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Study on Poisson Algebra and Automorphism of a Special Class of Solvable Lie Algebras

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Abstract: We define a four-dimensional Lie algebra g in this paper and then prove that this Lie algebra is solvable but not nilpotent. Due to the fact that g is a Lie algebra, $\forall x, y \in g, [x, y] = -[y, x]$, that is, the operation $[,]$ has anti symmetry. Symmetry is a very important law, and antisymmetry is also a very important law. We studied the structure of Poisson algebras on g using the matrix method. We studied the necessary and sufficient conditions for the automorphism of this class of Lie algebras, and give the decomposition of its automorphism group by $Aut(g) = G_3G_1G_2G_3G_4G_7G_8G_5$, or $Aut(g) = G_3G_1G_2G_3G_4G_7G_8G_5G_6$, or $Aut(g) = G_3G_1G_2G_3G_4G_7G_8G_5G_3$, where G_i is a commutative subgroup of $Aut(g)$. We give some subgroups of g's automorphism group and systematically studied the properties of these subgroups.

Keywords: Poisson algebra; solvable Lie algebra; isomorphism; isomorphic group

MSC: 17B30

1. Introduction

In the past 20 years, Poisson algebra , which has a wide and profound application, has attracted the interests of many researchers, see [1–6] for details.

In [1–3], the authors studied DG Poisson algebras, Poisson Hopf algebras, Poisson ore extensions and their universal envelope algebras. Jie Tong and Quanqin Jin studied non-commutative Poisson algebra structures on the Lie algebra $so_n(\widetilde{\mathbb{C}_Q})$ and $sl_n(\widetilde{\mathbb{C}_Q})$ in [4]. Poisson algebra structures on toroidal Lie algebras, Witt algebra, and Virasoro algebra were studied by researchers in [5,6]. In this paper, we studied the structure of Poisson algebras over four-dimensional Lie algebra g using the matrix method.

Scholars have obtained many profound results on the automorphism of Lie algebras. In [7–9], scholars studied the automorphisms of many kinds of Lie algebras, such as the Bianchi model Lie groups and matrix algebras over communicative rings. Automorphisms of some matrix algebras were discussed by scholars in [10–12]. Automorphisms of some triangular matrices over commutative rings were explored by researchers in [13–15]. In [16], Qiu Yu and Dengyin Wang and Shikun Ou studied the automorphism of standard Borel subalgebras of CM type Lie algebras over a co ring. In a word, many scholars have studied the automorphism of Lie algebras [17–21]. Determining the automorphism group $Aut(g)$ of a Lie algebra g is a basic problem in the study of the structure theory of Lie algebras. The structure problem of Lie algebras also occupies an irreplaceable position in the study of the structure theory of Lie algebras. The author of this paper has also studied the structure and representation of Lie algebras [22–26].

The set of all second-order square matrices on the complex field $\mathcal{C}$ is denoted as $gl(2, \mathcal{C})$. The definition of $gl(2, \mathcal{C})$ in the four-dimensional Lie algebra satisfies a very special lie oper-

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ation different from the general one. This operation also satisfies the bilinear, antisymmetric condition and the square bracket product of Jacobi constant equation as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$[A, B] = \begin{pmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} - b_{11}a_{12} - b_{12}a_{22} \\ 0 & 0 \end{pmatrix}.$$

A group of bases of $gl(2, \mathcal{C})$ is $e_{11}, e_{12}, e_{21}, e_{22}$ and satisfies the following formula:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$[e_{11}, e_{12}] = e_{12},\ [e_{11}, e_{21}] = 0,\ [e_{11}, e_{22}] = 0,$$
$$[e_{12}, e_{21}] = 0,\ [e_{12}, e_{22}] = e_{12},\ [e_{21}, e_{22}] = 0.$$

For convenience, Lie algebra $gl(2, \mathcal{C})$ is written as Lie algebra g, and represents its square bracket product [A,B] as function $F(A, B)$.

Let R be an elementary divisor ring or a local ring; [27] determined the automorphisms of the general Lie operation

$$[A, B] = AB - BA,$$

linear Lie algebra $sl(2, R)$and the general linear Lie algebra $gl(2, R)$. However, in this paper, we discussed the automorphisms of the Lie operation of $gl(2, \mathcal{C})$as:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$F(A, B) = \begin{pmatrix} 0 & a_{11}b_{12} + a_{12}b_{22} - b_{11}a_{12} - b_{12}a_{22} \\ 0 & 0 \end{pmatrix}.$$

2. Main Results

After calculation, it can be verified that $g^{(1)} = [g, g]$ is a one-dimensional ideal generated by e_{12}. So, g is a four-dimensional solvable Lie algebra. The center $Z(g)$ of g is a two-dimensional subspace generated by $e_{21}, e_{11} + e_{22}$. In order to save space, it is no longer verified.

Theorem 1. *g is not a nilpotent Lie algebra.*

Proof. According to the operation law between the basis vectors of g, the following formula can be obtained.

$$F(e_{11}, e_{12}) = e_{12},$$
$$F(e_{11}, F(e_{11}, e_{12})) = e_{12},$$
$$F(e_{11}, F(e_{11}, F(e_{11}, e_{12}))) = e_{12},$$
$$F(e_{11}, \cdots F(e_{11}, F(e_{11}, e_{12}))) = e_{12}.$$

Thus, g is not a nilpotent Lie algebra. □

Poisson algebra is defined below.

Definition 1. *Define the Poisson algebra $(g, *, [-,-])$ on the base field $\mathcal{C}$, which is a vector space g on $\mathcal{C}$, and has bilinear product $*$ and Lie algebra structure $[-,-]$, and the following Leibniz rule holds:*

$$F(z * x, y) = F(z, y) * x + z * F(x, y). \forall z, x, y \in g.$$

For any z, x, y in g, $*$ does not necessarily satisfy the associative law and commutative law.

Since $*$ is a bilinear binary operation,

$$x * (k_1 y + k_2 z) = k_1(x * y) + k_2(x * z), \forall x, y, z \in g, \forall k_1, k_2 \in \mathcal{C},$$

Thus, $*$ induces a left multiply linear transformation L_x. Since $\forall x \in g, x$ can be linearly represented by base $e_{11}, e_{12}, e_{21}, e_{22}$. We only need to calculate the matrices of the linear transformation

$$L_{e_{11}}, L_{e_{12}}, L_{e_{21}}, L_{e_{22}}$$

under the basis of

$$e_{11}, e_{12}, e_{21}, e_{22}.$$

When studying the Poisson algebra structure of Lie algebra g, we marke e_{11} as e_1, e_{12} as e_2, e_{21} as e_3, and e_{22} as e_4. Note that we only simplify the sign in this way when we study the Poisson algebra structure of the Lie algebra g. When we study the automorphism of the Lie algebra g, we still use the original sign, because

$$F(e_{11}, e_{12}) = e_{12}, F(e_{11}, e_{21}) = 0, F(e_{11}, e_{22}) = 0,$$

$$F(e_{12}, e_{21}) = 0, F(e_{12}, e_{22}) = e_{12}, F(e_{21}, e_{22}) = 0.$$

thus,

$$F(e_1, e_2) = e_2, F(e_1, e_3) = 0, F(e_1, e_4) = 0,$$
$$F(e_2, e_3) = 0, F(e_2, e_4) = e_2, F(e_3, e_4) = 0.$$

Theorem 2. *$(g, *, [-,-])$ is a Poisson algebra on $(g, [-,-])$, then:*

$$L_{e_1}(e_1, e_2, e_3, e_4) = (e_1 * e_1, e_1 * e_2, e_1 * e_3, e_1 * e_4) = (e_1, e_2, e_3, e_4)A,$$

$$A = \begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{31} & a_{41} \\ 0 & a_{22} & 0 & 0 \\ a_{13} & 0 & a_{33} & a_{43} \\ a_{14} & 0 & a_{34} & a_{44} \end{pmatrix}$$

$$L_{e_2}(e_1, e_2, e_3, e_4) = (e_2 * e_1, e_2 * e_2, e_2 * e_3, e_2 * e_4) = (e_1, e_2, e_3, e_4)B,$$

$$B = \begin{pmatrix} b_{11} & b_{21} & b_{31} & b_{41} \\ b_{12} & b_{22} & b_{32} & b_{42} \\ b_{13} & b_{23} & b_{33} & b_{43} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{12} & 0 & b_{32} & b_{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_{e_3}(e_1, e_2, e_3, e_4) = (e_3 * e_1, e_3 * e_2, e_3 * e_3, e_3 * e_4) = (e_1, e_2, e_3, e_4)C,$$

$$C = \begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & c_{43} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{31} & c_{41} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{43} \\ 0 & 0 & c_{34} & c_{44} \end{pmatrix}$$

$$L_{e_4}(e_1, e_2, e_3, e_4) = (e_4 * e_1, e_4 * e_2, e_4 * e_3, e_4 * e_4) = (e_1, e_2, e_3, e_4)D,$$

$$D = \begin{pmatrix} d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \\ d_{14} & d_{24} & d_{34} & d_{44} \end{pmatrix} = \begin{pmatrix} d_{11} & 0 & d_{31} & d_{41} \\ 0 & d_{22} & 0 & 0 \\ d_{13} & 0 & d_{33} & d_{43} \\ d_{14} & 0 & d_{34} & d_{44} \end{pmatrix}.$$

Proof. Since the Leibniz law is established, there are

$$F(z * x, y) = F(z, y) * x + z * F(x, y), \forall z, x, y \in g. \tag{1}$$

Since $*$ and $[,]$ are bilinear operations. If Z, x, y can only select e_1, e_2, e_3, e_4, so that there are (1) cases of $4^3 = 64$, one by one can be verified. When

$$z = e_1, x = e_1, y = e_1,$$

the following equation can be obtained from (1):

$$F(e_1 * e_1, e_1) = F(e_1, e_1) * e_1 + e_1 * F(e_1, e_1), \tag{2a}$$

left side of (2a)

$$= F(a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4, e_1) = F(a_{12}e_2, e_1) = -a_{12}e_2,$$

right side of (2a)

$$= F(e_1, e_1) * e_1 + e_1 * F(e_1, e_1) = 0,$$

thus

$$a_{12} = 0.$$

Because

$$F(e_1 * e_1, e_2) = F(e_1, e_2) * e_1 + e_1 * F(e_1, e_2), \tag{2b}$$

left side of (2b)

$$= F(a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4, e_2) = (a_{11} - a_{14})e_2,$$

right side of (2b)

$$= F(e_1, e_2) * e_1 + e_1 * F(e_1, e_2) = e_2 * e_1 + e_1 * e_2 = b_{11}e_1 + b_{12}e_2 + b_{13}e_3 + b_{14}e_4 + a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4,$$

thus

$$b_{11} + a_{21} = 0, b_{13} + a_{23} = 0, b_{14} + a_{24} = 0, b_{12} + a_{22} = (a_{11} - a_{14}).$$

Because

$$F(e_1 * e_1, e_3) = F(e_1, e_3) * e_1 + e_1 * F(e_1, e_3), \tag{2c}$$

left side of (2c) = right side of (2c), so Equation (2c) holds.

Because

$$F(e_1 * e_1, e_4) = F(e_1, e_4) * e_1 + e_1 * F(e_1, e_4), \tag{2d}$$

left side of (2d)

$$= F(a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4, e_4) = a_{12}e_2,$$

right side of (2d)

$$= F(e_1, e_4) * e_1 + e_1 * F(e_1, e_4) = 0 * e_1 + e_1 * 0 = 0,$$

thus,

$$a_{12} = 0.$$

Because

$$F(e_1 * e_2, e_1) = F(e_1, e_1) * e_2 + e_1 * F(e_2, e_1), \tag{2e}$$

left side of (2e)

$$= F(a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4, e_1) = F(a_{22}e_2, e_1) = -a_{22}e_2,$$

right side of (2e)

$$= F(e_1, e_1] * e_2 + e_1 * F(e_2, e_1) = 0 - e_1 * e_2 = -(a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4),$$

thus,

$$a_{21} = 0, a_{23} = 0, a_{24} = 0.$$

Because

$$F(e_1 * e_2, e_2) = F(e_1, e_2) * e_2 + e_1 * F(e_2, e_2), \tag{2f}$$

left side of (2f)

$$= F(a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4, e_2) = a_{21}e_2 - a_{24}e_2,$$

right side of (2f)

$$= e_2 * e_2 = b_{21}e_1 + b_{22}e_2 + b_{23}e_3 + b_{24}e_4,$$

thus,

$$b_{21} = 0, b_{23} = 0, b_{24} = 0, (a_{21} - a_{24}) = b_{22}.$$

Because

$$F(e_1 * e_2, e_3) = F(e_1, e_2) * e_3 + e_1 * F(e_2, e_3), \tag{2g}$$

left side of (2g) = right side of (2g) so Equation (2g) holds.

Because

$$F(e_1 * e_2, e_4) = F(e_1, e_4) * e_2 + e_1 * F(e_2, e_4), \tag{2h}$$

and it can be known from (2e):

$$a_{21} = 0, a_{23} = 0, a_{24} = 0.$$

left side of (2h)

$$F(a_{22}e_2, e_4) = a_{22}e_2,$$

right side of (2h)

$$F(e_1, e_4) * e_2 + e_1 * F(e_2, e_4) = 0 + e_1 * e_2 = a_{22}e_2,$$

So the equation holds.

By analogy from the remaining 56 cases:

$$a_{32} = 0, b_{31} = 0, b_{33} = 0, b_{34} = 0, (a_{31} - a_{34}) = b_{32}, a_{42} = 0,$$

$$b_{21} = b_{23} = b_{22} = b_{24} = 0, b_{41} = a_{21}, b_{42} - a_{22} = a_{41} - a_{44}, b_{43} = a_{23}, b_{44} = a_{24}.$$

$$b_{11} = b_{13} = b_{14} = 0, b_{31} = b_{33} = b_{34} = 0, b_{22} = b_{11} - b_{14}, b_{41} = 0, b_{43} = 0, b_{44} = 0,$$

$$c_{12} = 0, c_{21} = c_{23} = c_{24} = 0, c_{22} = c_{11} - c_{14} = 0, c_{22} = 0, c_{11} = 0, c_{13} = 0, c_{14} = 0, c_{24} = 0.$$

$$c_{32} = 0, c_{31} = c_{34}, c_{42} = 0, c_{41} - c_{44} = 0,$$

$$d_{12} = 0, d_{32} = 0, d_{21} = d_{23} = d_{24} = 0, d_{31} - d_{34} = -b_{32}, d_{42} = 0, d_{41} - d_{44} = -b_{42} - d_{22}.$$

Since $*$ is a bilinear binary operation,

$$(k_1 y + k_2 z) * x = k_1(y * x) + k_2(z * x), \forall x, y, z \in g, \forall k_1, k_2 \in \mathcal{C}.$$

Thus, $*$ induces a right multiply linear transformation R_x. Since any x in g can be linearly represented by base $e_{11}, e_{12}, e_{21}, e_{22}$, we only need to calculate the matrices of linear transformation

$$R_{e_{11}}, R_{e_{12}}, R_{e_{21}}, R_{e_{22}}$$

under base

$$e_{11}, e_{12}, e_{21}, e_{22}.$$

□

Since it is similar to Theorem 2, in order to save space, it will not be described again.

In addition, if any z, x, y in g, $*$ satisfies the associative law or the commutative law, there will be more strict requirements for the matrix A, B, C, D. In order to save space, we will not repeat it.

Definition 2. *Let g_1, g_2 be a Lie algebra over field F. If the linear mapping φ of g_1 to g_2 satisfies*

$$F(\varphi(x), \varphi(y)) = \varphi(F(x, y)), \forall x, y \epsilon g_1,$$

then 6φ is said to be a homomorphic mapping or homomorphism of g_1 to g_2.

Definition 3. *The homomorphism of a Lie algebra g to itself is called the endomorphism of g, and all endomorphisms of g are denoted as $End(g)$. The isomorphism from g to itself is called automorphism and all automorphisms of g form a group, which is called the automorphism group of g and is called $Aut(g)$.*

Theorem 3. *The linear mapping in the four-dimensional Lie algebra g is established as follows:*

$$\varphi(e_{11}, e_{12}, e_{21}, e_{22}) = (e_{11}, e_{12}, e_{21}, e_{22}) \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}. \end{pmatrix}.$$

If φ is an automorphism of Lie algebra g, then there must be

$$a_{12} = a_{32} = a_{42} = 0,$$

$$a_{23} = 0,$$

$$a_{44} - a_{14} = 1,$$

$$a_{11} - a_{41} = 1,$$

$$a_{43} - a_{13} = 0,$$

$$a_{24} + a_{21} = 0.$$

Proof. Let

$$\overline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Let φ be an automorphism on g, then φ must be a linear transformation on g.

$$\varphi(e_{11}, e_{12}, e_{21}, e_{22}) = (e_{11}, e_{12}, e_{21}, e_{22}) \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = (e_{11}, e_{12}, e_{21}, e_{22})\overline{A},$$

thus,

$$\varphi(e_{11}) = a_{11}e_{11} + a_{21}e_{12} + a_{31}e_{21} + a_{41}e_{22},$$

$$\varphi(e_{12}) = a_{12}e_{11} + a_{22}e_{12} + a_{32}e_{21} + a_{42}e_{22},$$
$$\varphi(e_{21}) = a_{13}e_{11} + a_{23}e_{12} + a_{33}e_{21} + a_{43}e_{22},$$
$$\varphi(e_{22}) = a_{14}e_{11} + a_{24}e_{12} + a_{34}e_{21} + a_{44}e_{22}.$$

According to the definition of isomorphism, $F(\varphi(e_{ij}), \varphi(e_{mn})) = \varphi(F(e_{ij}, e_{mn}))$, $(i, j, m, n = 1, 2)$,

(1) Because
$$F(e_{11}, e_{12}) = e_{12},$$
so
$$F(\varphi(e_{11}), \varphi(e_{12})) = \varphi(F(e_{11}, e_{12})) = \varphi(e_{12}),$$
$$F(a_{11}e_{11} + a_{21}e_{12} + a_{31}e_{21} + a_{41}e_{22}, a_{12}e_{11} + a_{22}e_{12} + a_{32}e_{21} + a_{42}e_{22}) = a_{12}e_{11} + a_{22}e_{12} + a_{32}e_{21} + a_{42}e_{22},$$
$$(a_{11}a_{22} - a_{12}a_{21} + a_{21}a_{42} - a_{41}a_{22})e_{12} = a_{12}e_{11} + a_{22}e_{12} + a_{32}e_{21} + a_{42}e_{22};$$
thus,
$$a_{12} = a_{32} = a_{42} = 0, a_{11}a_{22} - a_{12}a_{21} + a_{21}a_{42} - a_{41}a_{22} = a_{22}.$$

Therefore, there are
$$a_{11}a_{22} - a_{41}a_{22} = a_{22}.$$

(2) Because
$$F(e_{11}, e_{21}) = 0,$$
so
$$F(\varphi(e_{11}), \varphi(e_{21})) = \varphi(F(e_{11}, e_{21})) = 0,$$
$$F(a_{11}e_{11} + a_{21}e_{12} + a_{31}e_{21} + a_{41}e_{22}, a_{13}e_{11} + a_{23}e_{12} + a_{33}e_{21} + a_{43}e_{22}) = 0,$$
$$(a_{11}a_{23} - a_{21}a_{13} + a_{21}a_{43} - a_{41}a_{23})e_{12} = 0,$$
thus
$$a_{11}a_{23} - a_{21}a_{13} + a_{21}a_{43} - a_{41}a_{23} = 0.$$

(3) Because
$$F(e_{11}, e_{22}) = 0,$$
so
$$F(\varphi(e_{11}), \varphi(e_{22})) = \varphi(F(e_{11}, e_{22})) = 0,$$
$$F(a_{11}e_{11} + a_{21}e_{12} + a_{31}e_{21} + a_{41}e_{22}, a_{14}e_{11} + a_{24}e_{12} + a_{34}e_{21} + a_{44}e_{22}) = 0,$$
$$(a_{11}a_{24} - a_{21}a_{14} + a_{21}a_{44} - a_{41}a_{24})e_{12} = 0,$$
thus
$$a_{11}a_{24} - a_{21}a_{14} + a_{21}a_{44} - a_{41}a_{24} = 0.$$

(4) Because
$$F(e_{12}, e_{21}) = 0,$$
so
$$F(\varphi(e_{12}), \varphi(e_{21})) = \varphi(F(e_{12}, e_{21})) = 0,$$
$$F(a_{12}e_{11} + a_{22}e_{12} + a_{32}e_{21} + a_{42}e_{22}, a_{13}e_{11} + a_{23}e_{12} + a_{33}e_{21} + a_{43}e_{22}) = 0,$$
$$(a_{12}a_{23} - a_{22}a_{13} + a_{22}a_{43} - a_{42}a_{23})e_{12} = 0,$$
thus
$$a_{12}a_{23} - a_{22}a_{13} + a_{22}a_{43} - a_{42}a_{23} = 0.$$

(5) Because
$$F(e_{12}, e_{22}) = e_{12},$$

so

$$F(\varphi(e_{12}),\varphi(e_{22}))] = \varphi(F(e_{12},e_{22})) = \varphi(e_{12}),$$
$$F(a_{12}e_{11}+a_{22}e_{12}+a_{32}e_{21}+a_{42}e_{22}, a_{14}e_{11}+a_{24}e_{12}+a_{34}e_{21}+a_{44}e_{22}) = a_{12}e_{11}+a_{22}e_{12}+a_{32}e_{21}+a_{42}e_{22},$$
$$(a_{12}a_{24}-a_{22}a_{14}+a_{22}a_{44}-a_{42}a_{24})e_{12} = a_{12}e_{11}+a_{22}e_{12}+a_{32}e_{21}+a_{42}e_{22};$$

thus,

$$a_{12}=a_{32}=a_{42}=0, a_{12}a_{24}-a_{22}a_{14}+a_{22}a_{44}-a_{42}a_{24}=a_{22}.$$

Therefore, there are

$$-a_{22}a_{14}+a_{22}a_{44}=a_{22}.$$

(6) Because

$$F(e_{21},e_{22})=0,$$

so

$$F(\varphi(e_{21}),\varphi(e_{22})) = \varphi([e_{21},e_{22})) = 0,$$
$$F(a_{13}e_{11}+a_{23}e_{12}+a_{33}e_{21}+a_{43}e_{22}, a_{14}e_{11}+a_{24}e_{12}+a_{34}e_{21}+a_{44}e_{22}) = 0,$$
$$(a_{13}a_{24}-a_{23}a_{14}+a_{23}a_{44}-a_{43}a_{24})e_{12}=0;$$

thus,

$$a_{13}a_{24}-a_{23}a_{14}+a_{23}a_{44}-a_{43}a_{24}=0.$$

Based on the above six cases, the following equation holds:

$$a_{12}=a_{32}=a_{42}=0,$$
$$(a_{43}-a_{13})a_{22}=0,$$
$$a_{22}(1+a_{14}-a_{44})=0,$$
$$a_{22}(1+a_{41}-a_{11})=0,$$
$$(a_{11}-a_{41})a_{23}+(a_{43}-a_{13})a_{21}=0,$$
$$(a_{11}-a_{41})a_{24}+(a_{44}-a_{14})a_{21}=0,$$
$$(a_{13}-a_{43})a_{24}+(a_{44}-a_{14})a_{23}=0.$$

Since φ is isomorphic, $a_{12}=a_{32}=a_{42}=0$ is known from the previous reasoning, so there must be $a_{22}\neq 0$, otherwise:

$$\begin{vmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & 0 & -a_{21} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{11}-1 & 0 & a_{13} & a_{14}+1 \end{vmatrix} = 0;$$

it is an isomorphic contradiction with φ. Thus, $a_{22}\neq 0$, and the following equation holds:

$$a_{23}=0,$$
$$a_{44}-a_{14}=1,$$
$$a_{11}-a_{41}=1,$$
$$a_{43}-a_{13}=0,$$
$$a_{24}+a_{21}=0;$$

so:

$$\overline{A} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & -a_{21} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{11}-1 & 0 & a_{13} & a_{14}+1 \end{pmatrix},$$

$$|\overline{A}| = a_{22}(a_{13}(a_{34}+a_{31}) - a_{33}(a_{14}+a_{11})) \neq 0. \quad \square$$

Theorem 4. *Let*

$$G_1 = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1, \end{pmatrix} | \forall a_{21} \in \mathcal{C} \},$$

then g_1 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_1 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_1, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_1 & 1 & 0 & -m_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_2 & 1 & 0 & -m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_1+m_2 & 1 & 0 & -(m_1+m_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so $\gamma_1\gamma_2$ belongs to G_1, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$. Let

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_1 & 1 & 0 & -m_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1,$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -m_1 & 1 & 0 & m_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1.$$

So, G_1 is a commutative subgroup of $Aut(g)$. □

Theorem 5. *Let*

$$G_2 = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} | \forall a_{34} \in \mathcal{C} \};$$

then, G_2 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_2 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_2, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & m_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & m_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & m_1+m_2 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

so $\gamma_1\gamma_2$ belongs to G_2, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$.

$$\forall \gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & m_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_2, \forall m_1 \in \mathcal{C},$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -m_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_2;$$

so, G_2 is a commutative subgroup of $Aut(g)$. □

Theorem 6. *Let*

$$G_3 = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1, \end{pmatrix} | \forall a_{31} \in \mathcal{C} \},$$

then G_3 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_3 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_3, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m_1 + m_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

so, $\gamma_1\gamma_2$ belongs to G_3, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$.

$$\forall \gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1, \forall m_1 \in \mathcal{C},$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -m_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_3;$$

so, G_3 is a commutative subgroup of $Aut(g)$. □

Theorem 7. *Let*

$$G_4 = \{ \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix} | \forall a_{13} \in \mathcal{C} \},$$

then G_4 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_4 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_4, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & m_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & m_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & m_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & m_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & m_1 + m_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -(m_1 + m_2) & 1 \end{pmatrix};$$

so, $\gamma_1\gamma_2$ belongs to G_4, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$.

$$\forall \gamma_1 = \begin{pmatrix} 1 & 0 & m_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & m_1 & 1 \end{pmatrix} \in G_4, \forall m_1 \in \mathcal{C},$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & -m_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_1 & 1 \end{pmatrix} \in G_4;$$

so, G_4 is a commutative subgroup of $Aut(g)$. □

Theorem 8. *Let*

$$G_5 = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} | \forall a_{22} \neq 0, a_{33} \neq 0 \in \mathcal{C} \},$$

then G_5 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_5 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_5, $\forall a_{22} \neq 0, a_{33} \neq 0 \in \mathcal{C}$, $b_{22} \neq 0, b_{33} \neq 0 \in \mathcal{C}$, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22}b_{22} & 0 & 0 \\ 0 & 0 & a_{33}b_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

so, $\gamma_1\gamma_2$ belongs to G_5, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$.

$$\forall\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_5, \forall a_{22} \neq 0, a_{33} \neq 0, \in \mathcal{C},$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22}^{-1} & 0 & 0 \\ 0 & 0 & a_{33}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_5;$$

so, G_5 is a commutative subgroup of $Aut(g)$. □

Theorem 9. *Let*

$$G_6 = \{ \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \};$$

then G_6 is a second order cyclic subgroup of $Aut(g)$.

Proof. Obviously, $G_6 \subseteq Aut(g)$ holds.

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\forall\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in G_6,$$

$$\gamma_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in G_6;$$

so, G_6 is a second order cyclic subgroup of $Aut(g)$. □

Theorem 10. *Let*

$$G_7 = \{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} | \forall a_{11} \in \mathcal{C} \},$$

then G_7 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_7 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_7, $\forall a_{11} \neq 0, b_{11} \neq 0 \in \mathcal{C}$, there are

$$\gamma_1\gamma_2 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_{11}-1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}b_{11}-1 & 0 & 0 & 1 \end{pmatrix};$$

so, $\gamma_1\gamma_2$ belongs to G_7, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$. $\forall a_{11} \neq 0 \in \mathcal{C}$,

$$\gamma_1 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \in G_7,$$

$$\gamma_1^{-1} = \begin{pmatrix} a_{11}^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}^{-1}-1 & 0 & 0 & 1 \end{pmatrix} \in G_7;$$

so, G_7 is a commutative subgroup of $Aut(g)$. □

Theorem 11. *Let*

$$G_8 = \{ \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} | \forall a_{14} \in \mathcal{C} \};$$

then G_8 is a commutative subgroup of $Aut(g)$.

Proof. Obviously, $G_8 \subseteq Aut(g)$ holds. For any γ_1, γ_2 in G_8, $\forall a_{14} \neq -1, b_{14} \neq -1 \in \mathcal{C}$,

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & b_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+b_{14} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a_{14}b_{14}+a_{14}+b_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14}b_{14}+a_{14}+b_{14} \end{pmatrix};$$

so, $\gamma_1\gamma_2$ belongs to G_8, easy-to-know $\gamma_1\gamma_2$ is equal to $\gamma_2\gamma_1$. $\forall a_{14} \neq -1 \in \mathcal{C}$,

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \in G_8,$$

$$\gamma_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\frac{a_{14}}{1+a_{14}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\frac{a_{14}}{1+a_{14}} \end{pmatrix} \in G_8;$$

so, G_8 is a commutative subgroup of $Aut(g)$. □

Theorem 12. *G_1 and G_2 are interchangeable.*

Proof. $\forall \gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_2$, because

$$\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_2\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

so,

$$\gamma_1\gamma_2 = \gamma_2\gamma_1.$$

Thus, G_1 and G_2 are interchangeable. □

Definition 4. *Given that G_1 and G_2 are two subgroups of $Aut(g)$, let $G_1 \circ G_2 = \{xy | \forall x \in G_1, \forall y \in G_2\}$.*

Theorem 13. *$G_1 \circ G_2 = \{xy | \forall x \in G_1, \forall y \in G_2\}$ is a subgroup of $Autg$.*

Proof. Because $e \in G_1$ and $e \in G_2$, so $ee = e \in G_1 \circ G_2$.

$\forall x_1, x_2 \in G_1; y_1, y_2 \in G_2$, due to G_1 and G_2 is exchangeable, so

$$x_1y_1x_2y_2 = x_1x_2y_1y_2 \in G_1 \circ G_2,$$

and

$$(x_1y_1)^{-1} = y_1^{-1}x_1^{-1} = x_1^{-1}y_1^{-1} \in G_1 \circ G_2.$$

□

Theorem 14. *G_1 and G_3 are interchangeable.*

Proof. $\forall \gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_3$, because

$$\gamma_1\gamma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_3\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so

$$\gamma_1\gamma_3 = \gamma_3\gamma_1.$$

Thus, G_1 and G_3 are interchangeable. □

Theorem 15. *G_1 and G_4 are interchangeable.*

Proof. $\forall\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_4 = \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix} \epsilon G_4$, because

$$\gamma_1\gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix},$$

$$\gamma_4\gamma_1 = \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix};$$

so,

$$\gamma_1\gamma_4 = \gamma_4\gamma_1.$$

Thus, G_1 and G_4 are interchangeable. □

Theorem 16. *G_1 and G_5 are not necessarily interchangeable.*

Proof. $\forall\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_5$, because

$$\gamma_1\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & -a_{21} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_5\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21}a_{22} & a_{22} & 0 & -a_{21}a_{22} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

so, $\gamma_1\gamma_5$ is not necessarily equal to $\gamma_5\gamma_1$.

Thus, G_1 and G_5 are not necessarily interchangeable. □

Theorem 17. *G_1 and G_6 are interchangeable.*

Proof. $\forall\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_{61} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$

$\gamma_{62} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, because

$$\gamma_1\gamma_{61} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_{61}\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$

so,

$$\gamma_1\gamma_{61} = \gamma_{61}\gamma_1,$$

obviously,

$$\gamma_1\gamma_{62} = \gamma_{62}\gamma_1.$$

Thus, G_1 and G_4 are interchangeable. □

Theorem 18. *G_1 and G_7 are interchangeable.*

Proof. $\forall\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_7 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \epsilon G_7$, bcause

$$\gamma_1\gamma_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_7\gamma_1 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix};$$

so,

$$\gamma_1\gamma_7 = \gamma_7\gamma_1.$$

Thus, G_1 and G_7 are interchangeable. □

Theorem 19. *G_1 and G_8 are interchangeable.*

Proof. $\forall \gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \epsilon G_1, \gamma_8 = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \epsilon G_8$, because

$$\gamma_1\gamma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix},$$

$$\gamma_8\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix};$$

so,

$$\gamma_1\gamma_8 = \gamma_8\gamma_1.$$

Thus, G_1 and G_8 are interchangeable. □

Similarly, we can study the commutativity between $G_2, G_3, G_4, G_5, G_6, G_7$ and G_8. For example, we can prove that G_2 and G_3 can be exchanged, G_5 and G_7 can be exchanged, and G_5 and G_8 can be exchanged. Studying this commutativity is certainly helpful for the subsequent study of whether the decomposition of automorphism groups is unique.

Theorem 20. *The automorphism group $Aut(g) = g$ of Lie algebra g can be decomposed into the following form:*

(1) When $b_{11} \neq 0$, there are

$$G = G_3G_1G_2G_3G_4G_7G_8G_5;$$

(2) When $b_{11} = 0, b_{14} \neq 0$, there are

$$G = G_3G_1G_2G_3G_4G_7G_8G_5G_6;$$

(3) When $b_{11} = 0, b_{14} = 0, b_{13} \neq 0$, there are

$$G = G_3G_1G_2G_3G_4G_7G_8G_5G_3.$$

Proof. Take any B in G at $Aut(g)$; let

$$B = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31} & 0 & b_{33} & b_{34} \\ b_{11}-1 & 0 & b_{13} & b_{14}+1 \end{pmatrix}.$$

Using the undetermined coefficient method, for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a_{34} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_3, \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_2,$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_3 \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix} \in G_4,$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \in G_7, \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \in G_8, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_5,$$

it is advisable to set

$$B = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31} & 0 & b_{33} & b_{34} \\ b_{11}-1 & 0 & b_{13} & b_{14}+1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a_{34} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & -a_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{13} & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{11}-1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+a_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a_{34} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & a_{13} & a_{11}a_{14} \\ a_{21} & a_{22} & 0 & -a_{21} \\ a_{11}a_{31}+a_{11}a_{34}-a_{34} & 0 & a_{13}a_{31}+a_{13}a_{34}+1 & a_{11}a_{14}a_{31}+a_{11}a_{14}a_{34}+a_{34} \\ a_{11}-1 & 0 & a_{13} & a_{11}a_{14}+1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & 0 & a_{13} & a_{11}a_{14} \\ a_{21} & a_{22} & 0 & -a_{21} \\ a_{11}a_{31}-a_{34} & 0 & a_{13}a_{31}+1 & a_{11}a_{14}a_{31}+a_{34} \\ a_{11}-1 & 0 & a_{13} & a_{11}a_{14}+1 \end{pmatrix}.$$

Case 1:

$$b_{11} \neq 0,$$

because

$$B = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31} & 0 & b_{33} & b_{34} \\ b_{11}-1 & 0 & b_{13} & b_{14}+1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{11}a_{14} \\ a_{21} & a_{22} & 0 & -a_{21} \\ a_{11}a_{31}-a_{34} & 0 & a_{13}a_{31}+1 & a_{11}a_{14}a_{31}+a_{34} \\ a_{11}-1 & 0 & a_{13} & a_{11}a_{14}+1 \end{pmatrix},$$

so

$$a_{11} = b_{11},$$

$$a_{13} = b_{13},$$

$$a_{11}a_{14} = b_{14} \Rightarrow a_{14} = \frac{b_{14}}{b_{11}},$$

$$a_{21} = b_{21},$$

$$a_{22} = b_{22},$$

$$a_{11}a_{31} + 1 = b_{33} \Rightarrow a_{31} = \frac{b_{33}-1}{b_{11}},$$

$$a_{11}a_{14}a_{31} + a_{34} = b_{34} \Rightarrow a_{34} = b_{34} - b_{14}\frac{b_{33}-1}{b_{11}}.$$

Thus, the original proposition holds.
Case 2:

$$b_{11} = 0, b_{14} \neq 0,$$

Take

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in G_6,$$

$$\begin{pmatrix} 0 & 0 & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31} & 0 & b_{33} & b_{34} \\ -1 & 0 & b_{13} & b_{14}+1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -b_{14} & 0 & b_{13} & 0 \\ b_{21} & b_{22} & 0 & -b_{21} \\ -b_{34} & 0 & b_{33} & -b_{31} \\ -b_{14}-1 & 0 & b_{13} & 1 \end{pmatrix},$$

At this time, because $b_{14} \neq 0$, it is converted to case 1.
Case 3:

$$b_{11} = 0, b_{14} = 0, b_{13} \neq 0,$$

Take

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_3,$$

because

$$\begin{pmatrix} 0 & 0 & b_{13} & 0 \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31} & 0 & b_{33} & b_{34} \\ -1 & 0 & b_{13} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} b_{13} & 0 & b_{13} & 0 \\ b_{21} & b_{22} & 0 & -b_{21} \\ b_{31}+b_{33} & 0 & b_{33} & b_{34} \\ b_{13}-1 & 0 & b_{13} & 1 \end{pmatrix},$$

At this time, when $b_{13} \neq 0$, it is changed to case 1. □

3. Conclusions

This article cleverly utilized the elementary transformation of partitioned matrices to study the subgroups of a four-dimensional solvable Lie algebra g and obtain the necessary and sufficient conditions for its automorphism. It also characterized three decomposition scenarios of the automorphism group of g, presenting the structure of its automorphism group more clearly. This article added new methods to the study of low-dimensional Lie algebra automorphism, which can provide assistance for the structure of research of general Lie algebra automorphism groups.

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