsigma(\top))(\varpi''((\varsigma(\top))))^{\ell}\right)' + \frac{\kappa}{\mu}\widetilde{\Theta}_2(\top)\varpi^{\ell}(\Omega(\top)) \leq 0.$$

This ends the proof. □

Lemma 3. *Assume that* $\varpi(\top) > 0$*, and* $\varpi'(\top)$ *are positive and increasing. Then*

$$\top\varpi(\Omega(\top)) - K_0\Omega(\top)\varpi(\top) \geq 0,\ k_0 \in (0,1),\ \textit{eventually}. \tag{15}$$

Proof. Since $\varpi'(\top)$ is positive and increasing, we have

$$\varpi(\Omega(\top)) - \varpi(\top)\int_{\top}^{\Omega(\top)} \varpi'(s)\mathrm{d}s \geq \varpi'(\top)(\Omega(\top) - \top)$$

or

$$\frac{\varpi(\Omega(\top))}{\varpi(\top)} \geq \frac{\varpi'(\top)}{\varpi(\top)}(\Omega(\top) - \top) + 1. \tag{16}$$

Using the fact $\lim_{\top\to\infty}\varpi(\top) = \infty$, $\exists$ a $\top_1$ is large enough, such that

$$\begin{aligned} k_0\varpi(\top) &\leq \varpi(\top) - \varpi(\top_1) = \int_{\top_1}^{\top} \varpi'(s)\mathrm{d}s \\ &\leq \varpi'(\top)(\top - \top_1) \leq \varpi'(\top)\top,\ \text{for any } k_0 \in (0,1). \end{aligned}$$

That is

$$\top\varpi'(\top) \geq k_0\varpi(\top). \tag{17}$$

By using (17) in (16), we obtain

$$\begin{aligned} \frac{\varpi(\Omega(\top))}{\varpi(\top)} &\geq \frac{k_0(\Omega(\top) - \top)}{\top} + 1 \\ &\geq \frac{k_0\Omega(\top)}{\top}. \end{aligned}$$

Thus, the proof is complete. □

3. Main Results

In this section, we present some new criteria that ensure that property N holds.

Theorem 1. *Assume that* $\varsigma(\top) \geq \Omega(\top)$*. If*

$$\liminf_{\top\to\infty} \frac{1}{Q(\top)}\int_{\top}^{\infty} R\left(\Omega^{-1}(s)\right)Q^{1+1/\ell}(s)\mathrm{d}s > \frac{1}{(\ell+1)^{1+1/\ell}}, \tag{18}$$

where

$$Q(\top) = \frac{\kappa}{\mu}\int_{\top}^{\infty} \widetilde{\Theta}_1(s)\mathrm{d}s,$$

then property N holds.

Proof. *Let* $\varkappa > 0$ be a solution of (2), that is $\varpi > 0$ and satisfying Case (II). By (7), we have

$$(L_1\varpi(\top))' \leq -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top)\varpi^{\ell}(\top).$$

Define

$$w(\top) = \frac{L_1\varpi(\top)}{\varpi^{\ell}(\top)} > 0. \tag{19}$$

Since $\left(\Omega^{-1}(\top)\right)' > 0$, that is

$$w(\top) < r\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)^{\ell}\left(\frac{1}{\Omega_0}+\frac{p_0^{\ell}}{\Omega_0\varsigma_0}\right)\frac{1}{\varpi^{\ell}(\top)}$$

and

$$\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^{\ell}}w(\top) < \frac{r\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)^{\ell}}{\varpi^{\ell}(\top)}. \tag{20}$$

That is

$$w'(\top) = \frac{(L_1\varpi(\top))'}{\varpi^{\ell}(\top)} - \ell\frac{L_1\varpi(\top)}{\varpi^{\ell}(\top)}\frac{\varpi'(\top)}{\varpi(\top)}.$$

By (7), and taking into account that $\varpi'\left(\Omega^{-1}(\top)\right) \le \varpi'(\top)$, implies

$$\begin{aligned} w'(\top) &\le \frac{(L_1\varpi(\top))'}{\varpi^{\ell}(\top)} - \ell\frac{L_1\varpi(\top)}{\varpi^{\ell}(\top)}\frac{\varpi'\left(\Omega^{-1}(\top)\right)}{\varpi(\top)} \\ &\le -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top) - \ell w(\top)\frac{\varpi'\left(\Omega^{-1}(\top)\right)}{\varpi(\top)}. \end{aligned} \tag{21}$$

By using $\left(r(\top)(\varpi''(\top))^{\ell}\right)' \le 0$, we have

$$\begin{aligned} \varpi'(\top) &\ge \int_{\top_1}^{\top}\left(r(s)(\varpi''(s))^{\ell}\right)^{1/\ell}\frac{1}{r^{1/\ell}(s)}\mathrm{d}s \ge \left(r(\top)(\varpi''(\top))^{\ell}\right)^{1/\ell}\int_{\top_1}^{\top} r^{-1/\ell}(s)\mathrm{d}s \\ &\ge \left(r(\top)(\varpi''(\top))^{\ell}\right)^{1/\ell}R(\top). \end{aligned} \tag{22}$$

Hence, we obtain

$$\varpi'\left(\Omega^{-1}(\top)\right) \ge \left(r\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)^{\ell}\right)^{1/\ell}R\left(\Omega^{-1}(\top)\right). \tag{23}$$

From (21) and (23), we obtain

$$w'(\top) \le -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top) - \ell w(\top)\frac{r^{1/\ell}\left(\Omega^{-1}(\top)\right)\left(\varpi''\left(\Omega^{-1}(\top)\right)\right)}{\varpi(\top)}R\left(\Omega^{-1}(\top)\right).$$

By (20), we find that

$$w'(\top) \le -\left[\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^{\ell}}\right)^{1/\ell} w^{1+1/\ell}(\top)R\left(\Omega^{-1}(s)\right)\right].$$

Integrating from $\top$ to ∞, we obtain

$$w(\top) \ge Q(\top) + \int_{\top}^{\infty}\ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^{\ell}}\right)^{1/\ell} w^{1+1/\ell}(s)R\left(\Omega^{-1}(s)\right)\mathrm{d}s \tag{24}$$

or

$$\frac{w(\top)}{Q(\top)} \ge 1 + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^{\ell}}\right)^{1/\ell}\frac{1}{Q(\top)}\int_{\top}^{\infty} R\left(\Omega^{-1}(s)\right)Q^{1+1/\ell}(s)\left(\frac{w(s)}{Q(s)}\right)^{1+1/\ell}\mathrm{d}s.$$

According to the fact $w(\top) > Q(\top)$, we note that

$$\inf_{\top\ge\top_1} w(\top)/Q(\top) = \lambda \ge 1.$$

This implies

$$\frac{w(\top)}{Q(\top)} \geq 1 + \ell \left(\frac{\Omega_0 \varsigma_0}{\varsigma_0 + p_0^{\ell}} \right)^{1/\ell} \frac{\lambda^{1+1/\ell}}{Q(\top)} \int_{\top}^{\infty} R\left(\Omega^{-1}(s)\right) Q^{1+1/\ell}(s) \mathrm{d}s. \tag{25}$$

From (18), there exists $\eta > 0$, such that

$$\frac{1}{Q(\top)} \int_{\top}^{\infty} R\left(\Omega^{-1}(s)\right) Q^{1+1/\ell}(s) \mathrm{d}s > \eta > \frac{1}{(j+1)^{1+1/j}}, \tag{26}$$

where $j = \ell(\Omega_0 \varsigma_0)^{1/\ell} / \left(\varsigma_0 + p_0^{\ell}\right)^{1/\ell}$. Using (25) in (26), we have

$$\frac{w(\top)}{Q(\top)} \geq 1 + j\left(\lambda^{1+1/j}\right)\eta,$$

and then

$$\lambda \geq 1 + j\left(\lambda^{1+1/\ell}\right)\eta > 1 + \frac{j\left(\lambda^{1+1/\ell}\right)}{(\ell+1)^{1+1/\ell}}$$

or

$$\frac{1}{j+1} + \frac{1}{j+1} \frac{j\left(\lambda^{1+1/\ell}\right)}{(\ell+1)^{1+1/\ell}} - \frac{1}{j+1}\lambda < 0.$$

Set

$$g\left(\widehat{\lambda}\right) = \frac{1}{j+1} + \frac{J}{j+1}\widehat{\lambda}^{1+1/\ell} - \widehat{\lambda}.$$

This contradicts the fact that the function $g\left(\widehat{\lambda}\right)$ is positive for all $\widehat{\lambda} > 0$. This completes the proof. □

Corollary 1. *If*

$$\int_{\top_0}^{\infty} \widetilde{\Theta}_1(\top) \mathrm{d}s = \infty, \tag{27}$$

then property (N) holds.

Proof. The proof being the same as the proof of Theorem 1, we are led to the inequality (24). Thus, there is a contradiction with (27). The proof is complete. □

Corollary 2. *If*

$$\int_{\top_0}^{\infty} R\left(\Omega^{-1}(s)\right) Q(s)^{1+1/\ell} \mathrm{d}s = \infty, \tag{28}$$

then property N holds.

Proof. Using $w(\top) > Q(\top)$ and (24), we have

$$w(\top_1) \geq Q(\top_1) + \int_{\top_1}^{\infty} \ell \left(\frac{\Omega_0 \varsigma_0}{\varsigma_0 + p_0^{\ell}} \right)^{1/\ell} Q^{1+1/\ell}(s) R\left(\Omega^{-1}(s)\right) \mathrm{d}s.$$

This contradicts (28). □

Theorem 2. *Assume that (2) has property N. If*

$$\int_{\top_0}^{\infty} \int_{v}^{\infty} r^{-1/\ell}(u) \left(\int_{u}^{\infty} \widetilde{\Theta}_2(s) \mathrm{d}s \right)^{1/\ell} \mathrm{d}u \mathrm{d}v = \infty, \tag{29}$$

then every nonoscillatory solution $\varpi(\top)$ of (2) converges to zero as $\top \to \infty$.

Proof. Since property N holds, then $\varpi(\top)$ satisfies Case (I) and implies that

$$\lim_{\top\to\infty}\varpi(\top) = \varrho \geq 0. \tag{30}$$

We claim that $\varrho = 0$, if not, then $\varrho > 0$.

Let $\varrho > 0$. Integrating (8) from $\top$ to ∞ and using $\varpi(\Omega(\top)) > \varrho$, we obtain

$$L_2\varpi(\top) \geq \frac{\kappa}{\mu}\varrho^{\ell}\int_{\top}^{\infty}\widetilde{\Theta}_2(s)\mathrm{d}s. \tag{31}$$

Using $\varsigma(\top) > \top$, we have

$$\begin{aligned} L_2\varpi(\top) &= r(\top)(\varpi''(\top))^{\ell} + \frac{p_0^{\ell}}{\varsigma_0}r(\varsigma(\top))(\varpi''((\varsigma(\top))))^{\ell} \\ &\leq r(\top)(\varpi''(\top))^{\ell}\left(\frac{\varsigma_0 + p_0^{\ell}}{\varsigma_0}\right). \end{aligned}$$

Thus, (31) becomes

$$\varpi''(\top) \geq \left(\frac{\kappa\varsigma_0}{\mu\left(\varsigma_0 + p_0^{\ell}\right)}\right)^{\frac{1}{\ell}}\frac{\varrho}{r^{\frac{1}{\ell}}(\top)}\left(\int_{\top}^{\infty}\widetilde{\Theta}_2(s)\mathrm{d}s\right)^{\frac{1}{\ell}}. \tag{32}$$

Integrating (32) from $\top$ to ∞, we find

$$-\varpi'(\top) \geq \varrho\left(\frac{\kappa\varsigma_0}{\mu\left(\varsigma_0 + p_0^{\ell}\right)}\right)^{\frac{1}{\ell}}\int_{\top}^{\infty}\frac{1}{r^{1/\ell}(u)}\left(\int_{u}^{\infty}\widetilde{\Theta}_2(s)\mathrm{d}s\right)^{1/\ell}\mathrm{d}u.$$

Integrating again from $\top_1$ to ∞, we see that

$$\varpi(\top_1) \geq \varrho\left(\frac{\kappa\varsigma_0}{\mu\left(\Omega_0 + p_0^{\ell}\right)}\right)^{\frac{1}{\ell}}\int_{\top_1}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\ell}(u)}\left(\int_{u}^{\infty}\widetilde{\Theta}_2(s)\mathrm{d}s\right)^{1/\ell}\mathrm{d}u\mathrm{d}v.$$

There is a contradiction with (29). That is $\lim_{\top\to\infty}\varpi(\top) = 0$. $\square$

Now, for the next result, we define a sequence $\{A_n(\top)\}_{n=0}^{\infty}$ as follows:

$$A_0(\top) = Q(\top)$$

and

$$A_{n+1}(\top) = A_0(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^{\ell}}\right)^{1/\ell}\int_{\top}^{\infty}A_n^{1+1/\ell}(s)R(s)\mathrm{d}s, \quad n = 0, 1, \ldots. \tag{33}$$

Theorem 3. *Assume that there $\exists$ some $A_n(\top)$ such that*

$$\int_{\top_0}^{\infty}\widetilde{\Theta}_1(\top)\left(e^{\ell\left(\frac{\Omega_0\varsigma_0}{\Omega_0\varsigma_0 + p_0^{\ell}}\right)^{1/\ell}\int_{\top_0}^{\top}A_n^{1/\ell}(s)R(s)\mathrm{d}s}\right)\mathrm{d}\top = \infty, \tag{34}$$

for some $K \in (0,1)$. Then property N holds.

Proof. Let $\varkappa$ be a solution of (2) and $\varpi(\top) > 0$ satisfying Case (II). As in the proof of Theorem 1, we see that (24) holds. Using (24) and $A_0(\top)$, it is easy to note that $w(\top) \geq A_0(\top)$. Thus,

$$\begin{aligned} {}_1(\top) &= A_0(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell} \int_\top^\infty A_0^{1+1/\ell}(s)R(s)\mathrm{d}s \\ &\leq A_0(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell} \int_\top^\infty w^{1+1/\ell}(s)R(s)\mathrm{d}s \\ &\leq w(\top). \end{aligned}$$

By induction, we find that sequence $\{A_n(\top)\}_{n=0}^\infty$ is nondecreasing and $w(\top) \geq A_n(\top)$. So the sequence $\{A_n(\top)\}_{n=0}^\infty$ converges to $A(\top)$. Let $n \to \infty$, then, by means of the Lebesgue monotone theorem, (33) becomes

$$A(\top) = A_0(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell} \int_\top^\infty A^{1+1/\ell}(s)R(s)\mathrm{d}s.$$

Taking into account $A(\top) \geq A_n(\top)$, we obtain

$$\begin{aligned} A'(\top) &= -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top) - \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell} A^{1+1/\ell}(\top)R(\top) \\ &\leq -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top) - \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell} A(\top)A_n^{1/\ell}(\top)R(\top), \text{ for } \top \geq \top_1. \end{aligned}$$

That is,

$$\left[A(\top)\left(e^{\ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^\ell}\right)^{1/\ell}\int_{\top_1}^{\top} A_n^{1/\ell}(s)R(s)\mathrm{d}s}\right)\right]' \leq -\frac{\kappa}{\mu}\widetilde{\Theta}_1(\top)\left(e^{\ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^\ell}\right)^{1/\ell}\int_{\top_1}^{\top} A_n^{1/\ell}(s)R(s)\mathrm{d}s}\right).$$

Thus, we have

$$\begin{aligned} 0 &\leq A(\top)\left(e^{\ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^\ell}\right)^{1/\ell}\int_{\top_1}^{\top} A_n^{1/\ell}(s)R(s)\mathrm{d}s}\right) \\ &\leq A(\top_1) - \frac{\kappa}{\mu}\int_{\top_1}^{\top}\widetilde{\Theta}_1(u)\left(e^{\ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^\ell}\right)^{1/\ell}\int_{\top_1}^{\top} A_n^{1/\ell}(s)R(s)\mathrm{d}s}\right)\mathrm{d}u. \end{aligned}$$

This is a contradiction with (34). The proof is complete. □

Theorem 4. *Assume that there $\exists$ some $A_n(\top)$ such that*

$$\limsup_{\top\to\infty}\left[\int_{\top_1}^{\top}(R(s) - R(\top_1))\mathrm{d}s\right]^\ell A_n(\top) > 1. \tag{35}$$

Then property N holds.

Proof. Assume that $\varkappa(\top)$ is a solution of (2) and $\varpi(\top) > 0$ satisfying Case (6). Since $\top < \Omega(\top)$, by (22), we have

$$\varpi(\top) \geq r^{\frac{1}{\ell}}(\top)\varpi''(\top)\int_{\top_1}^{\top}\int_{\top_1}^{u} r^{-1/\ell}(s)\mathrm{d}s\mathrm{d}u. \tag{36}$$

On the other hand, combining (20) together with (36), we obtain

$$\frac{1}{w(\top)} = \frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\frac{1}{r(\top)}\left(\frac{\varpi(\top)}{\varpi''(\top)}\right)^\ell \geq \frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\left[\int_{\top_1}^{\top} R(s) - R(\top_1)\mathrm{d}s\right]^\ell.$$

Therefore,

$$\left[\int_{\top_1}^{\top}(R(s) - R(\top_1))\mathrm{d}s\right]^\ell A_n(\top) \leq \left[\int_{\top_1}^{\top}(R(s) - R(\top_1))\mathrm{d}s\right]^\ell w(\top) \leq 1.$$

This contradicts (35). □

The following corollaries are immediate by putting $n = 0$ and $n = 1$ in Theorem 4.

Corollary 3. *Assume that*

$$\limsup_{\top\to\infty}\left[\int_{\top 1}^{\top} R(s) - R(\top_1)\mathrm{d}s\right]^\ell \int_{\top}^{\infty}\widetilde{\Theta}_1(s)\mathrm{d}s > 1. \tag{37}$$

Then property N holds.

Corollary 4. *Assume that*

$$\limsup_{\top\to\infty}\left[\int_{\top_1}^{\top} R(s) - R(\top_1)\mathrm{d}s\right]^\ell \left[Q(\top) + \ell\left(\frac{\Omega_0\varsigma_0}{\varsigma_0 + p_0^\ell}\right)^{1/\ell}\int_{\top}^{\infty} Q^{1+1/\ell}(s)R(s)\right]\mathrm{d}s > 1.$$

Then property N holds.

4. Application

Example 1. *Consider the following third-order differential equations*

$$\left(\top\left((\varkappa(\top) + p\varkappa(\lambda\top))''\right)^3\right)' + \beta\top^{-6}\varpi^3(\lambda\top) = 0,\ \ \beta > 0,\ \ \lambda \geq 1,\ \ \top \geq 1. \tag{38}$$

That is

$$\begin{aligned}\widetilde{\Theta}_1(\top) &= \min\left\{\Theta\left(\Omega^{-1}(\top)\right), \Theta\left(\Omega^{-1}(\varsigma(\top))\right)\right\} \\ &= \frac{\beta}{\top^6}.\end{aligned}$$

and

$$Q(\top) = \frac{\kappa}{\mu}\int_{\top}^{\infty}\frac{\beta}{s^6}\mathrm{d}s = \frac{\kappa\beta}{5\mu\top^5}.$$

By means of Theorem 1, we see that (38) has property N if

$$\beta > \left(\frac{2}{3}\right)^{1/3}\left(\frac{5}{4}\right)^4\frac{\mu^{11/3}}{\kappa}\lambda.$$

On the other hand, by means of Theorem 2, we note that (29) holds. Hence, every nonoscillatory solution $\varpi(\top)$ of (38) converges to zero as $\top \to \infty$.

Example 2. *Consider the following differential equation*

$$\left(\top\left(\varpi''(\top)\right)^3\right)' + \frac{\beta}{\top^9}\varpi^3\left(\top^2\right) = 0,\ \beta > 0,\ \top \geq 1. \tag{39}$$

By means of Theorem 1, we see that (39) has property N if

$$\beta > \left(\frac{5}{4}\right)^4 \frac{8}{27}.$$

Remark 1. *Putting $p = 0$ in Example 1, we notice that the condition for property N depends mainly on the greatness of the advanced argument. We see that $\Omega(\top)$ in Example 2 is greater than in Example 1 and this allows the function $\Theta(\top)$ to be reduced.*

Example 3. *Consider the third-order nonlinear ADE*

$$\left(\top^2\left(\varpi''(\top)\right)^3\right)' + \frac{\beta}{\top^5}\varpi^3(\lambda\top) = 0,\ \beta > 0,\ \lambda \geq 1,\ \top \geq 1. \tag{40}$$

By Corollaries 3 and 4, we see that (40) has property N if

$$\beta > \frac{4^4}{(9\lambda)^3} \tag{41}$$

or

$$\beta > \frac{4^4}{9^3\lambda^3}\left(1 - \frac{9^4}{4^{16/3}}\lambda^4\beta^{4/3}\right), \tag{42}$$

respectively.

On the other hand, when applying Theorem 2, note that (29) holds. Hence, every nonoscillatory solution $\varpi(\top)$ of (39) converges to zero as $\top \to \infty$.

Remark 2. *Condition (42), (which was obtained for $n = 1$ in (35)), is better than condition (41), (which was obtained for $n = 0$ in (35)).*

Furthermore, all previous results are also correct for the Equation (3). To discuss these results, we provide the following example:

Example 4. *Consider the third-order ADE*

$$\left(\top^a\left|\varpi''(\top)\right|^{\ell-1}\varpi''(\top)\right)' + \frac{\beta}{\top^b}\left|\varpi(\top^c)\right|^{\ell-1}\varpi(\top^c) = 0,\ \top \geq 1, \tag{43}$$

where $\ell > a > 0$, $b, \beta > 0$ and $c \geq 1$. By applying Corollary 1, Corollary 2 and Theorem 1, respectively, (43) have property N if

(**a**) $1 - \ell \geq b - \ell c$.
(**b**) $1 - \ell < b - \ell c$,
and $\frac{1}{\ell} - \ell + 1 \geq \frac{1}{\ell+2}a + \left(\frac{1}{\ell} - 1\right)b - \ell c$.
(**c**) $1 - \ell < b - \ell c$,
$\frac{1}{\ell} - \ell + 1 < \frac{1}{\ell+2}a + \left(\frac{1}{\ell} - 1\right)b - \ell c$,
$\frac{1}{\ell} + 1 = \frac{a}{\ell} + \frac{b}{\ell} - c$
and $\frac{1}{(\ell-a)(b+\ell-\ell c-1)^{1+1/\ell}} > \frac{1}{\ell\beta^{1/\ell}(\ell+1)^{1+1/\ell}}$.

5. Conclusions

We, herein, presented a study on the monotonic properties and oscillatory behavior of Equation (2). We presented a number of relationships that link the solution of the

Equation (2) and the corresponding function. These relationships are applicable in the two cases of positive nonoscillatory solutions of the studied equation. Then, we used the relationships to obtain conditions that ensured that there were no nonoscillatory solutions of type (II). Through examples, we clarified the importance of our results.

It would be interesting to study Equation (2) in a more general form, such as:

$$\left(r(\top)\left((\varkappa(\top)+p(\top)\varkappa(\varsigma(\top)))''\right)^{\ell}\right)'+\sum_{\xi=1}^{m}\Theta_{\xi}(\top)f\left(\varkappa(\Omega_{\xi}(\top))\right)=0.$$

It would also be worthwhile to discuss obtaining the oscillation criteria of Equation (2) without condition $\varsigma\circ\Omega=\Omega\circ\varsigma$.

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