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On the Computation of the Codimension of Map Germs Using the Lie Algebra Associated with a Restricted Left–Right Group

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Abstract: The codimension is an important invariant, which measures the complexity of map germs and play an important role in classification and recognition problems. The restricted $\mathbb{A}$-equivalence was introduced to obtain a classification of reducible curves. The aim was to classify simple parameterized curves with two components, one of them being smooth with respect to the $\mathbb{A}$-equivalence in characteristic p. In characteristic 0, the corresponding classification was given by Kolgushkin and Sadykov. The aim of this article is to present an algorithm to compute the codimension of germs of singularities under a restricted left–right equivalence ($\mathbb{A}$-symmetry). We also give the implementation of this algorithm in the computer algebra system SINGULAR.

Keywords: map germ; $\mathcal{A}$-equivalence; codimension

MSC: 58Q05; 14H20

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1. Introduction

Let $\mathcal{F}$ be a field of characteristic 0 and $f : (\mathcal{F}^n, 0) \to (\mathcal{F}^p, 0)$ be a map germ. A symmetry of f is a pair (α, β), where α is a diffeomorphism germ of $(\mathcal{F}^n, 0)$ and β is a diffeomorphism germ of $(\mathcal{F}^p, 0)$, such that the following diagram commutes:

$$\begin{tikzcd} (\mathcal{F}^n,0) \arrow[r, "f"] & (\mathcal{F}^p,0) \arrow[d, "\beta"] \\ (\mathcal{F}^n,0) \arrow[u, "\alpha^{-1}"] \arrow[r, "f"'] & (\mathcal{F}^p,0) \end{tikzcd}$$

Let $\mathcal{F}[[z]]$, where $z = (z_1, \ldots, z_m)$ denotes the local ring of formal power series in m-indeterminates. Consider a local ordering $>$ on $\mathcal{F}[[z]]$ and we denote by $\gg$ the extension of ordering $>$ on $\mathcal{F}[[z]]^n = \sum_{i=1}^{n} \mathcal{F}[[z]]e_i$, where $e_i = (0, \ldots, 1, \ldots, 0)$, and defined as: $z^\alpha e_i \gg z^\beta e_j$, if $i < j$ or ($i = j$ and $z^\alpha > z^\beta$).

Let $S(m,n) = \langle z \rangle \mathcal{F}[[z]]^n$, $\mathcal{R} = Aut_{\mathcal{F}}(\mathcal{F}[[z]])$ and $\mathcal{L} = Aut_{\mathcal{F}}(\mathcal{F}[[y]])$, where $y = (y_1, \ldots, y_m)$. Define the left–right group $\mathbb{A} = \mathcal{L} \times \mathcal{R}$. The action of the group $\mathbb{A}$ on $S(m,n)$ is given as follows:

$$\mathbb{A} \times S(m,n) \to S(m,n)$$

such that

$$((\varphi_1, \varphi_2), h) \mapsto \varphi_2 \circ h \circ \varphi_1^{-1}.$$

Any two map germs $h_1, h_2 \in S(m,n)$ are said to be $\mathbb{A}$-equivalent ($h_1 \sim_{\mathbb{A}} h_2$) if they lie in the same orbit under the group action of $\mathbb{A}$. For $h \in S(m,n)$, the orbit map can be defined as: $\Phi_h : \mathbb{A} \to S(m,n)$ such that $\Phi_h(\varphi_1, \varphi_2) = \varphi_2 \circ h \circ \varphi_1^{-1}$. Particularly, $\Phi_h(id) = h$. The orbit of h under the group action of $\mathbb{A}$ is the image of Φ_h; we set $\mathrm{Img}(\Phi_h) = \mathbb{A}_h$. We denote by $\mathcal{T}_{\mathbb{A}_h,h}$ a tangent space which is the image of the tangent map

$$\mathcal{T}_{\mathbb{A}_h,id} : \mathcal{T}_{\mathbb{A},id} \to \mathcal{T}_{S(m,n),h}$$

to the orbit at h. Note that the orbit map is separable, since $\mathrm{char}(\mathcal{F}) = 0$. It is easy to see that

$$\mathcal{T}_{\mathbb{A}_h,h} = \langle z \rangle_{\mathcal{F}[[z]]} \langle \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_m} \rangle_{\mathcal{F}[[z]]} + \langle h_1, \ldots, h_m \rangle_{\mathcal{F}[[h_1,\ldots,h_n]]} \mathcal{F}[[h_1, \ldots, h_n]]^n.$$

and

$$codim_{\mathbb{A}}(h) = dim_{\mathcal{F}} \frac{S(m,n)}{\mathcal{T}_{\mathbb{A}_h,h}}.$$

Definition 1. *$f \in S(m,n)$ is $\mathbb{A}$-finitely determined if there exists a $k > 0$ such that for all $g \in S(m,n)$ with $jet(f,k) = jet(g,k)$, g is in the orbit of f under the action of $\mathbb{A}$.*

Definition 2. *Let $U_1 \subseteq \mathcal{F}[[z]]$ be a subspace of $\mathcal{F}$-vector space $\mathcal{F}[[z]]$ and $a > 0$ a local monomial ordering. A subset $U_2 \subseteq U_1$ is called a standard basis of U_1 if $L(U_1) = L(U_2)$. Here, $L(U_1)$ is the $\mathcal{F}$ vector space generated by the leading monomials of U_1 with respect to the ordering $>$.*

In the history of the theory of singularities of map germs, $\mathbb{A}$-equivalence has been the most natural equivalence among map germs from the view point of differential topology. Group $\mathbb{A}$, the tangent space to the orbit under the action of this group and its codimension play an important role in the classification of map germs (see [1–12]). In [13], the authors gave an algorithm to compute the codimension of map germs under an $\mathbb{A}$-equivalence. Our aim is to present a similar algorithm, which computes the codimension of map germs under an $\mathbb{A}_r$-equivalence (restricted $\mathbb{A}$-equivalence).

2. Computation of Codimension under Restricted Left–Right Action

Let $f_i : \mathcal{F}[[y_1, y_2, \ldots, y_n]] \to \mathcal{F}[[z]]$ define a germ of a parameterized curve singularity, $i = 1, 2, \ldots, k$. Let $\mathcal{R} = Aut_{\mathcal{F}}(\mathcal{F}[[z]])$, $\mathcal{L} = Aut_{\mathcal{F}}(\mathcal{F}[[y_1, y_2, \ldots, y_n]])$ and $\mathbb{A} = \mathcal{L} \times \mathcal{R}^k$. Let G act on the set $E = \{(f_1, f_2, \ldots, f_k) : f_i : \mathcal{F}[[y_1, y_2, \ldots, y_n]] \to \mathcal{F}[[z]], dim_{\mathcal{F}}(\mathcal{F}[[z]]/im(f_i)) < \infty\}$ by

$$(g, (h_1, h_2, \ldots, h_k)) \circ (f_1, f_2, \ldots, f_k) = (h_1^{-1} \circ f_1 \circ g, \ldots, h_k^{-1} \circ f_k \circ g).$$

Definition 3. *Let $(f_1, f_2, \ldots, f_k), (g_1, g_2, \ldots, g_k) \in E$. They are called $\mathbb{A}$-equivalent if they are in the same orbit under the action of G. We write in this case $(f_1, f_2, \ldots, f_k) \sim_{\mathbb{A}} (g_1, g_2, \ldots, g_k)$.*

Let us consider a special case. Let $f_1(t, 0, 0, \ldots, 0)$, $g_1(t, 0, 0, \ldots, 0)$, $f_2 = (x_1(t), x_2(t), \ldots, x_n(t))$ and $g_2 = (y_1(t), y_2(t), \ldots, y_n(t))$. Then, $(f_1, f_2) \sim_{\mathbb{A}} (g_1, g_2)$ if and only if, for suitable $(g, (h_1, h_2)) \in G$,

$$(g_1, g_2) = (h_1^{-1} \circ f_1 \circ g, h_2^{-1} \circ f_2 \circ g).$$

Let $g = (H_1(x_1, x_2, \ldots, x_n), H_2(x_1, x_2, \ldots, x_n), \ldots, H_1(x_1, x_2, \ldots, x_n))$; then, we must have $H_i(x_1, x_2, \ldots, x_n) = 0, i = 2, 3, \ldots, n$ and $\partial H_1 / \partial x_1(0) \neq 0$. This implies that the classification of parameterized curves with two components, one of them smooth, is equivalent to the classification of simple irreducible curves with respect to the action of the following group $G^r = L^r \times R$ (the action is as above for $k = 1$) with

$$L^r = \{\phi \in Aut_{\mathcal{F}}(\mathcal{F}[[x_1, x_2, \ldots, x_n]]) : \phi(x_i)(x_1, 0, \ldots, 0) = 0, i = 2, 3, \ldots, n\}.$$

In [10,14], the authors introduced a direct and natural generalization of $\mathbb{A}$-equivalence denoted by $\mathbb{A}_{\mathcal{G}}$-equivalence, where $\mathcal{G}$ was a subgroup of $\mathcal{L} = Aut_{\mathcal{F}}(\mathcal{F}[[y]])$. Define the group $\mathbb{A}_{\mathcal{G}} = \mathcal{G} \times \mathcal{R}$. The action of the group $\mathbb{A}_{\mathcal{G}}$ on $S(m,n)$ is defined as follows:

$$\mathbb{A}_{\mathcal{G}} \times S(m,n) \to S(m,n)$$

such that

$$((\varphi_1, \varphi_2), h) \mapsto \varphi_2 \circ h \circ \varphi_1^{-1}.$$

Any two map germs $h_1, h_2 \in S(m,n)$ are said to be $\mathbb{A}_{\mathcal{G}}$-equivalent ($h_1 \sim_{\mathbb{A}_{r\mathcal{G}}} h_2$) if they lie in the same orbit under the group action of $\mathbb{A}_{\mathcal{G}}$.

Example 1. *(1)* $(t^4, t^5, t^4 + t^6, t^8) \sim_{\mathbb{A}_r} (t^6, t^4 + t^6, t^5, t^8) \sim_{\mathbb{A}_r} (t^6, t^4 + t^6, t^5)$, *therefore* $\gamma =< 4,5,6 >= \{0,4,5,6,8,\ldots\}$ *and* $\gamma_r = \{0,4,5,8,\ldots\}$.
(2) $(0, t^4, t^6 + t^7, t^{13}) \sim_{\mathbb{A}_r} (0, t^4, t^6 + t^7, t^{15})$ *since* $13 \in \gamma(< t^4, t^6 + t^7 > F[[t^4, t^6 + t^7]])$.
(3) $(t^2, t^4, t^6 + t^7) \sim_{\mathbb{A}_r} (t^2, t^4, t^7)$ *since* $6 \in \gamma(< t^4, t^6 + t^7 > F[[t^2, t^4, t^6 + t^7]])$.

Proposition 1. *Let* $h \in S(m,n)$ *and* $\mathcal{G}$ *be a subgroup of* $Aut_{\mathcal{F}}(\mathcal{F}[[y]])$. *The tangent space with respect to the* $\mathbb{A}_{\mathcal{G}}$*-equivalence is*

$$\mathcal{T}_{\mathbb{A}_{\mathcal{G}h},h} =: \langle z \rangle \langle \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_m} \rangle_{\mathcal{F}[[z]]} + \{H(h_1, ..., h_m) : H \in Lie(\mathcal{G})\},$$

where $Lie(\mathcal{G})$ *denote the Lie algebra associated with group* $\mathcal{G}$*, i.e.,* $\mathcal{T}_{\mathbb{A}_{\mathcal{G}h},id} = Lie(\mathcal{G})$.

Proof. Let

$$\mathcal{R} = Aut_{\mathcal{F}}(\mathcal{F}[[z]]) = \{\varphi_1 = \begin{pmatrix} \varphi_{11} \\ . \\ . \\ . \\ \varphi_{1m} \end{pmatrix} : \varphi_{1i} \in \langle z \rangle \mathcal{F}[[z]], det(\frac{\partial \varphi_{1i}}{\partial z_j}(0)) \neq 0\},$$

$$\mathcal{G} \subseteq \mathcal{L} = Aut_{\mathcal{F}}(\mathcal{F}[[y]]) = \{\varphi_2 = \begin{pmatrix} \varphi_{21} \\ . \\ . \\ . \\ \varphi_{2n} \end{pmatrix} : \varphi_{2i} \in \langle y \rangle \mathcal{F}[[y]], det(\frac{\partial \varphi_{2i}}{\partial y_j}(0)) \neq 0\}$$

and $\mathbb{A}_{\mathcal{G}} = \mathcal{G} \times \mathcal{R}$. We have Lie$(\mathbb{A}_{\mathcal{G}})$ = Lie$(\mathcal{G}) \oplus$ Lie$(\mathcal{R})$ and

$$Lie(\mathcal{R}) = \{\begin{pmatrix} H_1 \\ . \\ . \\ . \\ H_n \end{pmatrix} : H_i \in \langle z \rangle \mathcal{F}[[z]]\},$$

$$Lie(\mathcal{G}) \subseteq \{\begin{pmatrix} L_1 \\ . \\ . \\ . \\ L_p \end{pmatrix} : L_i \in \langle y \rangle \mathcal{F}[[y]]\}.$$

Given $(\varphi_2, \varphi_1) \in \mathbb{A}_{\mathcal{G}}$, we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{F}[[y]] \arrow[r, "h"] & \mathcal{F}[[z]] \arrow[d, "\varphi_1"] \\ \mathcal{F}[[y]] \arrow[u, "\varphi_2^{-1}"] \arrow[r, "\varphi_1 \circ h \circ \varphi_2^{-1}"'] & \mathcal{F}[[z]] \end{tikzcd}$$

This way, $\mathbb{A}_{\mathcal{G}}$ acts on $S(m,n)$

$$\mathbb{A}_{\mathcal{G}} \times S(m,n) \to S(m,n)$$
$$((\varphi_2,\varphi_1),h) \mapsto \varphi_1 \circ h \circ \varphi_2^{-1}.$$

For $h \in S(m,n)$, we have the orbit map

$$\Phi_h : \mathbb{A}_{\mathcal{G}} \to S(m,n)$$

defined by

$$\Phi_h(\varphi_2,\varphi_1) = \varphi_1 \circ h \circ \varphi_2^{-1}.$$

The orbit map induces a map

$$\mathcal{T}_{\Phi_h,id} : \mathcal{T}_{\mathbb{A}_{\mathcal{G}},id} \to \mathcal{T}_{S(m,n),h},$$

where $\mathcal{T}_{\mathbb{A}_{\mathcal{G}},id} = \mathrm{Lie}(\mathcal{G}) \oplus \mathrm{Lie}(\mathcal{R})$ and $\mathcal{T}_{S(m,n),h} = \langle z \rangle \mathcal{F}[[z]]^n$. This gives

$$\mathcal{T}_{\Phi_h,id}(\begin{pmatrix} L_1 \\ . \\ . \\ . \\ L_p \end{pmatrix} + \begin{pmatrix} H_1 \\ . \\ . \\ . \\ H_n \end{pmatrix}) = \sum H_i \frac{\partial h}{\partial z_i} + \begin{pmatrix} L_1(h) \\ . \\ . \\ . \\ L_p(h) \end{pmatrix}.$$

Since the characteristic of $\mathcal{F}$ is zero, the image of $\mathcal{T}_{\Phi_h,id}$ is the tangent space to the orbit at f:

$$\mathcal{T}_{\mathbb{A}_{\mathcal{G}}h,h} =: \langle z \rangle \langle \frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_m} \rangle_{\mathcal{F}[[z]]} + \{H(h_1,\ldots,h_m) : H \in Lie(\mathcal{G})\}.$$

□

The following theorem is a generalization of a theorem of Du Plessis [9] (Corollary 3.10) and can be proved similarly to [13].

Theorem 1. *Let $h \in S(m,n)$ and assume that*

$$\langle z \rangle^p \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathcal{F}h,h} + \langle z \rangle^{p+1} \mathcal{F}[[z]]^n,$$
$$\langle z \rangle^q \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathbb{A}_{\mathcal{G}}h,h} + \langle z \rangle^{p+q} \mathcal{F}[[z]]^n.$$

Then, h is $(p+q)$-determined and $\langle z \rangle^q \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathbb{A}_{\mathcal{G}}h,h}$.

The theorem is the basis to pass to ℓ-jets. We assume that h is ℓ-determined and let $\mathbb{A}_{\mathcal{G}}^{(\ell)} = \mathrm{jet}(\mathbb{A}_{\mathcal{G}},\ell)$ and $S^{(\ell)}(m,n) = \mathrm{jet}(S(m,n),\ell)$. Then, we have an induced action of $\mathbb{A}_{\mathcal{G}}^{(\ell)}$ on $S^{(\ell)}(m,n)$. Moreover, we have

$$\mathcal{T}_{\mathbb{A}_{\mathcal{G}}^{(\ell)}h,h} = \frac{\mathbb{A}_{\mathcal{G}h,h}}{\langle z \rangle^{\ell+1} \mathcal{F}[[z]]^n}$$

and
$$codim_{\mathbb{A}_{\mathcal{G}}}(h) = dim_{\mathcal{F}} \frac{S(m,n)}{\langle z\rangle^{\ell+1}\mathcal{F}[[z]]^n} - dim_{\mathcal{F}} \mathcal{T}_{\mathbb{A}_{\mathcal{G}}^{(\ell)} h,h}.$$

Remark 1. *1. If $\mathcal{G} = \{id\}$, then $\mathbb{A}_{\mathcal{G}} = \mathcal{R}$, i.e., the right equivalence. In this case, the computation of the codimension of map germs is trivial.*

2. If $\mathcal{G} = \mathcal{L} = Aut_{\mathcal{F}}(\mathcal{F}[[y]])$, then $\mathbb{A}_{\mathcal{G}} = \mathbb{A}$, i.e., the left–right equivalence. For this case an algorithm to compute the codimension of map germs can be found in [13].

We consider the following case:

Let $\mathcal{G} = \mathcal{G}_r = \{\psi \in Aut_{\mathcal{F}}(\mathcal{F}[[y]]) : \psi(y_i) \in \langle y_2, \dots, y_n\rangle \mathcal{F}[[y]], i \geq 2\}$ then $\mathbb{A}_{\mathcal{G}} = \mathbb{A}_r$, i.e., the restricted left–right equivalence. This equivalence relation is considered in [15], where it reduces the $\mathbb{A}$-classification of simple multigerms into the $\mathbb{A}_r$-classification of irreducible simple germs. In this article, our aim was to give an implementation of an algorithm in the computer algebra system SINGULAR [16] to compute the $\mathbb{A}_r$-codimension of map germs.

Proposition 2. *If $\mathcal{G} = \mathcal{G}_r$, then $Lie(\mathcal{G}_r) = \{\psi \in \mathcal{F}[[y]]^n : \psi_i \in \langle y_2, \dots, y_n\rangle \mathcal{F}[[y]], i \geq 2\}$.*

Proof. If $\mathcal{G}$ is a subgroup of $Aut_{\mathcal{F}}(\mathcal{F}[[y]])$, then, by definition, $\mathrm{Lie}(\mathcal{G}) = \mathcal{T}_{\mathcal{G},id}$, the tangent space of the group $\mathcal{G}$ at the identity id ($id(y_i) = y_i, \forall i$). If we take any curve φ_t in $\mathcal{G}$ such that $\varphi_0 = id$, then $\frac{d\varphi_t}{dt}|_{t=0}$ gives a tangent vector. Thus, for the case $\mathcal{G} = \mathcal{G}_r$, such a curve is of type

$$\varphi_t(y) = \begin{pmatrix} \varphi_t^1(y) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_t^n(y) \end{pmatrix}$$

with $\varphi_t^i(y) = y_i + t\psi_i(y,t)$ and $\psi_i \in \langle y_2, \dots, y_n\rangle \mathcal{F}[[y]]$, if $i \geq 2$. This gives

$$\frac{d\varphi_t}{dt}|_{t=0} = \begin{pmatrix} \psi_1(y,0) \\ \cdot \\ \cdot \\ \cdot \\ \psi_n(y,0) \end{pmatrix}.$$

Therefore, we obtain the required result. □

Theorem 1 is the basis for the following algorithm (Algorithm 1):

Algorithm 1 (`codim`)

Input: $h = (h_1, \dots, h_n) \in \mathcal{F}[[z]]^n$ and N a $\mathcal{F}$-basis of $\mathrm{Lie}(\mathcal{G})$.

Output: $\mathcal{A}_r$-codimension of h.

1: Compute k, a bound for the determinacy of h such that $\langle z\rangle^k \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathbb{A}_{\mathcal{G}} h,h}$.

2: Compute an $\mathcal{F}$-basis $\{N_1, \dots, N_s\}$ of $(\mathrm{Lie}(\mathcal{G}) + \langle z\rangle^{k+1})\mathcal{F}[[z]]/\langle z\rangle^{k+1}\mathcal{F}[[z]]$.

3: Compute S, a standard basis of $\langle z\rangle\langle \frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_m}\rangle$.

4: Compute a reduced row echelon form $M = (M_1, \dots, M_t)$ of $\mathrm{NF}(N_1|S), \dots, \mathrm{NF}(N_s|S)$ by using the Gaussian algorithm.

5: Return $\dim \frac{\mathcal{F}[[z]]^n}{M(S)} - t$.

Remark 2. *An estimated value for the determinacy of h can be computed by using the code computeBound. We compute a standard basis of $\mathcal{T}_{\mathcal{F}h,h}$ and check the condition $\langle z\rangle^p \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathcal{F}h,h} + \langle z\rangle^{p+1}\mathcal{F}[[z]]^n$ case by case for computing the value of p satisfying $\langle z\rangle^p \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathcal{F}h,h} + \langle z\rangle^{p+1}\mathcal{F}[[z]]^n$. As an initial bound, it uses $q = 10$ and the value of the bound q increases as long as the condition $\langle z\rangle^q \mathcal{F}[[z]]^n \subseteq \mathcal{T}_{\mathbb{A}_{\mathcal{G}}h,h} + \langle z\rangle^{p+q}\mathcal{F}[[z]]^n$ has been satisfied. Then, Theorem 1 gives h as $(p+q)$-determined.*

3. Singular Examples

We give some examples.

```
ring R=0,t,ds;
> ideal I=t3,t5,t6+t7;
> coDimMap(I);
[1]:
13
[2]:
11


ring R=0,t,ds;
>  ideal I=t4,t7+t9,t17;
> coDimMap(I);
[1]:
44
[2]:
37


ring R=0,(x,y),ds;
> ideal I=x,xy+y4;
> coDimMap(I);
[1]:
13
[2]:
3
```

By using Algorithm 1, we computed the $\mathbb{A}_r$ codimension of different map germs. Moreover, Table 1 gives a comparison between the $\mathbb{A}$-codimension and $\mathbb{A}_r$-codimension of map germs from the plane to the plane.

Table 1. A comparison between the $\mathbb{A}$-codimension and $\mathbb{A}_r$-codimension of map germs from the plane to the plane

Normal Form	$\mathbb{A}$-Codimension	$\mathbb{A}_r$-Codimension
(x, y^2)	1	
$(x, y^3 \pm x^k y), k > 1, \pm$agree for odd k	$k+1$	$4k-3$
$(x, xy + y^3)$	2	1
$(x, xy + y^4)$	3	3
$(x, xy + y^5)$	5	6
$(x, xy + y^5 \pm y^7)$	4	4
$(x, xy + y^6)$	8	10
$(x, xy + y^6 + y^{14})$	7	8
$(x, xy + y^6 + y^9)$	6	8
$(x, xy + y^6 + y^8 + \alpha y^9)$	6	7
$(x, x^2 y + y^4 \pm y^5)$	5	8
$(x, x^2 y + y^4)$	6	10
$(x, y^4 + x^3 y^2 + x^l y), l \geq 5$	$l+4$	$3(l+4)$
$(x, y^4 + x^k y + x^l y^2), k = 4, 5\ k-1 \leq l < 2k-1$	$k+l+1$	$7k-6$
$(x, y^4 + x^k y + x^l y^2), k = 4\ l = 2k-1$	$k+l+1$	23
$(x, y^4 + x^k y + x^l y^2), k = 5\ l = 2k-1$	$k+l+1$	30
$(x, y^4 + x^2 y^2 + x^k y), k \geq 4$	$k+3$	$3k+6$
$(x, y^4 + x^3 y - \frac{3}{2} x^2 y^2 + x^k y), k \geq 6$	$k+3$	$k+12$
$(x, y^4 + x^3 y + \alpha x^2 y^2), \alpha \neq \frac{-3}{2}$	9	17
$(x, y^4 + x^3 y - \frac{3}{2} x^2 y^2 + x^4 y^2)$	8	17
$(x, y^4 + x^3 y + \alpha x^2 y^2 + x^4 y^2), \alpha \neq \frac{-3}{2}$	8	15
$(x, y^4 + x^3 y - \frac{3}{2} x^2 y^2 + x^3 y^2)$	7	16
$(x, y^4 + x^3 y + \alpha x^2 y^2 + x^3 y^2), \alpha \neq \frac{-3}{2}$	7	15

4. Conclusions

In [13], the authors computed the codimension of map germs with respect to the left–right equivalence and contact equivalence. In this work, we gave an algorithm to compute the codimension of map germs with respect to the restricted left–right equivalence. Moreover, this algorithm was implemented in the computer algebra system SINGULAR. In the future, one can find the codimension of map germs with respect to several other equivalence relations, such as the $\mathbb{B}$-equivalence.

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Data Availability Statement: The code used in this paper can be downloaded from the following link: https://www.mathcity.org/files/ahsan/classifyMapGerms2.txt (accessed on 1 April 2023).

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