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Research on the Symmetry of the Hamiltonian System under Generalized Operators

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Abstract: Generalized operators have recently been proposed with great potential applications. Here, we present research carried out on Noether figury and perturbation to Noether symmetry for Hamiltonian systems within generalized operators. There are four parts, and each part contains two kinds of generalized operator. Firstly, Hamilton equations are established. Secondly, the Noether symmetry method is used for finding the solutions to the differential equations of motion, and conserved quantities are obtained. Thirdly, perturbation to Noether symmetry and adiabatic invariants are further explored. In the end, two examples are given to illustrate the methods and results.

Keywords: generalized operator; Hamilton equation; Noether symmetry; conserved quantity; perturbation to Noether symmetry; adiabatic invariant

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1. Introduction

In 1788, Lagrange published his famous book Analytical Mechanics, in which he expressed the general equation of dynamics in the form of the Lagrange equation by introducing generalized coordinates. Then, in 1834, Hamilton developed *Analytical Mechanics*. The Hamilton principle and Hamilton canonical equation are the core of Hamiltonian mechanics. The Hamilton principle is highly universal and can be used for approximate calculation [1,2]. The Hamilton principle is also extended to holonomic nonconservative systems [3] and high-order systems [4]. As for the Hamilton canonical equation, it is not only simpler in form than the Lagrange equation, but it is also more convenient for general discussion when solving many complex mechanical problems, such as celestial mechanics and vibration theory. What is more, Hamiltonian mechanics also contributes to the formation and development of generalized Hamiltonian mechanics [5] and Birkhoffian mechanics [6]. Thanks to Hamiltonian mechanics, the rapid development of nonlinear science in the last century has been possible. Hamiltonian mechanics is still a keyword today.

Fractional calculus has been widely considered. The latest developments in science, bioengineering and applied mathematics show that the results obtained through fractional calculus are more accurate [7,8]. In order to deal with dissipative forces in nonconservative systems, Riewe [9,10] studied the fractional calculus of variational problems and established fractional Lagrangian and Hamiltonian mechanics. After that, fractional Hamiltonian mechanics was established on the basis of different fractional derivatives. For example, Song [11] studied fractional singular systems and fractional constrained Hamilton equations using mixed derivatives. Baleanu [12] established fractional Hamilton formalism within Caputo's derivatives. Rabei [13] achieved the passage from the Lagrangian, containing Riemann–Liouville fractional derivatives, to the Hamiltonian, and investigated the classical fields with fractional derivatives—he considered two discrete problems and one continuous to demonstrate the application of the formalism. Klimek [14] discussed the models described by fractional order derivatives of the Riemann–Liouville type in sequential form in Lagrangian and Hamiltonian formalism. Herzallah [15] presented fractional Euler–Lagrange equations and transversality conditions for fractional variational problems in the sense of Caputo and Riemann–Liouville fractional derivatives, and then he developed a fractional Hamiltonian formulation and some illustrative examples in detail. Muslih [16] presented a Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives. Agrawal [17] introduced a three-parameter fractional derivative, developed integration by parts formulae, and provided the corresponding fractional Hamiltonian formulations. Nawafleh [18] investigated Caputo fractional derivatives for classical field systems using fractional Hamiltonian formalism and provided two continuous examples to demonstrate the application of the formalism. Notably, in 2010, Agrawal [19] introduced three general fractional operators, which we call generalized operators. Generalized operators contain many special fractional operators, such as Riemann–Liouville fractional operators, Caputo fractional operators, Riesz–Riemann–Liouville fractional operators, Riesz–Caputo fractional operators, etc. In this paper, we detail the beginning of our research, which is to establish Hamiltonian mechanics based on these generalized operators.

After the fractional differential equations are established, the next step is to solve them. An integral is a conserved quantity; therefore, scholars are committed to finding all conserved quantities of mechanics systems. The Noether symmetry method is one of the most useful methods for finding solutions to the differential equations of motion.

Noether symmetry and conserved quantity, which are useful for revealing the inherent physical properties of the dynamic systems, were put forward by German mathematician Emmy Noether [20]. Noether symmetry, from which the conserved quantity can be directly derived, refers to the invariance of the Hamilton action under infinitesimal transformations. A series of important achievements on Noether symmetry and conserved quantity for constrained mechanics systems has already been obtained, such as classical Noether theorems [21–26], fractional Noether theorems [27–35], Noether theorems on time scales [36–38], Noether theorems with time delay [39], etc.

For general dynamic problems, we should study the invariance property of mechanics systems, and the impact of this invariance on the behavior of the mechanics systems is also increasingly being valued. Zhao [24] pointed out that symmetry is a very important and universal property of mechanics systems. There is a close relationship between the change in symmetry under the action of small disturbances and their invariants and the integrability of mechanics systems, so it is necessary to study this carefully. The adiabatic invariant belongs to this problem. The classical adiabatic invariant refers to a physical quantity that changes more slowly than the change of the system's changed parameter. Adiabatic refers to regardless the reasons of the cause of the parameter's change in mechanics system. When discussing the adiabatic invariant, the problem of a slow-changing parameter is often discussed, which can be transformed into a small perturbation problem to be studied. The existence of invariants in a mechanics system often corresponds to its symmetry. Although the adiabatic invariant refers to a quantity that is approximately constant under certain conditions, there should be some symmetry corresponding to it, and the response of the symmetry may not be changed or may be perturbed. In this paper, perturbation to Noether symmetry and the corresponding adiabatic invariant of the Hamiltonian system are to be discussed under generalized operators.

The structure of this paper is as follows. Section 2 briefly lists the definitions and properties of the generalized operators. The fractional variational problems are studied in Section 3. Noether symmetry and conserved quantity, perturbation to Noether symmetry and adiabatic invariants are investigated in Sections 4 and 5, respectively. Section 6 presents two examples to show the methods and results obtained in this paper. In Section 7, a conclusion is given.

2. Preliminaries

Generalized operators K, A, and B are introduced by Agrawal [19]. Here, we only list their definitions and integration by part formulae.

The operators K, A, and B are defined as:

$$K_M^\alpha f(t) = m\int_a^t \kappa_\alpha(t,\tau)f(\tau)\mathrm{d}\tau + \omega\int_t^b \kappa_\alpha(\tau,t)f(\tau)\mathrm{d}\tau,\ \alpha > 0 \tag{1}$$

$$A_M^\alpha f(t) = D^n K_M^{n-\alpha} f(t),\ n-1 < \alpha < n, \tag{2}$$

$$B_M^\alpha f(t) = K_M^{n-\alpha} D^n f(t),\ n-1 < \alpha < n, \tag{3}$$

where $f(t)$ is continuous and integrable, $a < t < b$, $M =< a,t,b,m,\omega >$ is a parameter set, m and ω are two real numbers, n is an integer, and $\kappa_\alpha(t,\tau)$ is a kernel that probably depend on a parameter α.

Remark 1. *Let $\kappa_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$. When the conditions are different, the results are different. For example, when $M = M_1 =< a,t,b,1,0 >$, we can obtain*:

$$A_M^\alpha f(t) = D^n K_M^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau = {}_a^{RL}D_t^\alpha f(t), \tag{4}$$

$$B_M^\alpha f(t) = K_M^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\mathrm{d}\tau = {}_a^{C}D_t^\alpha f(t), \tag{5}$$

i.e., the operator A reduces to the Riemann–Liouville fractional operator to the left, and the operator B reduces to the Caputo fractional operator to the left. When $M = M_3 = < a,t,b,1/2,1/2 >$, we can obtain:

$$A_M^\alpha f(t) = D^n K_M^{n-\alpha} f(t) = \frac{1}{2\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_b^a |t-\tau|^{n-\alpha-1} f(\tau)\mathrm{d}\tau = {}_a^{R}D_b^\alpha f(t), \tag{6}$$

$$B_M^\alpha f(t) = K_M^{n-\alpha} D^n f(t) = \frac{1}{2\Gamma(n-\alpha)}\int_a^b |t-\tau|^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n f(\tau)\mathrm{d}\tau = {}_a^{RC}D_b^\alpha f(t), \tag{7}$$

i.e., the operator A reduces to the Riesz–Riemann–Liouville fractional operator, and the operator B reduces to the Riesz–Caputo fractional operator.

The integrations by parts formulae of operators K, A and B are

$$\int_a^b g(t)K_M^\alpha f(t)\mathrm{d}t = \int_a^b f(t)K_{M^*}^\alpha g(t)\mathrm{d}t, \tag{8}$$

$$\int_a^b g(t)A_M^\alpha f(t)\mathrm{d}t = (-1)^n\int_a^b f(t)B_{M^*}^\alpha g(t)\mathrm{d}t + \sum_{j=0}^{n-1}(-D)^{n-1-j}g(t)A_M^{\alpha+j-n}f(t)\Bigg|_{t=a}^{t=b}, \tag{9}$$

$$\int_a^b g(t)B_M^\alpha f(t) = (-1)^n\int_a^b f(t)A_{M^*}^\alpha g(t)\mathrm{d}t + \sum_{j=0}^{n-1}(-1)^j A_{M^*}^{\alpha+j-n}g(t)D^{n-1-j}f(t)\Bigg|_{t=a}^{t=b}, \tag{10}$$

where $M^* =< a,t,b,\omega,m >,\ n-1 < \alpha < n$, and n is an integer.

It is noted that in the following text we set $n = 1$, so $0 < \alpha < 1$. The first thing we intend to study is the variational problem.

3. Hamilton Equations within Generalized Operators

3.1. Hamilton Equation within Generalized Operator A

Let $L_A = L_A\left(t, \mathbf{q_A}, \dot{\mathbf{q}}_\mathbf{A}, \mathbf{A_M^\alpha q_A}\right)$ be the Lagrangian within generalized operator A, $\mathbf{q_A} = (q_{A1}, q_{A2}, \cdots, q_{An})$, $\dot{\mathbf{q}}_\mathbf{A} = (\dot{q}_{A1}, \dot{q}_{A2}, \cdots, \dot{q}_{An})$ and $\mathbf{A_M^\alpha q_A} = (A_M^\alpha q_{A1}, A_M^\alpha q_{A2}, \cdots, A_M^\alpha q_{An})$, then the elements of the generalized moments $\mathbf{p_A} = (p_{A1}, p_{A2}, \cdots, p_{An})$ and $\mathbf{p_A^\alpha} = \left(p_{A1}^\alpha, p_{A2}^\alpha, \cdots, p_{An}^\alpha\right)$ are defined as $p_{Ai} = \partial L_A / \partial \dot{q}_{Ai}$ and $p_{Ai}^\alpha = \partial L_A / \partial A_M^\alpha q_{Ai}$, and the Hamiltonian $H_A = H_A\left(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p_A^\alpha}\right)$ can be expressed as $H_A = p_{Ai}\dot{q}_{Ai} + p_{Ai}^\alpha A_M^\alpha q_{Ai} - L_A$, $i = 1, 2, \cdots n$.

Hamilton action within generalized operator A has the form

$$S_A = \int_a^b \left[p_{Ai} \cdot \dot{q}_{Ai} + p_{Ai}^\alpha \cdot A_M^\alpha q_{Ai} - H_A(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p_A^\alpha})\right] \mathrm{d}t. \tag{11}$$

Then,

$$\delta \mathrm{S}_A = 0, \tag{12}$$

where δ refers to the isochronous variation, with the commutative conditions

$$\delta A_M^\alpha q_{Ai} = A_M^\alpha \delta q_{Ai}, i = 1, 2, \cdots, n, \tag{13}$$

and the boundary conditions

$$\mathbf{q_A}(a) = \mathbf{q_{Aa}},\ \mathbf{q_A}(b) = \mathbf{q_{Ab}}, \tag{14}$$

where $\mathbf{q_{Aa}} = (q_{Aa1}, q_{Aa2}, \cdots, q_{Aan})$, $\mathbf{q_{Ab}} = (q_{Ab1}, q_{Ab2}, \cdots, q_{Abn})$, is called the Hamilton principle within generalized operator A.

Using Equations (9), (13) and (14), we derive from Equation (12) that

$$\begin{aligned}
\delta S_A &= \int_a^b \Big[\delta p_{Ai} \cdot \dot{q}_{Ai} + p_{Ai} \cdot \delta \dot{q}_{Ai} + \delta p_{Ai}^\alpha \cdot A_M^\alpha q_{Ai} + p_{Ai}^\alpha \cdot \delta A_M^\alpha q_{Ai} \\
&\quad - \tfrac{\partial H_A}{\partial q_{Ai}} \cdot \delta q_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}} \cdot \delta p_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}^\alpha} \cdot \delta p_{Ai}^\alpha\Big] \mathrm{d}t \\
&= \int_a^b \Big[-\Big(B_{M^*}^\alpha p_{Ai}^\alpha + \dot{p}_{Ai} + \tfrac{\partial H_A}{\partial q_{Ai}} - m p_{Ai}^\alpha(b)\kappa_{1-\alpha}(b, t) + \omega p_{Ai}^\alpha(a)\kappa_{1-\alpha}(t, a)\Big)\delta q_{Ai} \\
&\quad + \Big(\dot{q}_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}}\Big)\delta p_{Ai} + \Big(A_M^\alpha q_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}^\alpha}\Big) \cdot \delta p_{Ai}^\alpha\Big] \mathrm{d}t = 0.
\end{aligned} \tag{15}$$

From the Hamiltonian $H_A = p_{Ai} \cdot \dot{q}_{Ai} + p_{Ai}^\alpha A_M^\alpha q_{Ai} - L_A$, the independence of δq_{Ai} and the arbitrariness of the interval $[a, b]$, we obtain:

$$\begin{gathered}
A_M^\alpha q_{Ai} = \tfrac{\partial H_A}{\partial p_{Ai}^\alpha},\ \dot{q}_{Ai} = \tfrac{\partial H_A}{\partial p_{Ai}}, \\
B_{M^*}^\alpha p_{Ai}^\alpha = -\dot{p}_{Ai} - \tfrac{\partial H_A}{\partial q_{Ai}} + m p_{Ai}^\alpha(b)\kappa_{1-\alpha}(b, t) - \omega p_{Ai}^\alpha(a)\kappa_{1-\alpha}(t, a).
\end{gathered} \tag{16}$$

Equation (16) is called the Hamilton equation within generalized operator A.

Remark 2. *Let $\kappa_\alpha(t, \tau) = (t - \tau)^{\alpha - 1}/\Gamma(\alpha)$, when $M = M_1$, $M = M_2$ and $M = M_3$. From Equation (16), we can obtain the Hamilton equations within the left Riemann–Liouville fractional operator, the right Riemann–Liouville fractional operator, and the Riesz–Riemann–Liouville fractional operator, respectively. These results are consistent with the ones in Ref. [29].*

3.2. Hamilton Equation within Generalized Operator B

Let $L_B = L_B\left(t, \mathbf{q_B}, \dot{\mathbf{q}}_\mathbf{B}, \mathbf{B_M^\alpha q_B}\right)$ be the Lagrangian within generalized operator B, $\mathbf{q_B} = (q_{B1}, q_{B2}, \cdots, q_{Bn})$, $\dot{\mathbf{q}}_\mathbf{B} = \left(\dot{q}_{B1}, \dot{q}_{B2}, \cdots, \dot{q}_{Bn}\right)$ and $\mathbf{B_M^\alpha q_B} = (B_M^\alpha q_{B1}, B_M^\alpha q_{B2}, \cdots, B_M^\alpha q_{Bn})$, then the elements of the generalized moments $\mathbf{p_B} = (p_{B1}, p_{B2}, \cdots, p_{Bn})$ and $\mathbf{p_B^\alpha} = \left(p_{B1}^\alpha, p_{B2}^\alpha, \cdots, p_{Bn}^\alpha\right)$ are defined as $p_{Bi} = \partial L_B / \partial \dot{q}_{Bi}$ and $p_{Bi}^\alpha = \partial L_B / \partial B_M^\alpha q_{Bi}$, and the Hamiltonian $H_B = H_B\left(t, \mathbf{q}_B, \mathbf{p}_B, \mathbf{p}_B^\alpha\right)$ can be expressed as $H_B = p_{Bi} \cdot \dot{q}_{Bi} + p_{Bi}^\alpha B_M^\alpha q_{Bi} - L_B$, $i = 1, 2, \cdots n$.

Hamilton action within generalized operator B can be expressed as

$$S_B = \int_a^b \left[p_{Bi} \cdot \dot{q}_{Bi} + p^{\alpha}_{Bi} \cdot B^{\alpha}_{M} q_{Bi} - H_B(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p^{\alpha}_B}) \right] \mathrm{d}t. \tag{17}$$

Then,

$$\delta S_B = 0, \tag{18}$$

with the commutative conditions

$$\delta B^{\alpha}_{M} q_{Bi} = B^{\alpha}_{M} \delta q_{Bi}, \ \ i = 1, 2, \cdots, n, \tag{19}$$

and the boundary conditions

$$\mathbf{q_B}(a) = \mathbf{q_{Ba}}, \ \mathbf{q_B}(b) = \mathbf{q_{Bb}}, \tag{20}$$

where $\mathbf{q_{Ba}} = (q_{Ba1}, q_{Ba2}, \cdots, q_{Ban})$, $\mathbf{q_{Bb}} = (q_{Bb1}, q_{Bb2}, \cdots, q_{Bbn})$, is called the Hamilton principle within generalized operator B.

Using Equations (10), (19) and (20), we derive from Equation (18) that

$$\begin{aligned} \delta S_B = & \int_a^b \Big[\delta p_{Bi} \cdot \dot{q}_{Bi} + p_{Bi} \cdot \delta \dot{q}_{Bi} + \delta p^{\alpha}_{Bi} \cdot B^{\alpha}_{M} q_{Bi} + p^{\alpha}_{Bi} \cdot \delta B^{\alpha}_{M} q_{Bi} - \tfrac{\partial H_B}{\partial q_{Bi}} \cdot \delta q_{Bi} \\ & - \tfrac{\partial H_B}{\partial p_{Bi}} \cdot \delta p_{Bi} - \tfrac{\partial H_B}{\partial p^{\alpha}_{Bi}} \cdot \delta p^{\alpha}_{Bi} \Big] \mathrm{d}t \\ = & \int_a^b \left[-\left(A^{\alpha}_{M^*} p^{\alpha}_{Bi} + \dot{p}_{Bi} + \tfrac{\partial H_B}{\partial q_{Bi}} \right) \delta q_{Bi} + \left(\dot{q}_{Bi} - \tfrac{\partial H_B}{\partial p_{Bi}} \right) \delta p_{Bi} + \left(B^{\alpha}_{M} q_{Bi} - \tfrac{\partial H_B}{\partial p^{\alpha}_{Bi}} \right) \delta p^{\alpha}_{Bi} \right] \mathrm{d}t = 0. \end{aligned} \tag{21}$$

From the Hamiltonian $H_B = p_{Bi} \cdot \dot{q}_{Bi} + p^{\alpha}_{Bi} B^{\alpha}_{M} q_{Bi} - L_B$, the independence of δq_{Bi} and the arbitrariness of the interval $[a, b]$, we obtain:

$$B^{\alpha}_{M} q_{Bi} = \frac{\partial H_B}{\partial p^{\alpha}_{Bi}}, \ \dot{q}_{Bi} = \frac{\partial H_B}{\partial p_{Bi}}, \ A^{\alpha}_{M^*} p^{\alpha}_{Bi} = -\dot{p}_{Bi} - \frac{\partial H_B}{\partial q_{Bi}}. \tag{22}$$

Equation (22) is called the Hamilton equation within generalized operator B.

Remark 3. *Let $\kappa_{\alpha}(t, \tau) = (t - \tau)^{\alpha - 1} / \Gamma(\alpha)$. When $M = M_1$, $M = M_2$ and $M = M_3$, from Equation (22), we can obtain the Hamilton equations in terms of the left Caputo fractional operator, the right Caputo fractional operator, and the Riesz–Caputo fractional operator, respectively. These results are consistent with the ones in Ref. [29].*

4. Noether Theorems within Generalized Operators

We already know that Noether symmetry means the invariance of the Hamilton action under infinitesimal transformations, and the conserved quantity of the system can be directly derived from Noether symmetry. We begin with the definition of conserved quantity.

Definition 1. *A quantity I is called a conserved quantity if and only if the condition $\mathrm{d}I/\mathrm{d}t = 0$ holds.*

4.1. Noether Theorem within Generalized Operator A

Firstly, we give the infinitesimal transformations in terms of generalized operator A as

$$\bar{t} = t + \Delta t, \ \bar{q}_{Ai}(\bar{t}) = q_{Ai}(t) + \Delta q_{Ai}, \ \bar{p}_{Ai}(\bar{t}) = p_{Ai}(t) + \Delta p_{Ai}, \ \bar{p}^{\alpha}_{Ai}(\bar{t}) = p^{\alpha}_{Ai}(t) + \Delta p^{\alpha}_{Ai}. \tag{23}$$

Expanding Equation (23), we have

$$\begin{aligned} \bar{t} = t + \theta_A \xi^0_{A0}(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^{\alpha}_A}) + o(\theta_A), \ & \bar{q}_{Ai}(\bar{t}) = q_{Ai}(t) + \theta_A \xi^0_{Ai}(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^{\alpha}_A}) + o(\theta_A), \\ & \bar{p}_{Ai}(\bar{t}) = p_{Ai}(t) + \theta_A \eta^0_{Ai}(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^{\alpha}_A}) + o(\theta_A), \\ & \bar{p}^{\alpha}_{Ai}(\bar{t}) = p^{\alpha}_{Ai}(t) + \theta_A \eta^{\alpha 0}_{Ai}(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^{\alpha}_A}) + o(\theta_A). \end{aligned} \tag{24}$$

where θ_A is an infinitesimal parameter, ξ^0_{A0}, ξ^0_{Ai}, η^0_{Ai} and $\eta^{\alpha 0}_{Ai}$ are called infinitesimal generators within generalized operator A, and $o(\theta_A)$ means the higher-order infinity small of θ_A.

Then, letting ΔS_A be the linear part of $\overline{S}_A - S_A$ and neglecting the higher-order infinity small of θ_A, we obtain:

$$\begin{aligned}
\Delta S_A = \overline{S}_A - S_A = \textstyle\int_{\overline{a}}^{\overline{b}} \Big[\overline{p}_{Ai} \cdot \dot{\overline{q}}_{Ai} + \overline{p}^{\alpha}_{Ai} \cdot A^{\alpha}_{\overline{M}} \overline{q}_{Ai} - H_A(t, \overline{\mathbf{q}}_{\mathbf{A}}, \overline{\mathbf{p}}_{\mathbf{A}}, \overline{\mathbf{p}}^{\boldsymbol{\alpha}}_{\mathbf{A}}) \mathrm{d}\overline{t} \\
- \textstyle\int_a^b \big[p_{Ai} \cdot \dot{q}_{Ai} + p^{\alpha}_{Ai} \cdot A^{\alpha}_M q_{Ai} - H_A\big(t, \mathbf{q}_{\mathbf{A}}, \mathbf{p}_{\mathbf{A}}, \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}\big) \big] \mathrm{d}t \\
= \textstyle\int_a^b \{ (p_{Ai} + \Delta p_{Ai})(\dot{q}_{Ai} + \Delta \dot{q}_{Ai}) + (p^{\alpha}_{Ai} + \Delta p^{\alpha}_{Ai}) \cdot \big[A^{\alpha}_M q_{Ai} + A^{\alpha}_M \delta q_{Ai} \\
+ \Delta t \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_M q_{Ai} + \omega \Delta b \cdot q_{Ai}(b) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) - m \Delta a \cdot q_{Ai}(a) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) \big] \\
- H_A(t + \Delta t, \mathbf{q}_{\mathbf{A}} + \boldsymbol{\Delta}\mathbf{q}_{\mathbf{A}}, \mathbf{p}_{\mathbf{A}} + \boldsymbol{\Delta}\mathbf{p}_{\mathbf{A}}, \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}} + \boldsymbol{\Delta}\mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}) \} \cdot \Big(1 + \tfrac{\mathrm{d}}{\mathrm{d}t} \Delta t \Big) \mathrm{d}t \\
- \textstyle\int_a^b \big[p_{Ai} \cdot \dot{q}_{Ai} + p^{\alpha}_{Ai} \cdot A^{\alpha}_M q_{Ai} - H_A\big(t, \mathbf{q}_{\mathbf{A}}, \mathbf{p}_{\mathbf{A}}, \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}\big) \big] \mathrm{d}t \\
= \textstyle\int_a^b \Big\{ p_{Ai} \dot{q}_{Ai} + p_{Ai} \Delta \dot{q}_{Ai} + \Delta p_{Ai} \cdot \dot{q}_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_M \delta q_{Ai} + p^{\alpha}_{Ai} \Delta t \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_M q_{Ai} \\
+ \Delta p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} + p^{\alpha}_{Ai} \cdot \Big[\omega \Delta b \cdot q_{Ai}(b) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) - m \Delta a \cdot q_{Ai}(a) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) \Big] \\
- H_A\big(t, \mathbf{q}_{\mathbf{A}}, \mathbf{p}_{\mathbf{A}}, \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}\big) - \tfrac{\partial H_A}{\partial t} \Delta t - \tfrac{\partial H_A}{\partial q_{Ai}} \Delta q_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}} \Delta p_{Ai} - \tfrac{\partial H_A}{\partial p^{\alpha}_{Ai}} \Delta p^{\alpha}_{Ai} + \big(p_{Ai} \cdot \dot{q}_{Ai} \\
+ p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} - H_A \big) \tfrac{\mathrm{d}}{\mathrm{d}t} \Delta t \Big\} \mathrm{d}t - \textstyle\int_a^b \big[p_{Ai} \cdot \dot{q}_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} - H_A\big(t, \mathbf{q}_{\mathbf{A}}, \mathbf{p}_{\mathbf{A}}, \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}\big) \big] \mathrm{d}t \\
= \theta_A \textstyle\int_a^b \Big\{ p_{Ai} \dot{\xi}^0_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_M \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0} \big) + \Big(p^{\alpha}_{Ai} \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_M q_{Ai} - \tfrac{\partial H_A}{\partial t} \Big) \xi^0_{A0} \\
+ \big(p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} - H_A \big) \dot{\xi}^0_{A0} - \tfrac{\partial H_A}{\partial q_{Ai}} \xi^0_{Ai} + \omega \cdot q_{Ai}(b) \cdot p^{\alpha}_{Ai} \xi^0_{A0}\big(b, \mathbf{q}_{\mathbf{A}}(b), \mathbf{p}_{\mathbf{A}}(b), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(b)\big) \\
\times \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) - m q_{Ai}(a) \cdot p^{\alpha}_{Ai} \xi^0_{A0}\big(a, \mathbf{q}_{\mathbf{A}}(a), \mathbf{p}_{\mathbf{A}}(a), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(a)\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) \Big\} \mathrm{d}t.
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
A^{\alpha}_{\overline{M}} \overline{q}_{Ai} = A^{\alpha}_M q_{Ai} + A^{\alpha}_M \delta q_{Ai} + \Delta t \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_M q_{Ai} + \omega \Delta b \cdot q_{Ai}(b) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) - m \Delta a \\
\times q_{Ai}(a) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a),\ \Delta \dot{q}_{Ai} = \theta_A \Big(\dot{\xi}^0_{Ai} - \dot{q}_{Ai} \dot{\xi}^0_{A0} \Big),\ \overline{M} = < \overline{a}, \overline{t}, \overline{b}, m, \omega > .
\end{aligned} \tag{26}$$

It follows from Noether symmetry ($\Delta S_A = 0$) that

$$\begin{aligned}
p_{Ai} \dot{\xi}^0_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_M \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0} \big) + \Big(p^{\alpha}_{Ai} \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_M q_{Ai} - \tfrac{\partial H_A}{\partial t} \Big) \xi^0_{A0} - \tfrac{\partial H_A}{\partial q_{Ai}} \xi^0_{Ai} + \omega p^{\alpha}_{Ai} \\
\times q_{Ai}(b) \xi^0_{A0}\big(b, \mathbf{q}_{\mathbf{A}}(b), \mathbf{p}_{\mathbf{A}}(b), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(b)\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) + \big(p^{\alpha}_{Ai} \cdot A^{\alpha}_M q_{Ai} - H_A \big) \dot{\xi}^0_{A0} \\
- m p^{\alpha}_{Ai} \cdot q_{Ai}(a) \cdot \xi^0_{A0}\big(a, \mathbf{q}_{\mathbf{A}}(a), \mathbf{p}_{\mathbf{A}}(a), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(a)\big) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) = 0,\ \ i = 1, 2, \cdots, n.
\end{aligned} \tag{27}$$

Equation (27) is called the Noether identity within generalized operator A.

Finally, the conserved quantity within generalized operator A deduced by the Noether symmetry is presented.

Theorem 1. *For the Hamiltonian system within generalized operator A (Equation (16)), if the infinitesimal generators ξ^0_{A0}, ξ^0_{Ai}, η^0_{Ai} and $\eta^{\alpha 0}_{Ai}$ satisfy the Noether identity (Equation (27)), then there exists a conserved quantity:*

$$\begin{aligned}
I_{A0} = \big(p^{\alpha}_{Ai} A^{\alpha}_M q_{Ai} - H_A \big) \xi^0_{A0} + \textstyle\int_a^t \{ p^{\alpha}_{Ai} A^{\alpha}_M \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0} \big) + \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0} \big) \\
\times \big[B^{\alpha}_{M^*} p^{\alpha}_{Ai} - m p_{Ai}(b) \kappa_{1-\alpha}(b,\tau) + \omega p_{Ai}(a) \kappa_{1-\alpha}(\tau,a) \big] \} \mathrm{d}\tau + p_{Ai} \xi^0_{Ai} \\
+ \omega \cdot q_{Ai}(b) \cdot \xi^0_{A0}\big(b, \mathbf{q}_{\mathbf{A}}(b), \mathbf{p}_{\mathbf{A}}(b), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(b)\big) \textstyle\int_a^t p^{\alpha}_{Ai}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \kappa_{1-\alpha}(b,\tau) \mathrm{d}\tau \\
- m \cdot q_{Ai}(a) \cdot \xi^0_{A0}\big(a, \mathbf{q}_{\mathbf{A}}(a), \mathbf{p}_{\mathbf{A}}(a), \mathbf{p}^{\boldsymbol{\alpha}}_{\mathbf{A}}(a)\big) \textstyle\int_a^t p^{\alpha}_{Ai}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \kappa_{1-\alpha}(\tau,a) \mathrm{d}\tau = \mathrm{const}.
\end{aligned} \tag{28}$$

Proof of Theorem 1. From Equations (16) and (27), we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_{A0} = &\Big(\dot{p}^{\alpha}_{Ai} A^{\alpha}_{M} q_{Ai} + p^{\alpha}_{Ai} \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_{M} q_{Ai} - \tfrac{\partial H_A}{\partial t} - \tfrac{\partial H_A}{\partial q_{Ai}} \cdot \dot{q}_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}} \dot{p}_{Ai} - \tfrac{\partial H_A}{\partial p^{\alpha}_{Ai}} \dot{p}^{\alpha}_{Ai} \Big) \xi^0_{A0} \\
&+ \big(p^{\alpha}_{Ai} A^{\alpha}_{M} q_{Ai} - H_A\big) \dot{\xi}^0_{A0} + p^{\alpha}_{Ai} A^{\alpha}_{M} \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) + \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) \cdot \big[B^{\alpha}_{M^*} p^{\alpha}_{Ai} \\
&- m p^{\alpha}_{Ai}(b) \kappa_{1-\alpha}(b,\tau) + \omega p^{\alpha}_{Ai}(a) \kappa_{1-\alpha}(\tau,a) \big] + \dot{p}_{Ai} \xi^0_{Ai} + p_{Ai} \dot{\xi}^0_{Ai} \\
&+ \omega p^{\alpha}_{Ai} \cdot q_{Ai}(b) \xi^0_{A0}\big(b, \mathbf{q_A}(b), \mathbf{p_A}(b), \mathbf{p^{\alpha}_A}(b)\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) \\
&- m p^{\alpha}_{Ai} \cdot q_{Ai}(a) \xi^0_{A0}\big(a, \mathbf{q_A}(a), \mathbf{p_A}(a), \mathbf{p^{\alpha}_A}(a)\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) \\
= &\tfrac{\partial H_A}{\partial q_{Ai}} \xi_{Ai} + \dot{p}_{Ai} \xi^0_{Ai} + \Big(\dot{p}^{\alpha}_{Ai} A^{\alpha}_{M} q_{Ai} - \tfrac{\partial H_A}{\partial q_{Ai}} \dot{q}_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}} \dot{p}_{Ai} - \tfrac{\partial H_A}{\partial p^{\alpha}_{Ai}} \dot{p}^{\alpha}_{Ai} \Big) \xi^0_{A0} \\
&+ \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) \cdot \big[B^{\alpha}_{M^*} p^{\alpha}_{Ai} - m p^{\alpha}_{Ai}(b) \kappa_{1-\alpha}(b,\tau) + \omega p^{\alpha}_{Ai}(a) \kappa_{1-\alpha}(\tau,a) \big] \\
= &\Big(A^{\alpha}_{M} q_{Ai} - \tfrac{\partial H_A}{\partial p^{\alpha}_{Ai}} \Big) \cdot \dot{p}^{\alpha}_{Ai} \xi^0_{A0} + \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) \cdot \Big[B^{\alpha}_{M^*} p^{\alpha}_{Ai} + \dot{p}_{Ai} + \tfrac{\partial H_A}{\partial q_{Ai}} \\
&- m p^{\alpha}_{Ai}(b) \kappa_{1-\alpha}(b,t) + \omega p^{\alpha}_{Ai}(a) \kappa_{1-\alpha}(t,a) \Big] = 0.
\end{aligned} \tag{29}$$

The proof is completed. □

If we let $\Delta S_A = -\int_a^b (\mathrm{d}/\mathrm{d}t)\big(\Delta G^0_A\big)\mathrm{d}t$, where $\Delta G^0_A = \theta_A G^0_A\big(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^{\alpha}_A}\big)$, then from Equation (25), we have

$$\begin{aligned}
&p_{Ai} \dot{\xi}^0_{Ai} + p^{\alpha}_{Ai} A^{\alpha}_{M} \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) + \Big(p^{\alpha}_{Ai} \tfrac{\mathrm{d}}{\mathrm{d}t} A^{\alpha}_{M} q_{Ai} - \tfrac{\partial H_A}{\partial t} \Big) \xi^0_{A0} - \tfrac{\partial H_A}{\partial q_A} \xi^0_{Ai} + \omega p^{\alpha}_{Ai} \cdot q_{Ai}(b) \\
&\times \xi^0_{A0}\big(b, \mathbf{q_A}(b), \mathbf{p_A}(b), \mathbf{p^{\alpha}_A}(b)\big) \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(b,t) + \big(p^{\alpha}_{Ai} \cdot A^{\alpha}_{M} q_{Ai} - H_A\big) \dot{\xi}^0_{A0} + \dot{G}^0_A \\
&- m p^{\alpha}_{Ai} \cdot q_{Ai}(a) \cdot \xi^0_{A0}\big(a, \mathbf{q_A}(a), \mathbf{p_A}(a), \mathbf{p^{\alpha}_A}(a)\big) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \kappa_{1-\alpha}(t,a) = 0, \quad i = 1, 2, \cdots, n.
\end{aligned} \tag{30}$$

Equation (30) is called the Noether quasi-identity within generalized operator A. In this case, we have the following theorem.

Theorem 2. *For the Hamiltonian system within the generalized operator A (Equation (16)), if there exists a function G^0_A such that the infinitesimal generators ξ^0_{A0}, ξ^0_{Ai}, η^0_{Ai} and $\eta^{\alpha 0}_{Ai}$ satisfy the Noether quasi-identity (Equation (30)), then there exists a conserved quantity*

$$\begin{aligned}
I_{AG0} = &\big(p^{\alpha}_{Ai} A^{\alpha}_{M} q_{Ai} - H_A\big) \xi^0_{A0} + p_{Ai} \xi^0_{Ai} + \int_a^t \big\{ p^{\alpha}_{Ai} A^{\alpha}_{M} \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) + \big(\xi^0_{Ai} - \dot{q}_{Ai} \xi^0_{A0}\big) \\
&\times \big[B^{\alpha}_{M^*} p^{\alpha}_{Ai} - m p_{Ai}(b) \kappa_{1-\alpha}(b,\tau) + \omega p_{Ai}(a) \kappa_{1-\alpha}(\tau,a) \big] \big\} \mathrm{d}\tau + \omega \cdot q_{Ai}(b) \\
&\times \xi^0_{A0}\big(b, \mathbf{q_A}(b), \mathbf{p_A}(b), \mathbf{p^{\alpha}_A}(b)\big) \int_a^t p^{\alpha}_{Ai}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \kappa_{1-\alpha}(b,\tau) \mathrm{d}\tau - m \cdot q_{Ai}(a) \\
&\times \xi^0_{A0}\big(a, \mathbf{q_A}(a), \mathbf{p_A}(a), \mathbf{p^{\alpha}_A}(a)\big) \int_a^t p^{\alpha}_{Ai}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \kappa_{1-\alpha}(\tau,a) \mathrm{d}\tau + G^0_A = \mathrm{const.}
\end{aligned} \tag{31}$$

Proof of Theorem 2. From Equations (16) and (30), we have $\mathrm{d}I_{AG0}/\mathrm{d}t = 0$.
The proof is completed. □

Remark 4. *Let $\kappa_{\alpha}(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$. When $M = M_1$, $M = M_2$ and $M = M_3$, from Equations (27) and (30) and Theorems 1 and 2, we can obtain Noether identities, Noether quasi-identities, and Noether theorems in terms of the left Riemann–Liouville fractional operator, the right Riemann–Liouville fractional operator, and the Riesz–Riemann–Liouville fractional operator, respectively. These results are consistent with the ones in Ref. [29].*

4.2. Noether Theorem within Generalized Operator B

The infinitesimal transformations in terms of generalized operator B are

$$\bar{t} = t + \Delta t,\ \bar{q}_{Bi}(\bar{t}) = q_{Bi}(t) + \Delta q_{Bi},\ \bar{p}_{Bi}(\bar{t}) = p_{Bi}(t) + \Delta p_{Bi},\ \bar{p}^{\alpha}_{Bi}(\bar{t}) = p^{\alpha}_{Bi}(t) + \Delta p^{\alpha}_{Bi}. \tag{32}$$

Expanding Equation (32), we have

$$
\begin{aligned}
\bar{t} = t + \theta_B \xi_{B0}^0(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) + o(\theta_B),\ \bar{q}_{Bi}(\bar{t}) = q_{Bi}(t) + \theta_B \xi_{Bi}^0(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) + o(\theta_B), \\
\bar{p}_{Bi}(\bar{t}) = p_{Bi}(t) + \theta_B \eta_{Bi}^0(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) + o(\theta_B), \\
\bar{p}_{Bi}^\alpha(\bar{t}) = p_{Bi}^\alpha(t) + \theta_B \eta_{Bi}^{\alpha 0}(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) + o(\theta_B),
\end{aligned}
\tag{33}
$$

where θ_B is an infinitesimal parameter, ξ_{B0}^0, ξ_{Bi}^0, η_{Bi}^0 and $\eta_{Bi}^{\alpha 0}$ are called infinitesimal generators within generalized operator B, and $o(\theta_B)$ means the higher-order infinity small of θ_B.

Similarly, letting ΔS_B be the linear part of $\overline{S}_B - S_B$ and neglecting the higher-order infinity small of θ_B, we obtain

$$
\begin{aligned}
\Delta S_B = \overline{S_B} - S_B = \int_{\bar{a}}^{\bar{b}} \left[\bar{p}_{Bi} \cdot \dot{\bar{q}}_{Bi} + \bar{p}_{Bi}^\alpha \cdot B_{\overline{M}}^\alpha \bar{q}_{Bi} - H_B(t, \overline{\mathbf{q}}_\mathbf{B}, \overline{\mathbf{p}}_\mathbf{B}, \overline{\mathbf{p}}_\mathbf{B}^\alpha) \right] \mathrm{d}\bar{t} \\
- \int_a^b \left[p_{Bi} \cdot \dot{q}_{Bi} + p_{Bi}^\alpha \cdot B_M^\alpha q_{Bi} - H_B(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) \right] \mathrm{d}t \\
= \int_a^b \Big\{ (p_{Bi} + \Delta p_{Bi})(\dot{q}_{Bi} + \Delta \dot{q}_{Bi}) + (p_{Bi}^\alpha + \Delta p_{Bi}^\alpha) \cdot \Big[B_M^\alpha q_{Bi} + B_M^\alpha \delta q_{Bi} + \Delta t \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} \\
+ \omega \Delta b \cdot \dot{q}_{Bi}(b) \kappa_{1-\alpha}(b, t) - m \Delta a \cdot \dot{q}_{Bi}(a) \kappa_{1-\alpha}(t, a) \Big] - H_B(t + \Delta t, \mathbf{q_B} + \Delta \mathbf{q_B}, \mathbf{p_B} \\
+ \Delta \mathbf{p_B},\ \mathbf{p_B^\alpha} + \Delta \mathbf{p_B^\alpha}) \Big\} \cdot \Big(1 + \tfrac{\mathrm{d}}{\mathrm{d}t} \Delta t \Big) \mathrm{d}t - \int_a^b \left[p_{Bi} \cdot \dot{q}_{Bi} + p_{Bi}^\alpha \cdot B_M^\alpha q_{Bi} - H_B(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) \right] \mathrm{d}t \\
= \int_a^b \Big\{ p_{Bi} \dot{q}_{Bi} + p_{Bi} \Delta \dot{q}_{Bi} + \Delta p_{Bi} \dot{q}_{Bi} + p_{Bi}^\alpha B_M^\alpha q_{Bi} + p_{Bi}^\alpha B_M^\alpha \delta q_{Bi} + p_{Bi}^\alpha \Delta t \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} \\
+ \Delta p_{Bi}^\alpha B_M^\alpha q_{Bi} + p_{Bi}^\alpha \cdot \left[\omega \Delta b \cdot \dot{q}_{Bi}(b) \kappa_{1-\alpha}(b, t) - m \Delta a \cdot \dot{q}_{Bi}(a) \kappa_{1-\alpha}(t, a) \right] \\
- H_B(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) - \tfrac{\partial H_B}{\partial t} \Delta t - \tfrac{\partial H_B}{\partial q_{Bi}} \Delta q_{Bi} - \tfrac{\partial H_B}{\partial p_{Bi}} \Delta p_{Bi} - \tfrac{\partial H_B}{\partial p_{Bi}^\alpha} \Delta p_{Bi}^\alpha + \big(p_{Bi} \dot{q}_{Bi} \\
+ p_{Bi}^\alpha B_M^\alpha q_{Bi} - H_B \big) \tfrac{\mathrm{d}}{\mathrm{d}t} \Delta t \Big\} \mathrm{d}t - \int_a^b \left[p_{Bi} \cdot \dot{q}_{Bi} + p_{Bi}^\alpha B_M^\alpha q_{Bi} - H_B(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) \right] \mathrm{d}t \\
= \theta_B \int_a^b \Big[p_{Bi} \dot{\xi}_{Bi}^0 + p_{Bi}^\alpha B_M^\alpha (\xi_{Bi}^0 - \dot{q}_{Bi} \xi_{B0}^0) + \Big(p_{Bi}^\alpha \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} - \tfrac{\partial H_B}{\partial t} \Big) \xi_{B0}^0 + \big(p_{Bi}^\alpha B_M^\alpha q_{Bi} \\
- H_{Bi} \big) \dot{\xi}_{B0}^0 - \tfrac{\partial H_B}{\partial q_{Bi}} \xi_{Bi}^0 + \omega \cdot \dot{q}_{Bi}(b) \cdot p_{Bi}^\alpha \xi_{B0}^0(b, \mathbf{q_B}(b), \mathbf{p_B}(b), \mathbf{p_B^\alpha}(b)) \kappa_{1-\alpha}(b, t) \\
- m \cdot \dot{q}_{Bi}(a) \cdot p_{Bi}^\alpha \xi_{B0}^0(a, \mathbf{q_B}(a), \mathbf{p_B}(a), \mathbf{p_B^\alpha}(a)) \kappa_{1-\alpha}(t, a) \Big] \mathrm{d}t,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
B_{\overline{M}}^\alpha \bar{q}_{Bi} = B_M^\alpha q_{Bi} + B_M^\alpha \delta q_{Bi} + \Delta t \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} + \omega \Delta b \cdot \dot{q}_{Bi}(b) \cdot \kappa_{1-\alpha}(b, t) - m \Delta a \\
\times \dot{q}_{Bi}(a) \cdot \kappa_{1-\alpha}(t, a),\ \Delta \dot{q}_{Bi} = \theta_B \left(\dot{\xi}_{Bi}^0 - \dot{q}_{Bi} \dot{\xi}_{B0}^0 \right),\ \overline{M} = < \bar{a}, \bar{t}, \bar{b}, m, \omega > .
\end{aligned}
\tag{35}
$$

Letting $\Delta S_B = 0$, we have

$$
\begin{aligned}
p_{Bi} \dot{\xi}_{Bi}^0 + p_{Bi}^\alpha B_M^\alpha (\xi_{Bi}^0 - \dot{q}_{Bi} \xi_{B0}^0) + \Big(p_{Bi}^\alpha \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} - \tfrac{\partial H_B}{\partial t} \Big) \xi_{B0}^0 - \tfrac{\partial H_B}{\partial q_{Bi}} \xi_{Bi}^0 \\
+ (p_{Bi}^\alpha \cdot B_M^\alpha q_{Bi} - H_B) \dot{\xi}_{B0}^0 + \omega p_{Bi}^\alpha \cdot \dot{q}_{Bi}(b) \cdot \xi_{B0}^0(b, \mathbf{q_B}(b), \mathbf{p_B}(b), \mathbf{p_B^\alpha}(b)) \kappa_{1-\alpha}(b, t) \\
- m p_{Bi}^\alpha \cdot \dot{q}_{Bi}(a) \cdot \xi_{B0}^0(a, \mathbf{q_B}(a), \mathbf{p_B}(a), \mathbf{p_B^\alpha}(a)) \cdot \kappa_{1-\alpha}(t, a) = 0,\ \ i = 1, 2, \cdots, n.
\end{aligned}
\tag{36}
$$

Equation (36) is called the Noether identity within generalized operator B.

If we let $\Delta S_B = -\int_a^b (\mathrm{d}/\mathrm{d}t)(\Delta G_B^0) \mathrm{d}t$, $\Delta G_B^0 = \theta_B G_B^0(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha})$, then from Equation (34), we have

$$
\begin{aligned}
p_{Bi} \dot{\xi}_{Bi}^0 + p_{Bi}^\alpha B_M^\alpha (\xi_{Bi}^0 - \dot{q}_{Bi} \xi_{B0}^0) + \Big(p_{Bi}^\alpha \tfrac{\mathrm{d}}{\mathrm{d}t} B_M^\alpha q_{Bi} - \tfrac{\partial H_B}{\partial t} \Big) \xi_{B0}^0 - \tfrac{\partial H_B}{\partial q_{Bi}} \xi_{Bi}^0 \\
+ (p_{Bi}^\alpha \cdot B_M^\alpha q_{Bi} - H_B) \dot{\xi}_{B0}^0 + \omega p_{Bi}^\alpha \cdot \dot{q}_{Bi}(b) \cdot \xi_{B0}^0(b, q_B(b), p_B(b), p_B^\alpha(b)) \kappa_{1-\alpha}(b, t) \\
- m p_{Bi}^\alpha \cdot \dot{q}_{Bi}(a) \cdot \xi_{B0}^0(a, q_B(a), p_B(a), p_B^\alpha(a)) \kappa_{1-\alpha}(t, a) + \dot{G}_B^0 = 0,\ \ i = 1, 2, \cdots, n.
\end{aligned}
\tag{37}
$$

Equation (37) is called the Noether quasi-identity within generalized operator B. Therefore, we have the following theorem.

Theorem 3. *For the Hamiltonian system within generalized operator B (Equation (22)), if the infinitesimal generators* ξ^0_{B0}, ξ^0_{Bi}, η^0_{Bi} *and* $\eta^{\alpha 0}_{Bi}$ *satisfy the Noether identity (Equation (36)), then there exists a conserved quantity*

$$\begin{aligned} I_{B0} = {} & (p^\alpha_{Bi} B^\alpha_M q_{Bi} - H_B)\xi^0_{B0} + \int_a^t [p^\alpha_{Bi} B^\alpha_M (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) + (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) \cdot A^\alpha_{M^*} p^\alpha_{Bi}] \mathrm{d}\tau \\ & + p_{Bi}\xi^0_{Bi} + \omega \cdot \dot{q}_{Bi}(b) \cdot \xi^0_{B0}(b, \mathbf{q_B}(b), \mathbf{p_B}(b), \mathbf{p^\alpha_B}(b)) \int_a^t p^\alpha_{Bi}(\tau)\kappa_{1-\alpha}(b,\tau)\mathrm{d}\tau \\ & - m \cdot \dot{q}_{Bi}(a) \cdot \xi^0_{B0}(a, \mathbf{q_B}(a), \mathbf{p_B}(a), \mathbf{p^\alpha_B}(a)) \int_a^t p^\alpha_{Bi}(\tau) \cdot \kappa_{1-\alpha}(\tau, a)\mathrm{d}\tau = \text{const.} \end{aligned} \tag{38}$$

Proof of Theorem 3. From Equations (22) and (36), we obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_{B0} = {} & \left(\dot{p}^\alpha_{Bi} B^\alpha_M q_{Bi} + p^\alpha_{Bi} \frac{\mathrm{d}}{\mathrm{d}t} B^\alpha_M q_{Bi} - \frac{\partial H_B}{\partial t} - \frac{\partial H_B}{\partial q_{Bi}} \cdot \dot{q}_{Bi} - \frac{\partial H_B}{\partial p_{Bi}} \dot{p}_{Bi} - \frac{\partial H_B}{\partial p^\alpha_{Bi}} \dot{p}^\alpha_{Bi}\right)\xi^0_{B0} \\ & + (p^\alpha_{Bi} B^\alpha_M q_{Bi} - H_B)\dot{\xi}^0_{B0} + p^\alpha_{Bi} B^\alpha_M (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) + (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) \cdot A^\alpha_{M^*} p^\alpha_{Bi} \\ & + \dot{p}_{Bi}\xi_{Bi} + p_{Bi}\dot{\xi}^0_{Bi} + \omega p^\alpha_{Bi} \cdot \dot{q}_{Bi}(b)\xi^0_{B0}(b, \mathbf{q_B}(b), \mathbf{p_B}(b), \mathbf{p^\alpha_B}(b))\kappa_{1-\alpha}(b,t) \\ & - m p^\alpha_{Bi} \cdot \dot{q}_{Bi}(a)\xi^0_{B0}(a, \mathbf{q_B}(a), \mathbf{p_B}(a), \mathbf{p^\alpha_B}(a))\kappa_{1-\alpha}(t,a) \\ = {} & \frac{\partial H_B}{\partial q_{Bi}}\xi^0_{Bi} + \dot{p}_{Bi}\xi^0_{Bi} + \left(\dot{p}^\alpha_{Bi} B^\alpha_M q_{Bi} - \frac{\partial H_B}{\partial q_{Bi}}\dot{q}_{Bi} - \frac{\partial H_B}{\partial p_{Bi}}\dot{p}_{Bi} - \frac{\partial H_B}{\partial p^\alpha_{Bi}}\dot{p}^\alpha_{Bi}\right)\xi^0_{B0} \\ & + (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) \cdot A^\alpha_{M^*} p^\alpha_{Ai} \\ = {} & \left(B^\alpha_M q_{Bi} - \frac{\partial H_B}{\partial p^\alpha_{Bi}}\right) \cdot \dot{p}^\alpha_{Bi}\xi^0_{B0} + (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) \cdot \left(A^\alpha_{M^*} p^\alpha_{Bi} + \dot{p}_{Bi} + \frac{\partial H_B}{\partial q_{Bi}}\right) = 0. \end{aligned} \tag{39}$$

The proof is completed. □

Theorem 4. *For the Hamiltonian system within generalized operator B (Equation (22)), if there exists a function* G^0_B *such that the infinitesimal generators* ξ_{B0}, ξ_{Bi}, η_{Bi} *and* η^α_{Bi} *satisfy the Noether quasi-identity (Equation (37)), then there exists a conserved quantity*

$$\begin{aligned} I_{BG0} = {} & (p^\alpha_{Bi} B^\alpha_M q_{Bi} - H_B)\xi^0_{B0} + \int_a^t [p^\alpha_{Bi} B^\alpha_M (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) + (\xi^0_{Bi} - \dot{q}_{Bi}\xi^0_{B0}) \cdot A^\alpha_{M^*} p^\alpha_{Bi}] \mathrm{d}\tau \\ & + p_{Bi}\xi^0_{Bi} + \omega \cdot \dot{q}_{Bi}(b) \cdot \xi^0_{B0}(b, \mathbf{q_B}(b), \mathbf{p_B}(b), \mathbf{p^\alpha_B}(b)) \int_a^t p^\alpha_{Bi}(\tau)\kappa_{1-\alpha}(b,\tau)\mathrm{d}\tau \\ & - m \cdot \dot{q}_{Bi}(a) \cdot \xi^0_{B0}(a, \mathbf{q_B}(a), \mathbf{p_B}(a), \mathbf{p^\alpha_B}(a)) \int_a^t p^\alpha_{Bi}(\tau) \cdot \kappa_{1-\alpha}(\tau, a)\mathrm{d}\tau + G^0_B = \text{const.} \end{aligned} \tag{40}$$

Proof of Theorem 4. From Equations (22) and (37), we have $\mathrm{d}I_{BG0}/\mathrm{d}t = 0$.
The proof is completed. □

Remark 5. *Let* $\kappa_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$. *When* $M = M_1$, $M = M_2$ *and* $M = M_3$ *from Equation (36) and Equation (37) and Theorems 3 and 4, we can obtain Noether identities, Noether quasi-identities, and Noether theorems in terms of the left Caputo fractional operator, the right Caputo fractional operator and the Riesz–Caputo fractional operator, respectively. These results are consistent with the ones in Ref. [29].*

5. Adiabatic Invariants within Generalized Operators

First, we give the definition of an adiabatic invariant.

Definition 2. *A quantity* I_z *is called an adiabatic invariant if* I_z *contains a parameter* ε, *whose highest power is z, and also satisfies that* $\mathrm{d}I_z/\mathrm{d}t$ *is in proportion to* ε^{z+1}.

When the systems (Equations (16) and (22)) are disturbed by small forces, the conserved quantities may also change.

Supposing that the Hamiltonian system (Equation (16)) is disturbed as

$$\begin{aligned} B^\alpha_{M^*} p^\alpha_{Ai} = -\dot{p}_{Ai} - \frac{\partial H_A}{\partial q_{Ai}} + m p^\alpha_{Ai}(b)\kappa_{1-\alpha}(b,t) - \omega p^\alpha_{Ai}(a)\kappa_{1-\alpha}(t,a) - \varepsilon_A W_{Ai}(t, \mathbf{q_A}, \mathbf{p_A}, \mathbf{p^\alpha_A}), \\ A^\alpha_M q_{Ai} = \frac{\partial H_A}{\partial p^\alpha_{Ai}},\ \dot{q}_{Ai} = \frac{\partial H_A}{\partial p_{Ai}}. \end{aligned} \tag{41}$$

In this case, if the function G_A and the infinitesimal generators ξ_{A0}, ξ_{Ai}, η_{Ai} and η^{α}_{Ai} of the disturbed system (Equation (41)) have the forms

$$\begin{gathered} G_A = G_A^0 + \varepsilon_A G_A^1 + \varepsilon_A^2 G_A^2 + \cdots = \varepsilon_A^s G_A^s,\ \xi_{A0} = \xi_{A0}^0 + \varepsilon_A \xi_{A0}^1 + \varepsilon_A^2 \xi_{A0}^2 + \cdots = \varepsilon_A^s \xi_{A0}^s, \\ \eta_{Ai} = \eta_{Ai}^0 + \varepsilon_A \eta_{Ai}^1 + \varepsilon_A^2 \eta_{Ai}^2 + \cdots = \varepsilon_A^s \eta_{Ai}^s,\ \eta_{Ai} = \eta_{Ai}^{\alpha 0} + \varepsilon_A \eta_{Ai}^{\alpha 1} + \varepsilon_A^2 \eta_{Ai}^{\alpha 2} + \cdots = \varepsilon_A^s \eta_{Ai}^{\alpha s}, \\ i = 1, 2, \cdots, n,\ s = 0, 1, 2, \cdots, \end{gathered} \tag{42}$$

then we have:

Theorem 5. *For the disturbed Hamiltonian system (Equation (41)), if there exists a function G_A^s such that the infinitesimal generators ξ_{A0}^s, ξ_{Ai}^s, η_{Ai}^s and $\eta_{Ai}^{\alpha s}$ satisfy*

$$\begin{gathered} p_{Ai}\dot{\xi}_{Ai}^s + p_{Ai}^{\alpha} A_M^{\alpha} \left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) + \left(p_{Ai}^{\alpha} \tfrac{\mathrm{d}}{\mathrm{d}t} A_M^{\alpha} q_{Ai} - \tfrac{\partial H_A}{\partial t}\right)\xi_{A0}^s - \tfrac{\partial H_A}{\partial q_{Ai}}\xi_{Ai}^s \\ +\left(p_{Ai}^{\alpha} \cdot A_M^{\alpha} q_{Ai} - H_A\right)\dot{\xi}_{A0}^s + \omega p_{Ai}^{\alpha} \cdot q_{Ai}(b) \cdot \xi_{A0}^s\left(b, \mathbf{q_A}(b), \mathbf{p_A}(b), \mathbf{p_A^{\alpha}}(b)\right) \tfrac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(b,t) + \dot{G}_A^s \\ -m p_{Ai}^{\alpha} \cdot q_{Ai}(a) \cdot \xi_{A0}^s\left(a, \mathbf{q_A}(a), \mathbf{p_A}(a), \mathbf{p_A^{\alpha}}(a)\right) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,a) - W_{Ai}\left(\xi_{Ai}^{s-1} - \dot{q}_{Ai}\xi_{A0}^{s-1}\right) = 0, \end{gathered} \tag{43}$$

where $\xi_{Ai}^{s-1} = \xi_{A0}^{s-1} = 0$ when $s = 0$, then there exists an adiabatic invariant

$$\begin{gathered} I_{AGz} = \sum_{s=0}^{z} \varepsilon_A^s \Big\{ \left(p_{Ai}^{\alpha} A_M^{\alpha} q_{Ai} - H_A\right)\xi_{A0}^s + \int_a^t \left\{ p_{Ai}^{\alpha} A_M^{\alpha}\left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) + \left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) \right. \\ \left. \times \left[B_{M^*}^{\alpha} p_{Ai}^{\alpha} - m p_{Ai}(b)\kappa_{1-\alpha}(b,\tau) + \omega p_{Ai}(a)\kappa_{1-\alpha}(\tau,a)\right] \right\} \mathrm{d}\tau + p_{Ai}\xi_{Ai}^s \\ +\omega \cdot q_{Ai}(b) \cdot \xi_{A0}^s\left(b, q_A(b), p_A(b), p_A^{\alpha}(b)\right) \int_a^t p_{Ai}^{\alpha}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau}\kappa_{1-\alpha}(b,\tau)\mathrm{d}\tau \\ -m \cdot q_{Ai}(a) \cdot \xi_{A0}^s\left(a, q_A(a), p_A(a), p_A^{\alpha}(a)\right) \int_a^t p_{Ai}^{\alpha}(\tau) \tfrac{\mathrm{d}}{\mathrm{d}\tau}\kappa_{1-\alpha}(\tau,a)\mathrm{d}\tau + G_A^s \Big\}. \end{gathered} \tag{44}$$

Proof of Theorem 5. From Equations (41) and (43), we have:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} I_{AGz} = & \sum_{s=0}^{z} \varepsilon_A^s \Big\{ \left(p_{Ai}^{\alpha} A_M^{\alpha} q_{Ai} - H_A\right)\dot{\xi}_{A0}^s + \Big(\dot{p}_{Ai}^{\alpha} A_M^{\alpha} q_{Ai} + p_{Ai}^{\alpha} \tfrac{\mathrm{d}}{\mathrm{d}t} A_M^{\alpha} q_{Ai} - \tfrac{\partial H_A}{\partial t} - \tfrac{\partial H_A}{\partial q_{Ai}}\dot{q}_{Ai} \\ & - \tfrac{\partial H_A}{\partial p_{Ai}}\dot{p}_{Ai} - \tfrac{\partial H_A}{\partial p_{Ai}^{\alpha}}\dot{p}_{Ai}^{\alpha}\Big)\xi_{A0}^s + p_{Ai}^{\alpha} A_M^{\alpha}\left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) + \left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) \cdot \big[B_{M^*}^{\alpha} p_{Ai}^{\alpha} \\ & - m p_{Ai}(b)\kappa_{1-\alpha}(b,\tau) + \omega p_{Ai}(a)\kappa_{1-\alpha}(\tau,a)\big] + \dot{p}_{Ai}\xi_{Ai}^s + p_{Ai}\dot{\xi}_{Ai}^s \\ & + \omega p_{Ai}^{\alpha} \cdot q_{Ai}(b) \cdot \xi_{A0}^s\left(b, q_A(b), p_A(b), p_A^{\alpha}(b)\right) \tfrac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(b,t) \\ & - m p_{Ai}^{\alpha} \cdot q_{Ai}(a) \cdot \xi_{A0}^s\left(a, q_A(a), p_A(a), p_A^{\alpha}(a)\right) \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}\kappa_{1-\alpha}(t,a) + \dot{G}_A^s \\ = & \sum_{s=0}^{z} \varepsilon_A^s \Big\{ \tfrac{\partial H_A}{\partial q_A}\xi_{Ai}^s + W_{Ai}\left(\xi_{Ai}^{s-1} - \dot{q}_{Ai}\xi_{A0}^{s-1}\right) - \left(\tfrac{\partial H_A}{\partial q_{Ai}}\dot{q}_{Ai} + \tfrac{\partial H_A}{\partial p_{Ai}}\dot{p}_{Ai}\right)\xi_{A0}^s + \dot{p}_{Ai}\xi_{Ai}^s \\ & + \left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) \cdot \left[B_{M^*}^{\alpha} p_{Ai}^{\alpha} - m p_{Ai}(b)\kappa_{1-\alpha}(b,\tau) + \omega p_{Ai}(a)\kappa_{1-\alpha}(\tau,a)\right] \Big\} \\ = & \sum_{s=0}^{z} \varepsilon_A^s \left[-\varepsilon_A W_{Ai}\left(\xi_{Ai}^s - \dot{q}_{Ai}\xi_{A0}^s\right) + W_{Ai}\left(\xi_{Ai}^{s-1} - \dot{q}_{Ai}\xi_{A0}^{s-1}\right)\right] = -\varepsilon_A^{z+1} W_{Ai}\left(\xi_{Ai}^z - \dot{q}_{Ai}\xi_{A0}^z\right). \end{aligned} \tag{45}$$

This proof is completed. □

We assume that the Hamiltonian system (Equation (22)) is disturbed as

$$A_{M^*}^{\alpha} p_{Bi}^{\alpha} = -\dot{p}_{Bi} - \frac{\partial H_B}{\partial q_{Bi}} - \varepsilon_B W_{Bi}(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^{\alpha}}),\ B_M^{\alpha} q_{Bi} = \frac{\partial H_B}{\partial p_{Bi}^{\alpha}},\ \dot{q}_{Bi} = \frac{\partial H_B}{\partial p_{Bi}}. \tag{46}$$

In this case, if the function G_B and the infinitesimal generators ξ_{B0}, ξ_{Bi}, η_{Bi} and η^{α}_{Bi} of the disturbed system (Equation (46)) have the forms

$$\begin{gathered} G_B = G_B^0 + \varepsilon_B G_B^1 + \varepsilon_B^2 G_B^2 + \cdots = \varepsilon_B^s G_B^s,\ \xi_{B0} = \xi_{B0}^0 + \varepsilon_B \xi_{B0}^1 + \varepsilon_B^2 \xi_{B0}^2 + \cdots = \varepsilon_B^s \xi_{B0}^s, \\ \eta_{Bi} = \eta_{Bi}^0 + \varepsilon_B \eta_{Bi}^1 + \varepsilon_B^2 \eta_{Bi}^2 + \cdots = \varepsilon_B^s \eta_{Bi}^s,\ \eta_{Bi} = \eta_{Bi}^{\alpha 0} + \varepsilon_A \eta_{Bi}^{\alpha 1} + \varepsilon_A^2 \eta_{Bi}^{\alpha 2} + \cdots = \varepsilon_B^s \eta_{Bi}^{\alpha s}, \\ i = 1, 2, \cdots, n,\ s = 0, 1, 2, \cdots, \end{gathered} \tag{47}$$

then we have:

Theorem 6. *For the disturbed Hamiltonian system (Equation(46)), if there exists a function G_B^s such that the infinitesimal generators ξ_{B0}^s, ξ_{Bi}^s, η_{Bi}^s and $\eta_{Bi}^{\alpha s}$ satisfy*

$$\begin{gathered}p_{Bi}\dot{\xi}_{Bi}^{s}+p_{Bi}^{\alpha}B_{M}^{\alpha}\left(\xi_{Bi}^{s}-\dot{q}_{Bi}\xi_{B0}^{s}\right)+\left(p_{Bi}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}B_{M}^{\alpha}q_{Bi}-\frac{\partial H_B}{\partial t}\right)\xi_{B0}^{s}-\frac{\partial H_B}{\partial q_{Bi}}\xi_{Bi}^{s}\\ +\left(p_{Bi}^{\alpha}\cdot B_{M}^{\alpha}q_{Bi}-H_B\right)\dot{\xi}_{B0}^{s}+\omega p_{Bi}^{\alpha}\cdot\dot{q}_{Bi}(b)\cdot\xi_{B0}^{s}\left(b,q_B(b),p_B(b),p_B^{\alpha}(b)\right)\kappa_{1-\alpha}(b,t)\\ -mp_{Bi}^{\alpha}\cdot\dot{q}_{Bi}(a)\cdot\xi_{B0}^{s}\left(a,q_B(a),p_B(a),p_B^{\alpha}(a)\right)\cdot\kappa_{1-\alpha}(t,a)-W_{Bi}\left(\xi_{Bi}^{s-1}-\dot{q}_{Bi}\xi_{B0}^{s-1}\right)+\dot{G}_B^{s}=0,\end{gathered}\tag{48}$$

where $\xi_{Bi}^{s-1}=\xi_{B0}^{s-1}=0$ when $s=0$, then there exists an adiabatic invariant

$$\begin{aligned}I_{BGz}=\sum_{s=0}^{z}\varepsilon_B^s\Big\{&\left(p_{Bi}^{\alpha}B_M^{\alpha}q_{Bi}-H_B\right)\xi_{B0}^s+\int_a^t\left[p_{Bi}^{\alpha}B_M^{\alpha}\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)+\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)\right.\\ &\left.\times A_{M^*}^{\alpha}p_{Bi}^{\alpha}\right]\mathrm{d}\tau+p_{Bi}\xi_{Bi}^s+\omega\cdot\dot{q}_{Bi}(b)\cdot\xi_{B0}^s\left(b,q_B(b),p_B(b),p_B^{\alpha}(b)\right)\\ &\times\int_a^t p_{Bi}^{\alpha}(\tau)\kappa_{1-\alpha}(b,\tau)\mathrm{d}\tau-m\cdot\dot{q}_{Bi}(a)\cdot\xi_{B0}^s\left(a,q_B(a),p_B(a),p_B^{\alpha}(a)\right)\\ &\times\int_a^t p_{Bi}^{\alpha}(\tau)\kappa_{1-\alpha}(\tau,a)\mathrm{d}\tau+G_B^s\Big\}.\end{aligned}\tag{49}$$

Proof of Theorem 6. From Equations (46) and (48), we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{BGz}&=\sum_{s=0}^{z}\varepsilon_B^s\Big\{\left(p_{Bi}^{\alpha}B_M^{\alpha}q_{Bi}-H_B\right)\dot{\xi}_{B0}^s+\left(\dot{p}_{Bi}^{\alpha}B_M^{\alpha}q_{Bi}+p_{Bi}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}B_M^{\alpha}q_{Bi}-\frac{\partial H_B}{\partial t}-\frac{\partial H_B}{\partial q_{Bi}}\dot{q}_{Bi}\right.\\ &\left.-\frac{\partial H_B}{\partial p_{Bi}}\dot{p}_{Bi}-\frac{\partial H_B}{\partial p_{Bi}^{\alpha}}\dot{p}_{Bi}^{\alpha}\right)\xi_{B0}^s+p_{Bi}^{\alpha}B_M^{\alpha}\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)+\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)\cdot A_{M*}^{\alpha}p_{Bi}^{\alpha}\\ &+\dot{p}_{Bi}\xi_{Bi}^s+p_{Bi}\dot{\xi}_{Bi}^s+\omega p_{Bi}^{\alpha}\cdot\dot{q}_{Bi}(b)\cdot\xi_{B0}^s\left(b,q_B(b),p_B(b),p_B^{\alpha}(b)\right)\kappa_{1-\alpha}(b,t)\\ &-mp_{Bi}^{\alpha}\cdot\dot{q}_{Bi}(a)\cdot\xi_{B0}^s\left(a,q_B(a),p_B(a),p_B^{\alpha}(a)\right)\cdot\kappa_{1-\alpha}(t,a)+\dot{G}_A^s\\ &=\sum_{s=0}^{z}\varepsilon_B^s\left[\frac{\partial H_B}{\partial q_B}\xi_{Bi}^s+W_{Bi}\left(\xi_{Bi}^{s-1}-\dot{q}_{Bi}\xi_{B0}^{s-1}\right)-\left(\frac{\partial H_B}{\partial q_{Bi}}\dot{q}_{Bi}+\frac{\partial H_B}{\partial p_{Bi}}\dot{p}_{Bi}\right)\xi_{B0}^s+\dot{p}_{Bi}\xi_{Bi}^s\right.\\ &\left.+\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)\cdot A_{M^*}^{\alpha}p_{Bi}^{\alpha}\right]\\ &=\sum_{s=0}^{z}\varepsilon_B^s\left[-\varepsilon_B W_{Bi}\left(\xi_{Bi}^s-\dot{q}_{Bi}\xi_{B0}^s\right)+W_{Bi}\left(\xi_{Bi}^{s-1}-\dot{q}_{Bi}\xi_{B0}^{s-1}\right)\right]=-\varepsilon_B^{z+1}W_{Bi}\left(\xi_{Bi}^z-\dot{q}_{Bi}\xi_{B0}^z\right).\end{aligned}\tag{50}$$

This proof is completed. □

Remark 6. *Let $\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$. When $M=M_1$, $M=M_2$ and $M=M_3$, we can obtain the adiabatic invariants in terms of the left Riemann–Liouville fractional operator, the right Riemann–Liouville fractional operator, the Riesz–Riemann–Liouville fractional operator, the left Caputo fractional operator, the right Caputo fractional operator, and the Riesz–Caputo fractional operator from Theorem 5 and Theorem 6, respectively. These results are consistent with the ones obtained in Ref. [29].*

Remark 7. *If we let $\alpha\to 1$, all the six cases in Remark 14 are simplified to the classical adiabatic invariant, which can also be found in Ref. [29].*

Remark 8. *When $z=0$, the conserved quantities of Theorems 2 and 4 can be obtained from the adiabatic invariants of Theorem 5 and Theorem 6, respectively.*

6. Examples

In this section, two examples are given to illustrate the results and methods.

Example 1. *The Hamiltonian system within the generalized operator A.*

For the Lagrangian,

$$L_A=\frac{1}{2}m\left[\left(A_M^{\alpha}q_{A1}\right)^2+\left(A_M^{\alpha}q_{A2}\right)^2+\dot{q}_{A1}^2+\dot{q}_{A2}^2\right],\tag{51}$$

try to study its conserved quantity and adiabatic invariant.

From Equation (51), we have

$$\begin{array}{c} p_{A1}^{\alpha}=\frac{\partial L_A}{\partial A_M^{\alpha} q_{A1}}=mA_M^{\alpha} q_{A1},\ p_{A2}^{\alpha}=\frac{\partial L_A}{\partial A_M^{\alpha} q_{A2}}=mA_M^{\alpha} q_{A2},\ p_{A1}=\frac{\partial L_A}{\partial \dot{q}_{A1}}=m\dot{q}_{A1}, \\ p_{A2}=\frac{\partial L_A}{\partial \dot{q}_{A2}}=m\dot{q}_{A2},\ H_A=\frac{1}{2m}\left[\left(p_{A1}^{\alpha}\right)^2+\left(p_{A2}^{\alpha}\right)^2+p_{A1}^2+p_{A2}^2\right]. \end{array} \tag{52}$$

Then, Equation (16) gives the Hamilton equation

$$\begin{array}{c} A_M^{\alpha} q_{A1}=\frac{1}{m}p_{A1}^{\alpha},\ A_M^{\alpha} q_{A2}=\frac{1}{m}p_{A2}^{\alpha},\ \dot{q}_{A1}=\frac{1}{m}p_{A1},\ \dot{q}_{A2}=\frac{1}{m}p_{A2}, \\ B_{M^*}^{\alpha} p_{A1}^{\alpha}=-\dot{p}_{A1}+mp_{A1}^{\alpha}(b)\kappa_{1-\alpha}(b,t)-\omega p_{A1}^{\alpha}(a)\kappa_{1-\alpha}(t,a), \\ B_{M^*}^{\alpha} p_{A2}^{\alpha}=-\dot{p}_{A2}+mp_{A2}^{\alpha}(b)\kappa_{1-\alpha}(b,t)-\omega p_{A2}^{\alpha}(a)\kappa_{1-\alpha}(t,a). \end{array} \tag{53}$$

Under the condition $(\mathrm{d}/\mathrm{d}t)\kappa_\alpha(t,\tau)=-(\mathrm{d}/\mathrm{d}\tau)\kappa_\alpha(t,\tau)$, we can verify that

$$\xi_{A0}^0=1,\ \xi_{A1}^0=\xi_{A2}^0=0,\ G_A^0=0 \tag{54}$$

satisfy the Noether quasi-identity (Equation (30)). Therefore, from Theorem 2, we have

$$\begin{aligned} I_{AG0}=&\ p_{A1}^{\alpha}\cdot A_M^{\alpha} q_{A1}+p_{A2}^{\alpha}\cdot A_M^{\alpha} q_{A2}-H_A-\int_a^t\Big\{p_{A1}^{\alpha}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}A_M^{\alpha} q_{A1}+\dot{q}_{A1}\cdot\big[B_{M^*}^{\alpha} p_{A1}^{\alpha} \\ &-mp_{A1}^{\alpha}(b)\kappa_{1-\alpha}(b,\tau)+\omega p_{A1}^{\alpha}(a)\kappa_{1-\alpha}(\tau,a)\big]+p_{A2}^{\alpha}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}A_M^{\alpha} q_{A2} \\ &+\dot{q}_{A2}\big[B_{M^*}^{\alpha} p_{A2}^{\alpha}-mp_{A2}^{\alpha}(b)\kappa_{1-\alpha}(b,\tau)+\omega p_{A2}^{\alpha}(a)\kappa_{1-\alpha}(\tau,a)\big]\Big\}\mathrm{d}\tau \\ =&\ \tfrac{1}{2m}\left[\left(p_{A1}^{\alpha}\right)^2+\left(p_{A2}^{\alpha}\right)^2-p_{A1}^2-p_{A2}^2\right]-\int_a^t\Big\{p_{A1}^{\alpha}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}A_M^{\alpha} q_{A1}+\dot{q}_{A1}\cdot\big[B_{M^*}^{\alpha} p_{A1}^{\alpha} \\ &-mp_{A1}^{\alpha}(b)\kappa_{1-\alpha}(b,\tau)+\omega p_{A1}^{\alpha}(a)\kappa_{1-\alpha}(\tau,a)\big]+p_{A2}^{\alpha}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}A_M^{\alpha} q_{A2} \\ &+\dot{q}_{A2}\big[B_{M^*}^{\alpha} p_{A2}^{\alpha}-mp_{A2}^{\alpha}(b)\kappa_{1-\alpha}(b,\tau)+\omega p_{A2}^{\alpha}(a)\kappa_{1-\alpha}(\tau,a)\big]\Big\}\mathrm{d}\tau=\mathrm{const}. \end{aligned} \tag{55}$$

When the system is disturbed by $-\varepsilon_A W_{A1}(t,\mathbf{q_A},\mathbf{p_A},\mathbf{p_A^\alpha})=-\varepsilon_A q_{A2}$ and $-\varepsilon_A W_{A2}(t,\mathbf{q_A},\mathbf{p_A},\mathbf{p_A^\alpha})=-\varepsilon_A q_{A1}$, then we can find that

$$\xi_{A0}^1=1,\ \xi_{A1}^1=\xi_{A2}^1=0,\ G_A^1=-q_{A1}q_{A2} \tag{56}$$

is a solution to Equation (43). Therefore, from Theorem 5, we obtain

$$I_{AG1}=I_{AG0}+\varepsilon_A(I_{AG0}-q_{A1}q_{A2}). \tag{57}$$

Specifically, let $\kappa_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $M=M_1$ (or $M=M_2$ or $M=M_3$) and $\alpha\to 1$, we have

$$I_{AG0C}=-H_A=\mathrm{const},\ I_{AG1C}=-H_A-\varepsilon_A(H_A+q_{A1}q_{A2}). \tag{58}$$

Example 2. *The Hamiltonian system within the generalized operator B.*

For the Lagrangian,

$$L_B=\frac{1}{2}\left[(B_M^{\alpha} q_B)^2+\dot{q}_B{}^2\right]-q_B, \tag{59}$$

try to find its conserved quantity and adiabatic invariant.

From Equation (59), we have

$$p_B^{\alpha}=\frac{\partial L_B}{\partial B_M^{\alpha} q_B}=B_M^{\alpha} q_B,\ p_B=\frac{\partial L_B}{\partial \dot{q}_B}=\dot{q}_B, H_B=\frac{1}{2}\left[(p_B^{\alpha})^2+p_B^2\right]+q_B. \tag{60}$$

Then, Equation (22) gives the Hamilton equation

$$B_M^{\alpha} q_B=p_B^{\alpha},\ \dot{q}_B=p_B,\ A_{M^*}^{\alpha} p_B^{\alpha}=-\dot{p}_B-1. \tag{61}$$

Under the condition $(\mathrm{d}/\mathrm{d}t)\kappa_\alpha(t,\tau) = -(\mathrm{d}/\mathrm{d}\tau)\kappa_\alpha(t,\tau)$, we can verify that

$$\xi_{B0}^0 = 1,\ \xi_{B1}^0 = 0,\ G_B^0 = 0 \tag{62}$$

satisfy the Noether quasi-identity (Equation (37)). Therefore, from Theorem 4, we have

$$\begin{aligned} I_{BG0} &= p_B^\alpha \cdot B_M^\alpha q_B - H_B - \int_a^t \left(\dot{q}_B A_{M^*}^\alpha p_B^\alpha + p_B^\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} B_M^\alpha q_B\right)\mathrm{d}\tau \\ &= \frac{1}{2}\left[\left(p_B^\alpha\right)^2 - p_B^2 - 2q_B\right] - \int_a^t \left(\dot{q}_B A_{M^*}^\alpha p_B^\alpha + p_B^\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} B_M^\alpha q_B\right)\mathrm{d}\tau = \mathrm{const.} \end{aligned} \tag{63}$$

When the system is disturbed by $-\varepsilon_B W_{B1}(t, \mathbf{q_B}, \mathbf{p_B}, \mathbf{p_B^\alpha}) = -\varepsilon_B(2q_{B1}+1)$, then we can find that

$$\xi_{B0}^1 = 1,\ \xi_{B1}^1 = 0,\ G_B^1 = -q_{B1}^2 - q_{B1} \tag{64}$$

is a solution to Equation (48). Therefore, from Theorem 6, we have

$$I_{BG1} = I_{BG0} + \varepsilon_B\left(I_{BG0} - q_{B1}^2 - q_{B1}\right). \tag{65}$$

Specifically, let $\kappa_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $M = M_1$ (or $M = M_2$ or $M = M_3$) and $\alpha \to 1$, we have

$$I_{BG0C} = -H_B = \mathrm{const},\ I_{BG1C} = -H_B - \varepsilon_A(H_B + q_{B1}^2 + q_{B1}). \tag{66}$$

Remark 9. *We only provide two illustrative examples to explain the obtained methods and results. In fact, the symmetry of the Hamiltonian system can be applied to many problems, such as the Lotka biochemical oscillator model, the Toda lattice with three particles, the Emden equation, etc. [22,23].*

7. Conclusions

On the basis of the generalized operators, fractional variational problems are studied, Hamilton equations are established, and several special cases of the Hamilton equations are presented. Some results are consistent with the existing ones, while some are new. In order to reduce the degrees of the freedom of the differential equations and to better analyze the dynamic behaviors of the system, Noether symmetry and conserved quantities as well as perturbation to Noether symmetry and the corresponding adiabatic invariants are investigated. Hamilton equations (Equations (16) and (22)), Noether theorems (Theorems 1–4) and adiabatic invariants (Theorems 5 and 6) are all new work.

However, only the Noether symmetry method is studied here. In fact, fractional symmetry analysis and conservation laws can be adopted for many specific equations [40–43]. Particularly, for constrained mechanics systems, except for the Noether symmetry method, the Lie symmetry method and the Mei symmetry method are also two useful methods for solving differential equations of motion. The Lie symmetry is a kind of invariance of the differential equations under the infinitesimal transformations of time and coordinates. The Mei symmetry is a kind of invariance under which the transformed dynamical functions still satisfy the original differential equations of motion. The relationships between the three symmetry methods can be read in Ref. [21]. Therefore, the Lie symmetry method and the Mei symmetry method are to be studied in the near future.

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