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Some Refinements of the Tensorial Inequalities in Hilbert Spaces

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Abstract: Hermite–Hadamard inequalities and their refinements have been investigated for a long period of time. In this paper, we obtained refinements of the Hermite–Hadamard inequality of tensorial type for the convex functions of self-adjoint operators in Hilbert spaces. The obtained inequalities generalize the previously obtained inequalities by Dragomir. We also provide useful Lemmas which enabled us to obtain the results. The examples of the obtained inequalities for specific convex functions have been given in the example and consequences section. Symmetry in the upper and lower bounds can be seen in the last Theorem of the paper given, as the upper and lower bounds differ by a constant.

Keywords: tensorial product; self-adjoint operators; convex functions

MSC: 26D05; 26D07; 26D20

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1. Introduction

The notion of a tensor has its origin in the 19th century, where it was formulated by Gibbs, although he did not formally use the word 'tensor' but 'dyadic'. In modern language, it can be seen as the origin of the tensor definition and its introduction to mathematics. The interplay of inequalities in mathematics is vast, and as such, it has applications in tensors as well. Mathematics and other fields of science are strongly affected by inequalities. There are many types of inequality, but those that involve Jensen, Ostrowski, Hermite–Hadamard and Minkowski inequalities are of particular importance. More about inequalities and their history can be found in the books [1,2]. One of the most celebrated inequalities is the Hermite–Hadamard (HH) inequality, which states the following: If $f^\star : \mathbb{I} \to \mathbb{R}$ is a convex function on an interval $\mathbb{I} \subset \mathbb{R}$ and $r_1, r_2 \in \mathbb{I}$ are such that $r_1 < r_2$, then

$$f^\star\left(\frac{r_1 + r_2}{2}\right) \leqslant \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} f^\star(\lambda) d\lambda \leqslant \frac{f^\star(r_1) + f^\star(r_2)}{2}.$$

Concerning the generalization of this inequality, see the following and references therein for more information [3–14]. Inequalities in Hilbert space were intensively worked on in the 20th century, but a special expansion occurred with the proof of Jensen-type inequality in Hilbert space, the so-called Mond–Pecaric inequality [15]. Let $f : [m, M] \to \mathbb{R}$ be a continuous convex function. If $x \in H, (x, x) = 1$, then for every operator C such that $mI \leqslant C \leqslant MI$, where I is identity operator and C is self-adjoint, holds,

$$f((Cx, x)) \leqslant (f(C)x, x).$$

After its introduction, there followed an expansion of a special type of inequalities of the MD (Mond–Pecaric) type in the Hilbert space [16,17]. The recent further development of inequalities in Hilbert space was followed by the definition of an operator convex function by Dragomir in 2011 [18], which is given by the following.

A real valued continuous function f on an interval $\mathbb{I}$ is said to be operator convex (*resp.* operator concave) if

$$f((1-\lambda)A+\lambda B) \leqslant (resp. \geq)(1-\lambda)f(A)+\lambda f(B)$$

in the operator order, for all $\lambda \in [0,1]$ and for every self-adjoint operator A and B on a Hilbert space H whose spectra are contained in I. By introducing the given definition of operator convexity the expansion of another branch of Hilbert inequalities began. You can find more about the class of operator convex functions in the following references [19,20]. For more information on inequalities in Hilbert spaces, see the references [21,22]. Recent advances concerning the theory of inequalities in Hilbert spaces will be shown to supplement the presentation of this work. Alomari [23] gave the following numerical radius inequality.

Let $A,B,C,D \in B(H)$. Let f be a positive, increasing and convex function on $\mathbb{R}$. If f is twice differentiable such that $f'' \geqslant \lambda > 0$, then

$$f(w(DCBA)) \leqslant \frac{1}{2}||f(A^*|B|^2A)+f(D|C^*|^2D^*)|| - \inf_{||x||=1}\eta(x),$$

where $\eta(x) := \frac{1}{8}\lambda\langle[A^*|B|^2A - D|C^*|^2D^*]x,x\rangle^2$.

The recent generalization of the Cauchy–Schwartz inequality and related inequalities has been given by Altwaijry et al. [24], one of the results given is the following.

If $T,S \in B(H), r \geqslant 1$ and $\lambda \in [0,1]$, then the inequality

$$w^{2r}(S^*T) \leqslant \frac{1}{2}|||T|^{4r}+|S|^{4r}|| - \frac{\lambda(1-\lambda)}{1+\lambda-\lambda^2}\Big(|||T|^{4r}+|S|^{4r}|| - w(S^*T)|||T|^{2r}+|S|^{2r}||\Big)$$

holds.

Recent research on the tensorial inequalities was carried out by Wang et al. [25], who gave the following inequality.
Let $A = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{C}^{[m,n]}$. If

$$(i)\ |a_{i\ldots i}| > r_i^{[i]}(A),\ for\ all\ i \in \langle n\rangle,$$

$$(ii)\ \Big(|a_{i\ldots i}| - r_i^{[i]}(A)\Big)\Big(|a_{j\ldots j}| - \overline{r_j^{[i]}}(A)\Big) > \overline{r_i^{[i]}}(A)r_j^{[i]}(A),\ for\ all\ i,j \in \langle n\rangle, i \neq j,$$

then A is nonsingular; that is, $0 \notin \sigma(A)$. In this paper, various inequalities of tensorial type will be obtained. The concept of symmetry can be seen in Theorem 4, which gives a generalization of the inequality obtained by Dragomir. Now, we begin with the introduction related to the tensorial product and Hilbert space topic of the paper.

Let $I_1^\star, \ldots, I_k^\star$ be intervals from $\mathbb{R}$ and let $f^\star : I_1^\star \times \ldots \times I_k^\star \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $Q_\star = (Q_{1\star}, \ldots, Q_{k\star})$ be a k-tuple of bounded self-adjoint operators on Hilbert spaces $H_1^\star, \ldots, H_k^\star$ such that the spectrum of $Q_{i\star}$ is contained in $I_i^\star$ for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of $f^\star$. If

$$Q_{i\star} = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $Q_{i\star}$ for $i = 1, \ldots, k$ by following [26], we define

$$f^\star(Q_{1\star}, \ldots, Q_{k\star}) := \int_{I_1^\star} \cdots \int_{I_k^\star} f^\star(\lambda_1, \ldots, \lambda_k) dE_1(\lambda_1) \tilde{\otimes}^\star \ldots \tilde{\otimes}^\star dE_k(\lambda_k)$$

as a bounded self-adjoint operator on the tensorial product $H_1^\star \otimes^{\bar{\star}} \ldots \otimes^{\bar{\star}} H_k^\star$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. Construction of this type extends the definition of Kornyi [27] for functions of two variables and has the property that

$$f^\star(Q_{1\star}, \ldots, Q_{k\star}) = f_1^\star(Q_{1\star}) \otimes^{\bar{\star}} \ldots \otimes^{\bar{\star}} f_k^\star(Q_{k\star}),$$

whenever $f^\star$ can be separated as a product $f^\star(t_1^\star, \ldots, t_k^\star) = f_1^\star(t_1^\star) \ldots f_k^\star(t_k^\star)$ of k functions each depending on only one variable.

2. Preliminaries

We begin this section with certain basic concepts and lemmas, which will be needed in the sequel.

Since we will be using tensorial products, we will define in the following what tensors and tensorial products are in short; for more, consult the following book [28].

Definition 1. *Let $\tilde{U}, \tilde{V}$ and $\tilde{W}$ be vector spaces over the same field $\tilde{F}$. A mapping $\Phi : \tilde{U} \times \tilde{V} \to \tilde{W}$ is called a bilinear mapping if it is linear in each variable separately. Namely, for all $u, u_1, u_2 \in \tilde{U}$, $v, v_1, v_2 \in \tilde{V}$ and $a, b \in \tilde{F}$,*

$\Phi(au_1 + bu_2, v) = a\Phi(u_1, v) + b\Phi(u_2, v)$,
$\Phi(u, av_1 + bv_2) = a\Phi(u, v_1) + b\Phi(u, v_2)$. If $\tilde{W} = \tilde{F}$, a bilinear mapping $\Phi : \tilde{U} \times \tilde{V} \to \tilde{F}$ is called a bilinear function.

Definition 2. *Let $\otimes^{\bar{\star}} : \tilde{U} \times \tilde{V} \to \tilde{W}$ be a bilinear mapping. The pair $(\tilde{W}, \otimes^{\bar{\star}})$ is called a tensor product space of $\tilde{U}$ and $\tilde{V}$ if it satisfies the following conditions:*

1. *$< Im\otimes^{\bar{\star}} >= \tilde{W}$ (Generating property);*
2. *$dim < Im\otimes^{\bar{\star}} >= dim\tilde{U} \cdot dim\tilde{V}$ (Maximal span property). If $\tilde{W} = \tilde{F}$, a bilinear mapping $\Phi : \tilde{U} \times \tilde{V} \to \tilde{F}$ is called a bilinear function.*

The member $w \in \tilde{W}$ is called a tensor, but not all tensors in $\tilde{W}$ are products of two vectors of the form $u \otimes^{\bar{\star}} v$. The notation $< Im\otimes^{\bar{\star}} >$ denotes the span.

Example 1. *Let $\boldsymbol{u} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\boldsymbol{v} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We can view u and v as column vectors. Namely,*

$$\boldsymbol{u} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are $m \times 1$ and $n \times 1$ matrices, respectively.
We define $\otimes^{\bar{\star}} : \mathbb{R}^m \times \mathbb{R}^n \to M_{m,n}$,

$$u \otimes^{\bar{\star}} v = uv^t = \begin{bmatrix} x_1y_1 \cdots x_1y_n \\ \vdots \\ x_my_1 \cdots x_my_n \end{bmatrix},$$

an $m \times n$ matrix with entries $A_{ij} = x_iy_j$. $(M_{m,n}, \otimes^{\bar{\star}})$ is a tensor product space of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Tensors do not need to be matrices. This is just one model given. For more, consult the following book [28].

Remember the following properties of the tensorial product:

$$(Q_\star L_\star) \otimes^{\bar{\star}} (W_\star R_\star) = (Q_\star \otimes^{\bar{\star}} W_\star)(L_\star \otimes^{\bar{\star}} R_\star)$$

that holds for any $Q_\star, W_\star, L_\star, R_\star \in B(H)$.

From the property, we can easily deduce the following consequences:

$$Q_\star^n \otimes^{\bar{\star}} W_\star^n = (Q_\star \otimes^{\bar{\star}} W_\star)^n, n \geqslant 0,$$

$$(Q_\star \otimes^{\bar{\star}} 1)(1 \otimes^{\bar{\star}} W_\star) = (1 \otimes^{\bar{\star}} W_\star)(Q_\star \otimes^{\bar{\star}} 1) = Q_\star \otimes^{\bar{\star}} W_\star,$$

which can be extended; for two natural numbers r, s, we have

$$(Q_\star \otimes^{\bar{\star}} 1)^s (1 \otimes^{\bar{\star}} W_\star)^r = (1 \otimes^{\bar{\star}} W_\star)^r (Q_\star \otimes^{\bar{\star}} 1)^s = Q_\star^s \otimes^{\bar{\star}} W_\star^r.$$

The current research concerning tensorial inequalities in Hilbert space can be seen in the following papers [29–33].

Lemma 1 ([34], p. 4)**.** *Assume that $Q_\star$ and $W_\star$ are self-adjoint operators with $Sp(Q_\star) \subset I$ and $Sp(W_\star) \subset J$. Let f, h be continuous on I, g, k continuous on J and ψ continuous on an interval K that contains the sum of the intervals $h(I) + k(J)$; then*

$$(f^\star(Q_\star) \otimes^{\bar{\star}} 1 + 1 \otimes^{\bar{\star}} g^\star(W_\star))\psi(h(Q_\star) \otimes^{\bar{\star}} 1 + 1 \otimes^{\bar{\star}} k(W_\star))$$
$$= \int_I \int_J (f(t) + g(s))\psi(h(t) + k(s)) dE_t \otimes^{\bar{\star}} dF(s),$$

where $Q_\star$ and $W_\star$ have the spectral resolutions,

$$Q_\star = \int_I t dE_t \text{ and } W_\star = \int_J s dF_s.$$

Theorem 1 ([35], p. 4)**.** *Assume that $Q_\star$ and $W_\star$ are self-adjoint operators with $Sp(Q_\star) \subset I$ and $Sp(W_\star) \subset J$. Let f be continuous on I; g continuous on J and continuous on an interval K that contains the product of the intervals $f(I)g(J)$, then*

$$\psi(f^\star(Q_\star) \otimes^{\bar{\star}} g^\star(W_\star)) = \int_I \int_J \psi(f(t)g(s)) dE_t \otimes^{\bar{\star}} dF(s),$$

where $Q_\star$ and $W_\star$ have the spectral resolutions

$$Q_\star = \int_I t dE_t, W_\star = \int_J s dF_s.$$

Theorem 2 ([35], p. 4)**.** *Assume that $Q_\star$ and $W_\star$ are self-adjoint operators with $Sp(Q_\star) \subset I$ and $Sp(W_\star) \subset J$. Let h be continuous on I, k continuous on J and ψ modulus on an interval U that contains the sum of the intervals $h(I) + k(J)$, then*

$$\psi(h(Q_\star) \otimes^{\bar{\star}} 1 + 1 \otimes^{\bar{\star}} k(W_\star)) = \int_I \int_J \psi(h(t) + k(s)) dE_t \otimes^{\bar{\star}} dF(s),$$

where $Q_\star$ and $W_\star$ have the spectral resolutions

$$Q_\star = \int_I t dE_t, W_\star = \int_J s dF_s.$$

In the following, we will obtain theorems that generalize the ones from Dragomir's paper [35]. In recent times, emphasis has been placed on inequalities in Hilbert space, which can be seen from the references to books on inequalities in Hilbert spaces given in the introduction. Motivation for this paper stems from the cited references and from the fact that the Hilbert space inequalities have many applications as indicated in the conclusion section.

Now we begin with the main results of this paper.

3. Main Results

The following Lemma generalizes the Lemma given by Dragomir in the paper ([35], p. 7) which will be instrumental in the obtained results.

Lemma 2. *Let $\psi : I \to \mathbb{R}$ be a convex function on the interval $I, q, w \in I^0$, the interior of I, with $q < w, 0 < \tau \leqslant 1$ and $\xi \in [0,1]$. Then,*

$$\left(\psi'_+\left((1-\xi)\frac{q}{\tau}+\xi w\right)-\psi'_-\left((1-\xi)\frac{q}{\tau}+\xi w\right)\right)\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right)$$

$$\leqslant (1-\xi)\psi\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

$$+w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)dt$$

$$\leqslant \left(\psi'_-\left(\frac{w}{\tau}\right)-\psi'_+\left(\frac{q}{\tau}\right)\right)\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right).$$

Proof. Since ψ is convex on I, it follows that the function is differentiable on I^0 except at a countable number of points; the lateral derivatives ψ' exist in each point of I^0—they are increasing on I^0 and $\psi'_- \leqslant \psi'_+$ on I^0. For any $x, y \in I^0$, we have the following relation:

$$\psi(x)=\psi\left(\frac{y}{\tau}\right)+\int_{\frac{y}{\tau}}^{x}\psi'(s)ds=\psi\left(\frac{y}{\tau}\right)+\left(x-\frac{y}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{y}{\tau}+tx\right)dt$$

Assume that $q < w$ and $\xi \in [0,1]$. Then we have

$$\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

$$=\psi\left(\frac{q}{\tau}\right)+\xi\left(w-\frac{q}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt$$

and

$$\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

$$=\psi\left(\frac{w}{\tau}\right)+\left(\frac{q}{\tau}(1-\xi)-w\left(\frac{1}{\tau}-1\right)\right)\int_0^1\psi'\left((1-t)\frac{w}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt.$$

If we multiply the first top inequality with $(1-\xi)$ and the second by ξ, and add the resulting equalities, we get

$$\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)=\psi(1-\xi)\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)$$

$$+\xi(1-\xi)\left(w-\frac{q}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt$$

$$+\xi\left(\frac{q}{\tau}(1-\xi)-w\left(\frac{1}{\tau}-1\right)\right)\int_0^1\psi'\left((1-t)\frac{w}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt.$$

From which we get

$$\psi(1-\xi)\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

$$+w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt$$

$$=\xi\left(-\frac{q}{\tau}(1-\xi)+w\left(\frac{1}{\tau}-1\right)\right)\int_0^1\psi'\left((1-t)\frac{w}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt$$

$$-\xi\left(-\frac{q}{\tau}(1-\xi)+w\left(\frac{1}{\tau}-1\right)\right)\int_0^1 \psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt;$$

an interesting equality in and of itself.
Since $q<w$ and $\xi\in[0,1]$, then $(1-\xi)\frac{q}{\tau}+\xi w\in(\frac{q}{\tau},w)$ and

$$(1-t)\frac{q}{\tau}+t\left((1-\xi)\frac{q}{\tau}+\xi w\right)\in\left[\frac{q}{\tau},(1-\xi)\frac{q}{\tau}+\xi w\right],$$

$$(1-t)\frac{w}{\tau}+t\left((1-\xi)\frac{q}{\tau}+\xi w\right)\in\left[(1-\xi)\frac{q}{\tau}+\xi w,\frac{w}{\tau}\right].$$

By the monotonicity of the derivative we have

$$\psi'_+\left((1-\xi)\frac{q}{\tau}+\xi w\right)\leqslant\psi'\left((1-t)\frac{w}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)\leqslant\psi'_-\left(\frac{w}{\tau}\right),$$

$$\psi'_+\left(\frac{q}{\tau}\right)\leqslant\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)\leqslant\psi'_-\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

for any $t\in[0,1]$.

By integrating the inequalities and subtracting them and making use of the identity developed before, we obtain the following inequality:

$$\begin{aligned}&\left(\psi'_+\left((1-\xi)\frac{q}{\tau}+\xi w\right)-\psi'_-\left((1-\xi)\frac{q}{\tau}+\xi w\right)\right)\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right)\\&\leqslant(1-\xi)\psi\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)\\&+w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w\right)dt\\&\leqslant\left(\psi'_-\left(\frac{w}{\tau}\right)-\psi'_+\left(\frac{q}{\tau}\right)\right)\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right).\end{aligned}$$

□

Corollary 1. *Letting $\tau\to1$ in the obtained Lemma, we obtain Lemma 1 given by Dragomir ([35], p. 7).*

$$\begin{aligned}&\left(\psi'_+((1-\xi)q+\xi w)-\psi'_-((1-\xi)q+\xi w)\right)\xi(w-q)(1-\xi)\\&\leqslant(1-\xi)\psi(q)+\xi\psi(w)-\psi((1-\xi)q+\xi w)\\&\leqslant(\psi'_-(w)-\psi'_+(q))\xi(w-q)(1-\xi).\end{aligned}$$

Corollary 2. *Considering the convex function $\psi:\mathbb{R}\to(0,+\infty)$, $\psi(x)=e^x$, then we have*

$$\begin{aligned}&(1-\xi)e^{\frac{q}{\tau}}+\xi e^{\frac{w}{\tau}}-e^{(1-\xi)\frac{q}{\tau}+\xi w}+\xi w\left(1-\frac{1}{\tau}\right)\int_0^1 e^{(1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)}dt\\&\leqslant(e^{\frac{w}{\tau}}-e^{\frac{q}{\tau}})\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right).\end{aligned}$$

Corollary 3. *From Lemma 2, we can observe that the term in the beginning of the chain of inequalities is positive if we assume ψ to be differentiable; therefore, we can observe the following:*

$$0\leqslant(1-\xi)\psi\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$

$$+w\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)dt$$
$$\leqslant \left(\psi'\left(\frac{w}{\tau}\right)-\psi'\left(\frac{q}{\tau}\right)\right)\xi\left(w\left(\frac{1}{\tau}-\xi\right)-\frac{q}{\tau}(1-\xi)\right).$$

In the following theorem, we generalize the result given by Dragomir.

Theorem 3. *Assume that ψ is a differentiable convex function on the interval I, and $Q_\star$ and $W_\star$ are self-adjoint operators with $Sp(Q_\star), Sp(W_\star) \subset I$; then, for all $\tau \in (0,1]$, we have*

$$0 \leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right)\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}}\psi\left(\frac{W_\star}{\tau}\right)-\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} W_\star\right)$$
$$+\xi\left(1-\frac{1}{\tau}\right)\int_0^1 (1\otimes^{\tilde{\star}} W_\star)\psi'\left(\frac{1-\xi u}{\tau}Q_\star\otimes^{\tilde{\star}} 1+u\xi 1\otimes^{\tilde{\star}} W_\star\right)du$$
$$\leqslant \left(\frac{\xi(1-\xi)}{\tau}Q_\star\otimes^{\tilde{\star}} 1-\xi\cdot\left(\frac{1}{\tau}-\xi\right)1\otimes^{\tilde{\star}} W_\star\right)\left(\psi'(Q_\star)\otimes^{\tilde{\star}} 1-1\otimes^{\tilde{\star}}\psi'(W_\star)\right).$$

In particular,

$$0 \leqslant \frac{\psi(2Q_\star)\otimes^{\tilde{\star}} 1+1\otimes^{\tilde{\star}}\psi(2W_\star)}{2}-\psi\left(Q_\star\otimes^{\tilde{\star}} 1+\frac{1\otimes^{\tilde{\star}} W_\star}{2}\right)$$
$$-\frac{1}{2}\int_0^1 (1\otimes^{\tilde{\star}} W_\star)\psi'\left((2-t)Q_\star\otimes^{\tilde{\star}} 1+\frac{t}{2}1\otimes^{\tilde{\star}} W_\star\right)dt$$
$$\leqslant \left(\frac{Q_\star\otimes^{\tilde{\star}} 1}{2}-\frac{3}{4}1\otimes^{\tilde{\star}} W_\star\right)\left(\psi'(Q_\star)\otimes^{\tilde{\star}} 1-1\otimes^{\tilde{\star}}\psi'(W_\star)\right).$$

Proof. Assume that $Q_\star$ and $W_\star$ have the spectral resolutions

$$Q_\star=\int_I t dE_t \text{ and } W_\star=\int_J s dF_s.$$

If we take the double integral $\int_I\int_I$ over $dE_t\otimes^{\tilde{\star}} dF(s)$ in the inequality we obtained in Corollary 3 , then we get

$$0 \leqslant \int_I\int_I\left((1-\xi)\psi\left(\frac{t}{\tau}\right)+\xi\psi\left(\frac{s}{\tau}\right)-\psi\left((1-\xi)\frac{t}{\tau}+\xi s\right)\right)dE(t)\otimes^{\tilde{\star}} dF(s)$$
$$+\int_I\int_I\left(s\xi\left(1-\frac{1}{\tau}\right)\int_0^1\psi'\left((1-u)\frac{t}{\tau}+u((1-\xi)\frac{t}{\tau}+\xi s)\right)du\right)dE(t)\otimes^{\tilde{\star}} dF(s)$$
$$\leqslant \int_I\int_I\left((\psi'(s)-\psi'(t))\xi\left(s\left(\frac{1}{\tau}-\xi\right)-\frac{t}{\tau}(1-\xi)\right)\right)dE(t)\otimes^{\tilde{\star}} dF(s).$$

Using the properties given in preliminaries, we obtain the following, where for the part that has the composite function we use Theorem 2, setting $h(t)=(1-\xi)t, \text{ and } k(s)=\xi s.$ We apply the Fubini's Theorem to the second term on the left hand side, and therefore all together we obtain

$$\int_I\int_I\psi\left((1-\xi)\frac{t}{\tau}+\xi s\right)dE(t)\otimes^{\tilde{\star}} dF(s)=\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} W_\star\right)$$

$$0 \leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right)\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}}\psi\left(\frac{W_\star}{\tau}\right)-\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} W_\star\right)$$
$$+\xi\left(1-\frac{1}{\tau}\right)\int_0^1 (1\otimes^{\tilde{\star}} W_\star)\psi'\left[\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+u\xi 1\otimes^{\tilde{\star}} W_\star\right]du.$$

$$\leqslant \xi\left(\frac{1}{\tau}-\xi\right)1\otimes^{\bar\star}\psi'(W_\star)B-\xi\left(\frac{1}{\tau}-\xi\right)\psi'(Q_\star)\otimes^{\bar\star}W_\star$$
$$-\frac{\xi(1-\xi)}{\tau}Q_\star\otimes^{\bar\star}\psi'(W_\star)+\frac{\xi(1-\xi)}{\tau}\psi'(Q_\star)Q_\star\otimes^{\bar\star}1.$$

By rewriting the terms using the tensorial properties in the following way,

$$(1\otimes^{\bar\star}W_\star)(1\otimes^{\bar\star}\psi'(W_\star))=1\otimes^{\bar\star}\psi'(W_\star)W_\star,\ \ \psi'(Q_\star)\otimes^{\bar\star}W_\star=(\psi'(Q_\star)\otimes^{\bar\star}1)(1\otimes^{\bar\star}W_\star),$$

and regrouping, we obtain the original inequality,

$$0\leqslant(1-\xi)\psi\left(\frac{Q_\star}{\tau}\right)\otimes^{\bar\star}1+\xi1\otimes^{\bar\star}\psi\left(\frac{W_\star}{\tau}\right)-\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\bar\star}1+\xi1\otimes^{\bar\star}W_\star\right)$$
$$+\xi\left(1-\frac{1}{\tau}\right)\int_0^1(1\otimes^{\bar\star}W_\star)\psi'\left(\frac{1-\xi u}{\tau}Q_\star\otimes^{\bar\star}1+u\xi1\otimes^{\bar\star}W_\star\right)du$$
$$\leqslant\left(\frac{\xi(1-\xi)}{\tau}Q_\star\otimes^{\bar\star}1-\xi\cdot\left(\frac{1}{\tau}-\xi\right)1\otimes^{\bar\star}W_\star\right)\left(\psi'(Q_\star)\otimes^{\bar\star}1-1\otimes^{\bar\star}\psi'(W_\star)\right).$$

For the particular value, let $\tau,\xi=\frac{1}{2}$. □

Corollary 4. *Setting $\tau=1$, we obtain the inequality given by Dragomir ([35], p. 9):*

$$0\leqslant(1-\xi)\psi(Q_\star)\otimes^{\bar\star}1+\xi1\otimes^{\bar\star}\psi(W_\star)-\psi\big((1-\xi)Q_\star\otimes^{\bar\star}1+\xi1\otimes^{\bar\star}W_\star\big)$$
$$\leqslant\xi(1-\xi)(Q_\star\otimes^{\bar\star}1-1\otimes^{\bar\star}W_\star)\big(\psi'(Q_\star)\otimes^{\bar\star}1-1\otimes^{\bar\star}\psi'(W_\star)\big).$$

Lemma 3. *Let $\psi:I\subset\mathbb{R}\to\mathbb{R}$ be a twice-differentiable function on the interval I^0 the interior of I. If there exist the constants c,C such that*

$$c\leqslant\psi''(t)\leqslant C\ for\ any\ t\in I^0,$$

then

$$\frac{1}{2}c\xi(1-\xi)\left(w-\frac{q}{\tau}\right)^2\leqslant(1-\xi)\psi\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$
$$-w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left(c\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)-\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)\right)dt$$
$$\leqslant(1-\xi)\psi\left(\frac{q}{\tau}\right)+\xi\psi\left(\frac{w}{\tau}\right)-\psi\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$
$$-w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left(C\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)-\psi'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)\right)dt$$
$$\leqslant\frac{1}{2}C\xi(1-\xi)\left(w-\frac{q}{\tau}\right)^2.$$

Proof. We consider the auxiliary function $\psi_C:I\subset\mathbb{R}\to\mathbb{R}$ defined by $\psi_C(x)=\frac{1}{2}x^2C-\psi(x)$. The function ψ_C is differentiable on I^0 and $\psi_C''(x)=C-\psi''(x)\geqslant0$, showing that ψ_C is a convex function on I^0. By the convexity of ψ_C, we have for any $q,w\in I^0$ and $\xi\in[0,1]$ that

$$0\leqslant(1-\xi)\psi_C\left(\frac{q}{\tau}\right)+\xi\psi_C\left(\frac{w}{\tau}\right)-\psi_C\left((1-\xi)\frac{q}{\tau}+\xi w\right)$$
$$+w\xi\left(1-\frac{1}{\tau}\right)\int_0^1\psi_C'\left((1-t)\frac{q}{\tau}+t((1-\xi)\frac{q}{\tau}+\xi w)\right)dt,$$

which, when rearranged, gives us the right hand side inequality. The left hand side inequality follows in a similar way by considering the auxiliary function $\psi_C(x) = \psi(x) - \frac{1}{2}cx^2$, which is twice-differentiable and convex on I^0. The middle inequality follows by comparing the integral parts of the left hand part of the right inequality and the right hand side of the left hand side inequality. If we take $\psi(x) = x^2$, then the inequality reduces to an equality $c = C = 2$. $\square$

Corollary 5. *Setting $\tau = 1$ in the Lemma, it reduces to the Lemma 2 given by Dragomir ([35], p. 11)*

$$\frac{1}{2}c\xi(1-\xi)(w-q)^2 \leqslant (1-\xi)\psi(q) + \xi\psi(w) - \psi((1-\xi)q + \xi w) \leqslant \frac{1}{2}C\xi(1-\xi)(w-q)^2.$$

Theorem 4. *Let $\psi : I \subset\to \mathbb{R}$ be a twice-differentiable function on the interval I^0, the interior of I. If there exist the constants c, C such that the condition holds, then for any self-adjoint operators $Q_\star, W_\star$ with $SP(Q_\star), SP(W_\star) \subset I$,*

$$\begin{aligned}
\frac{c}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star \otimes^{\tilde{\star}} 1 - 1 \otimes^{\tilde{\star}} W_\star\right)^2 &\leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right) \otimes^{\tilde{\star}} 1 + \xi 1 \otimes^{\tilde{\star}} \psi\left(\frac{W_\star}{\tau}\right) \\
&\quad -\psi\left(\frac{1-\xi}{\tau}Q_\star \otimes^{\tilde{\star}} 1 + \xi 1 \otimes^{\tilde{\star}} W_\star\right) \\
&\quad -\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \left[c\left(\frac{1-u\xi}{\tau}Q_\star \otimes^{\tilde{\star}} W_\star + u\xi(1 \otimes^{\tilde{\star}} W_\star)^2\right)\right. \\
&\quad \left. -(1 \otimes^{\tilde{\star}} W_\star)\psi'\left(\frac{(1-u\xi)}{\tau}Q_\star \otimes^{\tilde{\star}} 1 + u\xi 1 \otimes^{\tilde{\star}} W_\star\right)\right] du \\
&\leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right) \otimes^{\tilde{\star}} 1 + \xi 1 \otimes^{\tilde{\star}} \psi\left(\frac{W_\star}{\tau}\right) - \psi\left(\frac{1-\xi}{\tau}Q_\star \otimes^{\tilde{\star}} 1 + \xi 1 \otimes^{\tilde{\star}} W_\star\right) \\
&\quad -\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \left[C\left(\frac{1-u\xi}{\tau}Q_\star \otimes^{\tilde{\star}} W_\star + u\xi(1 \otimes^{\tilde{\star}} W_\star)^2\right)\right. \\
&\quad \left. -(1 \otimes^{\tilde{\star}} W_\star)\psi'\left(\frac{1-u\xi}{\tau}Q_\star \otimes^{\tilde{\star}} 1 + u\xi 1 \otimes^{\tilde{\star}} W_\star\right)\right] du \\
&\leqslant \frac{C}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star \otimes^{\tilde{\star}} 1 - 1 \otimes^{\tilde{\star}} W_\star\right)^2.
\end{aligned}$$

Proof. Assume that Q and W have spectral resolutions

$$Q = \int_I t dE_t \text{ and } W = \int_I s dF_s.$$

If we take the double integral $\int_I \int_I$ over $dE(t) \otimes^{\tilde{\star}} dF(s)$ in the inequality which we obtained in Lemma 3, then we get

$$\begin{aligned}
&\frac{1}{2}c\xi(1-\xi)\int_I\int_I\left(-s+\frac{t}{\tau}\right)^2 dE(t) \otimes^{\tilde{\star}} dF(s) \\
&\leqslant (1-\xi)\int_I\int_I \psi\left(\frac{t}{\tau}\right) dE(t) \otimes^{\tilde{\star}} dF(s) + \int_I\int_I \xi\psi\left(\frac{s}{\tau}\right) dE(t) \otimes^{\tilde{\star}} dF(s) \\
&\quad - \int_I\int_I \psi\left((1-\xi)\frac{t}{\tau} + \xi s\right) dE(t) \otimes^{\tilde{\star}} dF(s)
\end{aligned}$$

$$-\int_I\int_I w\xi\left(1-\frac{1}{\tau}\right)\int_0^1 d\left((1-u)\frac{t}{\tau} + u((1-\xi)\frac{t}{\tau} + \xi w)\right) du dE(t) \otimes^{\tilde{\star}} dF(s)$$

$$+\int_I\int_I w\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \psi'\left((1-u)\frac{t}{\tau}+u((1-\xi)\frac{t}{\tau}+\xi w)\right)dudE(t)\otimes^{\tilde{\star}} dF(s)$$

$$\leqslant (1-\xi)\int_I\int_I \psi\left(\frac{t}{\tau}\right)dE(t)\otimes^{\tilde{\star}} dF(s)+\int_I\int_I \xi\psi\left(\frac{s}{\tau}\right)dE(t)\otimes^{\tilde{\star}} dF(s)$$

$$-\int_I\int_I \psi\left((1-\xi)\frac{t}{\tau}+\xi s\right)dE(t)\otimes^{\tilde{\star}} dF(s)$$

$$-\int_I\int_I w\xi\left(1-\frac{1}{\tau}\right)\int_0^1 C\left((1-u)\frac{t}{\tau}+u((1-\xi)\frac{t}{\tau}+\xi w)\right)dudE(t)\otimes^{\tilde{\star}} dF(s)$$

$$+\int_I\int_I w\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \psi'\left((1-u)\frac{t}{\tau}+u((1-\xi)\frac{t}{\tau}+\xi w)\right)dudE(t)\otimes^{\tilde{\star}} dF(s)$$

$$\leqslant \frac{1}{2}C\xi(1-\xi)\int_I\int_I\left(-s+\frac{t}{\tau}\right)^2 dE(t)\otimes^{\tilde{\star}} dF(s),$$

since

$$\int_I\int_I\left(-s+\frac{t}{\tau}\right)^2 dE(t)\otimes^{\tilde{\star}} dF(s)=\int_I\int_I\left(\frac{t^2}{\tau^2}-\frac{2}{\tau}ts+s^2\right)dE(t)\otimes^{\tilde{\star}} dF(s)$$

$$=\int_I\int_I \frac{t^2}{\tau^2}dE(t)\otimes^{\tilde{\star}} dF(s)-\frac{2}{\tau}\int_I\int_I tsdE(t)\otimes^{\tilde{\star}} dF(s)+\int_I\int_I s^2dE(t)\otimes^{\tilde{\star}} dF(s)$$

$$=\frac{1}{\tau^2}Q_\star^2\otimes^{\tilde{\star}} 1-\frac{2}{\tau}Q_\star\otimes^{\tilde{\star}} W_\star+1\otimes^{\tilde{\star}} W_\star^2.$$

Using the tensorial properties we get

$$\int_I\int_I\left(-s+\frac{t}{\tau}\right)^2 dE(t)\otimes^{\tilde{\star}} dF(s)=\left(\frac{1}{\tau}Q_\star\otimes^{\tilde{\star}} 1-1\otimes^{\tilde{\star}} W_\star\right)^2.$$

Using Fubini's Theorem on the term with the integral under the spectral resolution, we obtain the original inequality,

$$\frac{c}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star\otimes^{\tilde{\star}} 1-1\otimes^{\tilde{\star}} W_\star\right)^2$$

$$\leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right)\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} \psi\left(\frac{W_\star}{\tau}\right)-\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} W_\star\right)$$

$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left[c\left(\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde{\star}} W_\star+u\xi(1\otimes^{\tilde{\star}} W_\star)^2\right)\right.$$

$$\left.-(1\otimes^{\tilde{\star}} W_\star)\psi'\left(\frac{(1-u\xi)}{\tau}Q_\star\otimes^{\tilde{\star}} 1+u\xi 1\otimes^{\tilde{\star}} W_\star\right)\right]du$$

$$\leqslant (1-\xi)\psi\left(\frac{Q_\star}{\tau}\right)\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} \psi\left(\frac{W_\star}{\tau}\right)-\psi\left(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+\xi 1\otimes^{\tilde{\star}} W_\star\right)$$

$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left[C\left(\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde{\star}} W_\star+u\xi(1\otimes^{\tilde{\star}} W_\star)^2\right)\right.$$

$$\left.-(1\otimes^{\tilde{\star}} W_\star)\psi'\left(\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde{\star}} 1+u\xi 1\otimes^{\tilde{\star}} W_\star\right)\right]du$$

$$\leqslant \frac{C}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star\otimes^{\tilde{\star}} 1-1\otimes^{\tilde{\star}} W_\star\right)^2.$$

□

4. Some Examples and Consequences

In the following sequel we give examples which demonstrate the obtained Theorems.

Example 2. *We consider the power function, $f(t) = t^z, t > 0$, which is convex for $z \in (-\infty, 0) \cup [1, +\infty)$. From Theorem 3 for $A, B > 0$, we obtain the following:*

$$0 \leqslant \frac{1}{\tau^z}((1-\xi)Q_\star^z \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star^z) - \left(\frac{1-\xi}{\tau}Q_\star \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star\right)^z$$
$$+z\xi\left(1-\frac{1}{\tau}\right)\int_0^1 (1 \otimes^{\bar\star} W_\star)\left(\frac{1-\xi u}{\tau}Q_\star \otimes^{\bar\star} 1 + u\xi 1 \otimes^{\bar\star} W_\star\right)^{z-1} du$$
$$\leqslant z\left(\frac{\xi(1-\xi)}{\tau}Q_\star \otimes^{\bar\star} 1 - \xi \cdot \left(\frac{1}{\tau} - \xi\right) 1 \otimes^{\bar\star} W_\star\right)\left(Q_\star^{z-1} \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star^{z-1}\right).$$

Corollary 6. *Letting $\tau = 1$, we obtain the example given by Dragomir ([35], p. 15):*

$$0 \leqslant (1-\xi)Q_\star^z \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star^z - \left((1-\xi)Q_\star \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star\right)^z$$
$$\leqslant z\xi(1-\xi)\left(Q_\star \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star\right)\left(Q_\star^{z-1} \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star^{z-1}\right).$$

Setting $\xi, \tau = \frac{1}{2}$, we get

$$0 \leqslant 2^{z-1}(Q_\star^z \otimes^{\bar\star} 1 + 1 \otimes^{\bar\star} W_\star^z) - \left(Q_\star \otimes^{\bar\star} 1 + \frac{1}{2} 1 \otimes^{\bar\star} W_\star\right)^z$$
$$-\frac{z}{2}\int_0^1 (1 \otimes^{\bar\star} W_\star)\left((2-u)Q_\star \otimes^{\bar\star} 1 + \frac{u}{2} 1 \otimes^{\bar\star} W_\star\right)^{z-1} du$$
$$\leqslant z\left(\frac{1}{2}Q_\star \otimes^{\bar\star} 1 - \frac{3}{4} 1 \otimes^{\bar\star} W_\star\right)\left(Q_\star^{z-1} \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star^{z-1}\right).$$

Using the inequality obtained in Theorem 4, we obtain

$$\frac{c}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star\right)^2$$
$$\leqslant \frac{1}{\tau^z}((1-\xi)Q_\star^z \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star^z) - \left(\frac{1-\xi}{\tau}Q_\star \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star\right)^z$$
$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \left[c\left(\frac{1-u\xi}{\tau}Q_\star \otimes^{\bar\star} W_\star + u\xi(1 \otimes^{\bar\star} W_\star)^2\right)\right.$$
$$\left.-z(1 \otimes^{\bar\star} W_\star)\left(\frac{(1-u\xi)}{\tau}Q_\star \otimes^{\bar\star} 1 + u\xi 1 \otimes^{\bar\star} W_\star\right)^{z-1}\right] du$$
$$\leqslant \frac{1}{\tau^z}((1-\xi)Q_\star^z \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star^z) - \left(\frac{1-\xi}{\tau}Q_\star \otimes^{\bar\star} 1 + \xi 1 \otimes^{\bar\star} W_\star\right)^z$$
$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1 \left[C\left(\frac{1-u\xi}{\tau}Q_\star \otimes^{\bar\star} W_\star + u\xi(1 \otimes^{\bar\star} W_\star)^2\right)\right.$$
$$\left.-z(1 \otimes^{\bar\star} W_\star)\left(\frac{(1-u\xi)}{\tau}Q_\star \otimes^{\bar\star} 1 + u\xi 1 \otimes^{\bar\star} W_\star\right)^{z-1}\right] du$$
$$\leqslant \frac{C}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star \otimes^{\bar\star} 1 - 1 \otimes^{\bar\star} W_\star\right)^2.$$

Example 3. *Let us consider now the convex function $f(t) = e^{\alpha t}, t, \alpha \in \mathbb{R}$ and $\alpha \neq 0$. From Theorem 3, we get the following for self-adjoint operators $Q_\star$ and $W_\star$:*

$$(1-\xi)e^{\frac{\alpha}{\tau}Q_\star} \otimes^{\tilde\star} 1 + \xi 1 \otimes^{\tilde\star} e^{\frac{\alpha}{\tau}W_\star} - e^{\alpha(\frac{1-\xi}{\tau}Q_\star \otimes^{\tilde\star} 1 + \xi 1 \otimes^{\tilde\star} W_\star)}$$

$$+\alpha\xi\left(1-\frac{1}{\tau}\right)\int_0^1 (1\otimes^{\tilde\star} W_\star)e^{\alpha(\frac{1-\xi u}{\tau}Q_\star\otimes^{\tilde\star}1+u\xi 1\otimes^{\tilde\star}W_\star)}du$$

$$\leqslant \alpha\left(\frac{\xi(1-\xi)}{\tau}Q_\star \otimes^{\tilde\star} 1 - \xi\cdot\left(\frac{1}{\tau}-\xi\right)1\otimes^{\tilde\star} W_\star\right)(e^{\alpha Q_\star}\otimes^{\tilde\star} 1 - 1\otimes^{\tilde\star} e^{\alpha W_\star}).$$

Corollary 7. *Setting $\tau = 1$ in the obtained inequality, we recover the example given by Dragomir ([35], p. 15)*

$$\begin{aligned} 0 &\leqslant (1-\xi)e^{\alpha Q_\star}\otimes^{\tilde\star} 1 + u1\otimes^{\tilde\star} e^{\alpha W_\star} - e^{\alpha((1-\xi)Q_\star\otimes^{\tilde\star}1+\xi 1\otimes^{\tilde\star}W_\star)} \\ &\leqslant \alpha\xi(1-\xi)\big(Q_\star\otimes^{\tilde\star}1 - 1\otimes^{\tilde\star}W_\star\big)(e^{\alpha Q_\star}\otimes^{\tilde\star}1 - 1\otimes^{\tilde\star}e^{\alpha W_\star}). \end{aligned}$$

Setting $\xi, \tau = \frac{1}{2}$, we obtain

$$\frac{1}{2}\left(e^{2\alpha Q_\star}\otimes^{\tilde\star} 1 + 1\otimes^{\tilde\star} e^{2\alpha W_\star}\right) - e^{\alpha(Q_\star\otimes^{\tilde\star}1+\frac{1}{2}1\otimes^{\tilde\star}W_\star)} - \frac{\alpha}{2}\int_0^1 (1\otimes^{\tilde\star}W_\star)e^{\alpha((2-u)Q_\star\otimes^{\tilde\star}1+\frac{u}{2}1\otimes^{\tilde\star}W_\star)}du$$

$$\leqslant z\left(\frac{1}{2}Q_\star\otimes^{\tilde\star}1 - \frac{3}{4}1\otimes^{\tilde\star}W_\star\right)(e^{\alpha Q_\star}\otimes^{\tilde\star}1 - 1\otimes^{\tilde\star}e^{\alpha W_\star}).$$

Using the inequality obtained in Theorem 4, we obtain

$$\frac{c}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star\otimes^{\tilde\star}1 - 1\otimes^{\tilde\star}W_\star\right)^2 \leqslant (1-\xi)e^{\frac{\alpha}{\tau}Q_\star}\otimes^{\tilde\star}1 + \xi 1\otimes^{\tilde\star}e^{\frac{\alpha}{\tau}W_\star} - e^{\alpha(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde\star}1+\xi 1\otimes^{\tilde\star}W_\star)}$$

$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left[c\left(\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde\star}W_\star + u\xi(1\otimes^{\tilde\star}W_\star)^2\right) - z(1\otimes^{\tilde\star}W_\star)e^{\left(\frac{(1-u\xi)}{\tau}Q_\star\otimes^{\tilde\star}1+u\xi 1\otimes^{\tilde\star}W_\star\right)}\right]du$$

$$\leqslant (1-\xi)e^{\frac{\alpha}{\tau}Q_\star}\otimes^{\tilde\star}1 + \xi 1\otimes^{\tilde\star}e^{\frac{\alpha}{\tau}W_\star} - e^{\alpha(\frac{1-\xi}{\tau}Q_\star\otimes^{\tilde\star}1+\xi 1\otimes^{\tilde\star}W_\star)}$$

$$-\xi\left(1-\frac{1}{\tau}\right)\int_0^1\left[C\left(\frac{1-u\xi}{\tau}Q_\star\otimes^{\tilde\star}W_\star + u\xi(1\otimes^{\tilde\star}W_\star)^2\right) - z(1\otimes^{\tilde\star}W_\star)e^{\left(\frac{(1-u\xi)}{\tau}Q_\star\otimes^{\tilde\star}1+u\xi 1\otimes^{\tilde\star}W_\star\right)}\right]du$$

$$\leqslant \frac{C}{2}\xi(1-\xi)\left(\frac{1}{\tau}Q_\star\otimes^{\tilde\star}1 - 1\otimes^{\tilde\star}W_\star\right)^2.$$

5. Conclusions

Tensors have become important in various fields, for example in physics, because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such, inequalities are crucial in numerical aspects. Reflected in this work is the generalized Dragomir's lemma, which, as a consequence, has new inequalities of the classical type. Using that lemma enabled us to generalize the results from Dragomir [35]. New HH inequalities are given, as well as the consequences showing our generalization. Examples of specific convex functions and their inequalities using our results are given in the section on examples and consequences. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond–Pecaric inequality, which we will work on.

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