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Investigating New Subclasses of Bi-Univalent Functions Associated with q-Pascal Distribution Series Using the Subordination Principle

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Abstract: In the real world, there are many applications that find the Pascal distribution to be a useful and relevant model. One of these is the normal distribution. In this work, we develop a new subclass of analytic bi-univalent functions by making use of the q-Pascal distribution series as a construction. These functions involve the q-Gegenbauer polynomials, and we use them to establish our new subclass. Moreover, we solve the Fekete–Szegö functional problem and analyze various different estimates of the Maclaurin coefficients for functions that belong to the new subclass.

Keywords: q-Pascal distribution series; Fekete–Szegö problem; q-calculus; bi-univalent functions; q-Gegenbauer polynomials

MSC: 30C45; 30C50

Citation: Alsoboh, A.; Amourah, A.; Darus, M.; Rudder, C.A. Investigating New Subclasses of Bi-Univalent Functions Associated with q-Pascal Distribution Series Using the Subordination Principle. *Symmetry* **2023**, *15*, 1109. https://doi.org/10.3390/sym15051109

Academic Editors: Junesang Choi, Charles F. Dunkl and Sergei D. Odintsov

Received: 24 January 2023
Revised: 24 February 2023
Accepted: 2 March 2023
Published: 18 May 2023

1. Introduction

Distributions of random variables, which represent the distribution of probabilities over the values of the random variable, play a fundamental role in statistics and probability and are widely used to describe and model a variety of real-life phenomena [1]. To emphasize the significance of certain distributions and the underlying random experiments, these distributions have been given specific names. We are interested in determining how many times we must repeat our random experiment before we achieve success. There are two different results and a geometric distribution for our random experiment (failure or success). The connection between this distribution and the geometric series gives it its name.

The symmetry and distributions of random variables are two important concepts in probability theory and statistics. Symmetry refers to the property of an object or system being unchanged after some transformation. In probability theory, symmetry is often used to describe the distribution of a random variable, which is a function that maps the outcomes of a random experiment to real numbers.

The negative binomial distribution, often known as the Pascal distribution, is the generalization of the geometric distribution. This distribution is known as a "negative binomial distribution" because it is associated with the expansion of the binomial series with a negative exponent. The random variable X in the Pascal distribution denotes the number of trials necessary to achieve r successes in successive independent Bernoulli trials.

Legendre first discovered orthogonal polynomials (OP) in 1784 [2]. OP are widely used to solve ordinary differential equations when certain model constraints are present. The OP also serve a crucial purpose in approximation theory [3].

Given two polynomials of orders α and β, which are $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$, respectively, then $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are OP over the interval $[a,b]$ if

$$\langle \mathcal{L}_\alpha,\ \mathcal{L}_\beta\rangle = \int_a^b \mathcal{L}_\alpha(\aleph)\mathcal{L}_\beta(\aleph)s(\aleph)d\aleph = 0, \qquad \text{for} \qquad \alpha \neq \beta,$$

where $s(\aleph)$ is a nonnegative function in (a,b); as a result, the integral of all finite order polynomials $\mathcal{L}_\alpha(\aleph)$ is well defined (see [2,3]).

In particular, Gegenbauer polynomials (GP) are a form of OP. According to [4], when using traditional algebraic formulations, the generating function of the GP and the integral representation of typically real functions are related to one another symbolically. This has led to several useful inequalities in the world of GP.

Many researchers are interested in quantum calculus (or q-calculus) due to its numerous applications in various branches of mathematics and physics, particularly geometric function theory. Because of the structure of q-calculus, the traditional complement method works better for various modules of OP and functions. The connection between the equilibrium states of differential formulae (equations, operators, and inequalities) and their solutions is one of the most useful and well-designed tools for analyzing the characteristics of special functions in mathematical analysis and mathematical physics. Euler and Jacobi pioneered q-calculus in the 18th century. Jackson [5,6] pioneered and systematically developed the application of q-calculus. The q-analogue of the Baskakov and Durrmeyer operator developed by Aral and Gupta [7,8] depends on quantum calculus. Recent applications of the q-operator can be found in [9–12].

We are reminded of the q-difference operator, which has applications in geometric function theory as well as other areas of research. In this section, we provide some fundamental definitions and properties of q-calculus that are applied throughout this investigation. These are based on the assumption that $\mathrm{q} \in (0,1)$ (for more details, see [13–15]).

Definition 1. *Let $0 < \mathrm{q} < 1$. Then, the* q*-factorial $[n]_\mathrm{q}!$ is defined by*

$$[n]_\mathrm{q}! = \begin{cases} \prod\limits_{s=1}^{n} [n]_\mathrm{q} = [n]_\mathrm{q}.[n-1]_\mathrm{q}\cdots[3]_\mathrm{q}.[2]_\mathrm{q}.[1]_\mathrm{q} & ,(n\in\mathbb{N}) \\ 1 & ,\quad n=0, \end{cases}$$

where $[\Bbbk]_\mathrm{q}$ denotes the basic (or q*-) number, defined by*

$$[\Bbbk]_\mathrm{q} = \begin{cases} \frac{1-\mathrm{q}^{\Bbbk}}{1-\mathrm{q}} & ,\ \Bbbk \in \mathbb{C}\backslash\{0\} \\ 0 & ,\ \Bbbk = 0 \\ 1+\mathrm{q}+\cdots+\mathrm{q}^{n-1} = \sum\limits_{i=0}^{n-1}\mathrm{q}^i & ,\ \Bbbk = n \in \mathbb{N}. \end{cases}$$

It is obvious from Definition 1 that $\lim\limits_{\mathrm{q}\to1^-}[n]_\mathrm{q} = \lim\limits_{\mathrm{q}\to1^-}\frac{1-\mathrm{q}^n}{1-\mathrm{q}} = n.$

Definition 2. *The Gaussian polynomial analogous (or* q*-binomial) coefficient is defined for non-negative integers x and r by*

$$\binom{\varkappa}{r}_\mathrm{q} = \frac{(\mathrm{q}^{\varkappa}-1)\,(\mathrm{q}^{\varkappa-1}-1)\,\cdots\,(\mathrm{q}^{\varkappa-r+1}-1)}{(\mathrm{q}-1)(\mathrm{q}^2-1)\,\cdots\,(\mathrm{q}^{r-1}-1)\,(\mathrm{q}^r-1)}, \qquad (r,n\in\mathbb{Z}_{\gg 0},\ 0\le r\le n)$$

or equivalently,

$$\binom{\varkappa}{r}_\mathrm{q} = \frac{[\varkappa]_{\mathrm{q};r}}{[r]_\mathrm{q}!} = \frac{[\varkappa]_\mathrm{q}!}{[r]_\mathrm{q}![\varkappa-r]_\mathrm{q}!}, \qquad (\varkappa\in\mathbb{C};\ r\in\mathbb{Z}_{\gg 0}),$$

where $[x]_{q;r}$ *is defined by*

$$[\varkappa]_{q;r} = [\varkappa]_q[\varkappa-1]_q \cdots [\varkappa-r+1]_q, \qquad (\varkappa \in \mathbb{C};\, r \in \mathbb{Z}_{\gg 0}).$$

Definition 3. *The q-derivative (or q-difference operator) of a function f is defined by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z-qz} & z \in \mathbb{C}\backslash\{0\} \\ f'(0) & z = 0. \end{cases}$$

We note that $\lim\limits_{q\to 1^-} \partial_q f(z) = f'(z)$, *if f is differentiable at $z \in \mathbb{C}$.*

The q-GP are a family of OP that are defined in terms of a weight function involving the q-analogue of the GP. These polynomials have many interesting properties and applications in mathematics, including the theory of OP, special functions, and quantum mechanics. The q-GP have been used to study the eigenfunctions of quantum mechanical operators and to approximate solutions to certain partial differential equations.

Putting Jackson's q-exponential into the form of a closed-form multiplicative series of regular exponentials with known coefficients is an important addition to the q-OP field made by Quesne [16]. In mathematics, the q-exponential is a type of nonstandard exponential function that is widely used in the study of q-OP and related areas. Particularly well-researched and commonly applied in the subject is Jackson's q-exponential, a specific case of the q-exponential. We use the aforementioned conclusion in particular to derive original nonlinear connection equations for q-GP in terms of their classical counterparts.

After the association of particular bi-univalent functions with q-GP, this paper examines several characteristics of the class under consideration. The foundation for the mathematical notations and definitions is established in the next section.

2. Preliminaries

Let $\mathfrak{A}$ be the class of functions f of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in \mathbb{U}), \tag{1}$$

which are analytic in the disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and gratify the normalization condition $f'(0) - 1 = 0 = f(0)$. Moreover, we represent by $\mathcal{S}$ the subclass of $\mathfrak{A}$ comprising functions of Equation (1), which are also univalent in $\mathbb{U}$.

Geometric function theory can benefit greatly from the powerful tools that differential subordination of analytical functions provides. Miller and Mocanu [17] introduced the first differential subordination problem; additionally, see [18]. The majority of the developments in the field have been compiled in Miller and Mocanu's book [19], along with the publication dates.

It is well known that there is an inverse function f^{-1} for every function $f \in \mathcal{S}$, which is defined by

$$f^{-1}(f(\xi)) = \xi \qquad (\xi \in \mathbb{U}),$$

and

$$\omega = f(f^{-1}(\omega)) \qquad \left(|\omega| < r_0(f);\, r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(\omega) = g(\omega) = \omega\Big(1 - a_2\omega + \omega^2(2a_2^2 - a_3) - \omega^3(5a_2^3 - 5a_2a_3 + a_4) + \cdots\Big). \tag{2}$$

A function is referred to as being bi-univalent in $\mathbb{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent. The class of bi-univalent functions in $\mathbb{U}$ given by (1) is denoted by Σ. Examples in Σ are

$$\frac{\xi}{1-\xi} \quad \text{and} \quad \log\left(\sqrt{\frac{1+\xi}{1-\xi}}\right).$$

The bi-univalent function class Σ was studied by Lewin [20], who showed that $|a_2| < 1.51$. Brannan and Clunie [21] then proposed the hypothesis that $|a_2| < \sqrt{2}$. On the other hand, Netanyahu [22] showed that $\max_{f \in \Sigma} |a_2| = 4/3$.

In 2005, Charalambides and Papadatos studied and introduced q-Pascal distribution [23]. When Y_k trials must be completed before the kth success occurs, the probability mass function is given by

$$\mathcal{P}_{Y_k}(\varkappa) = \binom{\varkappa - 1}{k-1}_{\mathrm{q}} \theta^{\varkappa - k} \prod_{i=1}^{k}\left(1 - \theta \mathrm{q}^{i-1}\right), \quad x = k, k+1, k+2, \cdots, \tag{3}$$

with $0 < \theta < 1, 0 < q < 1$.

By achieving $r-1$ successes in the first $x-1$ trials in any order and then achieving success on the xth trial, the probability of achieving the kth success on the xth trial is indicated by the probability density function above. If we replace y by $x+r$ in the probability density function (3), then we obtain

$$\mathcal{P}_{\mathrm{q}}(X = \varkappa) = \binom{\varkappa + r - 1}{r-1}_{\mathrm{q}} \theta^{\varkappa} \prod_{i=1}^{r}\left(1 - \theta \mathrm{q}^{i-1}\right), \quad \varkappa = 0, 1, 2, \cdots. \tag{4}$$

A power series using Pascal distribution probabilities as its coefficients was introduced as follows:

$$\mathcal{Q}_{\mathrm{q}}^{\theta,r}(\xi) = \xi + \sum_{n=2}^{\infty} \binom{n + r - 2}{r-1}_{\mathrm{q}} \theta^{n-1} \prod_{i=1}^{r}\left(1 - \theta \mathrm{q}^{i-1}\right) \xi^n, \quad (\xi \in \mathbb{U}, r \geq 1, 0 \leq \theta \leq 1). \tag{5}$$

The ratio test, it should be noted, led us to conclude that the radius of convergence of the above power series is infinite.

Now, we consider the linear operator $\mathcal{F}_{\mathrm{q}}^{\theta,r} : \mathfrak{A} \longrightarrow \mathfrak{A}$ defined by the convolution (or the Hadamard product) as

$$\mathcal{F}_{\mathrm{q}}^{\theta,r} f(\xi) = \mathcal{Q}_{\mathrm{q}}^{\theta,r} * f(\xi) = \xi + \sum_{n=2}^{\infty} \binom{n + r - 2}{r-1}_{\mathrm{q}} \theta^{n-1} \prod_{i=1}^{r}\left(1 - \theta \mathrm{q}^{i-1}\right) a_n \xi^n, \quad (\xi \in \mathbb{U}). \tag{6}$$

A family of polynomials that can be thought of as q-analogues of the GP was found in 1983 by Askey and Ismail [24]. Essentially, they are the polynomials $\mathfrak{G}_{\mathrm{q}}^{(\beth)}(\aleph, \xi)$.

$$\mathfrak{G}_{\mathrm{q}}^{(\beth)}(\aleph, \xi) = \sum_{n=0}^{\infty} \mathcal{C}_n^{(\beth)}(\aleph; \mathrm{q}) \xi^n. \tag{7}$$

The following recurrence relations can be used to interpret the class of polynomials found by Chakrabarti et al. [25] in 2006 as q-analogues of the GP:

$$\begin{aligned} &\mathcal{C}_0^{(\beth)}(\aleph; \mathrm{q}) = 1,\ \mathcal{C}_1^{(\beth)}(\aleph; \mathrm{q}) = [\beth]_{\mathrm{q}} C_1^1(\aleph) = 2[\beth]_{\mathrm{q}} \aleph, \\ &\mathcal{C}_2^{(\beth)}(\aleph; \mathrm{q}) = [\beth]_{\mathrm{q}^2} C_2^1(\aleph) - \frac{1}{2}\left([\beth]_{\mathrm{q}^2} - [\beth]_{\mathrm{q}}^2\right) C_1^2(\aleph) = 2\left([\beth]_{\mathrm{q}^2} + [\beth]_{\mathrm{q}}^2\right)\aleph^2 - [\beth]_{\mathrm{q}^2}, \end{aligned} \tag{8}$$

where $0 < \mathrm{q} < 1$, and $\beth \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

$\mathfrak{G}^{(\daleth)}(\aleph,\xi)$, where $\xi \in \mathbb{U}$ and $\aleph \in [-1,1]$, is the classical GP taken into consideration by Amourah et al. [26,27] in 2021. The function $\mathfrak{G}^{(\daleth)}$ is analytic in $\mathbb{U}$ for a fixed x, allowing it to be expanded in a Taylor series as

$$\mathfrak{G}^{(\daleth)}(\aleph,\xi) = \sum_{n=0}^{\infty} C_n^{\alpha}(\aleph)\xi^n,$$

where $C_n^{\alpha}(\aleph)$ is the classical GP of degree n.

Amourah et al. [28] showed three different subclasses of analytic and bi-univalent functions that make use of q-GP. However, by employing the q-GP in conjunction with a generalization of the neutrosophic Poisson distribution series, Alsoboh et al. [29] were able to determine the existence of a novel subclass of bi-univalent functions. It is possible to derive Fekete–Szegö inequalities for functions that correspond to these subclasses, together with the initial coefficient bounds $|a_2|$ and $|a_3|$.

Recently, several researchers started looking into subclasses of bi-univalent functions connected to OP. They discovered estimates for the initial coefficients of functions in them. However, as mentioned in a few places ([30–45]), the problem of sharp coefficient bounds for $|a_n|$, $(n = 3, 4, 5, \cdots)$ remains open.

Several researchers investigated certain subclasses of analytic functions using various probability distributions, such as the Pascal, Poisson, and Borel distributions, (see for example, [23,46–50]). There have been no previous studies that have investigated a bi-univalent class of functions using the q-Pascal distribution series in association with the q-GP using the subordination principle, to the best of the researchers' knowledge. This work's main objective is to begin investigating the characteristics of bi-univalent functions in relation to q-GP. The definitions given below begin this.

3. The Class $\mathcal{B}_{\Sigma}(\gamma,\beta,\mathfrak{G}_{\mathrm{q}}^{(\daleth)}(\aleph,\xi))$

In this section, we define and study a new subclass of bi-univalent functions in the symmetry open unit disk using the principle of subordination, using the q-Pascal distribution by constructed series (6) and q-analogues of the GP.

Definition 4. *A bi-univalent function f of the form (1) is referred to as being in the class $\mathcal{B}_{\Sigma}(\gamma,\beta,\mathfrak{G}_{\mathrm{q}}^{(\daleth)}(\aleph,\xi))$, if the following conditions of subordination are met:*

$$(1-\gamma)\frac{\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi)}{\xi} + \gamma\partial_{\mathrm{q}}\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi) + \beta\xi\partial_{\mathrm{q}}^{2}(\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi)) \prec \mathfrak{G}_{\mathrm{q}}^{(\daleth)}(\aleph,\xi), \tag{9}$$

$$(1-\gamma)\frac{\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)}{\omega} + \gamma\partial_{\mathrm{q}}\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega) + \beta\omega\partial_{\mathrm{q}}^{2}(\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)) \prec \mathfrak{G}_{\mathrm{q}}^{(\daleth)}(\aleph,\omega), \tag{10}$$

where $\daleth > 0$, $\gamma,\beta \geq 0$, $\aleph \in \left(\frac{1}{2},1\right]$, and the function $g = f^{-1}$ is given by (2).

Example 1. *A bi-univalent function f of the form (1) is referred to as being in the class $\mathcal{B}_{\Sigma}(\aleph,\daleth,\gamma,\beta) = \lim\limits_{\mathrm{q}\to 1^-} \mathcal{B}_{\Sigma}(\gamma,\beta,\mathfrak{G}_{\mathrm{q}}^{(\daleth)}(\aleph,\xi))$, if the following conditions of subordination are met:*

$$(1-\gamma)\frac{\mathcal{F}_{\theta}^{r}f(\xi)}{\xi} + \gamma(\mathcal{F}_{\theta}^{r}f(\xi))' + \beta\xi(\mathcal{F}_{\theta}^{r}f(\xi))'' \prec \mathfrak{G}^{\daleth}(\aleph,\xi), \tag{11}$$

$$(1-\gamma)\frac{\mathcal{F}_{\theta}^{r}g(\omega)}{\omega} + \gamma(\mathcal{F}_{\theta}^{r}g(\omega))' + \beta\xi(\mathcal{F}_{\theta}^{r}g(\omega))'' \prec \mathfrak{G}^{\daleth}(\aleph,\omega), \tag{12}$$

where $\daleth > 0$, $\gamma,\beta \geq 0$, $\aleph \in \left(\frac{1}{2},1\right]$.

Remark 1. *The class $\mathcal{B}_{\Sigma}(\aleph,\daleth,\gamma,\beta)$ was introduced by Amourah et al. [47].*

Example 2. *A bi-univalent function f of the form (1) is referred to as being in the class $\mathcal{B}_\Sigma(\gamma, 0, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi)) = \mathcal{B}_\Sigma(\gamma, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$, if the following conditions of subordination are met:*

$$(1-\gamma)\frac{\mathcal{F}_q^{\theta,r} f(\xi)}{\xi} + \gamma \partial_q \mathcal{F}_q^{\theta,r} f(\xi) \prec \mathfrak{G}_q^{(\daleth)}(\aleph, \xi), \tag{13}$$

$$(1-\gamma)\frac{\mathcal{F}_q^{\theta,r} g(\omega)}{\omega} + \gamma \partial_q \mathcal{F}_q^{\theta,r} g(\omega) \prec \mathfrak{G}_q^{(\daleth)}(\aleph, \omega), \tag{14}$$

where $\daleth > 0$, $\gamma \geq 0$, $\aleph \in \left(\frac{1}{2}, 1\right]$.

Example 3. *A bi-univalent function f of the form (1) is referred to as being in the class $\mathcal{B}_\Sigma(1, 0, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi)) = \mathcal{B}_\Sigma(1, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$, if the following conditions of subordination are met:*

$$\partial_q \mathcal{F}_q^{\theta,r} f(\xi) \prec \mathfrak{G}_q^{(\daleth)}(\aleph, \xi), \tag{15}$$

$$\partial_q \mathcal{F}_q^{\theta,r} g(\omega) \prec \mathfrak{G}_q^{(\daleth)}(\aleph, \omega), \tag{16}$$

where $\daleth > 0$, $\aleph \in \left(\frac{1}{2}, 1\right]$.

4. Main Results

To begin, we give some estimates for the coefficients that apply to the class $\mathcal{B}_\Sigma(\gamma, \beta, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$ that was defined in Definition 4.

Theorem 1. *Let the function f given by (1) belong to the class $\mathcal{B}_\Sigma(\gamma, \beta, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$; then,*

$$|a_2| \leq \frac{2|[\daleth]_q|\aleph\sqrt{2|[\daleth]_q|\aleph}}{\sqrt{\left|\mathfrak{L}(\daleth, \gamma, \theta, \beta, r; q)\aleph^2 + 2\left([r]_q\theta(1 + q\gamma + [2]_q\beta)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2 [\daleth]_{q^2}\right|}},$$

where

$$\begin{aligned}\mathfrak{L}(\daleth, \gamma, \theta, \beta, r; q) &= 4[\daleth]_q^2[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q} + q\gamma + [3]_q\beta\right)\prod_{i=1}^{r}(1-\theta q^{i-1}) \\ &\quad -2\left([r]_q\theta(1 + q\gamma + [2]_q\beta)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2\left([\daleth]_{q^2} + [\daleth]_q^2\right),\end{aligned}$$

and

$$\begin{aligned}|a_3| \leq &\frac{4[\daleth]_q^2\aleph^2}{\left([r]_q\theta(1 + q\gamma + [2]_q\beta)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2} \\ &+ \frac{2[\daleth]_q\aleph}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q} + q\gamma + [3]_q\beta\right)\prod_{i=1}^{r}(1-\theta q^{i-1})}.\end{aligned}$$

Proof. Let $f \in \mathcal{B}_\Sigma(\gamma, \beta, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$. From Definition 4, for some analytical Φ, Ψ such that $\Phi(0) = \Psi(0) = 0$ and $|\Phi(\xi)| < 1$, $|\Psi(\omega)| < 1$ for all $\xi, \omega \in \mathbb{U}$, we can write

$$(1-\gamma)\frac{\mathcal{F}_q^{\theta,r} f(\xi)}{\xi} + \gamma \partial_q \mathcal{F}_q^{\theta,r} f(\xi) + \beta\xi\partial_q^2(\mathcal{F}_q^{\theta,r} f(\xi)) = \mathfrak{G}_q^{(\daleth)}(\aleph, \Phi(\xi)), \tag{17}$$

and

$$(1-\gamma)\frac{\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)}{\omega}+\gamma\partial_{\mathrm{q}}\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)+\beta\omega\partial_{\mathrm{q}}^{2}(\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega))=\mathfrak{G}_{\mathrm{q}}^{(\gimel)}(\aleph,\Psi(\omega)). \tag{18}$$

From the equalities (17) and (18), we obtain that

$$\begin{aligned}&(1-\gamma)\frac{\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi)}{\xi}+\gamma\partial_{\mathrm{q}}\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi)+\beta\xi\partial_{\mathrm{q}}^{2}(\mathcal{F}_{\mathrm{q}}^{\theta,r}f(\xi))\\&=1+\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})c_{1}\xi+\left[\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})c_{2}+\mathrm{C}_{2}^{(\gimel)}(\aleph;\mathrm{q})c_{1}^{2}\right]\xi^{2}+\cdots,\end{aligned} \tag{19}$$

and

$$\begin{aligned}&(1-\gamma)\frac{\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)}{\omega}+\gamma\partial_{\mathrm{q}}\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega)+\beta\omega\partial_{\mathrm{q}}^{2}(\mathcal{F}_{\mathrm{q}}^{\theta,r}g(\omega))\\&=1+\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})d_{1}\omega+\left[\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})d_{2}+\mathrm{C}_{2}^{(\gimel)}(\aleph;\mathrm{q})d_{1}^{2}\right])\omega^{2}+\cdots.\end{aligned} \tag{20}$$

It is generally understood that if

$$|\Phi(\xi)|=\left|c_{1}\xi+c_{2}\xi^{2}+c_{3}\xi^{3}+\cdots\right|<1,\ \ (\xi\in\mathbb{U}),$$

and

$$|\Psi(\omega)|=\left|d_{1}\omega+d_{2}\omega^{2}+d_{3}\omega^{3}+\cdots\right|<1,\ \ (\omega\in\mathbb{U}),$$

then for all $(j\in\mathbb{N})$ (see [51]), we have

$$|c_{j}|\leq1 \text{ and } |d_{j}|\leq1. \tag{21}$$

The equivalent coefficients in (19) and (20) are so compared, and the result is

$$[r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)a_{2}=\mathrm{C}_{1}^{(\gimel)}(\aleph;q)c_{1}, \tag{22}$$

$$[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^{2}\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)a_{3}=\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})c_{2}+\mathrm{C}_{2}^{(\gimel)}(\aleph;q)c_{1}^{2}, \tag{23}$$

$$-[r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)a_{2}=\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})d_{1}, \tag{24}$$

and

$$[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^{2}\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\left(2a_{2}^{2}-a_{3}\right)=\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})d_{2}+\mathrm{C}_{2}^{(\gimel)}(\aleph;\mathrm{q})d_{1}^{2}. \tag{25}$$

It follows from (22) and (24) that

$$c_{1}=-d_{1}, \tag{26}$$

and

$$2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\right)^{2}a_{2}^{2}=\left[\mathrm{C}_{1}^{(\gimel)}(\aleph;\mathrm{q})\right]^{2}\left(c_{1}^{2}+d_{1}^{2}\right). \tag{27}$$

If we add (23) and (25), we obtain

$$[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)a_2^2=\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})(c_2+d_2)+\mathrm{C}_2^{(\beth)}(\aleph;\mathrm{q})(c_1^2+d_1^2). \tag{28}$$

By substituting the value of $\left(c_1^2+d_1^2\right)$ from (27) in the right hand side of (28), we deduce that

$$\begin{aligned}&2\left([r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\right.\\&\left.-\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\right)^2\frac{\mathrm{C}_2^{(\beth)}(\aleph;\mathrm{q})}{\left[\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})\right]^2}\right)a_2^2=\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})(c_2+d_2).\end{aligned}$$

Additionally, after performing certain computations using (8) and (4), we conclude that

$$|a_2|\le\frac{2|[\beth]_{\mathrm{q}}|\aleph\sqrt{2|[\beth]_{\mathrm{q}}|\aleph}}{\sqrt{\left|\mathfrak{L}(\beth,\gamma,\theta,\beta,r;\mathrm{q})\aleph^2+2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2[\beth]_{\mathrm{q}^2}\right|}},$$

where

$$\begin{aligned}\mathfrak{L}(\beth,\gamma,\theta,\beta,r;\mathrm{q})=&\,4[\beth]_{\mathrm{q}}^2[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\\&-2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2\left([\beth]_{\mathrm{q}^2}+[\beth]_{\mathrm{q}}^2\right).\end{aligned}$$

Currently, if we subtract (25) from (23), we obtain

$$\begin{aligned}&2[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\left(a_3-a_2^2\right)\\&\qquad=\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})(c_2-d_2)+\mathrm{C}_2^{(\beth)}(\aleph;\mathrm{q})(c_1^2-d_1^2).\end{aligned}$$

Then, in view of (27), the last equality becomes

$$\begin{aligned}a_3=&\frac{\left[\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})\right]^2}{2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2}\left(c_1^2+d_1^2\right)\\&+\frac{\mathrm{C}_1^{(\beth)}(\aleph;\mathrm{q})}{2[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}(c_2-d_2).\end{aligned}\tag{29}$$

Using (8) and (21), we therefore conclude that

$$|a_3|\le\frac{4[\beth]_{\mathrm{q}}^2\aleph^2}{\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2}+\frac{2|[\beth]_{\mathrm{q}}|\aleph}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}.$$

The theorem's proof is now complete. □

We explore the well-known Fekete–Szegö functional for functions in the class $\mathcal{B}_\Sigma(\gamma,\beta,\mathfrak{G}_{\mathrm{q}}^{(\beth)}(\aleph,\xi))$ in view of the Zaprawa [52] result.

Theorem 2. *Let the function f given by (1) belong to the class $\mathcal{B}_\Sigma(\gamma,\beta,\mathfrak{G}_{\mathrm{q}}^{(\beth)}(\aleph,\xi))$, then*

$$\left|a_3-\tau a_2^2\right| \leq \begin{cases} \dfrac{2\left|[\beth]_{\mathrm{q}}\right|\aleph}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{1}{[2]_{\mathrm{q}}}+\mathrm{q}\gamma+[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}, & |\tau-1|\leq\mathfrak{T}, \\ \left|\dfrac{16[\beth]_{\mathrm{q}}^3\aleph^3(1-\tau)}{\mathfrak{G}(\beta,\theta,\aleph,r,\beth;\mathrm{q})}\right| & |\tau-1|\geq\mathfrak{T}, \end{cases}$$

where

$$\mathfrak{G}(\beta,\theta,\aleph,r,\beth;\mathrm{q}) = 4[\beth]_{\mathrm{q}}^2\aleph^2[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)$$
$$-2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}\left(1-\theta\mathrm{q}^{i-1}\right)\right)^2\left(2\left([\beth]_{\mathrm{q}^2}+[\beth]_{\mathrm{q}}^2\right)\aleph^2-[\beth]_{\mathrm{q}^2}\right),$$

and

$$\mathfrak{T}=\left|1-\frac{2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2 C_2^{(\beth)}(\aleph;\mathrm{q})}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\left[C_1^{(\beth)}(\aleph;\mathrm{q})\right]^2}\right|.$$

Proof. From (27) and (29),

$$a_3-\tau a_2^2 = \frac{C_1^{(\beth)}(\aleph;\mathrm{q})}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}(c_2-d_2)$$
$$+\frac{(1-\tau)\left[C_1^{(\beth)}(\aleph;\mathrm{q})\right]^3}{\left([r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\left[C_1^{(\beth)}(\aleph;\mathrm{q})\right]^2 -2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2 C_2^{(\beth)}(\aleph;\mathrm{q})\right)}(c_2+d_2)$$
$$=C_1^{(\beth)}(\aleph;\mathrm{q})\left[\mathfrak{F}(\tau)+\frac{1}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}\right]c_2$$
$$+C_1^{(\beth)}(\aleph;\mathrm{q})\left[\mathfrak{F}(\tau)-\frac{1}{[r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})}\right]d_2,$$

where

$$\mathfrak{F}(\tau)=\frac{(1-\tau)\left[C_1^{(\beth)}(\aleph;\mathrm{q})\right]^2}{\left([r]_{\mathrm{q}}[r+1]_{\mathrm{q}}\theta^2\left(\frac{2}{[2]_{\mathrm{q}}}+2\mathrm{q}\gamma+2[3]_{\mathrm{q}}\beta\right)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\left[C_1^{(\beth)}(\aleph;\mathrm{q})\right]^2 -2\left([r]_{\mathrm{q}}\theta(1+\mathrm{q}\gamma+[2]_{\mathrm{q}}\beta)\prod_{i=1}^{r}(1-\theta\mathrm{q}^{i-1})\right)^2 C_2^{(\beth)}(\aleph;\mathrm{q})\right)}.$$

Then, in view of (8), we conclude that

$$\left|a_3 - \tau a_2^2\right| \le \begin{cases} \dfrac{\left|C_1^{(\beth)}(\aleph;q)\right|}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\gamma+[3]_q\beta\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)}, & |\mathfrak{F}(\tau)| \le \dfrac{1}{[r]_q[r+1]_q\theta^2\left(\frac{2}{[2]_q}+2q\gamma+2[3]_q\beta\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)}, \\ 2\left|C_1^{(\beth)}(\aleph;q)\right|\left|\mathfrak{F}(\tau)\right| & |\mathfrak{F}(\tau)| \ge \dfrac{1}{[r]_q[r+1]_q\theta^2\left(\frac{2}{[2]_q}+2q\gamma+2[3]_q\beta\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)}. \end{cases}$$

□

5. Corollaries and Consequences

Theorems 1 and 2 generate the following result, which roughly corresponds to Examples 1–3.

Corollary 1. *If the function f belongs to the class $\mathcal{B}_\Sigma(\aleph,\beth,\gamma,\beta)$, and for $1-\theta=p$, then*

$$|a_2| \le \frac{2|\beth|\aleph\sqrt{2|\beth|\aleph}}{\sqrt{r\theta^2 p^r\left(\left[(r+1)(1+2\gamma+6\beta)\beth - r\left(1+\gamma+2\beta\right)^2 p^r(1+\beth)\right]\aleph^2 + r\left(1+\gamma+2\beta\right)p^r\right)}},$$

$$|a_3| \le \frac{4\beth^2\aleph^2}{r^2\theta^2\left(1+\gamma+2\beta\right)^2(1-\theta)^{2r}} + \frac{4|\beth|\aleph}{r(r+1)\theta^2(1+2\gamma+6\beta)(1-\theta)^r},$$

and

$$\left|a_3 - \tau_1 a_2^2\right| \le \begin{cases} \dfrac{4|\beth|\aleph}{r(r+1)\theta^2(1+2\gamma+3\beta)(1-\theta)^r}, & |\tau_1 - 1| \le \mathfrak{T}, \\ \dfrac{8\beth^2\aleph^3(1-\tau_1)}{\left|2\beth\aleph^2(1+2\gamma+6\beta)(r+1) - r(1+\gamma+2\beta)^2(1-\theta)^r(2(1+\beth)\aleph^2-1)\right| r(1-\theta)^r\theta^2}, & |\tau_1 - 1| \ge \mathfrak{T}, \end{cases}$$

where

$$\mathfrak{T} = \left|1 - \frac{r(1+\gamma+2\beta)^2(1-\theta)^r\left(2(1+\beth)\aleph^2-1\right)}{2\beth\aleph^2(1+2\gamma+6\beta)(r+1)}\right|.$$

Corollary 2. *If the function f belongs to the class $\mathcal{B}_\Sigma(\gamma,\mathfrak{G}_q^{(\beth)}(\aleph,\xi))$, then*

$$|a_2| \le \frac{2\left|[\beth]_q\right|\aleph\sqrt{2\left|[\beth]_q\right|\aleph}}{\sqrt{\left|\mathfrak{L}(\beth,\gamma,\theta,r;q)\aleph^2 + 2\left([r]_q\theta(1+q\gamma)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)\right)^2[\beth]_{q^2}\right|}},$$

where

$$\mathfrak{L}(\beth,\gamma,\theta,r;q) = 4[\beth]_q^2[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\gamma\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right) \\ -2\left([r]_q\theta(1+q\gamma)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)\right)^2\left([\beth]_{q^2}+[\beth]_q^2\right),$$

$$|a_3| \le \frac{4[\beth]_q^2\aleph^2}{\left([r]_q\theta(1+q\gamma)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)\right)^2} + \frac{2[\beth]_q\aleph}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\gamma\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)},$$

and

$$\left|a_3-\sigma a_2^2\right| \leq \begin{cases} \dfrac{2\left|[\beth]_q\right|\aleph}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\gamma\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)}, & |\sigma-1| \leq \mathfrak{L}, \\ \left|\dfrac{16[\beth]_q^3\aleph^3(1-\sigma)}{\mathfrak{G}(\theta,\aleph,r,\beth;q)}\right| & |\sigma-1| \geq \mathfrak{L}, \end{cases}$$

where

$$\mathfrak{G}(\theta,\aleph,r,\beth;q) = 4[\beth]_q^2\aleph^2[r]_q[r+1]_q\theta^2\left(\frac{2}{[2]_q}+2q\gamma\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)$$
$$-2\left([r]_q\theta(1+q\gamma+[2]_q\beta)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)\right)^2\left(2\left([\beth]_{q^2}+[\beth]_q^2\right)\aleph^2-[\beth]_{q^2}\right),$$

and

$$\mathfrak{L} = \left|1-\frac{2\left([r]_q\theta(1+q\gamma)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2 C_2^{(\beth)}(\aleph;q)}{[r]_q[r+1]_q\theta^2\left(\frac{2}{[2]_q}+2q\gamma\right)\prod_{i=1}^{r}(1-\theta q^{i-1})\left[C_1^{(\beth)}(\aleph;q)\right]^2}\right|.$$

Corollary 3. *If the function f belongs to the class $\mathcal{B}_\Sigma(1,\mathfrak{G}_q^{(\beth)}(\aleph,\xi))$, then*

$$|a_2| \leq \frac{2\left|[\beth]_q\right|\aleph\sqrt{2\left|[\beth]_q\right|\aleph}}{\sqrt{\left|\mathfrak{L}(\beth,1,\theta,r;q)\aleph^2+2\left([r]_q\theta(1+q)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2[\beth]_{q^2}\right|}},$$

where

$$\mathfrak{L}(\beth,1,\theta,r;q) = 4[\beth]_q^2[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\right)\prod_{i=1}^{r}(1-\theta q^{i-1})$$
$$-2\left([r]_q\theta(1+q)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2\left([\beth]_{q^2}+[\beth]_q^2\right),$$

$$|a_3| \leq \frac{4[\beth]_q^2\aleph^2}{\left([r]_q\theta(1+q)\prod_{i=1}^{r}(1-\theta q^{i-1})\right)^2}+\frac{2[\beth]_q\aleph}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\right)\prod_{i=1}^{r}(1-\theta q^{i-1})},$$

and

$$\left|a_3-\Upsilon a_2^2\right| \leq \begin{cases} \dfrac{2\left|[\beth]_q\right|\aleph}{[r]_q[r+1]_q\theta^2\left(\frac{1}{[2]_q}+q\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)}, & |\Upsilon-1| \leq \mathfrak{L}, \\ \left|\dfrac{16[\beth]_q^3\aleph^3(1-\Upsilon)}{\mathfrak{G}(\theta,\aleph,r,\beth;q)}\right| & |\Upsilon-1| \geq \mathfrak{L}, \end{cases}$$

where

$$\mathfrak{G}(\theta,\aleph,r,\beth;q) = 4[\beth]_q^2\aleph^2[r]_q[r+1]_q\theta^2\left(\frac{2}{[2]_q}+2q\right)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)$$
$$-2\left([r]_q\theta(1+q+[2]_q\beta)\prod_{i=1}^{r}\left(1-\theta q^{i-1}\right)\right)^2\left(2\left([\beth]_{q^2}+[\beth]_q^2\right)\aleph^2-[\beth]_{q^2}\right),$$

and

$$\mathfrak{L} = \left| 1 - \frac{2\left([r]_q \theta(1+q) \prod_{i=1}^{r} \left(1 - \theta q^{i-1}\right)\right)^2 C_2^{(\daleth)}(\aleph; q)}{[r]_q [r+1]_q \theta^2 \left(\frac{2}{[2]_q} + 2q\right) \prod_{i=1}^{r} \left(1 - \theta q^{i-1}\right) \left[C_1^{(\daleth)}(\aleph; q)\right]^2} \right|.$$

6. Concluding Remarks

This article investigated three subclasses of bi-univalent functions, $\mathcal{B}_\Sigma(\gamma, \beta, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$, $\mathcal{B}_\Sigma(\gamma, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$, and $\mathcal{B}_\Sigma(1, \mathfrak{G}_q^{(\daleth)}(\aleph, \xi))$ on the symmetry disk $\mathbb{U}$. For functions belonging to each of these three bi-univalent function classes, we calculated estimates for the Fekete–Szegö functional problems and the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. By concentrating on the variables employed in our primary findings, several additional novel findings were made.

The study of bi-univalent functions is an important and active area of research in complex analysis and its applications. The investigation of these three subclasses provides deeper insights into the theory and applications of bi-univalent functions. The results obtained in this article can be generalized in the future using post-quantum calculus and other q-analogues of the fractional derivative operator.

Overall, this article contributes to the ongoing research in the field of complex analysis by providing a detailed study of three important subclasses of bi-univalent functions. Further research can be conducted to investigate more subclasses and their properties to enhance our understanding of the theory and applications of bi-univalent functions.

Author Contributions: Conceptualization, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); methodology, A.A. (Ala Amourah); validation, A.A. (Abdullah Alsoboh), A.A. (Ala Amourah), M.D. and C.A.R.; formal analysis, A.A. (Abdullah Alsoboh); investigation, A.A. (Abdullah Alsoboh), A.A. (Ala Amourah) and M.D.; writing—original draft preparation, A.A. (Abdullah Alsoboh) and A.A. (Ala Amourah); writing—review and editing, C.A.R. and A.A. (Abdullah Alsoboh); supervision, M.D. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (23UQU4320576DSR002).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare that there are no conflict of interest.

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