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Recursive Symmetries: Chemically Induced Combinatorics of Colorings of Hyperplanes of an 8-Cube for All Irreducible Representations

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Abstract: We outline symmetry-based combinatorial and computational techniques to enumerate the colorings of all the hyperplanes (q = 1–8) of the 8-dimensional hypercube (8-cube) and for all 185 irreducible representations (IRs) of the 8-dimensional hyperoctahedral group, which contains 10,321,920 symmetry operations. The combinatorial techniques invoke the Möbius inversion method in conjunction with the generalized character cycle indices for all 185 IRs to obtain the generating functions for the colorings of eight kinds of hyperplanes of the 8-cube, such as vertices, edges, faces, cells, tesseracts, and hepteracts. We provide the computed tables for the colorings of all the hyperplanes of the 8-cube. We also show that the developed techniques have a number of chemical, biological, chiral, and other applications that make use of such recursive symmetries.

Keywords: recursive symmetries; hyperplane colorings of 8-cubes; generalized character cycle indices for hyperplanes of 8-cube; 8D hyperoctahedral group; generating functions; chemical and biological applications

1. Introduction

N-dimensional hypercubes in general and the eight-dimensional hypercube (8-cube) in particular are of interest not only because they represent the potential energy surfaces of water clusters, including the octamer $(H_2O)_8$, but because they also have several novel applications in many other fields [1–42]. Moreover, hypercubes in general provide representations of the periodic table, encompassing superheavy elements and elements that are yet to be discovered [1–4]. Hypercubes are employed in quantum similarity measures, quantum chemistry, computational chemistry, chirality, image processing, quantitative measures of shapes and stereochemistry, and so forth [5–27]. There are several representations of hypercubes and the various hyperplanes of hypercubes. For example, the associated octeract, a representation of the 8-cube, is an isomorphic representation of the potential energy surfaces of a completely non-rigid water octamer, $(H_2O)_8$ [43], in which 256 vertices correspond to the potential minima in the potential energy surface. Consequently, the classifications of the rovibronic levels [43] of a fully nonrigid $(H_2O)_8$ require the automorphism group of the 8-cube, which is isomorphic with the eight-dimensional hyperoctahedral group or the wreath product group, $S_8[S_2]$, which contains 10,321,920 permutation operations for which S_8 is the permutation group of eight objects consisting of 8! operations. In a more general context, n-dimensional hypercubes, polycubes, recursive structures, and their properties have been studied over the years [6–43]. They find numerous applications in vast areas, such as the representation theory of nonrigid molecules, genetic regulatory networks, biological modeling, finite automata, isomerization reactions, computer graphics, DNA synthetic bases, chirality, protein–protein interactions, parallel computing, visualizations, big data, and so forth [7,11–29].

The double groups [44] of the wreath products are applicable to the classification of the potential energy surfaces of relativistic polymeric clusters [45–48] that could be made in supersonic expansions, for example, $(PoH_2)_n$, $(LvH_2)_8$, $(FlH_2)_8$, Sn_3, gallium arsenide

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clusters and their heavier analogs, and so forth. Relativistic effects [45–48] make significant contributions to such molecules that contain very heavy atoms; thus, the coupling of the spin and orbital angular momenta results in double group symmetries [44]. Combinatorial enumerations of the colorings of structures or enumerations under group actions, especially those that are pertinent to hypercubes, polytopes, and color symmetries, have been the subject matter of several studies [33–40,49–51]. Such enumerations have several molecular, biological, and other applications to phylogentic trees, pandemic trees, intrinsically disordered proteins, genetic regulatory networks, dynamic chirality, spontaneous generation of optical activity, configuration interaction computations in quantum chemistry, and so forth [52–65].

Combinatorial enumerations of the colorings of the various hyperplanes of the 8-cube have not been considered until now. Such enumerations are extremely challenging as the generating functions must be obtained for 185 IRs for each of the eight hyperplanes of the 8-cube. The 8-cube exhibits eight types of (8-q) hyperplanes, with q ranging from 1 to 8. For example, q = 1 corresponds to 16 hepteracts, q = 2 yields 112 hexeracts . . . q = 7 provides 1024 edges, and q = 8 is simply 256 vertices of the 8-cube. In fact, Pólya's theorem [20,21] of enumeration under group action becomes a special case of the combinatorial enumeration considered here when it is reduced to the totally symmetric A_1 representation of the hypercube. Several other variants of Pólya's theorem [36–39,50,51] have also been considered in the literature. Here, we have considered enumerations and generating functions for all 185 IRs and 185 conjugacy classes of the 8-cube. When the enumeration is generalized to all IRs, the results are extremely useful in a number of applications, for example, the enumeration of the nuclear spin statistics and nuclear spin species of the rovibronic levels of nonrigid molecules, multiple-quantum NMR spin functions and energy levels, the enumeration of isomerization reactions, dynamic chirality, mathematical methods pertinent to drug discovery, etc. [31,63–65]. Moreover, there exists no one-to-one correspondence between Pólya's cycle types and the conjugacy classes for the hyperoctahedral groups. We therefore employ the matrix representations of the conjugacy classes in combination with the Möbius inversion technique [35] to generate the cycle types of all eight hyperplanes of the 8-cube for all 185 IRs of the wreath product group $S_8[S_2]$. Consequently, for (8-q) hyperplanes (q = 1 to 8) and for each IR of the wreath group, we obtain a generalized character cycle index (GCCI) in order to derive the combinatorial generating functions for the colorings of (n-q) hyperplanes of the 8-cube. We construct the combinatorial enumeration tables for the 4-colorings and 2-colorings of the hyperplanes of the 8-cube for all 185 IRs.

2. Recursive Symmetries, Wreath Products, and Combinatorial and Computational Techniques for the 8-Cube

Hypercubes are recursive structures, and their symmetries are also recursively defined in terms of wreath product groups. Recursivity is a topic that has been explored at multiple levels, including its philosophical implications [66]. In the present context, recursivity implies that the symmetry of the larger system is constructed from the previous levels of smaller systems in a nested manner, as defined in the ensuing sentences. We define an n-dimensional hypercube whose graph is represented by Q_n recursively through the use of the cartesian product shown below:

$$Q_n = Q_{n-1} \times K_2 \text{ for } n \geq 2 \text{ with } Q_1 = K_2,$$

where K_2 is a graph with two vertices connected by an edge. The adjacency matrices of hypercubes are also generated recursively using the above recursive construction. The adjacency matrix of Q_{n+1}, A_{Qn+1}, is recursively expressible in terms of the corresponding matrices of Q_n, as follows:

$$A_{Q_{n+1}} = \begin{bmatrix} A_{Q_n} & I \\ I & A_{Q_n} \end{bmatrix}$$

where I is simply an identity matrix of the order $2^n \times 2^n$.

The automorphism group of a graph is defined as a set of permutations of the vertices of the graph that preserve the adjacency matrix. Subsequently, for n-cubes, the permutations of the vertices must not break or make any edges. The automorphism group of an n-cube is simply given by the wreath product group, $S_n[S_2]$.

We restrict ourselves to particular details concerning the 8-cube and the enumeration of the colorings of eight possible hyperplanes for 185 irreducible representations of the $S_8[S_2]$ group. The (8-q) hyperplanes ($1 \leq q \leq 8$) of the 8-cube are characterized by an 8×8 Coxeter's configuration matrix [67]:

$$:\begin{pmatrix} 256 & 8 & 28 & 56 & 70 & 56 & 28 & 8 \\ 2 & 1024 & 7 & 21 & 35 & 35 & 21 & 7 \\ 4 & 4 & 1792 & 6 & 15 & 20 & 15 & 6 \\ 8 & 12 & 6 & 1792 & 5 & 10 & 10 & 5 \\ 16 & 32 & 24 & 8 & 1120 & 4 & 6 & 4 \\ 32 & 80 & 80 & 40 & 10 & 448 & 3 & 3 \\ 64 & 192 & 240 & 160 & 60 & 12 & 112 & 2 \\ 128 & 448 & 672 & 560 & 280 & 84 & 4 & 16 \end{pmatrix}$$

The number of (n-q) hyperplanes for an nD-hypercube is given by

$$N_q = \binom{n}{q} 2^q$$

It can be seen that the diagonal elements of the 8×8 configuration matrix yield the number of hyperplanes, with the first row corresponding to the vertices and the last row providing the number of hepteracts. The off-diagonal element C_{ij} of the 8×8 configuration matrix provides the number of times hyperplane j occurs in hyperplane i of the 8-cube. Thus, $C_{51} = 16$ means that each penteract of the 8-cube contains 16 vertices, and so forth. Hence, the configuration matrix is fundamental to the combinatorial enumeration of the colorings of the hyperplanes of the 8-cube.

Figure 1 demonstrates the first row of the Coxeter [67] configuration matrix (q = 8) for the 8-cube, which shows 256 vertices and 1024 edges with $C_{12} = 8$, providing the degree of each vertex in Figure 1 taken from [68]. Transitive graphs, such as the one in Figure 1, have been considered by the author [69] as well as Balaban in the context of chemical isomerization graphs, such as the well-known Balaban cages [70,71]. Likewise, for the case under consideration, Figure 1 represents the isomerization rearrangements of the water octamer in the fully fluxional limit, where the breaking and remaking of all hydrogen bonds between any two water molecules can take place. This happens at a higher temperature at which facile rearrangements among the H-bonded water molecules lead to a totally nonrigid limit of the water octamer. The automorphism group of the graph in Figure 1 contains all permutations of the vertices such that no edges are made or broken, and it is provided by the wreath product group, $S_8[S_2]$. The generalized character cycle indices for all 185 IRs of the $S_8[S_2]$ group and the 8 hyperplanes of the 8-cube are constructed next using the generalized matrix cycle types of the permutations.

The Möbius inversion technique [30,55] is one of the most powerful methods for obtaining a number of generating functions; in the present case, it facilitates the construction of permutational cycle types for all eight hyperplanes of the 8-cube under the action of the $S_8[S_2]$ group. We accomplish this task by using the matrix generators for the conjugacy classes of the $S_8[S_2]$ group in combination with the Möbius inversion method. Table 1 shows 185 different 2×8 matrix cycle types for the conjugacy classes of the $S_8[S_2]$ group. The matrix cycle types are powerful generalizations of the Pólya cycle types used in the construction of the ordinary cycle index of a group. The matrix cycle types for the 185 conjugacy classes of the $S_8[S_2]$ group are constructed using a combinatorial technique that considers the group actions of the composing groups in the wreath product $S_8[S_2]$. Let a permutation $g \in S_8$ act on a set Ω of eight objects and generate a_1 cycles of length 1, a_2

cycles of length 2, a_3 cycles of length 3, ... , a_8 cycles of length 8; the group action can then be represented by $1^{a1}2^{a2}3^{a3}.\ldots.8^{a8}$ or a cycle type of g denoted as $T_g = (a_1, a_2, a_3, \ldots, a_8)$. From this structure, one obtains a 2 × 8 matrix cycle type for a conjugacy class of the wreath product in which the first row represents the action of {(g; *e*)} permutations for which *e* is the identity operation of the S_2 group, with $g \in S_8$, and the second row corresponds to the permutations {(g;π)} for π (12) ∈ S_2 and $g \in S_8$. Consequently, the matrix cycle type of any conjugacy class of the wreath product $S_8[S_2]$, denoted by T(g;π), is a 2 × 8 matrix generated using the orbit structure of $g \in S_n$ and the corresponding conjugacy class of S_2. Thus, for (g;π), the matrix types of the conjugacy classes of $S_8[S_2]$ are given by

$$T(g;\pi) = a_{ik}\ (1 \le i \le 2), (1 \le k \le 8), \quad (1)$$

To exemplify, take the conjugacy class {(1)(2)(3)(4)(5678);(12)} of $S_8[S_2]$ (class number 38 in Table 1) given by (2)

$$T[\{(1)(2)(3)(4)(5678);(12)\}] = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Likewise, the conjugacy class number 162 in Table 1, which corresponds to {(12345)(6)(7)(8);(12)}, is given by (3):

$$T[\{(12345)(6)(7)(8);(12)\}] = \begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

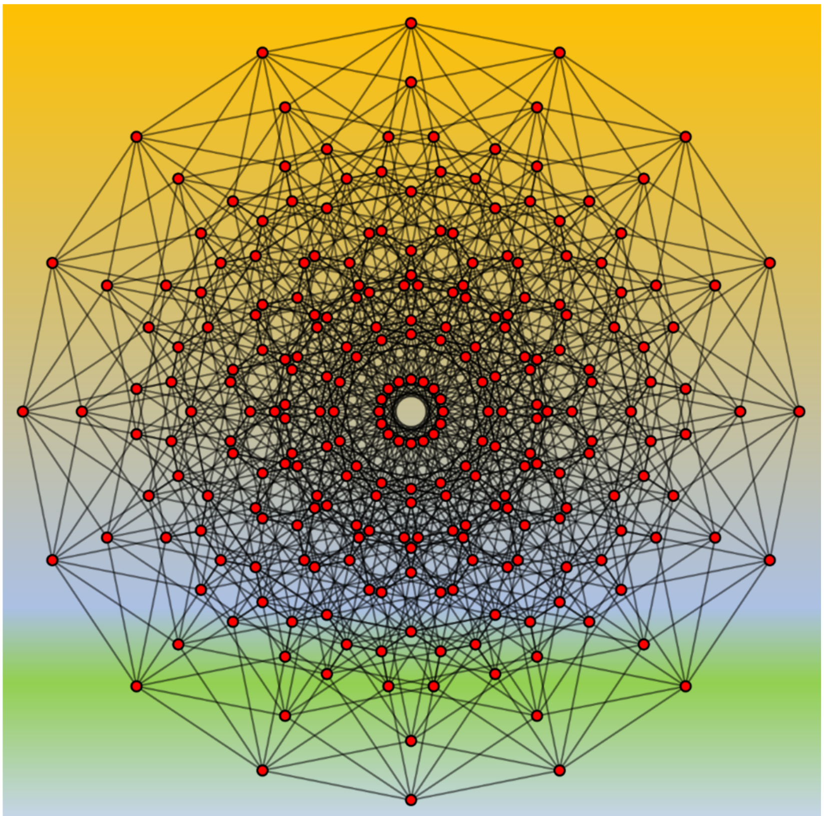

Figure 1. The octeract graph of an 8-cube, displaying the relationship among the 256 vertices of the 8-cube. The automorphism group of this graph is the 8-dimensional hyperoctahedral group or the wreath product $S_8[S_2]$ comprising 10,321,920 permutations and 185 irreducible representations. (Figure reproduced from Ref. [68]).

Table 1. Conjugacy Classes of the 8-dimensional hyperoctahedral group & cycle types for each hyperplane.

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
1	$\begin{bmatrix} 8 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1 R	F1 = $(1 + 2x)^8$	1^{16}	1^{112}	1^{448}	1^{1120}	1^{1792}	1^{1792}	1^{1024}	1^{256}
2	$\begin{bmatrix} 7 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8	F1 = $(1 + 2x)^7$ F2 = $(1 + 2x)^8$	1^{14} 2	1^{84} 2^{14}	$1^{280}2^{84}$	$1^{560}2^{280}$	$1^{672}2^{560}$	$1^{448}2^{672}$	$1^{128}2^{448}$	2^{128}
3	$\begin{bmatrix} 6 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	28 R	F1 = $(1 + 2x)^6$ F2 = $(1 + 2x)^8$	1^{12} 2^2	1^{60} 2^{26}	$1^{160}2^{144}$	$1^{240}2^{440}$	$1^{192}2^{800}$	$1^{64}2^{864}$	2^{512}	2^{128}
4	$\begin{bmatrix} 5 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56	F1 = $(1 + 2x)^5$ F2 = $(1 + 2x)^8$	1^{10} 2^3	1^{40} 2^{36}	$1^{80}2^{184}$	$1^{80}2^{520}$	$1^{32}2^{880}$	2^{896}	2^{512}	2^{128}
5	$\begin{bmatrix} 4 & 0 & 00 & 0 & 0 & 0 & 0 \\ 4 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	70 R	F1 = $(1 + 2x)^4$ F2 = $(1 + 2x)^8$	1^8 2^4	1^{24} 2^{44}	$1^{32}2^{208}$	$1^{16}2^{552}$	2^{896}	2^{896}	2^{512}	2^{128}
6	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 5 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56	F1 = $(1 + 2x)^3$ F2 = $(1 + 2x)^8$	1^6 2^5	1^{12} 2^{50}	$1^8 2^{220}$	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
7	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 6 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	28 R	F1 = $(1 + 2x)^2$ F2 = $(1 + 2x)^8$	1^4 2^6	1^4 2^{54}	2^{224}	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
8	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 7 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8	F1 = $(1 + 2x)$ F2 = $(1 + 2x)^8$	1^2 2^7	2^{56}	2^{224}	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
9	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 8 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1 R	F1 = 1 F2 = $(1 + 2x)^8$	2^8	2^{56}	2^{224}	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
10	$\begin{bmatrix} 4 & 2 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	F1 = $(1 + 2x)^4(1 + 2x^2)^2$ F2 = $(1 + 2x)^8$	$1^8 2^4$	$1^{28}2^{42}$	$1^{64}2^{192}$	$1^{116}2^{502}$	$1^{160}2^{816}$	$1^{160}2^{816}$	$1^{128}2^{448}$	$1^{64}2^{96}$
11	$\begin{bmatrix} 4 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1080	F1 = $(1 + 2x)^4(1 + 2x^2)$ F2 = $(1 + 2x)^6$ F4 = $(1 + 2x)^8$	$1^8 2^2 4$	$1^{26}2^{17}4^{13}$	$1^{48}2^{56}4^{72}$	$1^{64}2^{88}4^{220}$	$1^{64}2^{64}4^{400}$	$1^{32}2^{16}4^{432}$	4^{256}	4^{64}
12	$\begin{bmatrix} 3 & 2 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	F1 = $(1 + 2x)^3(1 + 2x^2)^2$ F2 = $(1 + 2x)^8$	$1^6 2^5$	$1^{16}2^{48}$	$1^{32}2^{208}$	$1^{52}2^{534}$	$1^{56}2^{868}$	$1^{48}2^{872}$	$1^{32}2^{496}$	2^{128}
13	$\begin{bmatrix} 3 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720 R	F1 = $(1 + 2x)^3(1 + 2x^2)$ F2 = $(1 + 2x)^6$ F4 = $(1 + 2x)^8$	$1^6 2^3 4$	$1^{14}2^{23}4^{13}$	$1^{20}2^{70}4^{72}$	$1^{24}2^{108}4^{220}$	$1^{16}2^{88}4^{400}$	$2^{32}4^{432}$	4^{256}	4^{64}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
14	$\begin{bmatrix} 2 & 2 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	5020 R	$F1 = (1 + 2x)^2(1 + 2x^2)^2$ $F2 = (1 + 2x)^8$	$1^4 2^6$	$1^8 2^{52}$	$1^{16} 2^{216}$	$1^{20} 2^{550}$	$1^{16} 2^{888}$	$1^{16} 2^{888}$	2^{512}	2^{128}
15	$\begin{bmatrix} 2 & 1 & 00 & 0 & 0 & 0 & 0 \\ 2 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080	$F1 = (1 + 2x)^2(1 + 2x^2)$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$1^4 2^4 4$	$1^6 2^{27} 4^{13}$	$1^8 2^{76} 4^{72}$	$1^8 2^{116} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
16	$\begin{bmatrix} 4 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	$F1 = (1 + 2x)^4$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	$1^8 4^2$	$1^{24} 4^{22}$	$1^{32} 4^{104}$	$1^{16} 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}
17	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x)^3$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	$1^6 2\ 4^2$	$1^{12} 2^6 4^{22}$	$1^8 2^{12} 4^{104}$	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}
18	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	5040 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	$1^4 2^2$ 4^2	$1^4 2^{10} 4^{22}$	$2^{16} 4^{104}$	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}
19	$\begin{bmatrix} 1 & 2 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x)(1 + 2x^2)^2$ $F2 = (1 + 2x)^8$	$1^2 2^7$	$1^4 2^{54}$	$1^8 2^{220}$	$1^4 2^{558}$	$1^8 2^{892}$	2^{896}	2^{512}	2^{128}
20	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 3 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720 R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$1^2 2^5 4$	$1^2 2^{29} 4^{13}$	$1^4 2^{78} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
21	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	$1^2 2^3$ 4^2	$2^{12} 4^{22}$	$2^{16} 4^{104}$	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}
22	$\begin{bmatrix} 0 & 2 & 00 & 0 & 0 & 0 & 0 \\ 4 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	$F1 = (1 + 2x^2)^2$ $F2 = (1 + 2x)^8$	2^8	$1^4 2^{54}$	2^{224}	$1^4 2^{558}$	2^{896}	2^{896}	2^{512}	2^{128}
23	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 4 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1080	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$2^6 4$	$1^2 2^{29} 4^{13}$	$2^{80} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
24	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 4 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	$F1 = 1$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	$2^4 4^2$	$2^{12} 4^{22}$	$2^{16} 4^{104}$	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
25	$\begin{bmatrix} 5 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	448 R	$F1 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^8$	$1^{10}\ 3^2$	$1^{40}3^{24}$	$1^{82}3^{122}$	$1^{100}3^{340}$	$1^{112}3^{560}$	$1^{160}3^{544}$	$1^{160}3^{288}$	$1^{64}3^{64}$
26	$\begin{bmatrix} 4 & 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	2240	$F1 = (1 + 2x)^4(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^7$ $F6 = (1 + 2x)^8$	$1^8 2$ 3^2	$1^{24}2^8$ $3^{20}6^2$	$1^{34}2^{24}$ $3^{82}6^{20}$	$1^{32}2^{34}$ $3^{176}6^{82}$	$1^{48}2^{32}$ $3^{208}6^{176}$	$1^{64}2^{48}$ $3^{128}\ 6^{208}$	$1^{32}2^{64}$ $3^{32}6^{128}$	$2^{32}6^{32}$
27	$\begin{bmatrix} 3 & 0 & 10 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	4480 R	$F1 = (1 + 2x)^3(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^6$ $F6 = (1 + 2x)^8$	$1^6 2^2$ 3^2	$1^{12}2^{14}$ $3^{16}6^4$	$1^{10}2^{36}$ $3^{50}6^{36}$	$1^{12}2^{44}$ $3^{76}6^{132}$	$1^{24}2^{44}$ $3^{56}6^{252}$	$1^{16}2^{72}$ $3^{16}6^{264}$	$2^{80}6^{144}$	$2^{32}6^{32}$
28	$\begin{bmatrix} 2 & 0 & 10 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	4480	$F1 = (1 + 2x)^2(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^5$ $F6 = (1 + 2x)^8$	$1^4 2^3$ 3^2	$1^4 2^{18}$ $3^{12}6^6$	$1^2 2^{40}$ $3^{26}6^{48}$	$1^8 2^{46}$ $3^{24}6^{158}$	$1^8 2^{52}$ $3^8 6^{276}$	$2^{80}6^{272}$	$2^{80}6^{144}$	$2^{32}6^{32}$
29	$\begin{bmatrix} 5 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	448	$F1 = (1 + 2x)^5$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^5$ $F6 = (1 + 2x)^8$	$1^{10}6$	$1^{40}6^{12}$	$1^{80}26^{61}$	$1^{80}2^{10}6^{170}$	$1^{32}2^{40}6^{280}$	$2^{80}6^{272}$	$2^{80}6^{144}$	$2^{32}6^{32}$
30	$\begin{bmatrix} 4 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	2240 R	$F1 = (1 + 2x)^4$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^4$ $F6 = (1 + 2x)^8$	$1^8 2$ 6	$1^{24}2^8 6^{12}$	$1^{32}2^{25}6^{61}$	$1^{16}2^{42}6^{170}$	$2^{56}6^{280}$	$2^{80}6^{272}$	$2^{80}6^{144}$	$2^{32}6^{32}$
31	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	4480	$F1 = (1 + 2x)^3$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^3$ $F6 = (1 + 2x)^8$	$1^6 2^2$ 6	$1^{12}2^{14}6^{12}$	$1^8 2^{37}6^{61}$	$2^{50}6^{170}$	$2^{56}6^{280}$	$2^{80}6^{272}$	$2^{80}6^{144}$	$2^{32}6^{32}$
32	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	4480 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^2$ $F6 = (1 + 2x)^8$	$1^4 2^3$ 6	$1^4 2^{18}6^{12}$	$2^{41}6^{61}$	$2^{50}6^{170}$	$2^{56}6^{280}$	$2^{80}6^{272}$	$2^{80}6^{144}$	$2^{32}6^{32}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
33	$\begin{bmatrix} 1 & 0 & 10 & 0 & 0 & 0 & 0 \\ 4 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	2240 R	$F1 = (1 + 2x)(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^4$ $F6 = (1 + 2x)^8$	$1^2 2^4$ 3^2	$2^{20} 3^8 6^8$	$1^2 2^{40} 3^{10} 6^{56}$	$1^4 2^{48}$ $3^4 6^{168}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
34	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 4 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	2240	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)$ $F6 = (1 + 2x)^8$	$1^2 2^4$ 6	$2^{20} 6^{12}$	$2^{41} 6^{61}$	$2^{50} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
35	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 5 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	448	$F1 = (1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^3$ $F6 = (1 + 2x)^8$	$2^5 3^2$	$2^{20} 3^4 6^{10}$	$1^2 2^{40}$ $3^2 6^{60}$	$2^{50} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
36	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 5 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	448 R	$F2 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^8$ $F1 = 1$ $F3 = 1$	$2^5 6$	$2^{20} 6^{12}$	$2^{41} 6^{61}$	$2^{50} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
37	$\begin{bmatrix} 4 & 0 & 01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x)^4(1 + 2x^4)$ $F2 = (1 + 2x)^4(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	$1^8 4^2$	$1^{24} 2^2 4^{21}$	$1^{32} 2^{16} 4^{96}$	$1^{18} 2^{49} 4^{251}$	$1^{16} 2^{72} 4^{408}$	$1^{48} 2^{56} 4^{408}$	$1^{64} 2^{32} 4^{224}$	$1^{32} 2^{16} 4^{48}$
38	$\begin{bmatrix} 4 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360 R	$F1 = (1 + 2x)^4$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	$1^8 8$	$1^{24} 8^{11}$	$1^{32} 8^{52}$	$1^{16} 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
39	$\begin{bmatrix} 3 & 0 & 01 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	13^{440} R	$F1 = (1 + 2x)^3(1 + 2x^4)$ $F2 = (1 + 2x)^4(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	$1^6 2$ 4^2	$1^{12} 2^8 4^{21}$	$1^8 2^{28} 4^{96}$	$1^2 2^{57} 4^{251}$	$1^{12} 2^{74} 4^{408}$	$1^{24} 2^{68} 4^{408}$	$1^{16} 2^{56} 4^{224}$	$2^{32} 4^{48}$
40	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	13^{440}	$F1 = (1 + 2x)^3$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	1^6 2 8	$1^{12} 2^6 8^{11}$	$1^8 2^{12} 8^{52}$	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
41	$\begin{bmatrix} 2 & 0 & 01 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	20,160	$F1 = (1 + 2x)^2(1 + 2x^4)$ $F2 = (1 + 2x)^4(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	$1^4 2^2$ 4^2	$1^4 2^{12} 4^{21}$	$2^{32} 4^{96}$	$1^2 2^{57} 4^{251}$	$1^8 2^{76} 4^{408}$	$1^8 2^{76} 4^{408}$	$2^{64} 4^{224}$	$2^{32} 4^{48}$
42	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	20,160 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	$1^4 2^2$ 8	$1^4 2^{10} 8^{11}$	$2^{16} 8^{52}$	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
43	$\begin{bmatrix} 1 & 0 & 01 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	13^{440} R	$F1 = (1 + 2x)(1 + 2x^4)$ $F2 = (1 + 2x)^4(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	$1^2 2^3$ 4^2	$2^{14} 4^{21}$	$2^{32} 4^{96}$	$1^2 2^{57} 4^{251}$	$1^4 2^{78} 4^{408}$	$2^{80} 4^{408}$	$2^{64} 4^{224}$	$2^{32} 4^{48}$
44	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	13^{440}	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	$1^2 2^3$ 8	$2^{12} 8^{11}$	$2^{16} 8^{52}$	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
45	$\begin{bmatrix} 0 & 0 & 01 & 0 & 0 & 0 & 0 \\ 4 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x^4)$ $F2 = (1 + 2x)^4(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	$2^4 4^2$	$2^{14} 4^{21}$	$2^{32} 4^{96}$	$1^2 2^{57} 4^{251}$	$2^{80} 4^{408}$	$2^{80} 4^{408}$	$2^{64} 4^{224}$	$2^{32} 4^{48}$
46	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 4 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360 R	$F1 = 1$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	$2^4 8$	$2^{12} 8^{11}$	$2^{16} 8^{52}$	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
47	$\begin{bmatrix} 1 & 2 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	$F1 = (1 + 2x)(1 + 2x^2)^2(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^4(1 + 2x^2)^2$ $F6 = (1 + 2x)^8$	$1^2 2^4$ 3^2	$1^4 2^{18}$ $3^8 6^8$	$1^{10} 2^{36}$ $3^{18} 6^{52}$	$1^8 2^{46}$ $3^{36} 6^{52}$	$1^{16} 2^{48}$ $3^{48} 6^{256}$	$1^{16} 2^{72}$ $3^{48} 6^{248}$	$1^8 2^{76}$ $3^{40} 6^{124}$	$1^{16} 2^{24}$ $3^{16} 6^{24}$
48	$\begin{bmatrix} 1 & 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	53,760	$F1 = (1 + 2x)(1 + 2x^2)(1 + 2x^3)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x)^4(1 + 2x^2)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	$1^2 2^2$ $3^2 4$	$1^2 2^5 3^8 4^7$ $6^4\ 12^2$	$1^6 2^2 3^{14} 4^{18}$ $6^{18}\ 12^{18}$	$1^4 2^4 3^{20} 4^{22}$ $6^{28}\ 12^{66}$	$1^4 2^{10} 3^{20} 4^{22}$ $6^{18}\ 12^{126}$	$1^8 2^4 3^8 4^{36}$ $6^4\ 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
49	$\begin{bmatrix} 0 & 2 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	$F1 = (1 + 2x^2)^2(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x^2)^2(1 + 2x)^3$ $F6 = (1 + 2x)^8$	$2^5 3^2$	$1^4 2^{18}$ $3^4 6^{10}$	$1^2 2^{40}$ $3^{10} 6^{56}$	$1^4 2^{48}$ $3^{16} 6^{162}$	$1^8 2^{52}$ $3^{16} 6^{272}$	$2^{80} 3^{16} 6^{264}$	$1^8 2^{76}$ $3^8 6^{140}$	$2^{32} 6^{32}$
50	$\begin{bmatrix} 0 & 1 & 10 & 0 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	53,760 R	$F1 = (1 + 2x^2)(1 + 2x^3)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x^2)(1 + 2x)^3$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	$2^3 3^2$ 4	$1^2 2^5$ $3^4 4^7$ $6^6\ 12^2$	$1^2 2^4$ $3^6 4^{18}$ $6^{22}\ 12^{18}$	$2^6 3^8 4^{22}$ $6^{34}\ 12^{66}$	$1^4 2^{10}$ $3^4 4^{22}$ $6^{26}\ 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
51	$\begin{bmatrix} 1 & 2 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	$F1 = (1 + 2x)(1 + 2x^2)^2$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)(1 + 2x^2)^2$ $F6 = (1 + 2x)^8$	$1^2 2^4$ 6	$1^4 2^{18} 6^{12}$	$1^8 2^{37} 6^{61}$	$1^4 2^{48} 6^{170}$	$1^8 2^{52} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
52	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	53,760 R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x)(1 + 2x^2)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	$1^2 2^2$ 4 6	$1^2 2^5 4^7$ $6^8\ 12^2$	$1^4 2^3 4^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
53	$\begin{bmatrix} 0 & 2 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	$F1 = (1 + 2x^2)^2$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x^2)^2$ $F6 = (1 + 2x)^8$	$2^5 6$	$1^4 2^{18} 6^{12}$	$2^{41} 6^{61}$	$1^4 2^{48} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
54	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	53,760	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x^2)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	$2^3 4$ 6	$1^2 2^5 4^7$ $6^8\ 12^2$	$2^5 4^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
55	$\begin{bmatrix}1&0&10&0&0&0&0\\0&2&00&0&0&0&0\end{bmatrix}$	26^{880} R	$F1 = (1 + 2x)(1 + 2x^3)$ $F2 = (1 + 2x)(1 + 2x^3)$ $F3 = (1 + 2x)^4$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^4$ $F12 = (1 + 2x)^8$	$1^2 3^2$ 4^2	$3^8 4^{10} 12^4$	$1^2 3^{10}$ $4^{20} 12^{28}$	$1^4 3^4$ $4^{24} 12^{84}$	$4^{28} 12^{140}$	$4^{40} 12^{136}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
56	$\begin{bmatrix}0&0&10&0&0&0&0\\1&2&00&0&0&0&0\end{bmatrix}$	26^{880}	$F1 = (1 + 2x^3)$ $F2 = (1 + 2x)(1 + 2x^3)$ $F3 = (1 + 2x)^3$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^4$ $F12 = (1 + 2x)^8$	$2\ 3^2$ 4^2	$3^4 4^{10}$ $6^2 12^4$	$1^2 3^2 4^{20}$ $6^4 12^{28}$	$2^2 4^{24}$ $6^2 12^{84}$	$4^{28} 12^{140}$	$4^{40} 12^{136}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
57	$\begin{bmatrix}1&0&00&0&0&0&0\\0&2&10&0&0&0&0\end{bmatrix}$	26^{880}	$F1 = (1 + 2x)$ $F2 = (1 + 2x)(1 + 2x^3)$ $F3 = (1 + 2x)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^4$ $F12 = (1 + 2x)^8$	$1^2 4^2$ 6	$4^{10} 6^4 12^4$	$2\ 4^{20}$ $6^5 12^{28}$	$2^2 4^{24}$ $6^2 12^{84}$	$4^{28} 12^{140}$	$4^{40} 12^{136}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
58	$\begin{bmatrix}0&0&00&0&0&0&0\\1&2&10&0&0&0&0\end{bmatrix}$	26^{880} R	$F1 = 1$ $F2 = (1 + 2x)(1 + 2x^3)$ $F3 = 1$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^4$ $F12 = (1 + 2x)^8$	$2\ 4^2$ 6	$4^{10} 6^4 12^4$	$2\ 4^{20}$ $6^5 12^{28}$	$2^2 4^{24}$ $6^2 12^{84}$	$4^{28} 12^{140}$	$4^{40} 12^{136}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
59	$\begin{bmatrix}1&0&11&0&0&0&0\\0&0&00&0&0&0&0\end{bmatrix}$	107,520	$F1 = (1 + 2x)(1 + 2x^3)(1 + 2x^4)$ $F2 = (1 + 2x)(1 + 2x^3)(1 + 2x^2)^2$ $F3 = (1 + 2x)^4(1 + 2x^4)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^4(1 + 2x^2)^2$ $F12 = (1 + 2x)^8$	$1^2 3^2$ 4^2	$2^2 3^8$ $4^9 12^4$	$1^2 2^4$ $3^{10} 4^{18}$ $6^4\ 12^{26}$	$1^6 2$ $3^4 4^{23}$ $6^{16}\ 12^{76}$	$1^4 2^6$ $3^4 4^{24}$ $6^{22}\ 12^{128}$	$2^8 3^{16} 4^{36}$ $6^{16}\ 12^{124}$	$1^4 2^2$ $3^{20} 4^{38}$ $6^{10}\ 12^{62}$	$1^8 2^4$ $3^8 4^{12}$ $6^4\ 12^{12}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
60	$\begin{bmatrix} 1 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520 R	F1 = $(1 + 2x)(1 + 2x^3)$ F2 = $(1 + 2x)(1 + 2x^3)$ F3 = $(1 + 2x)^4$ F4 = $(1 + 2x)(1 + 2x^3)$ F6 = $(1 + 2x)^4$ F8 = $(1 + 2x)^5(1 + 2x^3)$ F12 = $(1 + 2x)^4$ F24 = $(1 + 2x)^8$	$1^2 3^2$ 8	$3^8 8^5 24^2$	$1^2 3^{10}$ $8^{10} 24^{14}$	$1^4 3^4$ $8^{12} 24^{42}$	$8^{14} 24^{70}$	$8^{20} 24^{68}$	$8^{20} 24^{36}$	$8^8 24^8$
61	$\begin{bmatrix} 0 & 0 & 11 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520 R	F1 = $(1 + 2x^3)(1 + 2x^4)$ F2 = $(1 + 2x)(1 + 2x^3)(1 + 2x^2)^2$ F3 = $(1 + 2x)^3(1 + 2x^4)$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^4(1 + 2x^2)^2$ F12 = $(1 + 2x)^8$	2 $3^2 4^2$	$2^2 3^4 4^9$ $6^2\ 12^4$	$1^2 2^4$ $3^2 4^{18}$ $6^8\ 12^{26}$	$1^2 2^3$ $4^{23} 6^{18} 12^{76}$	$2^8 3^4 4^{24} 6^{22}$ 12^{128}	$2^8 3^8$ $4^{36} 6^{20}$ 12^{124}	$1^4 2^2$ $3^4 4^{38}$ $6^{18}\ 12^{62}$	$2^8 4^{12}$ $6^8 12^{12}$
62	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520	F1 = $(1 + 2x^3)$ F2 = $(1 + 2x)(1 + 2x^3)$ F3 = $(1 + 2x)^3$ F4 = $(1 + 2x)(1 + 2x^3)$ F6 = $(1 + 2x)^4$ F8 = $(1 + 2x)^5(1 + 2x^3)$ F12 = $(1 + 2x)^4$ F24 = $(1 + 2x)^8$	2 $3^2 8$	$3^4 6^2$ $8^5 24^2$	$1^2 3^2 6^4$ $8^{10} 24^{14}$	$2^2 6^2$ $8^{12} 24^{42}$	$8^{14} 24^{70}$	$8^{20} 24^{68}$	$8^{20} 24^{36}$	$8^8 24^8$
63	$\begin{bmatrix} 1 & 0 & 01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520 R	F1 = $(1 + 2x)(1 + 2x^4)$ F2 = $(1 + 2x)(1 + 2x^3)(1 + 2x^2)^2$ F3 = $(1 + 2x)(1 + 2x^4)$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^4(1 + 2x^2)^2$ F12 = $(1 + 2x)^8$	$1^2 4^2$ 6	$2^2 4^9$ $6^4 12^4$	$2^5 4^{18}$ $6^9 12^{26}$	$1^2 2^3$ $4^{23} 6^{18}\ 12^{76}$	$1^4 2^6 4^{24}$ $6^{24}\ 12^{128}$	$2^8 4^{36}$ $6^{24} 12^{124}$	$2^4 4^{38}$ $6^{20}\ 12^{62}$	$2^8 4^{12}$ $6^8\ 12^{12}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
64	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520	F1 = $(1 + 2x)$ F2 = $(1 + 2x)(1 + 2x^3)$ F3 = $(1 + 2x)$ F4 = $(1 + 2x)(1 + 2x^3)$ F6 = $(1 + 2x)^4$ F8 = $(1 + 2x)^5(1 + 2x^3)$ F12 = $(1 + 2x)^4$ F24 = $(1 + 2x)^8$	$1^2 6$ 8	$6^4 8^5 24^2$	$2\ 6^5$ $8^{10} 24^{14}$	$2^2 6^2$ $8^{12} 24^{42}$	$8^{14} 24^{70}$	$8^{20} 24^{68}$	$8^{20} 24^{36}$	$8^8 24^8$
65	$\begin{bmatrix} 0 & 0 & 01 & 0 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520	F1 = $(1 + 2x^4)$ F2 = $(1 + 2x)(1 + 2x^3)(1 + 2x^2)^2$ F3 = $(1 + 2x^4)$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^4(1 + 2x^2)^2$ F12 = $(1 + 2x)^8$	$2\ 4^2$ 6	$2^2 4^9$ $6^4 12^4$	$2^5 4^{18}$ $6^9 12^{26}$	$1^2 2^3 4^{23}$ $6^{18} 12^{76}$	$2^8 4^{24}$ $6^{24} 12^{128}$	$2^8 4^{36}$ $6^{24} 12^{124}$	$2^4 4^{38}$ $6^{20}\ 12^{62}$	$2^8 4^{12}$ $6^8\ 12^{12}$
66	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 11 & 0 & 0 & 0 & 0 \end{bmatrix}$	107,520 R	F1 = 1 F2 = $(1 + 2x)(1 + 2x^3)$ F3 = 1 F4 = $(1 + 2x)(1 + 2x^3)$ F6 = $(1 + 2x)^4$ F8 = $(1 + 2x)^5(1 + 2x^3)$ F12 = $(1 + 2x)^4$ F24 = $(1 + 2x)^8$	$2\ 6$ 8	$6^4 8^5$ 24^2	$2\ 6^5$ $8^{10} 24^{14}$	$2^2 6^2$ $8^{12} 24^{42}$	$8^{14} 24^{70}$	$8^{20} 24^{68}$	$8^{20} 24^{36}$	$8^8 24^8$
67	$\begin{bmatrix} 6 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56	F1 = $(1 + 2x)^6(1 + 2x^2)$ F2 = $(1 + 2x)^8$	1^{12} 2^2	$1^6 2^{225}$	$1^{184} 2^{132}$	$1^{360} 2^{380}$	$1^{512} 2^{640}$	$1^{544} 2^{624}$	$1^{38} 42^{320}$	$1^{128} 2^{64}$
68	$\begin{bmatrix} 6 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56 R	F1 = $(1 + 2x)^6$ F2 = $(1 + 2x)^6$ F4 = $(1 + 2x)^8$	$1^{12} 4$	$1^{60} 4^{13}$	$1^{160} 4^{72}$	$1^{240} 4^{220}$	$1^{192} 4^{400}$	$1^{64} 4^{432}$	4^{256}	4^{64}
69	$\begin{bmatrix} 5 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	336 R	F1 = $(1 + 2x)^5(1 + 2x^2)$ F2 = $(1 + 2x)^8$	1^{10} 2^3	$1^{42} 2^{35}$	$1^{100} 2^{174}$	$1^{160} 2^{480}$	$1^{192} 2^{800}$	$1^{160} 2^{816}$	$1^{64} 2^{480}$	2^{128}
70	$\begin{bmatrix} 5 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	336	F1 = $(1 + 2x)^5$ F2 = $(1 + 2x)^6$ F4 = $(1 + 2x)^8$	$1^{10} 2$ 4	$1^{40} 2^{10} 4^{13}$	$1^{80} 2^{40} 4^{72}$	$1^{80} 2^{80} 4^{220}$	$1^{32} 2^{80} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
71	$\begin{bmatrix} 4 & 1 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840	$F1 = (1 + 2x)^4(1 + 2x^2)$ $F2 = (1 + 2x)^8$	$1^8 2^4$	$1^{26} 2^{43}$	$1^{48} 2^{200}$	$1^{64} 2^{528}$	$1^{64} 2^{864}$	$1^{32} 2^{880}$	2^{512}	2^{128}
72	$\begin{bmatrix} 4 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	$F1 = (1 + 2x)^4$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$1^8 2^2$ 4	$1^{24} 2^{18} 4^{13}$	$1^{32} 2^{64} 4^{72}$	$1^{16} 2^{112} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
73	$\begin{bmatrix} 3 & 1 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1120 R	$F1 = (1 + 2x)^3(1 + 2x^2)$ $F2 = (1 + 2x)^8$	1^6 2^5	$1^{14} 2^{49}$	$1^{20} 2^{214}$	$1^{24} 2^{548}$	$1^{16} 2^{888}$	2^{896}	2^{512}	2^{128}
74	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1120	$F1 = (1 + 2x)^3$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	1^6 $2^3 4$	$1^{12} 2^{24} 4^{13}$	$1^8 2^{76} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
75	$\begin{bmatrix} 2 & 1 & 00 & 0 & 0 & 0 & 0 \\ 4 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840	$F1 = (1 + 2x)^2(1 + 2x^2)$ $F2 = (1 + 2x)^8$	1^4 2^6	$1^6 2^{53}$	$1^8 2^{220}$	$1^8 2^{556}$	2^{896}	2^{896}	2^{512}	2^{128}
76	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 4 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	840 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	1^4 $2^4 4$	$1^4 2^{28} 4^{13}$	$2^{80} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
77	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 5 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	336 R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^8$	1^2 2^7	$1^2 2^{55}$	$1^4 2^{222}$	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
78	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 5 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	336	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	1^2 $2^5 4$	$2^{30} 4^{13}$	$2^{80} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
79	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 6 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^8$	2^8	$1^2 2^{55}$	2^{224}	2^{560}	2^{896}	2^{896}	2^{512}	2^{128}
80	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 6 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	56 R	$F1 = 1$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$2^6 4$	$2^{30} 4^{13}$	$2^{80} 4^{72}$	$2^{120} 4^{220}$	$2^{96} 4^{400}$	$2^{32} 4^{432}$	4^{256}	4^{64}
81	$\begin{bmatrix} 3 & 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960	$F1 = (1 + 2x)^3(1 + 2x^2)(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^6(1 + 2x^2)$ $F6 = (1 + 2x)^8$	1^6 2^2 3^2	$1^{14} 2^{13}$ $3^{16} 6^4$	$1^{22} 2^{30}$ $3^{54} 6^{34}$	$1^{36} 2^{32}$ $3^{108} 6^{116}$	$1^{44} 2^{34}$ $3^{156} 6^{202}$	$1^{40} 2^{60}$ $3^{168} 6^{188}$	$1^{48} 2^{56}$ $3^{112} 6^{88}$	$1^{32} 2^{16}$ $3^{32}\ 6^{16}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
82	$\begin{bmatrix} 3 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960 R	F1 = $(1 + 2x)^3(1 + 2x^3)$ F2 = $(1 + 2x)^3(1 + 2x^3)$ F3 = $(1 + 2x)^6$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	1^6 $3^2 4$	$1^{12} 3^{16}$ $4^7 12^2$	$1^{10} 3^{50}$ $4^{18} 12^{18}$	$1^{12} 3^{76}$ $4^{22} 12^{66}$	$1^{24} 3^{56}$ $4^{22} 12^{126}$	$1^{16} 3^{16}$ $4^{36} 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
83	$\begin{bmatrix} 3 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960 R	F1 = $(1 + 2x)^3(1 + 2x^2)$ F2 = $(1 + 2x)^5(1 + 2x^3)$ F3 = $(1 + 2x)^3(1 + 2x^2)$ F6 = $(1 + 2x)^8$	1^6 $2^2 6$	$1^{14} 2^{13} 6^{12}$	$1^{20} 2^{31} 6^{61}$	$1^{24} 2^{38} 6^{170}$	$1^{16} 2^{48} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
84	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960	F1 = $(1 + 2x)^3$ F2 = $(1 + 2x)^3(1 + 2x^3)$ F3 = $(1 + 2x)^3$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	1^6 4 6	$1^{12} 4^7$ $6^8 12^2$	$1^8 24^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
85	$\begin{bmatrix} 2 & 1 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	F1 = $(1 + 2x)^2(1 + 2x^2)(1 + 2x^3)$ F2 = $(1 + 2x)^5(1 + 2x^3)$ F3 = $(1 + 2x)^5(1 + 2x^2)$ F6 = $(1 + 2x)^8$	1^4 2^3 3^2	$1^6 2^{17}$ $3^{12} 6^6$	$1^{10} 2^{36}$ $3^{30} 6^{46}$	$1^{16} 2^{42}$ $3^{48} 6^{146}$	$1^{12} 2^{50}$ $3^{60} 6^{250}$	$1^{16} 2^{72}$ $3^{48} 6^{248}$	$1^{16} 2^{72}$ $3^{16} 6^{136}$	$2^{32} 6^{32}$
86	$\begin{bmatrix} 2 & 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	F1 = $(1 + 2x)^2(1 + 2x^3)$ F2 = $(1 + 2x)^3(1 + 2x^3)$ F3 = $(1 + 2x)^5$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	$1^4 2$ 3^2 4	$1^4 2^4$ $3^{12} 4^7$ $6^2\ 12^2$	$1^2 2^4$ $3^{26} 4^{18}$ $6^{12}\ 12^{18}$	$1^8 2^2$ $3^{24} 4^{22}$ $6^{26}\ 12^{66}$	$1^8 2^8$ $3^8 4^{22}$ $6^{24}\ 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
87	$\begin{bmatrix} 2 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	F1 = $(1 + 2x)^2(1 + 2x^2)$ F2 = $(1 + 2x)^5(1 + 2x^3)$ F3 = $(1 + 2x)^2(1 + 2x^2)$ F6 = $(1 + 2x)^8$	1^4 $2^3 6$	$1^6 2^{17} 6^{12}$	$1^8 2^{37} 6^{61}$	$1^8 2^{46} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
88	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x)^2$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	$1^4 2$ 4 6	$1^4 2^4 4^7$ $6^8 12^2$	$2^5 4^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
89	$\begin{bmatrix} 1 & 1 & 10 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	$F1 = (1 + 2x)(1 + 2x^2)(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)^4(1 + 2x^2)$ $F6 = (1 + 2x)^8$	1^2 2^4 3^2	$1^2 2^{19}$ $3^8 6^8$	$1^6 2^{38}$ $3^{14} 6^{54}$	$1^4 2^{48}$ $3^{20} 6^{160}$	$1^4 2^{54}$ $3^{20} 6^{270}$	$1^8 2^{76}$ $3^8 6^{268}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
90	$\begin{bmatrix} 1 & 0 & 10 & 0 & 0 & 0 & 0 \\ 2 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	$F1 = (1 + 2x)(1 + 2x^3)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x)^4$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	1^2 2^2 $3^2 4$	$2^6 3^8 4^7$ $6^4 12^2$	$1^2 2^4$ $3^{10} 4^{18}$ $6^{20} 12^{18}$	$1^4 2^4$ $3^4 4^{22}$ $6^{36} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
91	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880} R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x)(1 + 2x^2)$ $F6 = (1 + 2x)^8$	1^2 $2^4 6$	$1^2 2^{19} 6^{12}$	$1^4 2^{39} 6^{61}$	$2^{50} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
92	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	26^{880}	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^3(1 + 2x^3)$ $F3 = (1 + 2x)$ $F4 = (1 + 2x)^5(1 + 2x^3)$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	1^2 $2^2 4$ 6	$2^6 4^7$ $6^8 12^2$	$2^5 4^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
93	$\begin{bmatrix} 0 & 1 & 10 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960 R	$F1 = (1 + 2x^2)(1 + 2x^3)$ $F2 = (1 + 2x)^5(1 + 2x^3)$ $F3 = (1 + 2x^2)(1 + 2x)^3$ $F6 = (1 + 2x)^8$	2^5 3^2	$1^2 2^{19}$ $3^4 6^{10}$	$1^2 2^{40}$ $3^6 6^{58}$	$2^{50} 3^8 6^{166}$	$1^4 2^{54}$ $3^4 6^{278}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
94	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 3 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960	F1 = $(1 + 2x^3)$ F2 = $(1 + 2x)^3(1 + 2x^3)$ F3 = $(1 + 2x)^3$ F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	2^3 3^2 4	$2^6 3^4$ $4^7 6^6$ 12^2	$1^2 2^4$ $3^2 4^{18}$ $6^{24} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
95	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960	F1 = $(1 + 2x^2)$ F2 = $(1 + 2x)^5(1 + 2x^3)$ F3 = $(1 + 2x^2)$ F6 = $(1 + 2x)^8$	2^5 6	$1^2 2^{19} 6^{12}$	$2^{41} 6^{61}$	$2^{50} 6^{170}$	$2^{56} 6^{280}$	$2^{80} 6^{272}$	$2^{80} 6^{144}$	$2^{32} 6^{32}$
96	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	8960 R	F1 = 1 F2 = $(1 + 2x)^3(1 + 2x^3)$ F3 = 1 F4 = $(1 + 2x)^5(1 + 2x^3)$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	2^3 4 6	$2^6 4^7$ $6^8 12^2$	$2^5 4^{18}$ $6^{25} 12^{18}$	$2^6 4^{22}$ $6^{38} 12^{66}$	$2^{12} 4^{22}$ $6^{28} 12^{126}$	$2^8 4^{36}$ $6^8 12^{132}$	$4^{40} 12^{72}$	$4^{16} 12^{16}$
97	$\begin{bmatrix} 2 & 1 & 01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	40,320 R	F1 = $(1 + 2x)^2(1 + 2x^2)(1 + 2x^4)$ F2 = $(1 + 2x)^4(1 + 2x^2)^2$ F4 = $(1 + 2x)^8$	1^4 2^2 4^2	$1^6 2^{11}$ 4^{21}	$1^8 2^{28} 4^{96}$	$1^{10} 2^{53}$ 4^{251}	$1^8 2^{76}$ 4^{408}	$1^{12} 2^7$ 44^{408}	$1^{16} 2^{56}$ 4^{224}	$1^{16} 2^{24}$ 4^{48}
98	$\begin{bmatrix} 2 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	40,320	F1 = $(1 + 2x)^2(1 + 2x^2)$ F2 = $(1 + 2x)^4$ F4 = $(1 + 2x)^4$ F8 = $(1 + 2x)^8$	1^4 $2^2 8$	$1^6 2^9$ 8^{11}	$1^8 2^{12}$ 8^{52}	$1^8 2^4$ 8^{138}	8^{224}	8^{224}	8^{128}	8^{32}
99	$\begin{bmatrix} 1 & 1 & 01 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640	F1 = $(1 + 2x)(1 + 2x^2)(1 + 2x^4)$ F2 = $(1 + 2x)^4(1 + 2x^2)^2$ F4 = $(1 + 2x)^8$	1^2 2^3 4^2	$1^2 2^{13}$ 4^{21}	$1^4 2^{30}$ 4^{96}	$1^2 2^{57}$ 4^{251}	$1^4 2^{78}$ 4^{408}	$1^4 2^{78}$ 4^{408}	$1^8 2^{60}$ 4^{224}	$2^{32} 4^{48}$
100	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640 R	F1 = $(1 + 2x)(1 + 2x^2)$ F2 = $(1 + 2x)^4$ F4 = $(1 + 2x)^4$ F8 = $(1 + 2x)^8$	1^2 2^3 8	$1^2 2^{11}$ 8^{11}	$1^4 2^{14}$ 8^{52}	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
101	$\begin{bmatrix}2&0&01&0&0&0&0\\0&1&00&0&0&0&0\end{bmatrix}$	40,320	$F1 = (1+2x)^2(1+2x^4)$ $F2 = (1+2x)^2(1+2x^2)^2$ $F4 = (1+2x)^8$	1^4 4^3	$1^4 2^2$ 4^{26}	$2^8 4^{108}$	$1^2 2^9$ 4^{275}	$1^8 2^4$ 4^{444}	$1^8 2^4$ 4^{444}	4^{256}	4^{64}
102	$\begin{bmatrix}2&0&00&0&0&0&0\\0&1&01&0&0&0&0\end{bmatrix}$	40,320 R	$F1 = (1+2x)^2$ $F2 = (1+2x)^2$ $F4 = (1+2x)^4$ $F8 = (1+2x)^8$	1^4 4 8	$1^4 4^5$ 8^{11}	$4^8 8^{52}$	$4^4 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
103	$\begin{bmatrix}1&0&01&0&0&0&0\\1&1&00&0&0&0&0\end{bmatrix}$	80,640 R	$F1 = (1+2x)(1+2x^4)$ $F2 = (1+2x)^2(1+2x^2)^2$ $F4 = (1+2x)^8$	1^2 2 4^3	$2^4 4^{26}$	$2^8 4^{108}$	$1^2 2^9$ 4^{275}	$1^4 2^6$ 4^{444}	$2^8 4^{444}$	4^{256}	4^{64}
104	$\begin{bmatrix}1&0&00&0&0&0&0\\1&1&01&0&0&0&0\end{bmatrix}$	80,640	$F1 = (1+2x)$ $F2 = (1+2x)^2$ $F4 = (1+2x)^4$ $F8 = (1+2x)^8$	1^2 2 4 8	$2^2 4^5$ 8^{11}	$4^8 8^{52}$	$4^4 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
105	$\begin{bmatrix}0&1&01&0&0&0&0\\2&0&00&0&0&0&0\end{bmatrix}$	40,320 R	$F1 = (1+2x^2)(1+2x^4)$ $F2 = (1+2x)^4(1+2x^2)^2$ $F4 = (1+2x)^8$	2^4 4^2	$1^2 2^{13}$ 4^{21}	$2^{32} 4^{96}$	$1^2 2^{57}$ 4^{251}	$2^{80} 4^{408}$	$1^4 2^{78}$ 4^{408}	$2^{64} 4^{224}$	$2^{32} 4^{48}$
106	$\begin{bmatrix}0&1&00&0&0&0&0\\2&0&01&0&0&0&0\end{bmatrix}$	40,320	$F1 = (1+2x^2)$ $F2 = (1+2x)^4$ $F4 = (1+2x)^4$ $F8 = (1+2x)^8$	2^4 8	$1^2 2^{11}$ 8^{11}	$2^{16} 8^{52}$	$2^8 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
107	$\begin{bmatrix}0&0&01&0&0&0&0\\2&1&00&0&0&0&0\end{bmatrix}$	40,320	$F1 = (1+2x^4)$ $F2 = (1+2x)^2(1+2x^2)^2$ $F4 = (1+2x)^8$	2^2 4^3	$2^4 4^{26}$	$2^8 4^{108}$	$1^2 2^9$ 4^{275}	$2^8 4^{444}$	$2^8 4^{444}$	4^{256}	4^{64}
108	$\begin{bmatrix}0&0&00&0&0&0&0\\2&1&01&0&0&0&0\end{bmatrix}$	40,320 R	$F1 = 1$ $F2 = (1+2x)^2$ $F4 = (1+2x)^4$ $F8 = (1+2x)^8$	2^2 4 8	$2^2 4^5$ 8^{11}	$4^8 8^{52}$	$4^4 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
109	$\begin{bmatrix}0&2&01&0&0&0&0\\0&0&00&0&0&0&0\end{bmatrix}$	40,320	$F1 = (1+2x^2)^2(1+2x^4)$ $F2 = (1+2x)^4(1+2x^2)^2$ $F4 = (1+2x)^8$	2^4 4^2	$1^4 2^{12}$ 4^{21}	$2^{32} 4^{96}$	$1^6 2^{55}$ 4^{251}	$2^{80} 4^{408}$	$1^8 2^{76}$ 4^{408}	$2^{64} 4^{224}$	$1^8 2^{28}$ 4^{48}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
110	$\begin{bmatrix} 0 & 1 & 01 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640 R	$F1 = (1 + 2x^2)(1 + 2x^4)$ $F2 = (1 + 2x)^2(1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	2^2 4^3	$1^2 2^3$ 4^{26}	$2^8 4^{108}$	$1^2 2^9$ 4^{275}	$2^8 4^{444}$	$1^4 2^6$ 4^{444}	4^{256}	4^{64}
111	$\begin{bmatrix} 0 & 2 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	40,320 R	$F1 = (1 + 2x^2)^2$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	2^4 8	$1^4 2^{10}$ 8^{11}	$2^{16} 8^{52}$	$1^4 2^6$ 8^{138}	8^{224}	8^{224}	8^{128}	8^{32}
112	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^2$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	2^2 4 8	$1^2 2$ $4^5 8^{11}$	$4^8 8^{52}$	$4^4 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
113	$\begin{bmatrix} 0 & 0 & 01 & 0 & 0 & 0 & 0 \\ 0 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	40,320	$F1 = (1 + 2x^4)$ $F2 = (1 + 2x^2)^2$ $F4 = (1 + 2x)^8$	4^4	$2^2 4^{27}$	4^{112}	$1^2 2$ 4^{279}	4^{448}	4^{448}	4^{256}	4^{64}
114	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 2 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	40,320 R	$F1 = 1$ $F2 = 1$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	$4^2 8$	$4^6 8^{11}$	$4^8 8^{52}$	$4^4 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
115	$\begin{bmatrix} 0 & 4 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1680 R	$F1 = (1 + 2x^2)^4$ $F2 = (1 + 2x)^8$	2^8	$1^8 2^{52}$	2^{224}	$1^{24} 2^{548}$	2^{896}	$1^{32} 2^{880}$	2^{512}	$1^{16} 2^{120}$
116	$\begin{bmatrix} 0 & 3 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720	$F1 = (1 + 2x^2)^3$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	$2^6 4$	$1^6 2^{27}$ 4^{13}	$2^{80} 4^{72}$	$1^{12} 2^{114}$ 4^{220}	$2^{96} 4^{400}$	$1^8 2^{28}$ 4^{432}	4^{256}	4^{64}
117	$\begin{bmatrix} 0 & 2 & 00 & 0 & 0 & 0 & 0 \\ 0 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080 R	$F1 = (1 + 2x^2)^2$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	2^4 4^2	$1^4 2^{10}$ 4^{22}	$2^{16} 4^{104}$	$1^4 2^6$ 4^{276}	4^{448}	4^{448}	4^{256}	4^{64}
118	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 3 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^2$ $F4 = (1 + 2x)^8$	2^2 4^3	$1^2 2$ 4^{27}	4^{112}	4^{280}	4^{448}	4^{448}	4^{256}	4^{64}
119	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 4 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	1680 R	$F1 = 1$ $F2 = 1$ $F4 = (1 + 2x)^8$	4^4	4^{28}	4^{112}	4^{280}	4^{448}	4^{448}	4^{256}	4^{64}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
120	$\begin{bmatrix} 2 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	17,920 R	$F1 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^8$	1^4 3^4	$1^4 3^{36}$	$1^4 3^{148}$	$1^{16} 3^{368}$	$1^{16} 3^{592}$	$1^4 3^{596}$	$1^{16} 3^{336}$	$1^{16} 3^{80}$
121	$\begin{bmatrix} 2 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	$F1 = (1 + 2x)^2(1 + 2x^3)$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^5$ $F6 = (1 + 2x)^8$	1^4 $3^2 6$	$1^4 3^{12}$ 6^{12}	$1^2 2$ $3^{26} 6^{61}$	$1^8 2^4$ $3^{24} 6^{172}$	$1^8 2^4$ $3^8 6^{292}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
122	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}$	17,920 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^2$ $F6 = (1 + 2x)^8$	1^4 6^2	$1^4 6^{18}$	$2^2 6^{74}$	$2^8 6^{184}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
123	$\begin{bmatrix} 1 & 0 & 20 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	$F1 = (1 + 2x)(1 + 2x^3)^2$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^7$ $F6 = (1 + 2x)^8$	1^2 2 3^4	$2^2 3^{28}$ 6^4	$1^4 3^{92}$ 6^{28}	$1^8 2^4$ $3^{184} 6^{92}$	$2^8 3^{224}$ 6^{184}	$1^4 3^{148}$ 6^{224}	$1^8 2^4$ $3^{40} 6^{148}$	$2^8 6^{40}$
124	$\begin{bmatrix} 1 & 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	71,680 R	$F1 = (1 + 2x)(1 + 2x^3)$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^4$ $F6 = (1 + 2x)^8$	1^2 2 $3^2 6$	$2^2 3^8$ 6^{14}	$1^2 2$ $3^{10} 6^{69}$	$1^4 2^6$ $3^4 6^{182}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
125	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)$ $F6 = (1 + 2x)^8$	1^2 2 6^2	$2^2 6^{18}$	$2^2 6^{74}$	$2^8 6^{184}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
126	$\begin{bmatrix} 0 & 0 & 20 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	17,920 R	$F1 = (1 + 2x^3)^2$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^6$ $F6 = (1 + 2x)^8$	2^2 3^4	$2^2 3^{20}$ 6^8	$1^4 3^{52}$ 6^{48}	$2^8 3^{80}$ 6^{144}	$2^8 3^{64}$ 6^{264}	$1^4 3^{20}$ 6^{288}	$2^8 6^{168}$	$2^8 6^{40}$
127	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	$F1 = (1 + 2x^3)$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x)^3$ $F6 = (1 + 2x)^8$	2^2 $3^2 6$	$2^2 3^4$ 6^{16}	$1^2 2$ $3^2 6^{73}$	$2^8 6^{184}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
128	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}$	17,920 R	F1 = 1 F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = 1 F6 = $(1 + 2x)^8$	2^2 6^2	$2^2 6^{18}$	$2^2 6^{74}$	$2^8 6^{184}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
129	$\begin{bmatrix} 0 & 1 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	F1 = $(1 + 2x^2)(1 + 2x^3)^2$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x^2)(1 + 2x)^6$ F6 = $(1 + 2x)^8$	2^2 3^4	$1^2 2$ $3^{20} 6^8$	$1^4 3^{60}$ 6^{44}	$2^8 3^{120}$ 6^{124}	$1^8 2^4$ $3^{168} 6^{212}$	$1^4 3^{180}$ 6^{208}	$2^8 3^{128}$ 6^{104}	$1^8 2^4$ $3^{40} 6^{20}$
130	$\begin{bmatrix} 0 & 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	71,680 R	F1 = $(1 + 2x^2)(1 + 2x^3)$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x^2)(1 + 2x)^3$ F6 = $(1 + 2x)^8$	2^2 $3^2 6$	$1^2 2$ $3^4 6^{16}$	$1^2 2$ $3^6 6^{71}$	$2^8 3^8$ 6^{180}	$1^4 2^6$ $3^4 6^{294}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
131	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840	F1 = $(1 + 2x^2)$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x^2)$ F6 = $(1 + 2x)^8$	2^{262}	$1^2 2$ 6^{18}	$2^2 6^{74}$	$2^8 6^{184}$	$2^8 6^{296}$	$2^2 6^{298}$	$2^8 6^{168}$	$2^8 6^{40}$
132	$\begin{bmatrix} 0 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840 R	F1 = $(1 + 2x^3)^2$ F2 = $(1 + 2x^3)^2$ F3 = $(1 + 2x)^6$ F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	3^4 4	$3^{20} 4$ 12^4	$1^4 3^{52}$ 12^{24}	$3^{80} 4^4$ 12^{72}	$3^{64} 4^4$ 12^{132}	$1^4 3^{20}$ 12^{144}	$4^4 12^{84}$	$4^4 12^{20}$
133	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	71,680	F1 = $(1 + 2x^3)$ F2 = $(1 + 2x^3)^2$ F3 = $(1 + 2x)^3$ F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	3^2 4 6	$3^4 4$ $6^8 12^4$	$1^2 2$ $3^2 6^{25} 12^{24}$	$4^4 6^{40}$ 12^{72}	$4^4 6^{32}$ 12^{132}	$2^2 6^{10}$ 12^{144}	$4^4 12^{84}$	$4^4 12^{20}$
134	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 20 & 0 & 0 & 0 & 0 \end{bmatrix}$	35,840 R	F1 = 1 F2 = $(1 + 2x^3)^2$ F3 = 1 F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^6$ F12 = $(1 + 2x)^8$	4 6^2	4 $6^{10} 12^4$	$2^2 6^{26}$ 12^{24}	$4^4 6^{40}$ 12^{72}	$4^4 6^{32}$ 12^{132}	$2^2 6^{10}$ 12^{144}	$4^4 12^{84}$	$4^4 12^{20}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
135	$\begin{bmatrix} 2 & 3 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x)^2(1 + 2x^2)^3$ $F2 = (1 + 2x)^8$	1^4 2^6	$1^{10}2^{51}$	$1^{24}2^{212}$	$1^{36}2^{542}$	$1^{48}2^{872}$	$1^{56}2^{868}$	$1^{32}2^{496}$	$1^{32}2^{112}$
136	$\begin{bmatrix} 2 & 2 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080 R	$F1 = (1 + 2x)^2(1 + 2x^2)^2$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	1^4 2^4 4	$1^8 2^{26}$ 4^{13}	$1^{16}2^{72}$ 4^{72}	$1^{20}2^{110}$ 4^{220}	$1^{16}2^{88}$ 4^{400}	$1^{16}2^{24}$ 4^{432}	4^{256}	4^{64}
137	$\begin{bmatrix} 2 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080	$F1 = (1 + 2x)^2(1 + 2x^2)$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	1^4 2^2 4^2	$1^6 2^9$ 4^{22}	$1^8 2^{12}$ 4^{104}	$1^8 2^4$ 4^{276}	4^{448}	4^{448}	4^{256}	4^{64}
138	$\begin{bmatrix} 1 & 3 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720 R	$F1 = (1 + 2x)(1 + 2x^2)^3$ $F2 = (1 + 2x)^8$	1^2 2^7	$1^6 2^{53}$	$1^{12}2^{218}$	$1^{12}2^{554}$	$1^{24}2^{884}$	$1^8 2^{892}$	$1^{16}2^{504}$	2^{128}
139	$\begin{bmatrix} 1 & 2 & 00 & 0 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	20,160	$F1 = (1 + 2x)(1 + 2x^2)^2$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	1^2 2^5 4	$1^4 2^{28}$ 4^{13}	$1^8 2^{76}$ 4^{72}	$1^4 2^{118}$ 4^{220}	$1^8 2^{92}$ 4^{400}	$2^{32}4^{432}$	4^{256}	4^{64}
140	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 1 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	20,160 R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	1^2 2^3 4^2	$1^2 2^{11}$ 4^{22}	$1^4 2^{14}$ 4^{104}	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}
141	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 3 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^2$ $F4 = (1 + 2x)^8$	1^4 4^3	$1^4 4^{27}$	4^{112}	4^{280}	4^{448}	4^{448}	4^{256}	4^{64}
142	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 3 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	6720	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^2$ $F4 = (1 + 2x)^8$	1^2 2 4^3	$2^2 4^{27}$	4^{112}	4^{280}	4^{448}	4^{448}	4^{256}	4^{64}
143	$\begin{bmatrix} 0 & 3 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360	$F1 = (1 + 2x^2)^3$ $F2 = (1 + 2x)^8$	2^8	$1^6 2^{53}$	2^{224}	$1^{12}2^{554}$	2^{896}	$1^8 2^{892}$	2^{512}	2^{128}
144	$\begin{bmatrix} 0 & 2 & 00 & 0 & 0 & 0 & 0 \\ 2 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080 R	$F1 = (1 + 2x^2)^2$ $F2 = (1 + 2x)^6$ $F4 = (1 + 2x)^8$	2^6 4	$1^4 2^{28}$ 4^{13}	$2^{80}4^{72}$	$1^4 2^{118}$ 4^{220}	$2^{96}4^{400}$	$2^{32}4^{432}$	4^{256}	4^{64}
145	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 2 & 2 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	10,080	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^4$ $F4 = (1 + 2x)^8$	2^4 4^2	$1^2 2^{11}$ 4^{22}	$2^{16}4^{104}$	$2^8 4^{276}$	4^{448}	4^{448}	4^{256}	4^{64}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
146	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 3 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	3360 R	F1 = 1 F2 = $(1 + 2x)^2$ F4 = $(1 + 2x)^8$	2^2 4^3	$2^2 4^{27}$	4^{112}	4^{280}	4^{448}	4^{448}	4^{256}	4^{64}
147	$\begin{bmatrix} 2 & 0 & 00 & 0 & 1 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	102,520	F1 = $(1 + 2x)^2(1 + 2x^6)$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x)^2(1 + 2x^2)^3$ F6 = $(1 + 2x)^8$	1^4 6^2	$1^4 3^2$ 6^{17}	$2^2 3^8$ 6^{70}	$2^8 3^{12}$ 6^{178}	$2^8 3^{16}$ 6^{288}	$1^2 2$ $3^{18} 6^{289}$	$1^8 2^4$ $3^8 6^{164}$	$1^8 2^4$ $3^8 6^{36}$
148	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 1 & 0 & 0 \end{bmatrix}$	102,520 R	F1 = $(1 + 2x)^2$ F2 = $(1 + 2x)^2$ F3 = $(1 + 2x)^2$ F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^2$ F12 = $(1 + 2x)^8$	1^4 12	$1^4 12^9$	$4\ 12^{37}$	$4^4 12^{92}$	$4^4 12^{148}$	$4\ 12^{149}$	$4^4 12^{84}$	$4^4 12^{20}$
149	$\begin{bmatrix} 1 & 0 & 00 & 0 & 1 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	205,040 R	F1 = $(1 + 2x)(1 + 2x^6)$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x)(1 + 2x^2)^3$ F6 = $(1 + 2x)^8$	1^2 2 6^2	$2^2 3^2$ 6^{17}	$2^2 3^4$ 6^{72}	$2^8 3^4$ 6^{182}	$2^8 3^8$ 6^{292}	$1^2 2$ $3^2 6^{297}$	$1^4 2^6$ $3^4 6^{166}$	$2^8 6^{40}$
150	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 1 & 0 & 0 \end{bmatrix}$	205,040	F1 = $(1 + 2x)$ F2 = $(1 + 2x)^2$ F3 = $(1 + 2x)$ F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^2$ F12 = $(1 + 2x)^8$	1^2 2 12	$2^2 12^9$	4 12^{37}	$4^4 12^{92}$	$4^4 12^{148}$	4 12^{149}	$4^4 12^{84}$	$4^4 12^{20}$
151	$\begin{bmatrix} 0 & 0 & 00 & 0 & 1 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	102,520	F1 = $(1 + 2x^6)$ F2 = $(1 + 2x)^2(1 + 2x^3)^2$ F3 = $(1 + 2x^2)^3$ F6 = $(1 + 2x)^8$	2^2 6^2	$2^2 3^2$ 6^{17}	$2^2 6^{74}$	$2^8 3^4$ 6^{182}	$2^8 6^{296}$	$1^2 2$ $3^2 6^{297}$	$2^8 6^{168}$	$2^8 6^{40}$
152	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 1 & 0 & 0 \end{bmatrix}$	102,520 R	F1 = 1 F2 = $(1 + 2x)^2$ F3 = 1 F4 = $(1 + 2x)^2(1 + 2x^3)^2$ F6 = $(1 + 2x)^2$ F12 = $(1 + 2x)^8$	2^2 12	$2^2 12^9$	4 12^{37}	$4^4 12^{92}$	$4^4 12^{148}$	4 12^{149}	$4^4 12^{84}$	$4^4 12^{20}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
153	$\begin{bmatrix} 0 & 1 & 00 & 0 & 1 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	215,040 R	$F1 = (1 + 2x^2)(1 + 2x^6)$ $F2 = (1 + 2x)^2(1 + 2x^3)^2$ $F3 = (1 + 2x^2)^4$ $F6 = (1 + 2x)^8$	2^2 6^2	$1^2 2$ $3^2 6^{17}$	$2^2 6^{74}$	$2^8 3^8$ 6^{180}	$2^8 6^{296}$	$1^2 2$ $3^{10} 6^{293}$	$2^8 6^{168}$	$1^4 2^6$ $3^4 6^{38}$
154	$\begin{bmatrix} 0 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 1 & 0 & 0 \end{bmatrix}$	215,040	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^2$ $F3 = (1 + 2x^2)$ $F4 = (1 + 2x)^2(1 + 2x^3)^2$ $F6 = (1 + 2x)^2$ $F12 = (1 + 2x)^8$	2^2 12	$1^2 2$ 12^9	$4\ 12^{37}$	$4^4 12^{92}$	$4^4 12^{148}$	$4\ 12^{149}$	$4^4 12^{84}$	$4^4 12^{20}$
155	$\begin{bmatrix} 0 & 0 & 00 & 0 & 1 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	215,040	$F1 = (1 + 2x^6)$ $F2 = (1 + 2x^3)^2$ $F3 = (1 + 2x^2)^3$ $F4 = (1 + 2x)^2(1 + 2x^3)^2$ $F6 = (1 + 2x)^6$ $F12 = (1 + 2x)^8$	4 6^2	$3^2 4$ $6^9 12^4$	$2^2 6^{26}$ 12^{24}	$3^4 4^4$ $6^{38} 12^{72}$	$4^4 6^{32}$ 12^{132}	$1^2 2 3^2$ $6^9 12^{144}$	$4^4 12^{84}$	$4^4 12^{20}$
156	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 1 & 0 & 0 \end{bmatrix}$	215,040 R	$F1 = 1$ $F2 = 1$ $F3 = 1$ $F4 = (1 + 2x)^2(1 + 2x^3)^2$ $F6 = 1$ $F12 = (1 + 2x)^8$	4 12	$4\ 12^9$	$4\ 12^{37}$	$4^4 12^{92}$	$4^4 12^{148}$	$4\ 12^{149}$	$4^4 12^{84}$	$4^4 12^{20}$
157	$\begin{bmatrix} 0 & 0 & 02 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640 R	$F1 = (1 + 2x^4)^2$ $F2 = (1 + 2x^2)^4$ $F4 = (1 + 2x)^8$	4^4	$2^4 4^{26}$	4^{112}	$1^4 2^{10}$ 4^{274}	4^{448}	$2^{16} 4^{440}$	4^{256}	$1^4 2^6$ 4^{60}
158	$\begin{bmatrix} 0 & 0 & 01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 01 & 0 & 0 & 0 & 0 \end{bmatrix}$	161,280	$F1 = (1 + 2x^4)$ $F2 = (1 + 2x^2)^2$ $F4 = (1 + 2x)^4$ $F8 = (1 + 2x)^8$	4^2 8	$2^2 4^5$ 8^{11}	$4^8 8^{52}$	$1^2 2$ $4^3 8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
159	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 02 & 0 & 0 & 0 & 0 \end{bmatrix}$	80,640 R	$F1 = 1$ $F2 = 1$ $F4 = 1$ $F8 = (1 + 2x)^8$	8^2	8^{14}	8^{56}	8^{140}	8^{224}	8^{224}	8^{128}	8^{32}

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
160	$\begin{bmatrix} 3 & 0 & 00 & 1 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	21,504 R	$F1 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^8$	1^6 5^2	$1^{12}5^{20}$	$1^8 5^{88}$	5^{224}	$1^2 5^{358}$	$1^{12}5^{356}$	$1^{24}5^{200}$	$1^{16}5^{48}$
161	$\begin{bmatrix} 2 & 0 & 00 & 1 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	64,512	$F1 = (1 + 2x)^2(1 + 2x^5)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^7$ $F10 = (1 + 2x)^8$	1^4 2 5^2	$1^4 2^4$ $5^{16}10^2$	$2^4 5^{56}$ 10^{16}	5^{112} 10^{56}	$1^2 5^{134}$ 10^{112}	$1^8 2^2$ $5^{88}10^{134}$	$1^8 2^8$ $5^{24}10^{88}$	$2^8 10^{24}$
162	$\begin{bmatrix} 3 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	21,504	$F1 = (1 + 2x)^3$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^3$ $F10 = (1 + 2x)^8$	1^6 10	$1^{12}10^{10}$	$1^8 10^{44}$	10^{112}	2 10^{179}	$2^6 10^{178}$	$2^{12}\ 10^{100}$	$2^8 10^{24}$
163	$\begin{bmatrix} 2 & 0 & 00 & 0 & 0 & 0 & 0 \\ 1 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	64,512 R	$F1 = (1 + 2x)^2$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^2$ $F10 = (1 + 2x)^8$	1^4 2 10	$1^4 2^4$ 10^{10}	$2^4 10^{44}$	10^{112}	2 10^{179}	$2^6 10^{178}$	$2^{12}\ 10^{100}$	$2^8 10^{24}$
164	$\begin{bmatrix} 1 & 0 & 00 & 1 & 0 & 0 & 0 \\ 2 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	64,512 R	$F1 = (1 + 2x)(1 + 2x^5)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^6$ $F10 = (1 + 2x)^8$	1^2 2^2 5^2	$2^6 5^{12}$ 10^4	$2^4 5^{32}$ 10^{28}	$5^{48}10^{88}$	$1^2 5^{38}$ 10^{160}	$1^4 2^4$ $5^{12}10^{172}$	$2^{12}10^{100}$	$2^8 10^{24}$
165	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 2 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	64,512	$F1 = (1 + 2x)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)$ $F10 = (1 + 2x)^8$	1^2 2^2 10	$2^6 10^{10}$	$2^4 10^{44}$	10^{112}	2 10^{179}	$2^6 10^{178}$	$2^{12}\ 10^{100}$	$2^8 10^{24}$
166	$\begin{bmatrix} 0 & 0 & 00 & 1 & 0 & 0 & 0 \\ 3 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	21,504	$F1 = (1 + 2x^5)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^5$ $F10 = (1 + 2x)^8$	2^3 5^2	$2^6 5^8$ 10^6	$2^4 5^{16}$ 10^{36}	$5^{16}10^{104}$	$1^2 5^6$ 10^{176}	$2^6 10^{178}$	$2^{12}10^{100}$	$2^8 10^{24}$
167	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 3 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	64,512 R	$F1 = 1$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = 1$ $F10 = (1 + 2x)^8$	2^3 10	$2^6 10^{10}$	$2^4 10^{44}$	10^{112}	2 10^{179}	$2^6 10^{178}$	$2^{12}10^{100}$	$2^8 10^{24}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
168	$\begin{bmatrix} 1 & 1 & 00 & 1 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	129,024	$F1 = (1 + 2x)(1 + 2x^2)(1 + 2x^5)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^6(1 + 2x^2)$ $F10 = (1 + 2x)^8$	1^2 2^2 5^2	$1^2 2^5$ $5^{12} 10^4$	$1^4 2^2$ $5^{36} 10^{26}$	$5^{72} 10^{76}$	$1^2 5^{102}$ 10^{128}	$1^4 2^4$ 5^{108} 10^{124}	$1^4 2^{10}$ $5^{76} 10^{62}$	$1^8 2^4$ $5^{24} 10^{12}$
169	$\begin{bmatrix} 1 & 0 & 00 & 1 & 0 & 0 & 0 \\ 0 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	129,024 R	$F1 = (1 + 2x)(1 + 2x^5)$ $F2 = (1 + 2x)(1 + 2x^5)$ $F4 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^6$ $F10 = (1 + 2x)^6$ $F20 = (1 + 2x)^8$	1^2 4 5^2	$4^3 5^{12}$ 20^2	$4^2 5^{32}$ 20^{14}	$5^{48} 20^{44}$	$1^2 5^{38}$ 20^{80}	$1^4 4^2$ 5^{12} 20^{86}	$4^6 20^{50}$	$4^4 20^{12}$
170	$\begin{bmatrix} 1 & 1 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	129,024 R	$F1 = (1 + 2x)(1 + 2x^2)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)(1 + 2x^2)$ $F10 = (1 + 2x)^8$	1^2 2^2 10	$1^2 2^5$ 10^{10}	$1^4 2^2$ 10^{44}	10^{112}	2 10^{179}	2^6 10^{178}	2^{12} 10^{100}	$2^8 10^{24}$
171	$\begin{bmatrix} 1 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 1 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	129,024	$F1 = (1 + 2x)$ $F2 = (1 + 2x)(1 + 2x^5)$ $F4 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)$ $F10 = (1 + 2x)^6$ $F20 = (1 + 2x)^8$	1^2 4 10	$4^3 10^6$ 20^2	$4^2 10^{16}$ 20^{14}	$10^{24} 20^{44}$	2 $10^{19} 20^{80}$	$2^2 4^2$ $10^6 20^{86}$	$4^6 20^{50}$	$4^4 20^{12}$
172	$\begin{bmatrix} 0 & 1 & 00 & 1 & 0 & 0 & 0 \\ 1 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	129,024 R	$F1 = (1 + 2x^2)(1 + 2x^5)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x^2)(1 + 2x)^5$ $F10 = (1 + 2x)^8$	2^3 5^2	$1^2 2^5$ $5^8 10^6$	$2^4 5^{20}$ 10^{34}	$5^{32} 10^{96}$	$1^2 5^{38}$ 10^{160}	$2^6 5^{32}$ 10^{162}	$1^4 2^{10}$ $5^{12} 10^{94}$	$2^8 10^{24}$
173	$\begin{bmatrix} 0 & 0 & 00 & 1 & 0 & 0 & 0 \\ 1 & 1 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	129,024	$F1 = (1 + 2x^5)$ $F2 = (1 + 2x)(1 + 2x^5)$ $F4 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x)^5$ $F10 = (1 + 2x)^6$ $F20 = (1 + 2x)^8$	2 4 5^2	$4^3 5^8$ $10^2 20^2$	$4^2 5^{16}$ $10^8 20^{14}$	$5^{16} 10^{16}$ 20^{44}	$1^2 5^6$ $10^{16} 20^{80}$	$2^2 4^2$ $10^6 20^{86}$	$4^6 20^{50}$	$4^4 20^{12}$

Table 1. *Cont.*

	Matrix Type	Order	$F_d(x)$	q = 1	q = 2	q = 3	q = 4	q = 5	q = 6	q = 7	q = 8
174	$\begin{bmatrix}0 & 1 & 00 & 0 & 0 & 0 & 0\\1 & 0 & 00 & 1 & 0 & 0 & 0\end{bmatrix}$	129,024	$F1 = (1 + 2x^2)$ $F2 = (1 + 2x)^3(1 + 2x^5)$ $F5 = (1 + 2x^2)$ $F10 = (1 + 2x)^8$	2^3 10	$1^2 2^5$ 10^{10}	$2^4 10^{44}$	10^{112}	$2\ 10^{179}$	$2^6 10^{178}$	$2^{12}\ 10^{100}$	$2^8 10^{24}$
175	$\begin{bmatrix}0 & 0 & 00 & 0 & 0 & 0 & 0\\1 & 1 & 00 & 1 & 0 & 0 & 0\end{bmatrix}$	129,024 R	$F1 = 1$ $F2 = (1 + 2x)(1 + 2x^5)$ $F4 = (1 + 2x)^3(1 + 2x^5)$ $F5 = 1$ $F10 = (1 + 2x)^6$ $F20 = (1 + 2x)^8$	2 4 10	$4^3 10^6$ 20^2	$4^2 10^{16}$ 20^{14}	$10^{24} 20^{44}$	2 $10^{19} 20^{80}$	$2^2 4^2$ $10^6 20^{86}$	$4^6 20^{50}$	$4^4 20^{12}$
176	$\begin{bmatrix}1 & 0 & 00 & 0 & 0 & 1 & 0\\0 & 0 & 00 & 0 & 0 & 0 & 0\end{bmatrix}$	368,640 R	$F1 = (1 + 2x)(1 + 2x^7)$ $F7 = (1 + 2x)^8$	1^2 7^2	7^{16}	7^{64}	7^{160}	7^{256}	7^{256}	$1^2 7^{146}$	$1^4 7^{36}$
177	$\begin{bmatrix}1 & 0 & 00 & 0 & 0 & 0 & 0\\0 & 0 & 00 & 0 & 0 & 1 & 0\end{bmatrix}$	368,640	$F1 = (1 + 2x)$ $F2 = (1 + 2x)(1 + 2x^7)$ $F7 = (1 + 2x)$ $F14 = (1 + 2x)^8$	1^2 14	14^8	14^{32}	14^{80}	14^{128}	14^{128}	2 14^{73}	$2^2 14^{18}$
178	$\begin{bmatrix}0 & 0 & 00 & 0 & 0 & 1 & 0\\1 & 0 & 00 & 0 & 0 & 0 & 0\end{bmatrix}$	368,640 R	$F1 = (1 + 2x^7)$ $F2 = (1 + 2x)(1 + 2x^7)$ $F7 = (1 + 2x)^7$ $F14 = (1 + 2x)^8$	2 7^2	$7^{12} 14^2$	$7^{40} 14^{12}$	$7^{80} 14^{40}$	$7^{96} 14^{80}$	$7^{64} 14^{96}$	$1^2 7^{18}$ 14^{64}	$2^2 14^{18}$
179	$\begin{bmatrix}0 & 0 & 00 & 0 & 0 & 0 & 0\\1 & 0 & 00 & 0 & 0 & 1 & 0\end{bmatrix}$	368,640	$F1 = 1$ $F2 = (1 + 2x)(1 + 2x^7)$ $F7 = 1$ $F14 = (1 + 2x)^8$	2 14	14^8	14^{32}	14^{80}	14^{128}	14^{128}	2 14^{73}	$2^2 14^{18}$
180	$\begin{bmatrix}0 & 0 & 00 & 0 & 0 & 0 & 1\\0 & 0 & 00 & 0 & 0 & 0 & 0\end{bmatrix}$	64,5120	$F1 = (1 + 2x^8)$ $F2 = (1 + 2x^4)^2$ $F4 = (1 + 2x^2)^4$ $F8 = (1 + 2x)^8$	8^2	$4^2 8^{13}$	8^{56}	$2^2 4^5$ 8^{137}	8^{224}	$4^8 8^{220}$	8^{128}	$1^2 2$ $4^3 8^{30}$

Table 1. *Cont.*

	Matrix Type	**Order**	$\mathbf{F_d(x)}$	**q = 1**	**q = 2**	**q = 3**	**q = 4**	**q = 5**	**q = 6**	**q = 7**	**q = 8**
181	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 1 \end{bmatrix}$	64,5120 R	F1 = 1 F2 = 1 F4 = 1 F8 = 1 F16 = $(1 + 2x)^8$	16	16^7	16^{28}	16^{70}	16^{112}	16^{112}	16^{64}	16^{16}
182	$\begin{bmatrix} 0 & 0 & 10 & 1 & 0 & 0 & 0 \\ 0 & 0 & 00 & 0 & 0 & 0 & 0 \end{bmatrix}$	172,032 R	F1 = $(1 + 2x^3)(1 + 2x^5)$ F3 = $(1 + 2x)^3(1 + 2x^5)$ F5 = $(1 + 2x^3)(1 + 2x)^5$ F15 = $(1 + 2x)^8$	3^2 5^2	$3^4 5^8$ 15^4	$1^2 3^2$ $5^{16} 15^{24}$	$5^{20} 15^{68}$	$1^2 5^{22}$ 15^{112}	$3^4 5^{32}$ 15^{108}	$3^8 5^{32}$ 15^{56}	$1^4 3^4$ $5^{12} 15^{12}$
183	$\begin{bmatrix} 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 00 & 1 & 0 & 0 & 0 \end{bmatrix}$	172,032	F1 = $(1 + 2x^3)$ F2 = $(1 + 2x^3)(1 + 2x^5)$ F3 = $(1 + 2x)^3$ F5 = $(1 + 2x^3)$ F6 = $(1 + 2x)^3(1 + 2x^5)$ F10 = $(1 + 2x^3)(1 + 2x)^5$ F15 = $(1 + 2x)^3$ F30 = $(1 + 2x)^8$	3^2 10	$3^4 10^4$ 30^2	$1^2 3^2$ $10^8 30^{12}$	$10^{10} 30^{34}$	2 $10^{11} 30^{56}$	6^2 $10^{16} 30^{54}$	6^4 $10^{16} 30^{28}$	$2^2 6^2$ $10^6 30^6$
184	$\begin{bmatrix} 0 & 0 & 00 & 1 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}$	172,032	F1 = $(1 + 2x^5)$ F2 = $(1 + 2x^3)(1 + 2x^5)$ F3 = $(1 + 2x^5)$ F5 = $(1 + 2x)^5$ F6 = $(1 + 2x)^3(1 + 2x^5)$ F10 = $(1 + 2x^3)(1 + 2x)^5$ F15 = $(1 + 2x)^5$ F30 = $(1 + 2x)^8$	5^2 6	$5^8 6^2$ 30^2	2 5^{16} 6 30^{12}	$5^{16} 10^2$ 30^{34}	$1^2 5^6$ 10^8 30^{56}	6^2 10^{16} 30^{54}	$6^4 10^{16}$ 30^{28}	$2^2 6^2$ $10^6 30^6$
185	$\begin{bmatrix} 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 1 & 0 & 0 & 0 \end{bmatrix}$	172,032 R	F1 = 1 F2 = $(1 + 2x^3)(1 + 2x^5)$ F3 = 1 F5 = 1 F6 = $(1 + 2x)^3(1 + 2x^5)$ F10 = $(1 + 2x^3)(1 + 2x)^5$ F15 = 1 F30 = $(1 + 2x)^8$	6 10	6^2 $10^4 30^2$	2 6 $10^8 30^{12}$	$10^{10} 30^{34}$	2 10^{11} 30^{56}	6^2 $10^{16} 30^{54}$	$6^4 10^{16}$ 30^{28}	$2^2 6^2$ $10^6 30^6$

Table 1 shows the results of all the matrix cycles thus obtained for all 185 conjugacy classes of $S_8[S_2]$. Moreover, Table 1 displays the order of each conjugacy class of the $S_8[S_2]$ group and the cycle types of the eight hyperplanes ($1 \leq q \leq 8$) obtained using the Möbius inversion method [30,55]. The label R is assigned to each conjugacy class in Table 1 if the symmetry operation is a proper rotation of the 8-cube. This facilitates the further discrimination of colorings into chiral or achiral colorings. The order of each conjugacy class is shown in the third column of Table 1, which is readily obtained from the corresponding 2×8 matrix cycle type shown in Table 1. Let P(n) represent the number of partitions of an integer n, given that P(0) = 1. The order any conjugacy class of $S_8[S_2]$ with the matrix type $T(g;\pi) = a_{ik}$ is given by (4):

$$|T(g;\pi)| = \frac{8!2^8}{\prod_{i,k} a_{ik}!(2k)^{a_{ik}}} \tag{4}$$

For example, for the conjugacy class in Equation (3) (conjugacy class 162 in Table 1), the order is given by (5):

$$\left|\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\right| = \frac{8!2^8}{3!(2.1)^3 1!(2.5)^1} = 21,504 \tag{5}$$

The above algorithm is iterated for all 185 conjugacy classes of the 8-cube to generate the orders of the conjugacy classes shown in the third column of Table 1. Each matrix type shown in Table 1 for the conjugacy class of the $S_8[S_2]$ group generates a permutation upon its action on the set of hyperplanes for each q of the 8-cube. These actions are computed in a single generating function through the Möbius inversion technique, as illustrated by Lemmis [30] for a 4D hypercube. Thus, the generating functions for the cycle types of the (8-q) hyperplanes are computed as coefficients of x^q in the $Q_p(x)$ polynomial obtained using the Möbius inversion method shown in (6):

$$Q_p(x) = \frac{1}{p}\sum_{d/p} \mu(p/d)F_d(x) \tag{6}$$

The summation is over all divisors d of p, and $\mu(p/d)$ is the Möbius function defined for any integer m as:

$\mu(m)$ = 1 if *one of m's prime factors is not a perfect square and m contains an even number of prime factors;*

$\mu(m) = -1$ if *m satisfies the same perfect-square condition as before but m contains odd number of prime factors;*

$\mu(m)$ = 0 if m has a perfect square as one of its factors. Consequently,

$$\mu(m) = 1, -1, -1, 0, -1, 1, -1, 0, 0, 1 \ldots \text{ for m = 1 to 10.} \tag{7}$$

In Equation (7), $F_d(x)$ is a polynomial in x obtained from the 2×8 matrix cycle types shown in the second column of Table 1. We consider only the nonzero columns of the matrix cycle types of the $S_8[S_2]$ group. Suppose p is the period of the matrix type shown in Table 1, and let g = gcd(k; p), p′ = k/g, h = gcd(2k; p). and p″ = 2k/h; then, the function $F_p(x)$ is given by (8):

$$\begin{aligned} F_p(x) &= \prod_k^{nc}\left(1+2x^{p'}\right)^{g a_{1k}}\left(1+2x^{p''}\right)^{\frac{h a_{2k}}{2}}, \ if\ h\ does\ not\ divide\ k; \\ F_p(x) &= \prod_k^{nc}\left(1+2x^{p'}\right)^{g a_{1k}}, \ if\ h\ divide\ k, \end{aligned} \tag{8}$$

where we take the product *only* over *nc* non-zero columns of the 2×8 matrix cycle type for each of the 185 classes displayed in Table 1. The coefficient of x^q in $Q_p(x)$ can be obtained by substituting the various F_d polynomials in the Möbius sum (7) in which ds are divisors of p. Thus, the cycle types of all eight hyperplanes of the 8-cube are obtained in a single generating function for each conjugacy class by collecting the coefficients of x^q to obtain the cycle types for (8-q) hyperplanes of the 8-cube. We now illustrate this with an example, using the matrix type in Equation (2) for conjugacy class 38 in Table 1 for the $S_8[S_2]$ group.

As can be seen from the matrix shown in Equation (2), the first and fourth columns of the matrix contain non-zero values; consequently, we consider only these two columns for computing the cycle types of the hyperplanes of the 8-cube. Thus, the maximum period to consider is 8, and the possible F polynomials are therefore F_8, F_4, F_2, and F_1, as divisors of 8 are 1, 2, 4, and 8. Applying the GCD, followed by the use of Equation (10), we obtain each of these polynomials as

$$F_1 = (1 + 2x)^4 \tag{9}$$

$$F_2 = (1 + 2x)^4 \tag{10}$$

$$F_4 = (1 + 2x)^4 \tag{11}$$

$$F_8 = (1 + 2x)^8 \tag{12}$$

The Q_p polynomials are obtained using the Möbius sum, (Equation (6)), as follows:

$$Q_1 = F_1 = 1 + 8x + 24x^2 + 32x^3 + 16x^4 \tag{13}$$

$$Q_2 = 1/2\,\{\mu(2)F_1 + \mu(1)F_2\} = 1/2\,\{F_2 - F_1\} = 1/2\,\{(1 + 2x)^4 - (1 + 2x)^4\} = 0 \tag{14}$$

$$Q_4 = 1/4\,\{\mu(1)F_4 + \mu(2)F_2 + \mu(4)F_1\} = 1/4\,\{F_4 - F_2\} = 0 \tag{15}$$

$$\begin{aligned} Q_8 &= 1/8\,\{\mu(1)F_8 + \mu(2)F_4 + \mu(4)F_2 + \mu(8)F_1\} = 1/8\,\{F_8 - F_4\} \\ &= x + 11x^2 + 52x^3 + 138x^4 + 224x^5 + 224x^6 + 128x^7 + 32x^8 \end{aligned} \tag{16}$$

The coefficients of the x^q terms thus obtained are sorted in a tabular form shown below for all possible Q_p polynomials. Once the coefficient of x^q is collected for each column, we obtain the cycle type of the (8-q) hyperplanes:

$\mathbf{Q_p}$	**x**	$\mathbf{x^2}$	$\mathbf{x^3}$	$\mathbf{x^4}$	$\mathbf{x^5}$	$\mathbf{x^6}$	$\mathbf{x^7}$	$\mathbf{x^8}$
Q_1	8	24	32	16				
Q_2								
Q_4								
Q_8	1	11	52	138	224	224	128	32
Cycle type	$1^8 8$	$1^{24}8^{11}$	$1^{32}8^{52}$	$1^{16}8^{138}$	8^{224}	8^{224}	8^{128}	8^{32}
Hyperplane	q = 1 (hepteracts)	q = 2 hexeracts	q = 3 penteracts	q = 4 tesseracts	q = 5 cubic cells	q = 6 faces	q = 7 edges	q = 8 vertices

The above technique was repeated for all 185 conjugacy classes of the 8-cube, and the results are shown in columns 5–12 (Table 1) for the (8-q) hyperplanes for q = 1 through 8, respectively. The generating functions for the colorings of the the hyperplanes for each irreducible representation are obtained using the generalized character cycle index (GCCI) for the character χ of the S_8 [S_2] group defined by

$$P_G^{\chi} = \frac{1}{|G|}\sum_{g\varepsilon G}\chi(g)s_1^{b_1}s_2^{b_2}\ldots\ldots s_n^{bn} \tag{17}$$

where the sum is taken over all permutations $g \in G = S_8$ [S_2] with cycle type $1^{b1}2^{b2}3^{b3}$ . . . $.8^{b8}$ upon its action on the (8-q) hyperplanes of the 8-cube. The GCCIs for all 185 irreducible representations and for each of the cycle types of the (8-q) hyperplanes (Table 1) are constructed for the S_8 [S_2] group. Multinomial generating functions are then computed for the colorings of the (8-q) hyperplanes of the 8-cube for all irreducible representations.

The generating function is defined as follows. Let $[n]$ be an ordered partition of eight (compositions of n = 8) into p parts such that $n_1 \geq 0$, $n_2 \geq 0$, ... , $n_p \geq 0$, $\sum_{i=1}^{p} n_i = n$. Then, a multinomial generating function in λs, in which λs represent arbitrary weights, is computed as

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_p)^n = \sum_{[n]}^{p} \binom{\mathrm{n}}{\mathrm{n_1 n_2 \ldots n_p}} \lambda_1^{\mathrm{n_1}} \lambda_2^{\mathrm{n_2}} \ldots\ldots.. \lambda_{\mathrm{p-1}}^{\mathrm{n_{p-1}}} \lambda_{\mathrm{p}}^{\mathrm{n_p}} \tag{18}$$

where $\binom{\mathrm{n}}{\mathrm{n_1 n_2 \ldots n_p}}$ are multinomials given by

$$\binom{\mathrm{n}}{\mathrm{n_1 n_2 \ldots n_p}} = \frac{\mathrm{n!}}{\mathrm{n_1! n_2! \ldots\ldots n_{p-1}! n_p!}} \tag{19}$$

Define D as the set of (8-q) hyperplanes which are to be colored, and let R be the set of different colors. Furthermore, let w_i be a weight assigned to each color r in R. Following Pólya, the weight of a function f from D to R can then be defined as

$$W(f) = \prod_{i=1}^{|R|} w(f(d_i)) \tag{20}$$

Hence, the generating functions for each of the 185 irreducible representations of the 8-cube with character χ are provided as follows:

$$GF^{\chi}(w_1, w_2 \ldots..w_p) = P_G^{\chi}\left\{s_k \to \left(w_1^k + w_2^k + \cdots . + w_{p-1}^k + w_p^k\right)\right\} \tag{21}$$

We compute the GFs for all irreducible representations of the 8-cube's group for all 8 hyperplanes, resulting in 1480 such combinatorial GFs for the 8-cube. The coefficient of a typical term, $w_1^{n1} w_2^{n2}, \ldots, w_p^{np}$, computes the number of colorings in the set R^D that transform according to the irreducible representation with character χ. Pólya's theorem becomes a special case for one of these 185 IRs; that is, for the totally symmetric irreducible representation A_1 which has the unit character values of all 185 conjugacy classes of the 8-cube. The sizes of the multinomial generators rapidly increase for such a large number of 8-cube hyperplanes, and we therefore consider only two colors in the set R for all cases except q = 1, for which 4-colorings are considered. All the computations were carried out in FORTRAN '95, invoking the quadruple precision arithmetic, which provides over 32 digits of accuracy for the computed results.

3. Results and Discussions

Tables 2–9 display our computed results for the colorings of eight different hyperplanes for q = 1–8, respectively. As there are 185 IRs, we do not show all the results for all IRs to prevent the Tables from becoming too large. Thus, we show the restricted results for the various hyperplanes. Table 2 shows the 4-colorings for the hepteracts (q = 1) for all one- and seven-dimensional irreducible representations. Tables 3–9 show the binomial colorings for the hyperplanes q = 8 through q = 2 in the order of vertices (Table 3) to hexeracts (Table 9). The reason for the restriction of the binomial colorings is that as q moves away from 1, too many partitions exist as the number of hyperplanes increases both binomially and exponentially.

Table 2. Generating functions for the one-dimensional and 7-dimensional IRs for coloring hepteracts (q = 1) for 4 colors.

[λ]	N(A_1)	N(A_3)	N(A_4)	N(A_5)	N(A_7)	N(A_8)	N(A_9)	N(A_{10})
16 0 0 0	1				0			0
15 1 0 0	1				1			1
14 2 0 0	2				2			1
13 3 0 0	2				3			2
12 4 0 0	3				4			2
11 5 0 0	3				5			3
10 6 0 0	4				6			3
9 7 0 0	4				7			4
8 8 0 0	5	1			7			4
14 1 1 0	2				3			3
13 2 1 0	3				6			5
12 3 1 0	4				9			7
11 4 1 0	5				12			9
10 5 1 0	6				15			11
9 6 1 0	7				18			13
8 7 1 0	8	1		1	20			15
12 2 2 0	6				12			8
11 3 2 0	7				18			13
10 4 2 0	10				25			16
9 5 2 0	11				31			21
8 6 2 0	14	1		1	37			24
7 7 2 0	14	1		2	39			27
10 3 3 0	10				28			20
9 4 3 0	13				39			27
8 5 3 0	16	1		1	49			34
7 6 3 0	18	1		1	56			39
8 4 4 0	19	1		1	55			36
7 5 4 0	21	1		2	67			46
6 6 4 0	24	1		2	73			48
6 5 5 0	24	1		2	78			54
13 1 1 1	4				9			9
12 2 1 1	7				18			16
11 3 1 1	9				27			23
10 4 1 1	12				37			30
9 5 1 1	14				46			37
8 6 1 1	17	1		2	55			44
7 7 1 1	18	3		5	58			48
11 2 2 1	12				36			30
10 3 2 1	17				56			45

Table 2. *Cont.*

[λ]	N(A_1)	N(A_3)	N(A_4)	N(A_5)	N(A_7)	N(A_8)	N(A_9)	N(A_{10})
9 4 2 1	22				77			60
8 5 2 1	27	1		2	97			75
7 6 2 1	31	3		7	111			87
9 3 3 1	24				88			70
8 4 3 1	33	1		2	124			96
7 5 3 1	39	3		7	152			119
6 6 3 1	43	3		9	166			129
7 4 4 1	44	3		7	171	1		132
6 5 4 1	51	3		9	202	2	1	156
5 5 5 1	54	3		9	219	3	3	171
10 2 2 2	24				75			57
9 3 2 2	32				117			91
8 4 2 2	45	1		2	165			122
7 5 2 2	52	3		9	202	1		154
6 6 2 2	59	6		15	221	1	1	166
8 3 3 2	48	1		2	189			146
7 4 3 2	64	3		9	262	2		201
6 5 3 2	75	6		18	311	5	3	240
6 4 4 2	87	6		18	354	9	5	266
5 5 4 2	91	6	1	21	386	14	10	297
7 3 3 3	71	3		9	303	2		237
6 4 3 3	95	6		21	411	11	6	319
5 5 3 3	102	10	1	30	450	17	14	354
5 4 4 3	117	10	2	34	516	28	20	402
4 4 4 4	138	15	3	45	594	43	30	456

Table 3. Generating functions for the binomial colorings of the vertices (q = 8) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	1	0
2	8	0	4	0
3	32	0	32	0
4	373	1	313	1
5	4647	119	4647	119
6	91028	13908	89722	13852
7	2074059	794855	2074059	794855
8	51107344	31177061	51073158	31174193
9	1245930065	965191516	1245930065	965191516
10	28900653074	25308942504	28899849712	25308861906
11	625715497344	583809974962	625715497344	583809974962
12	12562875567065	12113697810612	12562859327210	12113696094241
13	233750783834504	229300148849871	233750783834504	229300148849871
14	4038807303045625	3997804386459912	4038807021195271	3997804356178806
15	65003434860142353	64650435886116058	65003434860142353	64650435886116058
16	977872935273906860	975020351847385105	977872931016186973	975020351387480625
17	13795944871933252078	13774222202187964657	13795944871933252078	13774222202187964657
18	183113271146620771933	182956842576531615461	183113271089874321619	182956842570390661781
19	2293288191579045041618	2292219628052032359534	2293288191579045041618	2292219628052032359534
20	27172601104679810308230	27165657543252057055741	27172601104004619255357	27165657543178941093106
21	305350757901312578538505	305307729144669187569341	305350757901312578538505	305307729144669187569341
22	3261598821371763396803511	3261343947923445791656883	3261598821364520795113233	3261343947922661301616505
23	33182648967085223933532504	33181202911152729471133510	33182648967085223933532504	33181202911152729471133510

Table 3. *Cont.*

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
24	32214513315747442033301 6643	32213725965665045610518 4415	32214513315740380624886 5438	32213725965664280674049 4424
25	2989490904329310130221003214	2989449691542891602751761316	2989490904329310130221003214	2989449691542891602751761316
26	2656039764302244415702192 5451	2656018992418186924443493 3109	2656039764302181406660805 5393	2656018992418180098648048 0083
27	22625485918546088538465677 7972	22625384959718497543417195 2862	22625485918546088538465677 7972	22625384959718497543417195 2862
28	185043976741321338167630041 0605	185043502898936770110966723 9854	185043976741320820594833713 8938	185043502898936714041229443 0685

Table 4. Generating functions for the binomial colorings of edges (q = 7) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	0	0
2	14	0	4	0
3	216	0	153	0
4	13143	1285	12071	1274
5	1368944	579841	1357322	579593
6	179718686	129821262	179515891	129822963
7	23649956067	20897467555	23647435977	20897539088
8	2893910524721	2760276878890	2893871387345	2760280157691
9	321959734753775	316179169443213	321959246156199	316179239889735
10	32497054656052201	32271572199062564	32497047828556262	32271573699877343
11	2989231824380170346	2981221656585795852	2989231740547005833	2981221682328226491
12	252133297490831881715	251871989699732625444	252133296410532691241	251871990120247195309
13	19621374682741244060254	19613492123348498937956	19621374669883641851551	19613492129491101473396
14	1416769496997564229002044	1416548308266662098844372	1416769496843104025248968	1416548308351788104311581

Table 4. *Cont.*

$\mathbf{n_1}$	$\mathbf{N(A_1)}$	$\mathbf{N(A_2)}$	$\mathbf{N(A_3)}$	$\mathbf{N(A_4)}$
15	95391267226534335414344124	95385464119630857035122481	95391267224778314580202835	95385464120726678222923561
16	6015500298569544153993662343	6015357312352442891027589816	6015500298549822020138555174	6015357312365881101185952048
17	356681204098702555735994657211	356677882377176721030402271213	356681204098490475804398300875	356677882377332607124361022833
18	19954275394810936617446090999298	19954202386450506779938248001657	19954275394808706716919606533385	19954202386452238537914701133372

Table 5. Generating functions for the binomial colorings of faces (q = 6) for one-dimensional IRs of the 8-cube.

$\mathbf{n_1}$	$\mathbf{N(A_1)}$	$\mathbf{N(A_2)}$	$\mathbf{N(A_3)}$	$\mathbf{N(A_4)}$
0	1	0	0	0
1	1	0	0	0
2	17	0	3	0
3	502	8	325	16
4	71333	24354	63612	25469
5	17405405	12681606	17036302	12854742
6	4637637972	4212888299	4615175164	4228812207
7	1144154282643	1111013093508	1142833213165	1112152780844
8	252694411554530	250418661767733	252621360494651	250486917059594
9	49933006129016491	49793580872480751	49929316161439255	49797157352752701
10	8894464990395578492	8886752121307484548	8894294667459034186	8886919926687320706
11	1440472783586317238393	1440083733481783945901	1440465574963715354994	1440090890477948652349
12	213770312441853780512224	213752271869558123835303	213770031051943657246139	213752552262354430991274
13	29269240908200356802347666	29268466577556962305262111	29269230719024210548536477	29268476748627548133819831
14	3719250527778112615739134335	3719219584194378214567008273	3719250183675186524528404499	3719219927985383328745758341
15	^{440}853907156944454398222081566	440852750059160355141914968142	^{440}853896266243008648327879143	^{440}852760944756455945032626186
16	48962293141051882905412809207021	48962252475152977331838347830739	48962292816629300670347138945445	48962252799495751305458642354391

Table 6. Generating functions for the binomial colorings of cubic cells (q = 5) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	0	0
2	17	0	2	0
3	520	12	197	31
4	73190	23955	51521	33050
5	17658730	12519173	15593690	14108134
6	4670541393	4184082830	4460002235	4377662001
7	1147944547821	1107379772627	1128227990017	1126561130037
8	253070289097552	250047877926328	251418412879773	251684384321815
9	49965338520426373	49761394798932364	49841404217974456	49884926827502047
10	8896911981316427886	8884308904139556609	8888517541896747821	8892693620617688255
11	1^{440}638022063010631910	1439918584060756071731	1^{440}120252561468192126	1^{440}436135220492411513
12	213780388559484507191771	21374219769488529^{880}7844	213751077431815470552985	213771504232215034965055
13	29269801356205901555855379	29267906169133085641583032	29268268014083092931288337	29269439420444435412633656
14	3719279200778578000458180353	3719190911951887456836672038	3719204640030812101745972736	3719265471000672398359114098
15	440855265990629080980664890644	440851391239200521016160622915	^{440}851878646143680714866519271	^{440}854778553508389820473453710
16	48962353150087477418358489750594	48962192466353422956421447365043	48962208733183359241696473980451	48962336882746820615764869368554

Table 7. Generating functions for the binomial colorings of tesseracts (q = 4) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	0	0
2	14	0	1	0
3	271	0	44	10

Table 7. *Cont.*

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
4	18436	1755	6563	5422
5	2193367	878552	1310579	1462703
6	316843509	215441296	244997541	277130468
7	45385903112	38264656379	39915467721	43430408845
8	6032761198299	5577971407013	5656630241981	5945904368147
9	730228248478031	703689117001293	706979787447992	726738931053597
10	80357795001409842	78935801832625872	79059004752041424	80230215709062022
11	8072061169923281815	8001805117194789623	8005990292904955004	8067787788823615438
12	744350299080496007833	741136858948772220225	741267190289028028069	744218331077926971583
13	63372848789530529909508	63236254250272123668115	63240005325330608561992	63369069612560717297062
14	5008288556152866356535055	5002872178636514348211030	5002972617541373275912117	5008187668616345917852635
15	369178651343422851704804726	368977568670922245142254944	368980084480516795405801544	369176128846755941869855695
16	25492963441401967651387305634	25485950009552151553394643036	25486009235978749090972053270	25492904121555310159685722022

Table 8. Generating functions for the binomial colorings of penteracts (q = 3) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	0	0
2	9	0	0	0
3	83	0	1	0
4	1752	3	117	63
5	53631	2959	9571	13129
6	2206678	478390	765645	1186490
7	100831126	43280345	52757897	76988844

Table 8. *Cont.*

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
8	4655630517	2869803113	3126093034	4145344517
9	206159944002	155185722872	161095026870	196198260213
10	8552744534853	7214285370452	7333825654985	8374506027325
11	329462356516427	297042684545365	299205065844492	326523850764241
12	11762768558587306	11035832262497428	11071306010739393	11717875989405083
13	389788558944470907	374644306545546766	375177718301384463	389149723370338717
14	12022008920943629364	11727821396700243253	11735235648893519051	12013501431327418719
15	346243083080541945354	340895758256810378725	340991677499412431387	346136617589280701403
16	9343358594200843070436	9252109766228033928271	9253271398670725746098	9342101964148242810759
17	237003980460152929758378	235537560933998269545497	235550794325997486640218	236989945316393042010695
18	5668492308224642598946637	5646233383620513247159980	5646375791809126585493380	5668343540316085225671505

Table 9. Generating functions for the binomial colorings of hexeracts (q = 2) for one-dimensional IRs of the 8-cube.

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
0	1	0	0	0
1	1	0	0	0
2	5	0	0	0
3	19	0	0	0
4	97	0	1	0
5	523	0	7	0
6	3364	12	113	100
7	23495	468	1841	2900
8	177163	11566	26634	49872
9	1381168	202782	340001	641523

Table 9. *Cont.*

n_1	$N(A_1)$	$N(A_2)$	$N(A_3)$	$N(A_4)$
10	10815219	2764805	3817127	6806701
11	82876768	31064151	38036025	62678809
12	610666743	298841335	339544103	515757045
13	4277230169	2526305247	2739178116	3859880393
14	28291585986	19112520435	20122716541	26567032313
15	176133023346	131124054411	135519699197	169412236898
16	1030800460602	823890406748	841577934079	1006017212919
17	5671131339771	4777193964007	4843480433125	5584418861048
18	29353498288379	25714973280273	25947719387923	29064956439595
19	143105988701069	129121616153798	129891161595220	142191025038525
20	658063307862705	607198937700202	609605218691334	655293632298405
21	2858457998939006	2683015733790681	2690157639471948	2850441762739161
22	11746243530321863	11171273635072019	11191457507488421	11724027454457595
23	45730471397391867	43936844692101295	43991306801661762	45671434719950791
24	168913836578625999	163578906149612869	163719543899932333	168763207445340396
25	592737125111757011	577583393796842496	577931661515814991	592367656948196051
26	1978559832758151399	1937393462655462502	1938221987321702846	1977687567195144310
27	6289941346098637974	6182840640010272518	6184737193473689864	6287956989620704957
28	19065234710132878894	18798043092395561933	18802226237533170532	19060880013556511457
29	55155926406439783261	54515970479244152862	54524872037731254044	55146698500000142941
30	152448370819991091907	150975194909849530476	150993490533551033564	152429470569138848128
31	402931631210964655490	399668880173423084086	399705237549199768826	402894182587955370920
32	1019267441120735541170	1012308316233298718335	1012378237193154664252	1019195601065459214731

As can be seen from Table 2, among four one-dimensional IRs (A_1–A_4), only three appear in the table for the 4-colorings of the hepteracts of the 8-cube as $N(A_2)$ becomes zero for all 4-color partitions. Moreover, for the A_4 representation, only the last few rows of the color partitions give rise to nonzero values. The $N(A_1)$ numbers simply yield Pólya's equivalence classes for four colors. As can be seen from Table 2, there are two classes for the [2,14] and [3,13] color partitions. Among the 2-colorings, there are five equivalence classes for the [8,8] color partition, or there are five inequivalent ways to color the hepteracts of the 8–cube with eight black and eight white colors. Likewise, there are 138 inequivalent ways to color the hepteracts of the 8-cube with four blue colors, four red colors, four yellow colors, and four white colors (Table 2). Thus, the frequency for each color partition shown in Table 2 provides the numbers of IRs contained in the set of all the colorings of the hepteracts of the 8-cube that transform in accordance with that IR. The representations labeled A_5 to A_8 in Table 2 are seven-dimensional IRs of the wreath product group $S_8[S_2]$. The A_6 and A_7 seven-dimensional IRs appear less frequently in colorings compared to A_5 and A_8. This is analogous to the one-dimensional A_2 and A_4 IRs, which are less frequent, and A_4 appears only for the highly distributed color partitions, while A_2 does not appear at all for all four-colorings of the sixteen hepteracts.

Table 3 displays the 2-colorings of the vertices of the 8-cube (q = 8). As there are 256 vertices, the color partitions are of the form [n_1, 256-n_1]; thus, in Table 2, we have shown only n_1 or, for example, the number of black colors remaining as 256-n_1 whites. As all the vertices of the 8-cube are equivalent, $N(A_1)$ becomes 1 for n_1 = 1. As can be seen from Table 2, there are 8 ($N(A_1)$) inequivalent ways to color the vertices of the 8-cube with 2 black colors, 32 ways for 3 black colors, 373 ways for 4 black colors, and so forth. The maximum is reached at [128,128], which is too large to be shown accurately, even with quadruple precision. Among the results shown in Table 3, there are 1850439767413213381676300410605 inequivalent ways to color with 28 black colors. Although the A_2 IR does not present for the hepteracts, all one-dimensional IRs appear for most of the binomial colorings of the vertices of the 8-cube. The only exceptions are 1–3 black colors for the A_2 and A_4 IRs. As the number of black colors increases, almost every one-dimensional IR appears with nearly the same frequency. This also implies that almost every coloring becomes chiral as the number of black colors increases, approaching the peak at the [128,128] color partition. There is a similar pairing of the A_1 and A_3 IRs, while A_2 and A_4 exhibit similar pairs (Table 3).

Table 4 shows the edge colorings of the 8-cube for a set of two colors. As there are 1024 edges, the number of colorings for different IRs increases quite rapidly. As can be seen from Table 4, there are 14 inequivalent ways to color with 2 black colors, 216 ways for 3 black colors, 13,143 ways to color with 4 black colors, and so forth for the edge colorings of the 8-cube. The first nonzero frequency for the A_2 IR occurs for a minimum of four black colors for the A_2 IR, while for A_3 and A_4, one would need two and four black colors, respectively. A common feature of the binomial distribution is exhibited by all edge colorings for all IRs peaking at the [512,512] color partition.

Tables 5–9 show the colorings of the faces, cells, tesseracts, penteracts, and hexeracts of the 8-cube for two colors. For example, there are 17 inequivalent ways to color the faces of the 8-cube with 2 black colors, while the corresponding numbers are 17, 14, 9, and 5 for the cubic cells, tesseracts, penteracts, and hexeracts, respectively. A similar pattern is exhibited by these hyperplanes for other numbers of black colors. Likewise, the similarity of the (A_1, A_3) and (A_2, A_4) pairs is shared by all hyperplanes (Tables 5–9). When comparing the frequencies of the A_2 IR among hyperplanes, the hexeracts and hepteracts stand out, as A_2 does not appear in any 2-colorings of the hepteracts, while at least six black colors are needed to produce a coloring that contains the A_2 IR for the hexeracts. An important point for all hyperplanes is that there is only one equivalence class for the A_1 representation when only one black color is used. Consequently, all hyperplanes of the 8-cube or any n-cube are equivalent. This follows from the highly symmetric, vertex-, edge-, and arc-transitive natures of the n-cube. Thus, the combinatorial numbers are consistent with these general features of the n-cube.

4. Chemical, Biological and other Applications of the Colorings of 8-Cube

The colorings of the hyperplanes of n-cubes have several applications in a variety of fields, such as chemistry, biology, image processing, latent symmetries in computational psychiatry, phylogenetic networks, pandemic networks, Bethe lattices, Cayley trees, and so forth. In this section, we focus on chemical and biological applications, with slight coverage of other types of applications. An obvious connection of the 8-cube is to Boolean strings of a length of 8 for which each such string is represented by the vertex of the 8-cube, giving rise to 256 Boolean strings of a length of 8. An important chemical and spectroscopic application of the 8-cube is to the water octamer, $(H_2O)_8$. Figure 1 displays the graph of the octeract with 256 vertices and 1024 edges. If every hydrogen bond of the water octmer is allowed to be broken and remade, which happens at higher temperatures, then the cluster becomes fully nonrigid. At that limit, the graph in Figure 1 would represent the isomerization graph of the water octamer. The dynamic stereochemistry, isomerization pathways, and their intimate relations to graph theory through finite topologies and borel fields were explored earlier in the context of rapid internal rotations around the C-C single bonds [52]. The isomerization graphs provide insights into reaction pathways and phenomena such as the spontaneous generation of chirality and dynamic or transient chirality. This is attributed to the existence of *dl*-edges or edges between chiral pairs in the dynamic isomerization graphs.

The nuclear spin statistics and combinatorics of nuclear spin functions that have a wide-range of applications, from multiple quantum NMR spectroscopy to predicting the tunneling splittings of rovibronic levels, are critical to the interpretation of the observed spectra of nonrigid molecules including water clusters and the water octamer in particular. That is, as the molecule becomes fluxional as it surmounts several potential energy barriers, its overall symmetry is described in the nonrigid molecular group, which becomes the automorphism group of the hypercube for water clusters. For fluxional complexes such as water clusters, ammonia clusters, and methane clusters, the tunneling splittings of the rovibronic levels are obtained through the use of induced representations from the irreducible representations of the rigid molecular group to the ones in the symmetry group of the nonrigid molecule. While the magnitudes of the tunneling splittings would depend on the extent of the fluxionality, the tunneling levels are predicted using the induced representations from the smaller rigid group to the larger fluxional group. Moreover, nuclear spin functions and the nuclear spin populations of the rovibronic tunneling levels are predicted from the colorings of the hyperplanes of the n-cube.

The nuclear spin functions can be envisaged as colorings from the set of the nuclei in the molecule to a set of spin colors which correspond to the various nuclear spin orientations. The possible nuclear spin orientations depend on the isotopes of the nuclei present in the molecule and the overall nuclear spin quantum number of the isotope. For example, the naturally abundant isotope of bismuth, ^{209}Bi, exhibits a spin of 9/2 with 10 different spin orientations or 10 colors; therefore, the set of different colors would have a cardinality of 10. On the other hand, for the normal water clusters, the proton nuclei exhibit a spin of 1/2, and thus the two spin orientations of each proton of the octamer become two distinct colors; this corresponds to the 2-coloring of the hyperplanes of the 8-cube considered herein. The most stable naturally occurring isotope of oxygen, ^{16}O, carries no nuclear spin, although another naturally occurring ^{17}O isotope carries a nuclear spin of 5/2, giving rise to six nuclear spin orientations or colors. The ^{17}O NMR is the technique of choice for probing the dynamics of oxygen-containing molecules. Consequently, the combinatorial numbers enumerated in Table 2 for the 4-colorings of hepteracts contain the information needed to classify the protonic nuclear spin functions of the water octamer in the nonrigid limit. That is, there are 2^{16} proton nuclear spin functions for the water octamer, and together, they transform as reducible representations in the symmetry group of the 8-cube. The irreducible representations contained in the set of protonic nuclear spin functions are enumerated by the combinatorial numbers in Table 2 when they are extended for all 185 IRs of the 8-cube. Likewise, when the technique is extended for 6 colors and applied to all 185 IRs, we obtain the frequencies of the IRs of the nuclear spin functions for the ^{17}O isotopes of $(H_2{}^{17}O)_8$. The set of such ^{17}O nuclear functions has a cardinality of 6^8 for $(H_2{}^{17}O)_8$. Consequently, the combinatorial numbers enumerated in Tables contain rich information pertinent to a significant amount of spectroscopically important information, for example, nuclear spin species, the nuclear spin statistical weights of the rovibronic levels, and nuclear spin multiplets and thus the intensities and hyperfine structures of the rovibronic spectra of the water octamer.

In order to demonstrate the utility of the combinatorial colorings of the hyperplanes of the 8-cube, let us consider the deuterated water octamer, $(D_2O)_8$. As deuterium is a spin-1 nucleus, its bosonic spin functions can be denoted by λ, μ, and ν. Thus, there are 3^{16} nuclear spin functions corresponding to the 3-colorings of the hepteracts of the 8-cube, and the results are contained in Table 2 for eight of the IRs. That is, all the color partitions with three or fewer parts correspond to 3-colorings, and the subsets of numbers in Table 3 therefore provide the frequencies of the corresponding IRs in the set of the 3^{16} nuclear spin functions of $(D_2O)_8$. As yet another example, consider the 185th IR, which has a dimension of 672 in the hyperoctahedral group of the 8-cube; it shall be denoted as $\Gamma_{672\text{-}2}$ as it is the second IR with a dimension of 672 in the character table of $S_8[S_2]$. The generating function for $\Gamma_{672\text{-}2}$ for $(D_2O)_8$ is shown in Table 10. In Table 10, $N(\Gamma_{672\text{-}2})$ is the frequency of $\Gamma_{672\text{-}2}$ in the set of bosonic spin functions of the deuterium nuclei of $(D_2O)_8$ that encompass 3^{16} nuclear spin functions. An ordered partition of [16], for example, [7 6 3], provides the spin distribution $\lambda^7\mu^6\,\nu^3$. From Table 10, it can be readily seen that the frequency of $\Gamma_{672\text{-}2}$ for the color partition [7 6 3] spin distribution is 62. In this manner, the generating function obtained using the GCCI for $\Gamma_{672\text{-}2}$ then yields all frequencies of all spin distributions. One can sort these frequencies into spin multiplets; hence, we obtain the following nuclear spin multiplets with frequencies in parentheses for the $\Gamma_{672\text{-}2}$, as shown below:

$^{1}\Gamma_{672\text{-}2}(17)$, $^{3}\Gamma_{672\text{-}2}(46)$, $^{5}\Gamma_{672\text{-}2}(64)$, $^{7}\Gamma_{672\text{-}2}(67)$, $^{9}\Gamma_{672\text{-}2}(59)$, $^{11}\Gamma_{672\text{-}2}(44)$, $^{13}\Gamma_{672\text{-}2}(28)$, $^{15}\Gamma_{672\text{-}2}(15)$, $^{17}\Gamma_{672\text{-}2}(6)$, $^{19}\Gamma_{672\text{-}2}(2)$

We iterate the above process to generate the nuclear spin multiplets for each of the 185 IRs of the hyperoctahedral group of the 8-cube. Finally, we use either the Fermi–Dirac or Bose–Einstein stipulations for the overall wave function's symmetry. In this case, as all deuterium nuclei are bosons, we stipulate that the direct product of the rovibronic wave function and the nuclear spin function must be totally symmetric, which would then give the overall nuclear spin statistical weights for each tunneling level of the $(D_2O)_8$.

Table 10. Generating function for the second IR with dimension 672 for $(D_2O)_8$ where three bosnic colors are shown as ordered partitions of [16] into 3 parts for 3 colors.

$\mathbf{N(\Gamma_{672\text{-}2})}$	**[16]**	$\mathbf{N(\Gamma_{672\text{-}2})}$	**[16]**
1	10 5 1	19	8 2 6
2	9 6 1	62	7 3 6
3	8 7 1	116	6 4 6
3	7 8 1	142	5 5 6
2	6 9 1	116	4 6 6
1	5 10 1	62	3 7 6
1	11 3 2	19	2 8 6
5	10 4 2	2	1 9 6
12	9 5 2	3	8 1 7

Table 10. *Cont.*

$N(\Gamma_{672\text{-}2})$	[16]	$N(\Gamma_{672\text{-}2})$	[16]
19	8 6 2	22	7 2 7
22	7 7 2	62	6 3 7
19	6 8 2	99	5 4 7
12	5 9 2	99	4 5 7
5	4 10 2	62	3 6 7
1	3 11 2	22	2 7 7
1	11 2 3	3	1 8 7
8	10 3 3	3	7 1 8
24	9 4 3	19	6 2 8
46	8 5 3	46	5 3 8
62	7 6 3	60	4 4 8
62	6 7 3	46	3 5 8
46	5 8 3	19	2 6 8
24	4 9 3	3	1 7 8
8	3 10 3	2	6 1 9
1	2 11 3	12	5 2 9
5	10 2 4	24	4 3 9
24	9 3 4	24	3 4 9
60	8 4 4	12	2 5 9
99	7 5 4	2	1 6 9
116	6 6 4	1	5 1 10
99	5 7 4	5	4 2 10
60	4 8 4	8	3 3 10
24	3 9 4	5	2 4 10
5	2 10 4	1	1 5 10
1	10 1 5	1	3 2 11
12	9 2 5	1	2 3 11
46	8 3 5		
99	7 4 5		
142	6 5 5		
142	5 6 5		
99	4 7 5		
46	3 8 5		
12	2 9 5		
1	1 10 5		
2	9 1 6		

We expect the 8-cube to be applicable to relativistic measures of time [10] as hypercubes serve as time representation holders. Relativistic effects are known to make significant contributions to the electronic states and spectroscopic properties of molecules that possess very heavy atoms [44–46]. Consequently, the incorporation of spin–orbit coupling changes the nonrigid molecular symmetry into double groups of hypercube groups. The double group symmetry corresponds to the coupling of the spin angular and orbital angular momenta; hence, two states with the same symmetry in the double group mix. Such topics can be the subject matter of future studies.

Another important application of the colorings of n-cube hyperplanes is in chirality and transient chirality. One can define an object to be chiral if it does not possess an improper axis of rotation. For this reason, we assign a symbol R to those conjugacy classes in Table 1 to designate proper rotations. The absence of R would then imply that the operation is an improper rotation. This is important in determining the chirality of the enumerated coloring. The symbol R (Table 1) is assigned for each conjugacy class by stipulating that a conjugacy class with the matrix type $[a_{ik}]$ is *a proper rotation* if and only if the sum shown below is even:

$$\sum_{k}^{even} a_{1k} + \sum\nolimits_{k}^{odd} a_{2k}$$

where the first sum is restricted to even columns, while the second sum is restricted to odd columns. Consequently, the developed computer code carries out the above sum for each of the 185 conjugacy classes of the 8-cube to determine if the sum is even or odd, and then the code assigns the symbol R if the sum is even to designate the operation as a proper rotation. Consequently, a coloring of the (8-q) hyperplane of the 8-cube is chiral if and only if the coloring function transforms as the chiral *irreducible representation* for a given ordered color partition, $[n_1\ n_2]$. The chiral irreducible representation of the 8-cube is as a one-dimensional IR with +1 character value for *all proper rotations* or the ones that carry the label R in Table 1 and -1 for *all improper rotations*. The A_2 irreducible representation of the $S_8[S_2]$ group is chiral; hence, it can be seen from Tables 2–9 that the number of chiral colorings for the (8-q) hyperplanes corresponds to the frequencies of the A_2 irreducible representation in Tables 2–9. As can be inferred from Table 2, the A_2 representation does not occur at all for the 2-colorings or for up to four colors, suggesting that there are no chiral colorings for the hepteracts of the 8-cube if one uses up to four kinds of colors (blue, yellow, red, and white). However, there are chiral colorings for some of the other hyperplane colorings, including vertex colorings and edge colorings, as can be seen from Tables 3–9.

There are several biological and biochemical applications of the colorings of n-cube hyperplanes. In particular, the 2-colorings of the vertices of the 8-cube are critical to the combinatorics of genetic networks [41], as well as the combinatorics of the colorings of phylogenic trees [59] and pandemic trees [60]. The symmetries involved in both these applications are recursive in nature, and they are expressible as wreath product groups. Each level of the phylogenetic tree involves a recursive construction relative to the previous level. On the other hand, genetic regulatory networks are important combinatorial characterizations of the evolutionary processes which are also recursive; therefore, they are represented by hypercubes [59]. Consequently, the colorings of the vertices (q = 8) of the hypercube provide information about the equivalence classes that result in considerable simplification in computing the properties that are pertinent to the genetic regulatory networks. The equivalence classes are enumerated by the totally symmetric A_1 colorings of the vertices among the 185 IRs that were considered here for the 8-cube. Wallace [53,54] has illustrated several applications of hypercube symmetries for understanding the spontaneous symmetry-breaking in intrinsically disordered proteins and the dynamics of proteins in general. Furthermore, the moonlighting functions of such proteins can be better understood through the use of the recursive symmetries displayed by hypercubes. It would be interesting to explore the applications of the chiral colorings of the hyperplanes of the hypercubes, which are enumerated by the generating functions obtained herein for the chiral IR. This aspect could become the subject of future studies.

Moreover, the colorings of the 8-cube hyperplanes have applications in completely different areas, such as X-ray diffraction patterns, neutron scattering studies, quarks, magnetic symmetry, and other physics applications [72–75]. As shown by several investigators [72–75], the analysis of complex X-ray patterns, magnetic structures, their symmetries, and the neutron diffractions of materials exhibiting distortions and understanding the dynamics of different phases could be benefited by the insights derived from wreath product and other types of approaches. There are several other chemical and material applications of dynamic symmetries, for example, to O_8 clusters [76,77] and polytwistane type nanomaterials [78]. Finally such combinatorial enumerations pertinent to the nuclear spin statistics of donut type polyaromatic structures [79] and holey nanographenes should be of future interest.

5. Conclusions

In the present study on recursive symmetries arising from the wreath products of the 8-cube, we have employed combinatorial and computational techniques to seek generating functions for the colorings of 8 hyperplanes of the 8-cube for all 185 IRs. These techniques combined Möbius inversion with the generalized character cycle indices and computer codes in order to enumerate the colorings of (8-q) hyperplanes (for q = 1–8). Several applications were outlined for the prediction of the tunneling splittings of rovibronic levels and their nuclear spin species and nuclear spin statistics. A few biological and material science applications were also pointed out. It is hoped that the present study will generate further interest and applications, especially in dynamic chirality, transient chirality, and the observed spontaneous generation of optical activity and other dynamic-symmetry-induced phenomena. Graph theoretical and combinatorial techniques can be especially useful in enumerating the number of phases and the number of chiral phases that are generated during phase transitions. The dynamic chirality is such an interesting phenomenon because an achiral system could separate into distinct enantiomorphic phases during phase transitions. Such applications would require the juxtaposition of the colorings of the hyperplanes of hypercubes enumerated in higher symmetries to various subgroups that would correspond to the symmetries of different phases. Such exciting topics that involve the combinatorics of hypercubes to lower symmetries could be the subject matter of future studies.

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