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Highly Dispersive Optical Solitons in the Absence of Self-Phase Modulation by Lie Symmetry

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Abstract: The paper revisits highly dispersive optical solitons that are addressed by the aid of Lie symmetry followed by the implementation of the Riccati equation approach and the improved modified extended tanh-function approach. The soliton solutions are recovered and classified. The conservation laws are also recovered and the corresponding conserved quantities are enlisted.

Keywords: solitons; Lie symmetry analysis; Riccati equation method; conservation laws

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1. Introduction

The key principle of soliton transmission across intercontinental distances is the sustainment of a delicate balance between chromatic dispersion (CD) and self-phase modulation (SPM) [1–26]. Occasionally it may so happen that CD carries a low count. This would lead to a catastrophic collapse during soliton transmission across long distances. One of the measures to replenish this low count is to provide additional sources of dispersive effects. This can be done by including higher-order dispersion terms. In this case, inter-modal dispersion (IMD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD), and sixth-order dispersion effects are introduced. These would make up for the low count of CD. However, one of the cons that these additional dispersion terms would introduce is the presence of pronounced soliton radiation. This effect is neglected so the focus of the paper would be the successful applicability of Lie symmetry to recover soliton solutions to the model.

Another twist to the model is the absence of the necessary SPM. This absence is however replaced by the perturbation terms that provide the necessary balance through the nonlinear effects stemming from the perturbed terms. Thus, the modified model is still the well-known nonlinear Schrodinger's equation (NLSE) but it comes with six dispersion terms and no SPM; instead, three perturbation terms provide the effect of nonlinearity that maintains the balance. This is referred to as highly dispersive (HD) NLSE or HD-NLSE. This model will be the focus of attention in the paper.

The integration approach is Lie symmetry analysis. This would reduce the governing partial differential equation (PDE) to a pair of ordinary differential equations (ODEs). These ODEs will be addressed by the usage of a pair of integration schemes. These are the Riccati equation approach and the improved modified extended tanh-function approach. The two approaches would lead to the soliton solutions and plane waves although such waves are a byproduct of the two schemes and are not applicable in Fiber Optics. The soliton solutions are classified, and their respective existence criteria are also enlisted as parameter constraints. Finally, the conservation laws are computed for the scheme by the usage of the multipliers approach. The conserved quantity is derived from the retrieved conserved density with the aid of the bright soliton solution of the model. The results are exhibited in the rest of the paper after a succinct intro to the model.

Governing Equation

The modified model that comes with six dispersion terms and no SPM is indicated below [1–3]

$$\begin{aligned} &iq_t + ia_1q_x + a_2q_{xx} + ia_3q_{xxx} + a_4q_{xxxx} + ia_5q_{xxxxx} + a_6q_{xxxxxx} \\ &+i\left[\lambda\left(|q|^2q\right)_x + \mu\left(|q|^2\right)_x q + \sigma|q|^2q_x\right] = 0. \end{aligned} \tag{1}$$

Here, the first term depicts the linear–temporal evolution, where $i = \sqrt{-1}$, while $q(x,t)$ describes soliton molecule, where x and t are spatial and temporal variables in sequence. a_l $(l = 1 - 6)$ comes from IMD, CD, 3OD, 4OD, 5OD and 6OD in sequence. Lastly, λ, μ and σ stem from the perturbation terms, where λ yields self-steepening, μ gives self-frequency shift and σ arises from nonlinear dispersion. Biswas et al. [1] derived the singular and bright-singular combo solitons of the governing model using the exp-function method. The bright and dark solitons of model (1) were obtained by González-Gaxiola et al. [2] using a numerical algorithm, namely the Laplace–Adomian decomposition method. Ullah et al. [3] obtained some exact solutions to model (1) using the generalized tanh method and Kudryashov's method. These findings are significant as they provide a good understanding of the behavior of the model. For a deeper understanding, here we identify new solutions and symmetries of the governing model using Lie symmetry and then followed by the implementation of the Riccati equation approach and the improved modified extended tanh-function approach. Moreover, the conservation laws are also recovered, and the corresponding conserved quantities are enlisted from the retrieved conserved density with the aid of some obtained solution. These findings contribute to understanding the behavior of solitons in nonlinear models and can be useful in the design of optical communication systems.

Higher-order dispersion terms refer to the third-, fourth-, and higher-order derivatives of the refractive index with respect to frequency in the Taylor expansion of the refractive index. These higher-order terms can have a significant impact on the propagation of ultrashort pulses of light in optical fibers, waveguides, and other nonlinear optical media. The second-order dispersion term, which is proportional to the frequency squared, is the dominant dispersion term in most optical systems and is responsible for pulse broadening and distortion. However, higher-order dispersion terms become more important as the pulse duration decreases, leading to additional pulse broadening and other effects. The third-order dispersion term, which is proportional to the frequency cubed, can cause pulse splitting and the formation of multiple pulses during propagation. It can also lead to the formation of highly dispersive optical solitons (HDOS), which rely on the balance between higher-order dispersion and nonlinear effects to maintain their shape during propagation. The fourth-order dispersion term, which is proportional to the frequency of the fourth power, can cause pulse compression and can play a role in the generation of supercontinuum spectra.

Inter-modal dispersion in nonlinear optics is a phenomenon that occurs in optical fibers or waveguides, where different modes of the electromagnetic field (e.g., different polarizations or transverse modes) propagate at different speeds due to their different group velocities. This results in a temporal broadening and distortion of an optical pulse, which can limit the performance of optical communication systems or other optical devices that rely on the accurate transmission of a narrow pulse of light. In the presence of nonlinear effects, such as self-phase modulation or four-wave mixing, inter-modal dispersion can be even more complex and can lead to new phenomena such as mode conversion, mode locking, or inter-modal solitons. These effects arise due to the nonlinear interactions between the different modes of the electromagnetic field and can lead to novel applications in nonlinear optics. Overall, inter-modal dispersion is an important consideration in the design and performance of optical communication systems and other optical devices that rely on the transmission of narrow optical pulses. It is a complex phenomenon that requires careful attention and mitigation strategies to optimize the performance of these systems.

Self-steepening is a nonlinear effect that occurs in optics when a short, intense pulse of light propagates through a medium. In this phenomenon, the leading edge of the pulse experiences a higher intensity than the trailing edge due to the nonlinear interaction between the pulse and the medium, leading to a self-induced steepening of the pulse. Self-steepening is an important phenomenon in ultrafast optics and laser physics, as it can lead to the generation of very short pulses of light with high peak power and broad spectral bandwidth. These properties make self-steepening a useful tool for a variety of applications, such as laser micromachining, spectroscopy, and optical communications.

Self-frequency shift is a nonlinear effect that occurs in optical fibers or other nonlinear optical media when intense light propagates through them. In this phenomenon, the frequency of light changes due to the nonlinear interaction between the light and the medium. The self-frequency shift effect arises due to a combination of two nonlinear effects: self-phase modulation (SPM) and the Kerr effect. SPM causes a broadening of the spectrum of the pulse, and the Kerr effect causes a change in the refractive index of the medium proportional to the intensity of the light. This change in the refractive index leads to a change in the velocity of the light, causing the frequency to shift. Self-frequency shift can lead to a variety of interesting and useful phenomena, such as the generation of frequency combs or the formation of solitons. For example, when a pulse of light propagates through a fiber with anomalous dispersion, the self-frequency shift can balance the dispersion and cause the pulse to form a soliton, which is a stable, self-contained pulse that maintains its shape and frequency during propagation. Self-frequency shift can also be used in applications such as optical communications, where it can be used to extend the transmission distance of optical fibers by compensating for the dispersion of the fiber. However, self-frequency shift can also limit the performance of optical devices if not properly controlled, as it can lead to spectral broadening, pulse distortion, and other undesirable effects.

To derive Equation (1), it dates back to the earlier days of Maxwell's equation from Electromagnetics. With the application of multiple scales, one can easily derive the governing NLSE. This is for a very idealistic situation. There are several forms of departure from this idealistic scenario and that too for several practical reasons. If CD comes out to be problematic, the classical solitons give way to dispersion-managed solitons whose details have been extensively studied about two decades ago. The other form of problematic situation is when CD runs low. This therefore would lead to distortion to the delicate balance between CD and SPM, a much-needed symbiotic necessity for solitons to sustain the transmission across intercontinental distances. The current paper addresses the situation when both CD and SPM run low. In such an extreme situation, CD is compensated with additional dispersion terms and they are IMD, 3OD, 4OD, 5OD, and 6OD. However, the SPM effect is completely replaced by the nonlinear effects that stem from the three perturbation terms that are incorporated, namely the self-steepening, self-steepening, and the nonlinear dispersion. This gave way to the current model that is being addressed for the first time in this paper by Lie symmetry. Earlier this model was addressed numerically including Laplace–Adomian decomposition scheme and other mathematical formalisms [27–32].

2. Lie Symmetry Analysis

Lie symmetry analysis is a mathematical method used to study the invariance properties of differential equations. It involves the use of Lie groups, which are mathematical structures that represent continuous symmetries of a system, to analyze the equations and identify invariant solutions and transformations. In simple terms, Lie symmetry analysis is a tool for identifying solutions of a differential equation that remain unchanged under certain transformations. These transformations are represented by Lie groups, which are collections of transformations that leave the differential equation invariant. The basic idea of Lie symmetry analysis is to look for a set of transformations that map solutions of the differential equation to other solutions of the same equation. This set of transformations is called the symmetry group of the differential equation, and it plays a crucial role in understanding the structure and properties of the solutions. Lie symmetry analysis has many applications in physics, engineering, and other fields where differential equations are used to model complex systems. It provides a powerful tool for understanding the structure and behavior of solutions to differential equations, and for identifying new solutions and symmetries that were previously unknown.

In this portion, we will apply the Lie group analysis [4–7] to Equation (1) for possible symmetry reduction. First, we express $q(x,t)$ as follows:

$$q(x,t) = u(x,t) + iv(x,t). \tag{2}$$

This turns Equation (1) into the following real and imaginary equations as:

$$\begin{aligned}&-v_t - a_1 v_x + a_2 u_{xx} - a_3 v_{xxx} + a_4 u_{xxxx} - a_5 v_{xxxxx} + a_6 u_{xxxxxx} - \lambda\big[(2uu_x + 2vv_x)v \\ &+ \big(u^2+v^2\big)v_x\big] - \mu(2uu_x + 2vv_x)v - \sigma\big(u^2+v^2\big)v_x = 0,\end{aligned} \tag{3}$$

and

$$\begin{aligned}&u_t + a_1 u_x + a_2 v_{xx} + a_3 u_{xxx} + a_4 v_{xxxx} + a_5 u_{xxxxx} + a_6 v_{xxxxxx} + \lambda\big[(2uu_x + 2vv_x)u \\ &+ \big(u^2+v^2\big)u_x\big] + \mu(2uu_x + 2vv_x)u + \sigma\big(u^2+v^2\big)u_x = 0.\end{aligned} \tag{4}$$

To derive the symmetries of system of Equations (3) and (4), let us introduce the Lie group of point transformations

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u,v) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u,v) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u,v) + O(\epsilon^2), \\ v^* &= v + \epsilon\phi(x,t,u,v) + O(\epsilon^2), \end{aligned} \tag{5}$$

where ϵ is a continuous parameter, while ξ, τ, η and ϕ represent the infinitesimal, which must be determined. For the group of transformations (5),

$$V = \xi\partial_x + \tau\partial_t + \eta\partial_u + \phi\partial_v, \tag{6}$$

is the corresponding vector field. Using the sixth prolongation formula [5,6] for system of Equations (3) and (4), we obtain the invariance conditions as

$$\begin{aligned}&a_6\eta^{xxxxxx} - a_5\phi^{xxxxx} + a_4\eta^{xxxx} - a_3\phi^{xxx} + a_2\eta^{xx} - \phi^x\big[a_1 + 2v^2(\lambda+\mu) + (u^2+v^2)(\lambda+\sigma)\big] \\ &-2uv(\lambda+\mu)\eta^x - \phi^t - 2\eta[vu_x(\lambda+\mu) + uv_x(\lambda+\sigma)] - 2\phi[(uu_x + 2vv_x)(\lambda+\mu) + vv_x(\lambda+\sigma)] = 0,\end{aligned} \tag{7}$$

and

$$\begin{aligned}&a_6\phi^{xxxxxx} + a_5\eta^{xxxxx} + a_4\phi^{xxxx} + a_3\eta^{xxx} + a_2\phi^{xx} + \eta^x\big[a_1 + 2u^2(\lambda+\mu) + (u^2+v^2)(\lambda+\sigma)\big] \\ &+2uv(\lambda+\mu)\phi^x + \eta^t + 2\phi[uv_x(\lambda+\mu) + vu_x(\lambda+\sigma)] + 2\eta[(2uu_x + vv_x)(\lambda+\mu) + uu_x(\lambda+\sigma)] = 0,\end{aligned} \tag{8}$$

where $\eta^t, \phi^t, \eta^x, \phi^x$ etc. are the extended infinitesimals. We obtain a system of partial differential equations by replacing the values of the extended infinitesimals and equating the coefficient of various derivative components to zero. The system's solution yields the symmetries as

$$\xi = C_1, \quad \tau = C_2, \quad \eta = C_3 v, \quad \phi = -C_3 u. \tag{9}$$

The corresponding infinitesimal generators are given by

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}. \tag{10}$$

By considering the vector field $V_1 + \nu V_2 + \kappa V_3$, the similarity variables are obtained as follows:

$$q(x,t) = F(\xi)e^{i(\kappa t + G(\xi))}, \quad \xi = x - \nu t, \tag{11}$$

where F and G are new dependent variables. Now substituting (11) into Equation (1) and assuming $G(\xi) = B\xi$, we obtain the reduced system of ODEs as

$$\begin{aligned} &a_6 F^{(6)} + \left(-15B^2 a_6 - 5Ba_5 + a_4\right)F^{(4)} + \left(15B^4 a_6 + 10B^3 a_5 - 6B^2 a_4 - 3Ba_3 + a_2\right)F'' \\ &-B(\sigma + \lambda)F^3 - \left(B^6 a_6 + B^5 a_5 - B^4 a_4 - B^3 a_3 + B^2 a_2 + (-\nu + a_1)B + \kappa\right)F = 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &(6Ba_6 + a_5)F^{(5)} + \left(-20B^3 a_6 - 10B^2 a_5 + 4Ba_4 + a_3\right)F''' + (2\mu + \sigma + 3\lambda)F^2 F' \\ &+\left(6B^5 a_6 + 5B^4 a_5 - 4B^3 a_4 - 3B^2 a_3 + 2Ba_2 - \nu + a_1\right)F' = 0. \end{aligned} \tag{13}$$

From Equation (13), one has the restrictions as

$$a_5 = -6Ba_6, \tag{14}$$

$$a_3 = -40B^3 a_6 - 4Ba_4, \tag{15}$$

$$\sigma = -2\mu - 3\lambda, \tag{16}$$

and the velocity ν of the soliton is

$$\nu = 96B^5 a_6 + 8B^3 a_4 + 2Ba_2 + a_1. \tag{17}$$

Now we can substitute the values of (14)–(17) into Equation (12) to obtain

$$a_6 F^{(6)} + \left(15B^2 a_6 + a_4\right)F^{(4)} + \left(75B^4 a_6 + 6B^2 a_4 + a_2\right)F'' + 2B(\mu + \lambda)F^3 + \left(61B^6 a_6 + 5a_4 B^4 + a_2 B^2 - \kappa\right)F = 0. \tag{18}$$

3. Integration Schemes and Optical Solitons

Optical solitons are self-sustaining waves that maintain their shape and intensity while propagating over long distances in certain nonlinear media. Unlike traditional waves, which spread out and become weaker as they propagate, solitons retain their shape due to a balance between the dispersive and nonlinear effects in the medium. The dispersive effect refers to the tendency of waves to spread out over time, caused by differences in the speed of propagation for different frequencies. In contrast, the nonlinear effect arises when the refractive index of the medium varies with the intensity of the light. This nonlinear refractive index causes a self-focusing effect, where regions of higher intensity attract more light, further increasing their intensity. Optical solitons can be formed in various types of nonlinear media, such as optical fibers or photonic crystals. They have a wide range of applications in telecommunications, where they can transmit signals over long distances without degradation, and in nonlinear optics, where they are used to manipulate light at the nanoscale [33–44].

One of the most famous examples of an optical soliton is the so-called “bright soliton”, which is a pulse of light that maintains its shape and travels without distortion. Another type of soliton is the “dark soliton”, which is a localized region of low intensity that propagates through a background of higher intensity. In nonlinear optics, a singular soliton is a special type of soliton that has a singularity at its center. It is a self-sustained wave packet of light that has a highly localized intensity profile, with an infinitely narrow peak at its center. The singularity is also known as an optical vortex or phase singularity, and it arises due to the nonlinear interactions between the light and the medium. Unlike regular solitons, which have a smooth profile and propagate without changing shape, singular solitons are characterized by their highly localized intensity and their ability to create and sustain a phase singularity. This singularity gives the wavefront of the light a twisted, helical shape, such as a corkscrew or a tornado. Singular solitons are also known as optical vortices, because of this twisting, helical shape. Singular solitons have a wide range of applications in nonlinear optics, including optical communications, optical trapping and manipulation, and quantum optics. They can be generated in a variety of ways, such as using specially designed optical fibers, waveguides, or photonic crystals, or using holographic techniques to imprint a phase singularity onto a laser beam. Singular solitons are also of fundamental interest in the study of nonlinear phenomena in optics and the understanding of the role of singularities in wave physics. A straddled soliton is a type of soliton that consists of two soliton pulses of opposite polarity, which are separated by a small region of zero amplitude. The two pulses are said to be “straddling” the zero-amplitude region, hence the name straddled soliton. In nonlinear optics, straddled solitons can arise in certain types of fiber optic systems, where the interplay between dispersion and nonlinearity can lead to the formation of two soliton pulses that attract each other and eventually merge into a single pulse. However, if the initial separation between the two pulses is small enough, they can remain separate and form a straddled soliton. Straddled solitons have some interesting properties that make them useful in various applications. For example, they can be used to generate ultrashort pulses of light with high peak power and high energy, which can be used for precision laser machining, material processing, and medical applications. They can also be used to create optical frequency combs, which are precise sources of evenly spaced frequencies that are used in frequency metrology and spectroscopy. Straddled solitons are an active area of research in nonlinear optics, as they are relatively new and not fully understood. Researchers are investigating ways to optimize their generation, control their properties, and use them in new and exciting applications.

3.1. Riccati Equation Method

In this subsection, first, we provide a brief overview of the Riccati equation method [8,9] and then present the solution of the model (1).

Consider the model equation

$$K(w, w_x, w_t, w_{tx}, w_{xx}, \dots) = 0, \tag{19}$$

where $w = w(x,t)$ is the dependent variable, while x and t are independent variables. The wave transformation

$$\xi = kx - vt, \quad w(x,t) = F(\xi), \tag{20}$$

reduces (19) to

$$P(F, -vF', kF', k^2F'', \dots) = 0, \tag{21}$$

where k and v are arbitrary constants. Consider (21) has a solution as

$$G(\xi) = A_0 + A_1R(\xi) + A_2R(\xi)^2 + \cdots + A_NR(\xi)^N, \tag{22}$$

where the unknown parameters A_i's, $(i = 0, 1, \ldots, N)$ are to be determined, while N emerges from the homogeneous balance principle [25,26], by balancing the highest order derivative and nonlinear terms appearing in ODE (21). More precisely, we define the degree of $F(\xi)$ as $D[F(\xi)] = N$, which gives way to the degree of other expressions as follows

$$D\left[\frac{d^pF}{d\xi^p}\right] = N + p, \qquad D\left[F^q\left(\frac{d^pF}{d\xi^p}\right)^s\right] = qN + s(N+p). \tag{23}$$

Hence, we can derive the value of N. In (22), $R(\xi)$ satisfies the Riccati equation

$$R'(\xi) = B_0 + B_1R(\xi) + B_2R(\xi)^2, \tag{24}$$

where B_0, B_1 and B_2 are constants. Equation (24) generates the hyperbolic, trigonometric, and rational solutions, which are given as

$$\begin{aligned}
R(\xi) &= -\frac{B_1}{2B_2} - \frac{\sqrt{\delta}}{2B_2}\tanh\left(\frac{\sqrt{\delta}}{2}\xi + \xi_0\right), \quad \delta > 0,\\
R(\xi) &= -\frac{B_1}{2B_2} - \frac{\sqrt{\delta}}{2B_2}\coth\left(\frac{\sqrt{\delta}}{2}\xi + \xi_0\right), \quad \delta > 0,\\
R(\xi) &= -\frac{B_1}{2B_2} + \frac{\sqrt{-\delta}}{2B_2}\tan\left(\frac{\sqrt{-\delta}}{2}\xi + \xi_0\right), \quad \delta < 0,\\
R(\xi) &= -\frac{B_1}{2B_2} - \frac{\sqrt{-\delta}}{2B_2}\cot\left(\frac{\sqrt{-\delta}}{2}\xi + \xi_0\right), \quad \delta < 0,\\
R(s) &= -\frac{B_1}{2B_2} - \frac{1}{B_2s + s_0}, \quad \delta = 0,
\end{aligned} \tag{25}$$

where $\delta = B_1^2 - 4B_0B_2$. By plugging (22) along with (24) into (21), and extracting a system of algebraic equations, we can find the unknown constants k, v, A_i's, $(i = 0, 1, \ldots, N)$.

Now, we will perform the Riccati equation method to solve Equation (18). According to (22) and using the balance principle, the solution of (18) can be taken as

$$F(\xi) = A_0 + A_1R(\xi) + A_2R(\xi)^2 + A_3R(\xi)^3, \tag{26}$$

where the unknown parameters A_i's, $(i = 0, 1, 2, 3)$ are to be determined. Substituting (26) along with (24) into (18) and after equating the coefficients of the same power of R to zero, we have a system of algebraic equations, which yields the following results:

Result 1:

$$\begin{aligned}
A_0 &= \frac{A_2(6B_0B_2 - B_1^2)}{6B_2^2}, \quad A_1 = \frac{2A_2B_0}{B_1}, \quad A_3 = \frac{2A_2B_2}{3B_1}, \quad \lambda = -\frac{22680B_1^2B_2^4a_6 + B\mu A_2^2}{BA_2^2},\\
a_4 &= -a_6\left(15B^2 - 332B_0B_2 + 83B_1^2\right),\\
a_2 &= a_6\left(15B^4 - 1992B^2B_0B_2 + 498B^2B_1^2 + 15136B_0^2B_2^2 - 7568B_0B_1^2B_2 + 946B_1^4\right),\\
\kappa &= a_6\Big(B^6 - 332B^4B_0B_2 + 83B^4B_1^2 + 15136B^2B_0^2B_2^2 - 7568B^2B_0B_1^2B_2 + 946B^2B_1^4\\
&\quad + 80640B_0^3B_2^3 - 60480B_0^2B_1^2B_2^2 + 15120B_0B_1^4B_2 - 1260B_1^6\Big).
\end{aligned} \tag{27}$$

Substituting (27) along with (26) into (11), we arrive at soliton wave profiles:

Dark and singular solitons are, respectively, indicated below

$$q(x,t) = \left\{\frac{A_2(\delta)^{3/2}}{12B_1B_2^2}\tanh\left(\frac{\sqrt{\delta}}{2}(x - vt) + \xi_0\right)\left(3 - \tanh\left(\frac{\sqrt{\delta}}{2}(x - vt) + \xi_0\right)^2\right)\right\}e^{i(\kappa t + B(x - vt))}, \tag{28}$$

and

$$q(x,t) = \left\{ \frac{A_2(\delta)^{3/2}}{12B_1B_2^2} \coth\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right)\left(3- \coth\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right)^2\right)\right\} e^{i(\kappa t+B(x-\nu t))}, \tag{29}$$

where $\delta = B_1^2 - 4B_0B_2 > 0$.

The rational wave is also introduced below

$$\begin{aligned} q(x,t) = &\left\{ \frac{A_2B_0}{3B_2} + \frac{2A_2B_0}{B_1}\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right) + A_2\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right)^2 \right. \\ &\left. + \frac{2A_2B_2}{3B_1}\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right)^3 \right\} e^{i(\kappa t+B(x-\nu t))}, \end{aligned} \tag{30}$$

where $B_1^2 - 4B_0B_2 = 0$.

Result 2:

$$\begin{aligned} &A_0 = \frac{A_2B_0}{3B_2}, \quad A_1 = \frac{A_2(2B_0B_2 + B_1^2)}{3B_1B_2}, \quad A_3 = \frac{2A_2B_2}{3B_1}, \quad \lambda = -\frac{22680B_1^2B_2^4a_6 + B\mu A_2^2}{BA_2^2}, \\ &a_4 = -15B^2a_6, \quad a_2 = 3a_6\left(5B^4 - 112B_0^2B_2^2 + 56B_0B_1^2B_2 - 7B_1^4\right), \\ &\kappa = a_6\Big(B^6 - 336B^2B_0^2B_2^2 + 168B^2B_0B_1^2B_2 - 21B^2B_1^4 + 1280B_0^3B_2^3 - 960B_0^2B_1^2B_2^2 \\ &\quad +240B_0B_1^4B_2 - 20B_1^6\Big). \end{aligned} \tag{31}$$

Inserting (31) along with (26) into (11), we arrive at optoelectronic wave fields: Dark-bright soliton and singular soliton are, respectively, extracted as

$$q(x,t) = -\left\{ \frac{A_2(\delta)^{3/2}}{12B_1B_2^2} \tanh\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right) \operatorname{sech}^2\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right)\right\} e^{i(\kappa t+B(x-\nu t))}, \tag{32}$$

and

$$q(x,t) = \left\{ \frac{A_2(\delta)^{3/2}}{12B_1B_2^2} \coth\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right) \operatorname{csch}^2\left(\frac{\sqrt{\delta}}{2}(x-\nu t)+\xi_0\right)\right\} e^{i(\kappa t+B(x-\nu t))}, \tag{33}$$

where $\delta = B_1^2 - 4B_0B_2 > 0$.

Rational wave is also presented as

$$\begin{aligned} q(x,t) = &\left\{ \frac{A_2B_0}{3B_2} + \frac{A_2(2B_0B_2 + B_1^2)}{3B_1B_2}\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right) + A_2\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right)^2 \right. \\ &\left. + \frac{2A_2B_2}{3B_1}\left(-\frac{B_1}{2B_2} - \frac{1}{B_2(x-\nu t)+\xi_0}\right)^3 \right\} e^{i(\kappa t+B(x-\nu t))}, \end{aligned} \tag{34}$$

where $B_1^2 - 4B_0B_2 = 0$.

3.2. Improved Modified Extended Tanh-Function Method

In this subsection, first, we provide a brief overview of the improved modified extended tanh-function method [10,11] and then present the solution of the model (1).

Consider (21) has a solution as

$$G(\xi) = \sum_{i=0}^{N} A_N R(\xi)^N + \sum_{j=1}^{N} B_N R(\xi)^{-N}, \tag{35}$$

where the unknown parameters A_i's, $(i = 0, 1, \ldots, N)$ and B_j's, $(j = 1, 2, \ldots, N)$ are to be determined, while N emerges from the balancing algorithm. Here $R(\xi)$ satisfies the equation

$$R'(\xi) = \varepsilon\sqrt{c_0 + c_1R(\xi) + c_2R(\xi)^2 + c_3R(\xi)^3 + c_4R(\xi)^4}, \tag{36}$$

where $\varepsilon = \pm 1$ and c_i's, $(i = 0, 1, \ldots, 4)$ are constants. Equation (36) generates a variety of essential solutions [10]. Plug (35) along with (36) into (21), providing a system of algebraic equations that gives us a way to the unknown constants k, v, A_i's, $(i = 0, 1, \ldots, N), B_j$'s, $(j = 1, 2, \ldots, N)$.

Now, we will perform the improved modified extended tanh-function method to solve Equation (18). According to (35) and using the balance principle, the solution of (18) can be taken as

$$F(\xi) = A_0 + A_1R(\xi) + A_2R(\xi)^2 + A_3R(\xi)^3 + B_1R(\xi)^{-1} + B_2R(\xi)^{-2} + B_3R(\xi)^{-3}, \tag{37}$$

where the unknown parameters A_i's, $(i = 0, 1, 2, 3)$ and B_j's, $(j = 1, 2, 3)$ are to be determined, and $R(\xi)$ holds (36). By putting $F(\xi)$ and its derivatives together with Equation (36) into (18), and setting all the coefficients of $R(\xi)^k, k = \{-9, 9\}$ to equal zero, we obtain a system of equations. After solving these equations for the unknown parameters, we have the following cases:

Case 1: $c_0 = c_1 = c_3 = 0,\ c_2 > 0,\ c_4 < 0.$

Result 1:

$$\begin{aligned} &A_0 = A_1 = A_2 = B_1 = B_2 = B_3 = 0, \quad A_3 = 12c_4\sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}}, \\ &a_2 = a_6\left(15B^4 + 498B^2c_2 + 1891c_2^2\right), \quad a_4 = -15B^2a_6 - 83c_2a_6, \\ &\kappa = a_6\left(B^6 + 83B^4c_2 + 1891B^2c_2^2 + 11025c_2^3\right). \end{aligned} \tag{38}$$

Thus, a bright soliton is structured as

$$q(x,t) = \left\{12c_4m\left(\sqrt{-\frac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}(x - \nu t)\right)\right)^3\right\}e^{i(\kappa t + B(x - \nu t))}, \tag{39}$$

where $m = \sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}}$.

Result 2:

$$\begin{aligned} &A_0 = A_2 = B_1 = B_2 = B_3 = 0, \quad A_1 = \frac{144c_2}{17}\sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}}, \quad A_3 = 12c_4\sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}}, \\ &a_2 = \frac{a_6\left(4335B^4 - 59262B^2c_2 + 92659c_2^2\right)}{289}, \quad a_4 = -15B^2a_6 + \frac{581}{17}c_2a_6, \\ &\kappa = \frac{a_6\left(289B^6 - 9877B^4c_2 + 92659B^2c_2^2 + 102825c_2^3\right)}{289}. \end{aligned} \tag{40}$$

In this case, a bright soliton is extracted as

$$q(x,t) = \left\{\frac{144c_2}{17}m\left(\sqrt{-\frac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}(x - \nu t)\right)\right) + 12c_4m\left(\sqrt{-\frac{c_2}{c_4}}\operatorname{sech}\left(\sqrt{c_2}(x - \nu t)\right)\right)^3\right\}e^{i(\kappa t + B(x - \nu t))}, \tag{41}$$

where $m = \sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}}$.

Case 2: $c_0 = c_1 = c_2 = c_3 = 0,\ c_4 > 0.$
Result 1:

$$A_0 = A_1 = A_2 = B_1 = B_2 = B_3 = 0, \quad A_3 = 12c_4\sqrt{-\frac{70c_4a_6}{B\lambda + B\mu}},$$
$$a_2 = 15B^4a_6, \quad a_4 = -15B^2a_6, \quad \kappa = B^6a_6. \tag{42}$$

Therefore, the rational wave is arranged as

$$q(x,t) = -\left\{\frac{\varepsilon A_3}{(\sqrt{c_4}(x-\nu t))^3}\right\}e^{i(\kappa t + B(x-\nu t))}, \tag{43}$$

where $m = \sqrt{-\frac{70c_4a_6}{B\lambda+B\mu}}$.
Case 3: $c_1 = c_3 = 0, c_0 = \frac{c_2^2}{c_4},\ c_2 < 0,\ c_4 > 0.$
Result 1:

$$A_0 = A_1 = A_2 = A_3 = B_2 = 0, \quad a_2 = a_6\left(15B^4 - 996B^2c_2 + 3784c_2^2\right),$$
$$a_4 = -a_6\left(15B^2 - 166c_2\right), \quad \kappa = a_6\left(B^6 - 166B^4c_2 + 3784B^2c_2^2 + 10080c_2^3\right), \tag{44}$$
$$B_1 = \frac{6B_3c_4}{c_2}, \quad \lambda = -\frac{2B\mu B_3^2c_4^3 + 315a_6c_2^6}{2B_3^2Bc_4^3}.$$

Hence, a singular optoelectronic wave field is cast as

$$q(x,t) = \left\{\frac{6B_3c_4}{c_2}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-1} + B_3\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-3}\right\}e^{i(\kappa t + B(x-\nu t))}. \tag{45}$$

Result 2:

$$A_0 = A_2 = B_1 = B_2 = B_3 = 0, \quad a_2 = a_6\left(15B^4 - 996B^2c_2 + 3784c_2^2\right),$$
$$a_4 = -a_6\left(15B^2 - 166c_2\right), \quad \kappa = a_6\left(B^6 - 166B^4c_2 + 3784B^2c_2^2 + 10080c_2^3\right), \tag{46}$$
$$A_1 = \frac{3A_3c_2}{2c_4}, \quad \lambda = -\frac{B\mu A_3^2 + 10080a_6c_4^3}{BA_3^2}.$$

Consequently, a dark nonlinear waveform is recovered as

$$q(x,t) = \left\{\frac{3A_3c_2}{2c_4}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right) + A_3\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^3\right\}e^{i(\kappa t + B(x-\nu t))}. \tag{47}$$

Result 3:

$$A_0 = A_2 = B_2 = 0, \quad A_1 = \frac{2B_1c_4}{c_2}, \quad A_3 = \frac{68B_1c_4^2}{3c_2^2}, \quad B_3 = \frac{17B_1c_2}{6c_4},$$
$$a_2 = \frac{a_6\left(4335B^4 + 118524B^2c_2 + 370636c_2^2\right)}{289}, \quad a_4 = -\frac{a_6\left(255B^2 + 1162c_2\right)}{17}, \tag{48}$$
$$\kappa = \frac{a_6\left(289B^6 + 19754B^4c_2 + 370636B^2c_2^2 - 822600c_2^3\right)}{289}, \quad \lambda = -\frac{289B\mu B_1^2c_4 + 5670a_6c_2^4}{289c_4BB_1^2}.$$

As a result, a dark-singular straddled soliton is introduced below

$$\begin{aligned} q(x,t) =& \left\{ \frac{2B_1c_4}{c_2}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right) + \frac{68B_1c_4^2}{3c_2^2}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^3 \right. \\ & \left. +B_1\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-1} + \frac{17B_1c_2}{6c_4}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-3} \right\} \\ & \times \mathrm{e}^{i(\kappa t+B(x-\nu t))}. \end{aligned} \tag{49}$$

Result 4:

$$\begin{aligned} & A_0 = A_2 = B_2 = 0, \quad A_1 = -\frac{36B_3c_4^2}{c_2^2}, \quad A_3 = -\frac{8B_3c_4^3}{c_2^3}, \quad B_1 = \frac{18B_3c_4}{c_2}, \\ & a_2 = a_6\left(15B^4 - 3984B^2c_2 + 60544c_2^2\right), \quad a_4 = -a_6\left(15B^2 - 664c_2\right), \\ & \kappa = a_6\left(B^6 - 664B^4c_2 + 60544B^2c_2^2 + 645120c_2^3\right), \quad \lambda = -\frac{2B\mu B_3^2c_4^3 + 315a_6c_2^6}{2B_3^2Bc_4^3}. \end{aligned} \tag{50}$$

Thus, a dark-singular straddled optoelectronic wave field is structured as

$$\begin{aligned} q(x,t) =& \left\{ -\frac{36B_3c_4^2}{c_2^2}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right) - \frac{8B_3c_4^3}{c_2^3}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^3 \right. \\ & \left. +\frac{18B_3c_4}{c_2}\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-1} + B_3\left(\varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}(x-\nu t)\right)\right)^{-3} \right\} \\ & \times \mathrm{e}^{i(\kappa t+B(x-\nu t))}. \end{aligned} \tag{51}$$

Case 4: $c_0 = c_1 = c_4 = 0,\ c_2 > 0.$
Result 1:

$$\begin{aligned} & A_1 = A_2 = A_3 = B_1 = B_2 = 0, \quad a_2 = 15B^4a_6 + 84B^2c_2a_6 + 49c_2^2a_6, \quad B_3 = \frac{16A_0c_2^3}{5c_3^3}, \\ & a_4 = -15B^2a_6 - 14a_6c_2, \quad \lambda = -\mu, \quad \kappa = B^6a_6 + 14B^4a_6c_2 + 49B^2c_2^2a_6 + 36c_2^3a_6. \end{aligned} \tag{52}$$

In this case, the singular optoelectronic wave field is arranged as

$$q(x,t) = \left\{ A_0 - \frac{16A_0}{5}\,\mathrm{sech}^{-6}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right)\right\}\mathrm{e}^{i(\kappa t+B(x-\nu t))}. \tag{53}$$

Result 2:

$$\begin{aligned} & A_1 = A_2 = A_3 = B_2 = B_3 = 0, \quad \lambda = -\mu, \quad B_1 = \frac{2A_0c_2}{c_3}, \\ & \kappa = 61B^6a_6 + 75B^4c_2a_6 + 5a_4B^4 + 15B^2c_2^2a_6 + 6B^2a_4c_2 + c_2^3a_6 + a_2B^2 + a_4c_2^2 + a_2c_2. \end{aligned} \tag{54}$$

Therefore, the singular nonlinear waveform is presented below

$$q(x,t) = \left\{ A_0 - 2A_0\,\mathrm{sech}^{-2}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right)\right\}\mathrm{e}^{i(\kappa t+B(x-\nu t))}. \tag{55}$$

Result 3:

$$\begin{aligned} &A_1 = A_2 = A_3 = B_3 = 0, \quad \lambda = -\mu, \quad B_1 = \frac{8A_0c_2}{c_3}, \quad B_2 = \frac{8A_0c_2^2}{c_3^2}, \\ &\kappa = \left(61B^4a_6 + 56c_2B^2a_6 + 5B^2a_4 + 16a_6c_2^2 + 4c_2a_4 + a_2\right)\left(B^2 + 4c_2\right). \end{aligned} \tag{56}$$

Hence, the singular nonlinear wave profile is recovered as

$$q(x,t) = \left\{A_0 - 8A_0\,\mathrm{sech}^{-2}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right) + 8A_0\,\mathrm{sech}^{-4}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right)\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{57}$$

Result 4:

$$\begin{aligned} &A_1 = A_2 = A_3 = 0, \quad B_1 = \frac{18A_0c_2}{c_3}, \quad B_2 = \frac{48A_0c_2^2}{c_3^2}, \quad B_3 = \frac{32A_0c_2^3}{c_3^3}, \\ &\lambda = -\mu, \quad \kappa = \left(61B^4a_6 + 126c_2B^2a_6 + 5B^2a_4 + 81a_6c_2^2 + 9c_2a_4 + a_2\right)\left(B^2 + 9c_2\right). \end{aligned} \tag{58}$$

Consequently, a singular nonlinear soliton is presented below

$$q(x,t) = \left\{A_0 - 18A_0\,\mathrm{sech}^{-2}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right) + 48A_0\,\mathrm{sech}^{-4}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right) - 32A_0\,\mathrm{sech}^{-6}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right)\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{59}$$

Result 5:

$$\begin{aligned} &A_1 = A_2 = A_3 = B_1 = 0, \quad a_2 = a_6\left(15B^4 + 84B^2c_2 + 49c_2^2\right), \quad B_3 = \frac{2c_2\left(8A_0c_2^2 + 3B_2c_3^2\right)}{5c_3^3}, \\ &a_4 = -15B^2a_6 - 14a_6c_2, \quad \lambda = -\mu, \quad \kappa = a_6\left(B^2 + 9c_2\right)\left(B^2 + 4c_2\right)\left(B^2 + c_2\right). \end{aligned} \tag{60}$$

As a result, a singular nonlinear soliton is constructed as below

$$q(x,t) = \left\{A_0 + \frac{B_2c_2^2}{c_3^2}\,\mathrm{sech}^{-4}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right) - \frac{2\left(8A_0c_2^2 + 3B_2c_3^2\right)}{5c_2^2}\,\mathrm{sech}^{-6}\left(\frac{\sqrt{c_2}}{2}(x-\nu t)\right)\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{61}$$

Case 5: $c_0 = c_1 = c_2 = c_4 = 0$.
Result 1:

$$A_1 = A_2 = A_3 = B_2 = B_3 = 0, \quad a_2 = \frac{9B^6a_6 + 6\kappa}{B^2}, \quad a_4 = -\frac{14B^6a_6 + \kappa}{B^4}, \quad \lambda = -\mu. \tag{62}$$

Thus, the rational wave is cast as

$$q(x,t) = \left\{A_0 + \frac{4B_1}{c_3(x-\nu t)^2}\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{63}$$

Case 6: $c_0 = c_1 = c_2 = 0,\ c_4 < 0$.

$$A_1 = A_2 = A_3 = B_2 = B_3 = 0, \quad a_4 = -\frac{75B^4a_6 + a_2}{6B^2}, \quad \lambda = -\mu, \quad \kappa = -\frac{B^2\left(9B^4a_6 - a_2\right)}{6}. \tag{64}$$

In this case, solitary and rational waves are, respectively, extracted as

$$q(x,t) = \left\{ A_0 + \frac{c_3 B_1}{2c_4\, \mathrm{e}^{\left(\frac{\varepsilon c_3}{2\sqrt{-c_4}}(x-\nu t)\right)}} \right\} \mathrm{e}^{i(\kappa t + B(x-\nu t))}, \tag{65}$$

and

$$q(x,t) = \left\{ A_0 + \frac{4c_3 B_1}{c_3^2 (x-\nu t)^2 - 4c_4} \right\} \mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{66}$$

Case 7: $c_3 = c_4 = 0, c_0 = \frac{c_1^2}{4c_2},\ c_2 > 0.$
Result 1:

$$\begin{aligned} &A_1 = A_2 = A_3 = B_1 = 0, \quad a_2 = a_6\left(15B^4 + 498B^2 c_2 + 946c_2^2\right), \quad a_4 = -a_6\left(15B^2 + 83c_2\right), \\ &A_0 = -6c_2^2\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \quad B_2 = 9c_1^2\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \quad B_3 = \frac{3c_1^3}{c_2}\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \\ &\kappa = a_6\left(B^6 + 83B^4 c_2 + 946B^2 c_2^2 - 1260c_2^3\right). \end{aligned} \tag{67}$$

Therefore, a solitary wave is indicated below

$$q(x,t) = \left\{ -6c_2^2 m + 9c_1^2 m\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-2} + \frac{3c_1^3 m}{c_2}\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-3} \right\} \mathrm{e}^{i(\kappa t + B(x-\nu t))}, \tag{68}$$

where $m = \sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}$.
Result 2:

$$\begin{aligned} &A_0 = A_1 = A_2 = A_3 = 0, \quad a_2 = 3a_6\left(5B^4 - 7c_2^2\right), \quad a_4 = 3a_6\left(5B^4 - 7c_2^2\right), \\ &B_1 = 6c_1 c_2\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \quad B_2 = 9c_1^2\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \quad B_3 = \frac{3c_1^3}{c_2}\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}, \\ &\kappa = a_6\left(B^6 - 21B^2 c_2^2 - 20c_2^3\right). \end{aligned} \tag{69}$$

Hence, a solitary wave is indicated below

$$\begin{aligned} q(x,t) = &\sqrt{-\frac{35a_6}{2Bc_2(\mu+\lambda)}}\left\{ 6c_1 c_2\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-1} + 9c_1^2\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-2} \right. \\ &\left. + \frac{3c_1^3}{c_2}\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-3} \right\} \mathrm{e}^{i(\kappa t + B(x-\nu t))}. \end{aligned} \tag{70}$$

Result 3:

$$A_1 = A_2 = A_3 = 0, \quad A_0 = \frac{879\sqrt{70}\,c_2}{2686B(\mu+\lambda)}\left((\mu+\lambda)Bc_2m_1 + \frac{3m_2}{293}\right),$$

$$B_1 = \frac{8937\sqrt{70}\,c_1}{2686B(\mu+\lambda)}\left((\mu+\lambda)Bc_2m_1 + \frac{m_2}{993}\right), \quad B_2 = \frac{9\sqrt{70}\,m_1c_1^2}{2}, \quad B_3 = \frac{3\sqrt{70}\,m_1c_1^3}{2c_2},$$

$$a_2 = \frac{747m_1m_2c_2}{1343}\left(B^2 - \frac{626891c_2}{668814}\right), \quad a_4 = -\frac{15a_6}{m_1B(\mu+\lambda)}\left(B(\mu+\lambda)\left(B^2 - \tfrac{24319c_2}{40290}\right)m_1 - \frac{83m_2}{13430}\right), \tag{71}$$

$$\kappa = \frac{(1107540m_1m_2 - 40482540a_6)c_2^3}{3607298} + \frac{B^2(-1880673m_1m_2 + 44697365a_6)c_2^2}{3607298}$$

$$- \frac{24319B^4c_2}{2686}\left(-\frac{3m_1m_2}{293} + a_6\right) + B^6a_6,$$

where $m_1 = \sqrt{-\frac{a_6}{Bc_2(\mu+\lambda)}}$ and $m_2 = \sqrt{2399}\,\sqrt{(\mu+\lambda)Ba_6c_2}$. This result yields the solitary wave

$$\begin{aligned} q(x,t) = &\left\{ A_0 + B_1\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-1} + B_2\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-2} \right.\\ &\left. + B_3\left(-\frac{c_1}{2c_2} + \mathrm{e}^{\left(\varepsilon\sqrt{c_2}(x-\nu t)\right)}\right)^{-3}\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \end{aligned} \tag{72}$$

Case 8: $c_1 = c_3 = c_4 = 0,\ c_0 > 0,\ c_2 > 0.$

Result 1:

$$\begin{aligned} &A_0 = A_1 = A_2 = A_3 = B_1 = B_2 = 0, \quad a_2 = a_6\left(15B^4 + 498B^2c_2 + 1891c_2^2\right),\\ &B_3 = 12c_0\sqrt{-\frac{70a_6c_0}{B\lambda + B\mu}}, \quad a_4 = -15B^2a_6 - 83c_2a_6, \quad \kappa = a_6\left(B^6 + 83B^4c_2 + 1891B^2c_2^2 + 11025c_2^3\right). \end{aligned} \tag{73}$$

This result provides the singular soliton

$$q(x,t) = \left\{12c_0\sqrt{-\frac{70a_6c_0}{B\lambda + B\mu}}\left(\varepsilon\sqrt{\frac{c_0}{c_2}}\sinh\left(\sqrt{-c_2}(x-\nu t)\right)\right)^{-3}\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{74}$$

Result 2:

$$\begin{aligned} &A_0 = A_1 = A_2 = A_3 = B_2 = 0, \quad B_1 = \frac{144c_2}{17}\sqrt{-\frac{70a_6c_0}{B\lambda + B\mu}}, \quad B_3 = 12\sqrt{-\frac{70a_6c_0}{B\lambda + B\mu}},\\ &a_2 = \frac{a_6\left(4335B^4 - 59262B^2c_2 + 92659c_2^2\right)}{289}, \quad a_4 = -15B^2a_6 + \frac{581}{17}c_2a_6,\\ &\kappa = \frac{a_6\left(289B^6 - 9877B^4c_2 + 92659B^2c_2^2 + 102825c_2^3\right)}{289}. \end{aligned} \tag{75}$$

Thus, singular soliton appears as

$$q(x,t) = \sqrt{-\frac{70a_6c_0}{B\lambda + B\mu}}\left\{\frac{144c_2}{17}\left(\varepsilon\sqrt{\frac{c_0}{c_2}}\sinh\left(\sqrt{-c_2}(x-\nu t)\right)\right)^{-1} + 12\left(\varepsilon\sqrt{\frac{c_0}{c_2}}\sinh\left(\sqrt{-c_2}(x-\nu t)\right)\right)^{-3}\right\}\mathrm{e}^{i(\kappa t + B(x-\nu t))}. \tag{76}$$

Case 9: $c_0 = c_1 = 0,\ c_2 > 0,\ c_4 > 0.$

Result 1:

$$A_1 = A_2 = A_3 = B_2 = B_3 = 0, \quad A_0 = \frac{B_1 c_3}{2c_2},$$
$$\lambda = -\mu, \quad \kappa = \left(61B^4 a_6 + 14c_2 B^2 a_6 + 5B^2 a_4 + a_6 c_2^2 + c_2 a_4 + a_2\right)\left(B^2 + c_2\right). \tag{77}$$

In this case, dark–bright straddled soliton comes out as

$$q(x,t) = \left\{ \frac{B_1 c_3}{2c_2} + B_1 \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{c_2 \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right) \right\} e^{i(\kappa t + B(x - \nu t))}. \tag{78}$$

Result 2:

$$A_1 = A_2 = A_3 = B_3 = 0, \quad A_0 = \frac{B_1\left(4c_2 c_4 + c_3^2\right)}{8c_2 c_3}, \quad B_2 = \frac{B_1 c_2}{c_3},$$
$$\lambda = -\mu, \quad \kappa = \left(B^2 + 4c_2\right)\left(61B^4 a_6 + 56c_2 B^2 a_6 + 5B^2 a_4 + 16a_6 c_2^2 + 4c_2 a_4 + a_2\right). \tag{79}$$

Therefore, dark–bright straddled soliton turns out to be

$$\begin{aligned} q(x,t) = \Bigg\{ & \frac{B_1\left(4c_2 c_4 + c_3^2\right)}{8c_2 c_3} + B_1 \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{c_2 \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right) \\ & + \frac{B_1}{c_3} \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right)^2 \Bigg\} e^{i(\kappa t + B(x - \nu t))}. \end{aligned} \tag{80}$$

Result 3:

$$A_1 = A_2 = A_3 = 0, \quad A_0 = \frac{\left(12c_2 c_4 + c_3^2\right)B_2}{48c_2^2}, \quad B_1 = \frac{B_2\left(4c_2 c_4 + 3c_3^2\right)}{8c_2 c_3}, \quad B_3 = \frac{2B_2 c_2}{3c_3},$$
$$\lambda = -\mu, \quad \kappa = \left(61B^4 a_6 + 126c_2 B^2 a_6 + 5B^2 a_4 + 81a_6 c_2^2 + 9c_2 a_4 + a_2\right)\left(B^2 + 9c_2\right). \tag{81}$$

Hence, dark–bright straddled soliton shapes up as

$$\begin{aligned} q(x,t) = \Bigg\{ & \frac{\left(12c_2 c_4 + c_3^2\right)B_2}{48c_2^2} + \frac{B_2\left(4c_2 c_4 + 3c_3^2\right)}{8c_2 c_3} \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{c_2 \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right) \\ & + B_2 \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{c_2 \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right)^2 + \frac{2B_2 c_2}{3c_3} \left(\frac{2\varepsilon\sqrt{c_2 c_4}\tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right) - c_3}{c_2 \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)} \right)^3 \Bigg\} \\ & \times e^{i(\kappa t + B(x - \nu t))}. \end{aligned} \tag{82}$$

Case 10: $c_0 = c_1 = 0, c_3 = 2\varepsilon\sqrt{c_2 c_4},\ c_2 > 0,\ c_4 > 0.$
Result 1:

$$A_1 = B_1 = B_2 = B_3 = 0, \quad a_2 = 15B^4a_6 + 498B^2a_6c_2 + 946a_6c_2^2, \quad a_4 = -15B^2a_6 - 83a_6c_2,$$
$$A_0 = -\frac{3c_2\sqrt{c_2c_4}}{c_4}\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \quad A_2 = 18\sqrt{c_2c_4}\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \quad A_3 = 12c_4\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \tag{83}$$
$$\kappa = a_6\left(B^6 + 83B^4c_2 + 946B^2c_2^2 - 1260c_2^3\right).$$

As a result, dark soliton evolves as

$$\begin{aligned} q(x,t) = &\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}\left\{-\frac{3c_2\sqrt{c_2c_4}}{c_4} + 18\sqrt{c_2c_4}\left(\frac{\varepsilon}{2}\sqrt{\frac{c_2}{c_4}}\left(1 + \tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)\right)\right)^2\right. \\ &\left.+12c_4\left(\frac{\varepsilon}{2}\sqrt{\frac{c_2}{c_4}}\left(1 + \tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)\right)\right)^3\right\}e^{i(\kappa t + B(x - \nu t))}. \end{aligned} \tag{84}$$

Result 2:

$$A_0 = B_1 = B_2 = B_3 = 0, \quad a_2 = 3a_6\left(5B^4 - 7c_2^2\right), \quad a_4 = -15B^2a_6,$$
$$A_1 = 6c_2\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \quad A_2 = 18\sqrt{c_2c_4}\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \quad A_3 = 12c_4\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}, \tag{85}$$
$$\kappa = B^6a_6 - 21B^2a_6c_2^2 - 20a_6c_2^3.$$

Therefore, the dark soliton stands as

$$\begin{aligned} q(x,t) = &\sqrt{-\frac{70a_6c_4}{B\lambda + B\mu}}\left\{6c_2\left(\frac{\varepsilon}{2}\sqrt{\frac{c_2}{c_4}}\left(1 + \tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)\right)\right) + 18\sqrt{c_2c_4}\left(\frac{\varepsilon}{2}\sqrt{\frac{c_2}{c_4}}\left(1 + \tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)\right)\right)^2\right. \\ &\left.+12c_4\left(\frac{\varepsilon}{2}\sqrt{\frac{c_2}{c_4}}\left(1 + \tanh\left(\frac{\sqrt{c_2}}{2}(x - \nu t)\right)\right)\right)^3\right\}e^{i(\kappa t + B(x - \nu t))}. \end{aligned} \tag{86}$$

Remark 1. *We may also generate trigonometric solutions using both the Riccati equation approach and the improved modified extended tanh-function method. However, we did not include such solutions because they are less significant in optical fiber.*

4. Conservation Laws

Conservation laws are fundamental principles in physics that describe the behavior of physical systems under different circumstances. In nonlinear optics, several conservation laws play a crucial role in understanding the behavior of light in nonlinear media. Conservation of Energy: This law states that the total energy of a closed system remains constant over time. In nonlinear optics, this law is important because it ensures that the energy of the incident light is conserved as it interacts with the nonlinear medium. Nonlinear optical processes can change the frequency and intensity of the incident light, but the total energy must remain constant. Conservation of Momentum: This law states that the total momentum of a closed system remains constant over time. In nonlinear optics, this law is important because it governs the direction and magnitude of the changes in the polarization of light as it passes through a nonlinear medium. Changes in polarization can lead to changes in the direction and speed of light. Conservation of Angular Momentum: This law states that the total angular momentum of a closed system remains constant over time. In nonlinear optics, this law is important because it governs the behavior of light as it interacts with a nonlinear medium that possesses a certain symmetry. Changes in angular momentum can lead to changes in the direction and polarization of the light. Conservation of Phase Matching: This law states that for a nonlinear optical process to occur efficiently, the phase velocities of the interacting waves must match. This law is important because it determines the efficiency of nonlinear optical processes such as second harmonic generation and optical parametric amplification. Understanding these conservation laws is essential for understanding the behavior of light in nonlinear media and for designing nonlinear optical devices with specific functionalities.

The local conservation laws of the system (3) and (4) are derived by the multiplier approach [12–14].

Consider the multipliers of the form $\Lambda i(t, x, u, v)$, where $i = 1, 2$. The simplified determining equations for multipliers are

$$\begin{aligned}
&\frac{\partial}{\partial x}\Lambda 2(x,t,u,v) = 0, \frac{\partial}{\partial t}\Lambda 2(x,t,u,v) = 0,\\
&\frac{\partial}{\partial u}\Lambda 2(x,t,u,v) = \frac{\Lambda 2(x,t,u,v)}{u}, \frac{\partial}{\partial v}\Lambda 2(x,t,u,v) = 0,\\
&\Lambda 1(x,t,u,v) = -\frac{\Lambda 2(x,t,u,v)v}{u}.
\end{aligned} \tag{87}$$

The solution of the above system yields the following multiplier

$$\Lambda 1(x,t,u,v) = -C_1 v, \Lambda 2(x,t,u,v) = C_1 u, \tag{88}$$

where C_1 is arbitrary constant.

Using the direct method, the conservation fluxes [12] are achieved. Thus, the conserved vectors for the multipliers (88) are

$$\begin{aligned}
T^t &= \frac{u^2+v^2}{2}\\
T^x &= \frac{3}{4}\lambda\, u^4 + \frac{3}{2}\lambda\, v^2u^2 + \frac{1}{2}\mu\, u^4 + \mu\, v^2u^2 + \frac{1}{4}\sigma\, u^4 + \frac{1}{2}\sigma\, v^2u^2 + \frac{1}{2}a1\, u^2 + \frac{1}{4}\sigma\, v^4 + \frac{3}{4}\lambda\, v^4\\
&+ \frac{1}{2}\mu\, v^4 + \frac{1}{2}a_1\, v^2 + a_2\, uv_x - a_2\, u_x\, v, + a_3uu_{xx} + a_3vv_{xx} - a_4vu_{xxx} + a_4uv_{xxx}\\
&+ a_5uu_{xxxx} + a_5vv_{xxxx} - a_6vuxxxxx + a_6uv_{xxxxx}.
\end{aligned} \tag{89}$$

Therefore, we have

$$\Phi_1^t = \frac{1}{2}|q|^2. \tag{90}$$

Thus, from soliton solution (39), the corresponding conserved quantity is given by

$$P_1 = \int_{-\infty}^{\infty}\Phi_1^t dx = \int_{-\infty}^{\infty}|q|^2 dx = \frac{16A^2}{15B}, \tag{91}$$

where $A = -144\frac{c_2^3 m^2}{c_4}$ and $B = \sqrt{c_2}$, which represents the power P_1.

From soliton solution (41), the corresponding conserved quantity is given by

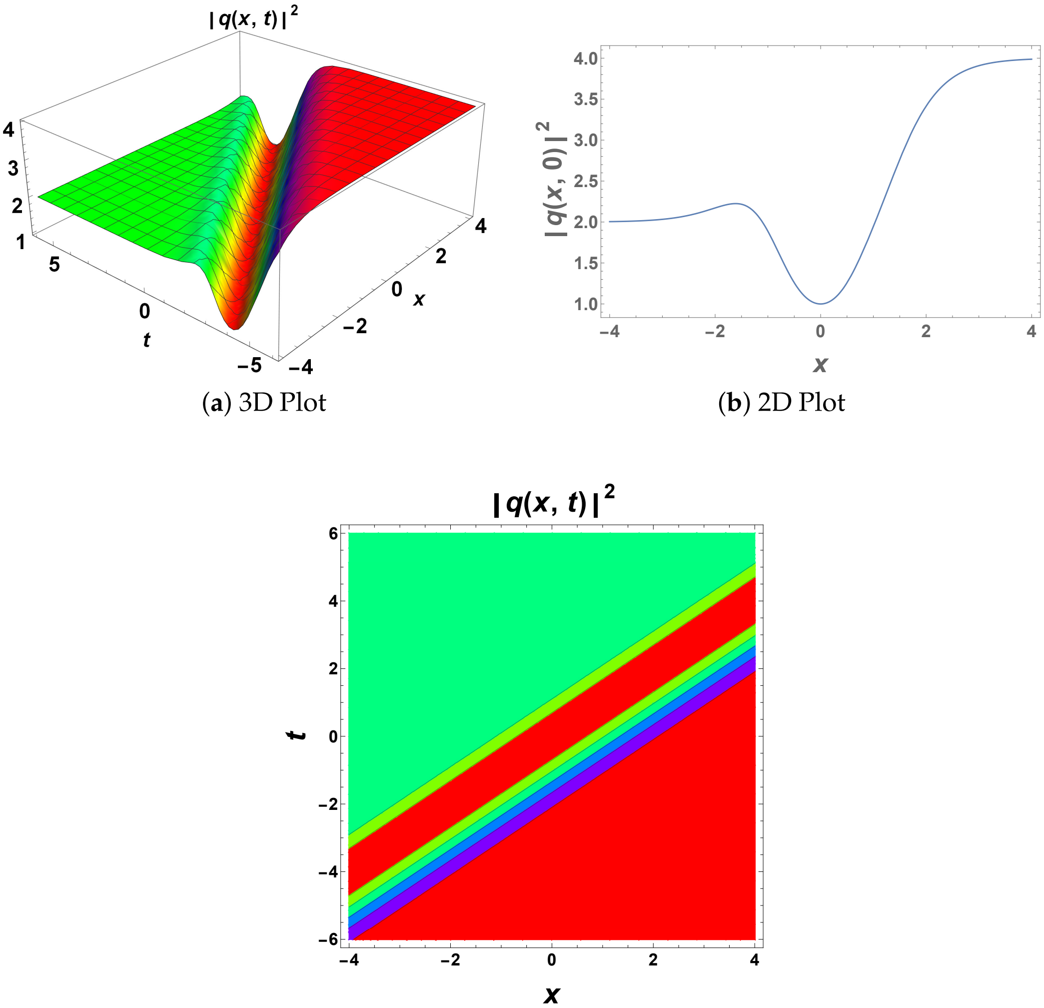
$$P_2 = \int_{-\infty}^{\infty}\Phi_1^t dx = \int_{-\infty}^{\infty}|q|^2 dx = \frac{2(15C^2 + 20CD + 8D^2)}{15E}, \tag{92}$$

where $C = \frac{144c_2 m}{17}\sqrt{-\frac{c_2}{c_4}}$, $D = 12c_4 m(-\frac{c_2}{c_4})^{\frac{3}{2}}$ and $E = \sqrt{c_2}$, which represents the power P_2.

Similarly, we can obtain the corresponding conserved quantities from other soliton solutions.

5. Physical Interpretation

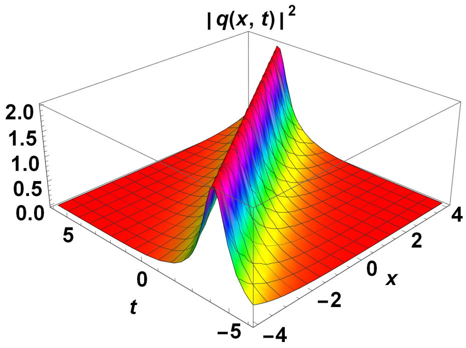
Here, we illustrate the graphical representation of our revealed optoelectronic wave field in relation to the suggested Equation (1), along with its corresponding physical significance. The interpretation of graphical plots is crucial for understanding the results in physics. Since the geometrical composition determines how investigative solutions behave, this section will depict various soliton solutions, including dark, bright, and dark-bright straddled solitons, using 3D, contour, and 2D plots. Each of the graphics in Figures 1–3 consists of three subfigures: (a) shows the 3D plot, while (b) and (c) show the 2D and contour plots, respectively. We begin by plotting the soliton solution $|q(x,t)|^2$, as defined by Equation (28), in various formats. These plots depict the dark optoelectronic wave field for the given parameters: $B_0 = -1$, $B_2 = 1$, $B_1 = 1$, $A_2 = 1$, $\xi_0 = 1$, $B = 1$, $a_6 = 1$, $a_4 = 1$, $a_2 = 1$, and $a_1 = 1$, over the ranges $x \in [-4, 4]$ and $t \in [-6, 6]$. Figure 1 shows these plots. The plot of the soliton solution $|q(x,t)|^2$, as given by Equation (41), represents the bright optoelectronic wave field over the range $x \in [-4, 4]$ and $t \in [-6, 6]$ for the parameters $c_2 = 1$, $c_4 = -1$, $a_6 = 1$, $B = 1$, $\lambda = 1$, $\mu = 1$, and $a_1 = 1$. This plot is shown in Figure 2. Figure 3 depicts the soliton solution $|q(x,t)|^2$ as characterized by Equation (82), which represents a dark-bright straddled soliton over the range $x \in [-4, 4]$ and $t \in [-6, 6]$ for the following parameters: $c_2 = 1$, $c_4 = 1$, $c_3 = 1$, $B_2 = 1$, $\varepsilon = 1$, $B = 1$, $a_6 = 1$, $a_4 = 1$, $a_2 = 1$, and $a_1 = 1$.



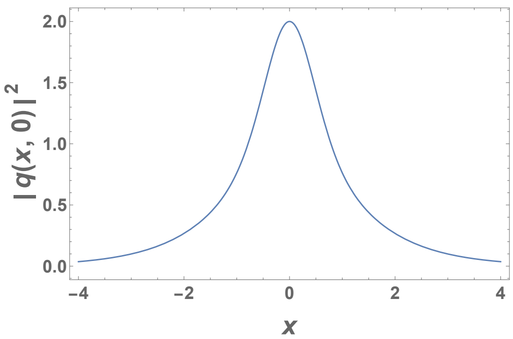
(**a**) 3D Plot (**b**) 2D Plot

(**c**) Contour Plot

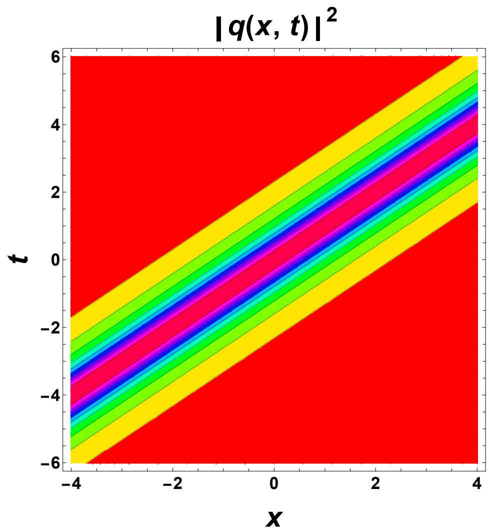
Figure 1. Profiles of a dark optoelectronic wave field (28).



(**a**) 3D Plot

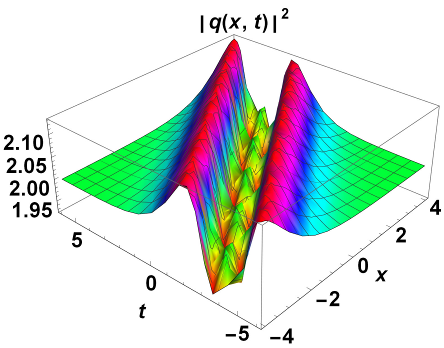


(**b**) 2D Plot

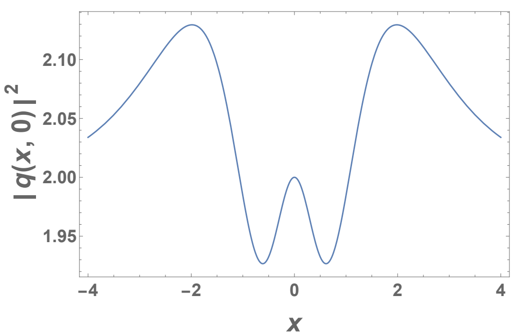


(**c**) Contour Plot

Figure 2. Profiles of a bright optoelectronic wave field (41).

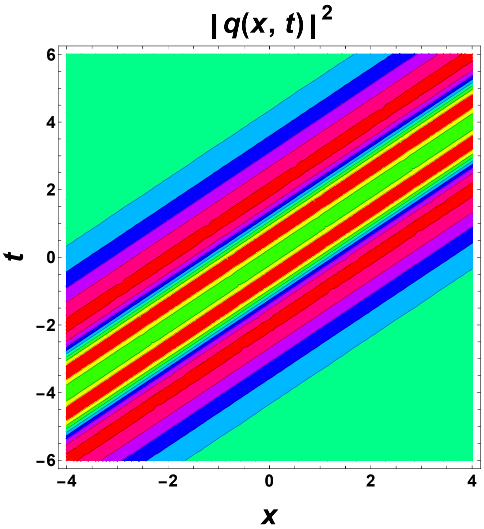


(**a**) 3D Plot



(**b**) 2D Plot

Figure 3. *Cont.*



(**c**) Contour Plot

Figure 3. Profiles of a dark–bright straddled optoelectronic wave field (82).

6. Conclusions

The paper addressed the recovery of optical soliton solutions to HD—NLSE, i.e., with the absence of SPM. The perturbation terms carry the effect of nonlinearity that provides the balance between CD and the effect of SPM and thus solitons sustain throughout. The soliton solutions are recovered and classified. The conservation laws are obtained and enlisted too.

The results of the paper provide a strong foundation for further exploration of the model. The model will be later explored with the nonlinear terms having maximum intensity. This would be a generalized version of the model whose integrability to obtain an exact soliton solution would pose a challenge. The consideration of the model in birefringent fibers and its extension to dispersion-flattened fibers would be the icing on the cake. Those results are currently awaited and will be reported in due time.

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