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On a Class of Analytic Functions Related to Robertson's Formula Involving Crescent Shaped Domain and Lemniscate of Bernoulli

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Abstract: In this paper, we introduce and study the class of analytic functions in the unit disc, which are derived from Robertson's analytic formula for starlike functions with respect to a boundary point combined with a subordination involving lemniscate of Bernoulli and crescent shaped domains. Using their symmetry property, the basic geometrical and analytical properties of the introduced classes were proved. Early coefficients and the Fekete–Szegö functional were estimated. Results for both classes were also obtained by applying the theory of differential subordinations.

Keywords: univalent function; starlike function of order α; starlike function with respect to a boundary point; lemniscate of Bernoulli

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1. Introduction

Let us denote by $\mathcal{H}$ the family of all holomorphic functions defined in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions h normalized by $h(0) = 0$ and $h'(0) = 1$, i.e., of having power series

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of univalent functions. A function $f \in \mathcal{H}$ is said to be subordinate to a function $g \in \mathcal{H}$ if there is a function $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f(z) = g(\omega(z))$ for every $z \in \mathbb{D}$. We write then $f \prec g$. If we assume that g is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let $\mathcal{P}$ stand for the subclass of $\mathcal{H}$ of all functions p normalized by $p(0) = 1$ and such that $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$, which is called as the Carathéodory class. Let $\mathcal{P}^*(1)$ be the subclass of $\mathcal{P}$ of all functions ϕ such that $\phi(0) = 1$, $\phi'(0) > 0$, ϕ is univalent in $\mathbb{D}$, and $\phi(\mathbb{D})$ is a set symmetric with respect to the real axis and starlike with respect to 1. Thus, every function $\phi \in \mathcal{P}^*(1)$ is of the form

$$\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \quad z \in \mathbb{D},$$

with $B_1 > 0$. The class $\mathcal{P}^*(1)$ is the basic general tool for defining classes of analytic functions, first proposed by Ma and Minda [1]. For example, given $\phi \in \mathcal{P}^*(1)$, let $\mathcal{S}^*(\phi)$ be the class of all $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \phi(z)$ for $z \in \mathbb{D}$. Classes defined in this way are called starlike of the Ma and Minda type.

Two widely used subclasses of $\mathcal{A}$, are the class of starlike and convex functions of order α, where $\alpha \in [0,1)$, introduced by Robertson [2] and given, respectively, by

$$\mathcal{S}^*(\alpha) := \left\{ h \in \mathcal{A} : \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > \alpha,\ z \in \mathbb{D} \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ h \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha,\ z \in \mathbb{D} \right\}.$$

It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}$ and $\mathcal{K}(\alpha) \subset \mathcal{S}$ for every $\alpha \in [0,1)$. It is clear that $\mathcal{S}^*(\alpha) = \mathcal{S}^*(\phi)$ with $\phi(z) := (1 + (1-2\alpha)z)/(1-z)$, $z \in \mathbb{D}$. When $\alpha = 0$ the classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the well-known classes of normalized starlike and convex univalent functions, respectively. Further ideas on the convexity of real functions and others related to them see, e.g., [3–5].

In this paper, we are interested in the following two functions:

$$\phi_L(z) := \sqrt{1+z}, \quad z \in \mathbb{D},\ \phi_L(0) := 1,$$

and

$$\phi_c(z) := z + \sqrt{1+z^2}, \quad z \in \mathbb{D},\ \phi_c(0) := 1.$$

Note that $\phi_L(\mathbb{D})$ is a domain bounded by Bernoulli lemniscate $(x^2+y^2)^2 - 2(x^2-y^2) = 0$, so it is a domain symmetric with respect to the real axis. This fact can be confirmed also by a simple observation that

$$\phi_L\left(\overline{\mathrm{e}^{\mathrm{i}\theta}}\right) = \phi_L\left(\mathrm{e}^{-\mathrm{i}\theta}\right) = \sqrt{1+\mathrm{e}^{-\mathrm{i}\theta}} = \overline{\phi_L\left(\mathrm{e}^{\mathrm{i}\theta}\right)}$$

for $\theta \in \mathbb{R}$. Clearly, $\operatorname{Re}\phi_L(z) > 0$ for $z \in \mathbb{D}$. Observe now that $\phi_L - 1$ is a starlike function since

$$\begin{aligned} \operatorname{Re}\frac{z(\phi_L(z)-1)'}{\phi_L(z)-1} &= \operatorname{Re}\frac{z}{2\sqrt{1+z}\left(\sqrt{1+z}-1\right)} = \operatorname{Re}\frac{\sqrt{1+z}+1}{2\sqrt{1+z}} \\ &= \frac{1}{2} + \frac{1}{2}\operatorname{Re}\sqrt{\frac{1}{1+z}} > 0, \quad z \in \mathbb{D}. \end{aligned}$$

Thus, $(\phi_L - 1)(\mathbb{D})$ is a starlike domain with respect to the origin, so $\phi_L(\mathbb{D})$ is a starlike domain with respect to 1. Consequently, $\phi_L \in \mathcal{P}^*(1)$. The class $\mathcal{S}^*(\phi_L)$ was introduced in [6] and further studied in [7–9].

On the other hand, $\phi_c(\mathbb{D})$ is a "crescent" domain bounded by two circular arcs $\Gamma_1 \subset \mathbb{T}(1,\sqrt{2})$, and $\Gamma_2 \subset \mathbb{T}(-1,\sqrt{2})$ with common end-points at i and $-$i, both arcs lying in the closed right half-plane. In addition, the arcs Γ_1 and Γ_2 intersect the real axis at $\sqrt{2}+1$ and $\sqrt{2}-1$, respectively, (for details see ([10] pp. 974–975) and ([11] pp. 356–357)). Here $\mathbb{T}(w_0,r_0) := \{w \in \mathbb{C} : |w-w_0| = r_0\}$ with $w_0 \in \mathbb{C}$ and $r_0 > 0$. Thus, $\phi_c(\mathbb{D})$ is a set symmetric with respect to the real axis, which can also be confirmed by observing that

$$\begin{aligned} \phi_c\left(\overline{\mathrm{e}^{\mathrm{i}\theta}}\right) &= \phi_c\left(\mathrm{e}^{-\mathrm{i}\theta}\right) = \mathrm{e}^{-\mathrm{i}\theta} + \sqrt{1+\mathrm{e}^{-2\mathrm{i}\theta}} = \mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta/2}\sqrt{2\cos\theta} \\ &= \overline{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta/2}\sqrt{2\cos\theta}} = \overline{\phi_c\left(\mathrm{e}^{\mathrm{i}\theta}\right)} \end{aligned}$$

for $\theta \in \mathbb{R}$. In ([11] p. 357) the authors mentioned that $\phi_c(\mathbb{D})$ is the set starlike with respect to 1, however without the proof. Now we complete it. Note that the arc Γ_2 can be parameterized as follows: $w = w(t) = -1 + \sqrt{2}\mathrm{e}^{\mathrm{i}t}$ for $t \in [-\pi/4, \pi/4]$. Thus, $w'(\pi/4) = -1 + \mathrm{i}$ is the vector tangent to Γ_2 at $w = \mathrm{i}$, and hence the tangent line l_1 to Γ_2 at $w = \mathrm{i}$ is given by the equation $l_1 : v = v(s) = \mathrm{i} + (-1+\mathrm{i})s$ for $s \in \mathbb{R}$. Since $v(-1) = 1$, it follows that the line segment $[1, \mathrm{i}]$ lies on the tangent line l_1. Similarly, the line segment $[1, -\mathrm{i}]$ lies on the tangent line l_2 to $\varGamma_2$ at $w = -\mathrm{i}$. Thus, the set bounded by the circular arc $\varGamma_2$ and the segments $[1, \mathrm{i}]$ and $[1, -\mathrm{i}]$ lies in $\phi_c(\mathbb{D})$, which allows us to conclude that the set $\phi_c(\mathbb{D})$ is starlike with respect to 1. The univalence of ϕ_c was shown in ([10] Theorem 2.1, p. 974). Consequently, $\phi_c \in \mathcal{P}^*(1)$. The corresponding class $\mathcal{S}^*(\phi_c)$ was introduced in [12] (for further results see [10,11]). The symmetry property of domains $\phi_L(\mathbb{D})$ and $\phi_c(\mathbb{D})$ is at the basis of the obtained results for both classes.

We say that $h \in \mathcal{H}$ is close-to-convex if and only if there is a function $\varPhi \in \mathcal{K}$ such that

$$\mathrm{Re}\left(\frac{h'(z)}{\varPhi'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

The class of close-to-convex functions was introduced by Kaplan [13].

At this point, it should be noted that the concept of starlikeness of a given order has been extensively studied by many authors, while less is known about the class of univalent functions g in $\mathcal{H}$ that map the unit disc $\mathbb{D}$ onto domains Ω starlike with respect to a boundary point. This important geometrical idea was introduced by Robertson [14], where he defined the subclass $\mathcal{G}^*$ of $\mathcal{H}$ of functions g such that $g(0) = 1$, $g(1) := \lim_{r \to 1^-} g(r) = 0$, g maps univalently $\mathbb{D}$ onto a domain starlike with respect to the origin and there exists $\delta \in \mathbb{R}$ such that $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\delta} g(z)) > 0$ for $z \in \mathbb{D}$. Let $I \equiv 1$ be the constant function. Robertson conjectured that the class $\mathcal{G}^* \cup \{I\}$ is identical with the class $\mathcal{G}$ of all $g \in \mathcal{H}$ of the form.

$$g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

such that

$$\mathrm{Re}\left(\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right) > 0, \quad z \in \mathbb{D}, \tag{2}$$

proving that $\mathcal{G} \subset \mathcal{G}^*$. Robertson's conjecture was shown by Lyzzaik ([15] p. 109) in 1984, who proved that $\mathcal{G}^* \subset \mathcal{G}$. In ([14] Theorem 3, p. 332), Robertson proved also that if $g \in \mathcal{G}$ and $g \neq I$, then g is close-to-convex and univalent in $\mathbb{D}$. It should be noted that the analytical condition (2) was known to Styer [16] much earlier.

In [17], Lecko proposed an alternative analytic characterization of starlike functions with respect to a boundary point and proved the necessity. The sufficiency was shown by Lecko and Lyzzaik [18] and in this way they confirmed this new analytic characterizations (see also [19] Chapter VII). Based on Robertson's idea, Aharanov et al. [20] introduced the class of spiral-like functions with respect to a boundary point (see also [21–23]).

A class closely related to $\mathcal{G}$ is the class $\mathcal{G}(M)$, $M > 1$, whose elements are functions $g \in \mathcal{H}$ of the form (1) such that

$$\mathrm{Re}\left(\frac{2zg'(z)}{g(z)} + \frac{z\mathrm{P}'(z;M)}{\mathrm{P}(z;M)}\right) > 0, \quad z \in \mathbb{D},$$

introduced by Jakubowski in [24]. Here,

$$\mathrm{P}(z;M) := \frac{4z}{\left(\sqrt{(1-z)^2 + 4z/M} + 1 - z\right)^2}, \quad z \in \mathbb{D},\ \sqrt{1} := 1,$$

stands for the Pick function. In [24], it was also defined the class

$$\mathcal{G}(1) := \left\{g \in \mathcal{H} : g(0) = 1,\ \mathrm{Re}\left(\frac{2zg'(z)}{g(z)} + 1\right) > 0,\ z \in \mathbb{D}\right\}.$$

Todorov [25] studied a functional $\mathbb{D} \ni z \mapsto f(z)/(1-z)$ over the class $\mathcal{G}$, and obtained a structural formula and coefficient estimates. Obradovič and Owa [26] and Silverman and Silvia [27] independently defined the families $\mathcal{G}_\alpha$, where $\alpha \in [0,1)$, of all $g \in \mathcal{H}$ of the form (1) such that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right) > 0, \quad z \in \mathbb{D}.$$

Silverman and Silvia ([27] Lemma 1, p. 296) proved that $\mathcal{G}_\alpha \subset \mathcal{G}^*$ for every $\alpha \in [0,1)$. Clearly, $\mathcal{G}_{1/2} = \mathcal{G}$. Further results on the class $\mathcal{G}$ were obtained by Abdullah et al. [28]. In [29], Jakubowski and Włodarczyk introduced the class $\mathcal{G}(A,B)$ for $-1 < A \leq 1$ and $-A < B \leq 1$, of all $g \in \mathcal{H}$ of the form (1) satisfying

$$\operatorname{Re}\left(\frac{2zg'(z)}{g(z)} + Q(z;A,B)\right) > 0, \quad z \in \mathbb{D},$$

where

$$Q(z;A,B) := \frac{1+Az}{1-Bz}, \quad z \in \mathbb{D}. \tag{3}$$

By using Ma and Minda's idea, Mohd and Darus in [30] defined the class $\mathcal{S}_b^*(\phi)$, where $\phi \in \mathcal{P}^*(1)$, of all $g \in \mathcal{H}$ of the form (1) such that

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec \phi(z), \quad z \in \mathbb{D}.$$

Combining the aforementioned concept from [29] with Ma and Minda's idea in [31] the class $\mathcal{G}(\phi;A,B)$ was introduced and examined.

Define

$$\mathcal{P}_L := \{p \in \mathcal{H} : p \prec \phi_L\},$$

and

$$\mathcal{P}_c := \{p \in \mathcal{H} : p \prec \phi_c\}.$$

The main goal of this paper is to define and investigate the following two classes of functions.

Definition 1. *Let $\mathcal{G}_c$ denote the class of all $g \in \mathcal{H}$ of the form* (1) *such that*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec \phi_c(z), \quad z \in \mathbb{D}. \tag{4}$$

Definition 2. *Let $\mathcal{G}_L$ denote the class of all $g \in \mathcal{H}$ of the form* (1) *such that*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec \phi_L(z), \quad z \in \mathbb{D}. \tag{5}$$

Remark 1. *1. In [32], the class $\mathcal{G}_e$ was introduced in a similar way as in Definitions 1 and 2, with the exponential function being the dominant, i.e., $\phi(z) := \exp(z)$ for $z \in \mathbb{D}$.*

2. Notice that the conditions (4) *and* (5) *are well defined, as the function*

$$p(z) := \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{6}$$

is analytic in $\mathbb{D}$.

Lemma 1.

$$\mathcal{G}_c \subset \mathcal{G}^*, \quad \mathcal{G}_L \subset \mathcal{G}^*.$$

2. Representation and Growth Theorems

Let us start with some examples.

Example 1. *1. Given $a \in \mathbb{R}$, define*

$$p_a(z) := 1 + az, \quad z \in \mathbb{D},$$

and

$$g(z) := (1-z)\exp\left(\frac{az}{2}\right), \quad z \in \mathbb{D}. \tag{7}$$

Then $p_a \prec \phi_c$ if and only if $|a| \le 2 - \sqrt{2}$ ([11] Theorem 3.2). Since $g \in \mathcal{H}$, $g(0) = 1$ and

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} = 1 + az = p_a(z), \quad z \in \mathbb{D},$$

we see that $g \in \mathcal{G}_c$ for $|a| \le 2 - \sqrt{2}$.

2. Given $-1 < A \le 1$ and $-A < B < 1$, let

$$p_{A,B}(z) := \frac{1+Az}{1-Bz} = Q(z; A, B), \quad z \in \mathbb{D},$$

where Q is defined by (3). Observe that $p_{A,B}(\mathbb{D})$ is an open disk symmetrical with respect to the real axis centered at $(1+AB)/(1-B^2)$ of radius $(A+B)/(1-B^2)$. Then $p_{A,B}(\mathbb{D}) \subset \phi_c(\mathbb{D})$ if and only if

$$p_{A,B}(-1) = \frac{1-A}{1+B} \ge \sqrt{2} - 1, \quad p_{A,B}(1) = \frac{1+A}{1-B} \le \sqrt{2} + 1, \tag{8}$$

so then $p_{A,B} \in \mathcal{P}_c$. Thus, a function $g \in \mathcal{H}$ with $g(0) = 1$ defined by

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} = \frac{1+Az}{1-Bz} = p_{A,B}(z), \quad z \in \mathbb{D},$$

with A and B satisfying the inequalities (8), belongs to the class $\mathcal{G}_c$ i.e., the function (7) with $a := A$ in the case when $B = 0$, and the function

$$g(z) = \frac{1-z}{(1-Bz)^{(A+B)/2B}}, \quad z \in \mathbb{D},$$

in the case when $B \neq 0$. Particularly, $p_{A,A} \in \mathcal{P}_c$ if and only if $A \in \left[1-\sqrt{2}, -1+\sqrt{2}\right]$ ([11] p. 358). In this case, the function (8) has the form

$$g(z) = \frac{1-z}{1-Az}, \quad z \in \mathbb{D},$$

and belongs to $\mathcal{G}_c$.

The representation theorem formulated below is a useful tool to construct functions in the class $\mathcal{G}_c$.

Theorem 1. *$g \in \mathcal{G}_c$ if and only if there exists $p \in \mathcal{P}_c$ such that*

$$g(z) = (1-z)\exp\left(\frac{1}{2}\int_0^z \frac{p(\zeta)-1}{\zeta} d\zeta\right), \quad z \in \mathbb{D}. \tag{9}$$

Proof. Assume that $g \in \mathcal{G}_c$. Then a function p defined by (6) is holomorphic and satisfies $p \prec \phi_c$, i.e., $p \in \mathcal{P}_c$. Note that (6) can be equivalently written as

$$\frac{p(z)-1}{z} = \frac{1}{z}\left(\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} - 1\right) = \frac{2g'(z)}{g(z)} + \frac{2}{1-z}, \quad z \in \mathbb{D}, \tag{10}$$

which by an integration yields

$$\log\frac{(g(z))^2}{(1-z)^2}=\int_0^z\frac{p(\zeta)-1}{\zeta}d\zeta,\quad z\in\mathbb{D},\ \log 1:=0.$$

Hence

$$(g(z))^2=(1-z)^2\exp\left(\int_0^z\frac{p(\zeta)-1}{\zeta}d\zeta\right),\quad z\in\mathbb{D},$$

which leads to the Formula (9).

Assume now that $p\in\mathcal{P}_c$ and a function g is defined by (9). As $p(0)=1$, we see that g is holomorphic in $\mathbb{D}$. A simple computation shows that g satisfies (10), so (6). Thus, $g\in\mathcal{G}_c$, which completes the proof. □

Define h_c as a holomorphic solution of the differential equation

$$\frac{zh_c'(z)}{h_c(z)}=\phi_c(z),\quad z\in\mathbb{D},\ h_c(0)=0,\ h_c'(0)=1,$$

i.e.,

$$h_c(z)=z\exp\left(\int_0^z\frac{\phi_c(\zeta)-1}{\zeta}d\zeta\right)=z\exp\left(\int_0^z\frac{\zeta+\sqrt{1+\zeta^2}-1}{\zeta}d\zeta\right),\quad z\in\mathbb{D}.$$

Since

$$\begin{aligned}&\int_0^z\frac{\zeta+\sqrt{1+\zeta^2}-1}{\zeta}d\zeta=z+\int_0^z\frac{\sqrt{1+\zeta^2}-1}{\zeta}d\zeta\\&=z+\int_0^z\frac{1+\zeta^2-1}{\zeta(\sqrt{1+\zeta^2}+1)}d\zeta=z+\int_0^z\frac{\zeta}{\sqrt{1+\zeta^2}+1}d\zeta\\&=z+\left[\sqrt{1+\zeta^2}-\log\left(1+\sqrt{1+\zeta^2}\right)\right]_0^z\\&=z+\sqrt{1+z^2}-\log\left(1+\sqrt{1+z^2}\right)-1+\log 2\\&=z-1+\sqrt{1+z^2}+\log\frac{2}{1+\sqrt{1+z^2}},\quad z\in\mathbb{D},\end{aligned}$$

it follows that

$$h_c(z)=\frac{2z}{1+\sqrt{1+z^2}}\exp\left(z-1+\sqrt{1+z^2}\right),\quad z\in\mathbb{D}.\tag{11}$$

Since the representation theorem for the class $\mathcal{G}_L$ is similar to that of the class $\mathcal{G}_c$ we omit the details involved.

Theorem 2. *$g\in\mathcal{G}_L$ if and only if there exists a function $p\in\mathcal{P}_L$ such that*

$$g(z)=(1-z)\exp\left(\frac{1}{2}\int_0^z\frac{p(\zeta)-1}{\zeta}d\zeta\right),\quad z\in\mathbb{D}.$$

Define h_L as a holomorphic solution of the differential equation

$$\frac{zh_L'(z)}{h_L(z)}=\phi_L(z),\quad z\in\mathbb{D},\ h_L(0)=0,\ h_L'(0)=1,$$

i.e.,

$$h_L(z)=z\exp\left(\int_0^z\frac{\phi_L(\zeta)-1}{\zeta}d\zeta\right)=z\exp\left(\int_0^z\frac{\sqrt{1+\zeta}-1}{\zeta}d\zeta\right),\quad z\in\mathbb{D}.$$

Since

$$\begin{aligned}
&\int_0^z \frac{\sqrt{1+\zeta}-1}{\zeta} d\zeta = \int_0^z \frac{1+\zeta-1}{\zeta\left(\sqrt{1+\zeta}+1\right)} d\zeta \\
&= \int_0^z \frac{1}{\sqrt{1+\zeta}+1} d\zeta = \left[2\sqrt{1+\zeta} - 2\log\left(1+\sqrt{1+\zeta}\right)\right]_0^z \\
&= 2\left(\sqrt{1+z}-1\right) + 2\log\frac{2}{1+\sqrt{1+z}}, \quad z \in \mathbb{D},
\end{aligned}$$

it follows that

$$h_L(z) = \frac{4z\exp\left(2\sqrt{1+z}-2\right)}{(1+\sqrt{1+z})^2}, \quad z \in \mathbb{D}.$$

We construct examples for the class $\mathcal{G}_L$ by the virtue of Theorem 2.

Example 2. One can easily see that the function

$$p(z) := \frac{1+az}{1+bz}, \quad z \in \mathbb{D},$$

for suitable chosen $0 \le b < a \le 1$ belongs to $\mathcal{P}_L$. For example, we can take $a = 2/3$ and $b = 1/3$. By Theorem 2, the function

$$g(z) := (1-z)\sqrt{1+\frac{z}{3}}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{G}_L$.

Theorem 3. *Let* $0 < r < 1$.

(i) If $g \in \mathcal{G}_c$, *then*

$$\sqrt{\frac{-h_c(-r)}{r}}(1-r) \le |g(z)| \le \sqrt{\frac{h_c(-r)}{r}}(1+r), \quad |z| = r. \tag{12}$$

(ii) If $g \in \mathcal{G}_L$, *then*

$$\sqrt{\frac{-h_L(-r)}{r}}(1-r) \le |g(z)| \le \sqrt{\frac{h_L(-r)}{r}}(1+r), \quad |z| = r. \tag{13}$$

Proof. (i) Let $g \in \mathcal{G}_c$ and define

$$h(z) := \frac{z(g(z))^2}{(1-z)^2}, \quad z \in \mathbb{D}. \tag{14}$$

Clearly, h is holomorphic in $\mathbb{D}$ and a simple calculation shows that

$$\begin{aligned}
\frac{h'(z)}{h(z)} - \frac{1}{z} &= \left(\log\frac{h(z)}{z}\right)' = \left(\log\left(\frac{1}{z}\cdot\frac{z(g(z))^2}{(1-z)^2}\right)\right)' \\
&= \left(\log\frac{(g(z))^2}{(1-z)^2}\right)' = (2\log g(z) - 2\log(1-z))' = \\
&= 2\frac{g'(z)}{g(z)} + \frac{2}{1-z}, \quad z \in \mathbb{D}.
\end{aligned}$$

Hence,

$$\frac{zh'(z)}{h(z)} = 2\frac{zg'(z)}{g(z)} + \frac{2z}{1-z} + 1 = \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

Since $g \in \mathcal{G}_c$,

$$\frac{zh'(z)}{h(z)} \prec \phi_c(z), \quad z \in \mathbb{D}. \tag{15}$$

Using the result of Ma and Minda ([1] Corollary 1′) we deduce that

$$-h_c(-r) \leq |h(z)| \leq h_c(r), \quad |z| = r,$$

i.e., by (14),

$$-h_c(-r) \leq \left| \frac{z(g(z))^2}{(1-z)^2} \right| \leq h_c(r), \quad |z| = r,$$

which yields (12).

(ii) By a similar argument, we can also prove (13) and therefore the details are omitted. □

Theorem 4. *Let* $0 < r < 1$.

(i) If $g \in \mathcal{G}_c$, *then*

$$\left| \arg \frac{g(z_0)}{(1-z_0)^2} \right| \leq \frac{1}{2} \max_{|z|=r} \arg \frac{h_c(z)}{z}, \quad |z_0| = r, \ \arg 1 := 0. \tag{16}$$

(ii) If $g \in \mathcal{G}_L$, *then*

$$\left| \arg \frac{g(z_0)}{(1-z_0)^2} \right| \leq \frac{1}{2} \max_{|z|=r} \arg \frac{h_L(z)}{z}, \quad |z_0| = r, \ \arg 1 := 0. \tag{17}$$

Proof. (i) Let $g \in \mathcal{G}_c$. Then by (15) a function h defined by (14) belongs to $\mathcal{S}^*(\phi_c)$. Thus, in view of a result from Ma and Minda ([1] Corollary 3′) the following inequality holds

$$\left| \arg \frac{h(z_0)}{z_0} \right| \leq \max_{|z|=r} \arg \frac{h_c(z)}{z}, \quad |z_0| = r,$$

where h_c is defined by (11). A substitution of (14) yields (16).

(ii) By a similar argument, we can also prove (17) and therefore the details are omitted. □

3. Initial Coefficient Bounds for the Class $\mathcal{G}_L$ and $\mathcal{G}_c$

By making use of the following lemmas, we compute a few coefficient estimates for $g \in \mathcal{G}_c$ and for $g \in \mathcal{G}_L$. Let $\mathcal{B} := \{\omega \in \mathcal{H} : |\omega(z)| \leq 1, \ z \in \mathbb{D}\}$ and $\mathcal{B}_0$ be the subclass of $\mathcal{B}$ of all ω such that $\omega(0) = 0$. The elements of $\mathcal{B}_0$ are known as Schwarz functions.

We will apply two lemmas below to prove the main theorem of this section. The first one was shown by Keogh and Merkes ([33] Inequality 7, p. 10).

Lemma 2 ([33])**.** *If* $\omega \in \mathcal{B}_0$ *is of the form*

$$\omega(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathbb{D}, \tag{18}$$

then for $\nu \in \mathbb{C}$,

$$|w_2 - \nu w_1^2| \leq \max\{1, |\nu|\}. \tag{19}$$

The following lemma was shown by Prokhorov and Szynal ([34] Lemma 2, p. 128).

Lemma 3 ([34])**.** *If* $\omega \in \mathcal{B}$, *then for any real numbers* q_1 *and* q_2, *the following sharp estimate holds:*

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq H(q_1, q_2), \tag{20}$$

where

$$H(q_1,q_2) := \begin{cases} 1, & (q_1,q_2) \in D_1 \cup D_2, \\ |q_2|, & (q_1,q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}}, & (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}}, & (q_1,q_2) \in (D_{10} \cup D_{11}) \setminus \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, & (q_1,q_2) \in D_{12}, \end{cases} \tag{21}$$

and the sets D_k, $k = 1,2,\dots,12$, *are defined in* [34]. *Particularly,*

$$\begin{aligned} D_1 &:= \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le 1\right\}, \\ D_2 &:= \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\ \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \le q_2 \le 1\right\}. \end{aligned} \tag{22}$$

Now we demonstrate upper bounds for early coefficients and for the Fekete–Szegö functional in the classes $\mathcal{G}_c$ and $\mathcal{G}_L$.

Theorem 5. *If* $g \in \mathcal{G}_c$ *is of the form* (1), *then*

$$|d_1 + 1| \le \frac{1}{2}, \tag{23}$$

$$|d_1| \le \frac{3}{2}, \tag{24}$$

$$\left|2d_2 - d_1^2 + 1\right| \le \frac{1}{2}, \tag{25}$$

$$|d_2| \le \frac{3}{4}, \tag{26}$$

$$\left|3d_3 - 3d_1d_2 + d_1^3 + 1\right| \le \frac{1}{2}, \tag{27}$$

$$|d_3| \le \frac{5}{12} \tag{28}$$

and for $\delta \in \mathbb{C}$,

$$|d_2 - \delta d_1^2| \le \frac{1}{4}(\max\{1, |1-\delta|\} + 2|1-2\delta| + 4|\delta|). \tag{29}$$

Inequalities (23)–(27) *are sharp.*

Proof. By (4) there exists $w \in \mathcal{B}_0$ of the form (18) such that

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} = \phi_c(w(z)) = w(z) + \sqrt{1+(w(z))^2}, \quad z \in \mathbb{D}. \tag{30}$$

Taking into account (1) we obtain

$$\begin{aligned} \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} =& 1 + 2(d_1+1)z + 2(2d_2 - d_1^2 + 1)z^2 \\ &+ 2\left(3d_3 - 3d_1d_2 + d_1^3 + 1\right)z^3 + \dots, \quad z \in \mathbb{D}. \end{aligned} \tag{31}$$

By (18), for $z \in \mathbb{D}$ we have

$$w(z) + \sqrt{1 + (w(z))^2} = 1 + w_1 z + \left(w_2 + \frac{1}{2}w_1^2\right)z^2 + (w_3 + w_1 w_2)z^3 + \ldots \tag{32}$$

Now, by comparing the corresponding coefficients in (30)–(32), we obtain

$$\begin{aligned} 2(d_1 + 1) &= w_1, \\ 2(2d_2 - d_1^2 + 1) &= w_2 + \frac{1}{2}w_1^2, \\ 2(3d_3 - 3d_1 d_2 + d_1^3 + 1) &= w_3 + w_1 w_2. \end{aligned} \tag{33}$$

Since

$$|w_1| \le 1, \tag{34}$$

(e.g., [35] Vol. I, p. 85) from the first equation in (33) results in (23) and (24) follow easily.

The second equation in (33) together with (19) yields

$$\left|2(2d_2 - d_1^2 + 1)\right| = \left|w_2 + \frac{1}{2}w_1^2\right| \le 1,$$

i.e., the inequality (25).

By substituting the first formula in (33) for d_1 into the second formula in (33) we obtain

$$4d_2 = w_2 + w_1^2 - 2w_1. \tag{35}$$

Hence by using (19) and (34) we obtain

$$4|d_2| \le |w_2 + w_1^2| + 2|w_1| \le 3,$$

which yields (26).

Since $(1, 0) \in D_2$, where D_2 is defined by (22), it follows from (21) that $H(1, 0) = 1$. Thus, by applying (20) on the third equation in (33) yields

$$\left|6d_3 - 6d_1 d_2 + 2d_1^3 + 2\right| = |w_3 + w_1 w_2| \le H(1, 0) = 1,$$

i.e., the inequality (27).

By substituting the formulas of d_1 in (33), and d_2 in (35) into the third formula in (33) we obtain

$$6d_3 = w_3 + \frac{7}{4}w_1 w_2 + \frac{1}{2}w_1^3 - \frac{3}{2}w_2 - \frac{3}{2}w_1^2$$

Since $(7/4, 1/2) \in D_2$, it follows from (21) that $H(7/4, 1/2) = 1$. Therefore by applying (19), (20) and (34) we obtain

$$6|d_3| \le H\left(\frac{7}{4}, \frac{1}{2}\right) + \frac{3}{2} = \frac{5}{2},$$

which yields (28).

By using (35) and the formula for d_1, and by applying (19) and (34), for $\delta \in \mathbb{C}$ we obtain

$$\begin{aligned} |d_2 - \delta d_1^2| &= \frac{1}{4}\left|w_2 + (1-\delta)w_1^2 + 2(2\delta - 1)w_1 - 4\delta\right| \\ &\le \frac{1}{4}\left(\left|w_2 + (1-\delta)w_1^2\right| + 2|1 - 2\delta| + 4|\delta|\right) \\ &\le \frac{1}{4}(\max\{1, |1-\delta|\} + 2|1 - 2\delta| + 4|\delta|), \end{aligned}$$

which leads to the inequality (29).

Equalities in (23) and (24) hold for the function $g \in \mathcal{G}_c$ satisfying (30) with $w(z) := -z$ for $z \in \mathbb{D}$. Equalities in (25) and (26) hold for the function $g \in \mathcal{G}_c$ satisfying (30) with $w(z) := z^2$ and $w(z) = -z$ for $z \in \mathbb{D}$, respectively. Equality in (27) holds for the function $g \in \mathcal{G}_c$ satisfying (30) with $w(z) := z^3$ for $z \in \mathbb{D}$. $\square$

Now we discuss the class $\mathcal{G}_L$. By (5) there exists $w \in \mathcal{B}_0$ of the form (18) such that

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} = \phi_L(w(z)) = \sqrt{1+w(z)}, \quad z \in \mathbb{D}. \tag{36}$$

Taking into account (18) for $z \in \mathbb{D}$ we have

$$\sqrt{1+w(z)} = 1 + \frac{w_1}{2}z + \frac{1}{2}\left(w_2 - \frac{w_1^2}{4}\right)z^2 + \frac{1}{2}\left(w_3 - \frac{w_1 w_2}{2} + \frac{w_1^3}{8}\right)z^3 + \dots. \tag{37}$$

By virtue of (36) and (37) by comparing corresponding coefficients we obtain

$$\begin{aligned}
d_1 + 1 &= \frac{w_1}{4}, \\
2d_2 - d_1^2 + 1 &= \frac{1}{4}\left(w_2 - \frac{w_1^2}{4}\right), \\
3d_3 - 3d_1 d_2 + d_1^3 + 1 &= \frac{1}{4}\left(w_3 - \frac{w_1 w_2}{2} + \frac{w_1^3}{8}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
d_1 &= \frac{1}{4}w_1 - 1, \\
d_2 &= \frac{1}{8}(w_2 - 2w_1), \\
d_3 &= \frac{1}{12}\left(w_3 - \frac{1}{8}w_1 w_2 + \frac{1}{16}w_1^3 - \frac{3}{2}w_2\right).
\end{aligned}$$

By similar computations as in Theorem 5 we can formulate the following theorem.

Theorem 6. *If $g \in \mathcal{G}_L$ is of the form (1), then*

$$|d_1 + 1| \le \frac{1}{4}, \tag{38}$$

$$|d_1| \le \frac{5}{4}, \tag{39}$$

$$\left|2d_2 - d_1^2 + 1\right| \le \frac{1}{4}, \tag{40}$$

$$|d_2| \le \frac{1}{4}, \tag{41}$$

$$\left|3d_3 - 3d_1 d_2 + d_1^3 + 1\right| \le \frac{1}{4} \tag{42}$$

and

$$|d_3| \le \frac{5}{24}$$

and for $\delta \in \mathbb{C}$,

$$|d_2 - \delta d_1^2| \le \frac{1}{4}\left(\frac{1}{2}\max\left\{1, \frac{1}{2}|\delta|\right\} + |1-2\delta| + 4|\delta|\right).$$

Inequalities (38)–(42) are sharp.

4. Differential Subordination Results Involving $\mathcal{G}_c$ and $\mathcal{G}_L$

In this section, we obtain a few differential subordination results. For the proofs, we need the following lemma (see [36] Theorem 3.4h, p. 132).

Lemma 4. *Let q be univalent in $\mathbb{D}$, θ and φ be holomorphic in a domain D containing $q(\mathbb{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Let $Q(z) := zq'(z)\varphi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$ for $z \in \mathbb{D}$.*

Suppose that either

(i) *Q is starlike univalent in $\mathbb{D}$, or*
(ii) *h is convex univalent $\mathbb{D}$.*

Assume also that

(iii)

$$\mathrm{Re}\,\frac{zh'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

If $p \in \mathcal{H}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad z \in \mathbb{D},$$

then $p \prec q$ and q is the best dominant.

Theorem 7. *If $g \in \mathcal{H}$ with $g(0) = 1$ and satisfies*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec 1 + \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{D}, \tag{43}$$

then

$$p(z) := \left(\frac{g(z)}{1-z}\right)^2 \prec \phi_c(z), \quad z \in \mathbb{D}, \tag{44}$$

i.e., $p \in \mathcal{P}_c$.

Proof. I. Let $\theta(w) := 1$, $w \in \mathbb{C}$, and $\varphi(w) := 1/w$, $w \in \mathbb{C} \setminus \{0\}$. Note that $\phi_c(\mathbb{D}) \subset D := \mathbb{C} \setminus \{0\}$ and θ and φ are holomorphic in D. Thus,

$$Q(z) := z\phi_c'(z)\varphi(\phi_c(z)) = \frac{z\phi_c'(z)}{\phi_c(z)} = \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{D}, \tag{45}$$

is well defined and holomorphic. Since $Q'(0) = 1$ and

$$\mathrm{Re}\,\frac{zQ'(z)}{Q(z)} = \mathrm{Re}\,\frac{1}{1+z^2} > 0, \quad z \in \mathbb{D}, \tag{46}$$

it follows that Q is a univalent starlike function ([37] p. 41, see also [38] Theorem 2.2, p. 92). Hence, for a function $h(z) := \theta(\phi_c(z)) + Q(z) = 1 + Q(z)$, $z \in \mathbb{D}$, we obtain

$$\mathrm{Re}\,\frac{zh'(z)}{Q(z)} = \mathrm{Re}\,\frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

Thus, for any function $p \in \mathcal{H}$ with $p(0) = \phi_c(0) = 1$ such that $p(\mathbb{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathbb{D}$, from Lemma 4 it follows that the subordination

$$1 + \frac{zp'(z)}{p(z)} \prec 1 + \frac{z\phi_c'(z)}{\phi_c(z)} = 1 + \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{D}, \tag{47}$$

yields the subordination $p \prec \phi_c$.

II. Let now take $g \in \mathcal{H}$ with $g(0) = 1$ and $g(z) \neq 0$ for $z \in \mathbb{D}$ satisfying (43). Define the function p as in (44). We see that $p(0) = \phi_c(0) = 1$, $p(z) \neq 0$ for $z \in \mathbb{D}$ and p is holomorphic. Since

$$1 + \frac{zp'(z)}{p(z)} = \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

from (47), (44) follows, which completes the proof. □

Theorem 8. *If $g \in \mathcal{H}$ with $g(0) = 1$ and satisfies*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec \frac{2+3z}{2(1+z)}, \quad z \in \mathbb{D}, \tag{48}$$

then $p \prec \phi_L$, i.e., $p \in \mathcal{P}_L$, where p is defined in (44).

Proof. Let $\theta(w) := 1$, $w \in \mathbb{C}$, and $\varphi(w) := 2/w$, $w \in \mathbb{C} \setminus \{0\}$. Note that $\phi_L(\mathbb{D}) \subset D := \mathbb{C} \setminus \{0\}$ and θ and φ are holomorphic in D. Thus

$$Q(z) := z\phi_L'(z)\varphi(\phi_L(z)) = \frac{2z\phi_L'(z)}{\phi_L(z)} = \frac{z}{1+z}, \quad z \in \mathbb{D}, \tag{49}$$

is well defined and holomorphic. Since $Q'(0) = 1$ and

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \frac{1}{1+z} > 0, \quad z \in \mathbb{D}, \tag{50}$$

it follows that Q is a univalent starlike function. Hence, for the function $h(z) := \theta(\phi_L(z)) + Q(z) = 1 + Q(z)$, $z \in \mathbb{D}$, we obtain

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

Thus, for any function $p \in \mathcal{H}$ with $p(0) = \phi_L(0) = 1$ such that $p(\mathbb{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathbb{D}$, from Lemma 4 it follows that the subordination

$$1 + \frac{zp'(z)}{p(z)} \prec 1 + \frac{z\phi_L'(z)}{\phi_L(z)} = \frac{2+3z}{2(1+z)}, \quad z \in \mathbb{D}, \tag{51}$$

yields the subordination $p \prec \phi_L$.

II. Let now take $g \in \mathcal{H}$ with $g(0) = 1$ and $g(z) \neq 0$ for $z \in \mathbb{D}$ satisfying (48). Define a function p as in (44). We see that $p(0) = \phi_L(0) = 1$, $p(z) \neq 0$ for $z \in \mathbb{D}$ and p is holomorphic. Since

$$1 + \frac{zp'(z)}{p(z)} = \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

from (51) it follows that $p \prec \phi_L$, which completes the proof. □

Theorem 9. *If $g \in \mathcal{H}$ with $g(0) = 1$ and satisfies*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec z + \sqrt{1+z^2} + \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{D}, \tag{52}$$

then

$$p(z) := z\left(\frac{g(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \prec \phi_c(z), \quad z \in \mathbb{D}, \tag{53}$$

i.e., $p \in \mathcal{P}_c$.

Proof. I. Let $\theta(w) := w$, $w \in \mathbb{C}$, and $\varphi(w) := 1/w$, $w \in \mathbb{C} \setminus \{0\}$. Note that $\phi_c(\mathbb{D}) \subset D := \mathbb{C} \setminus \{0\}$ and θ and φ are holomorphic in D. Thus, the function Q defined by (45) is holomorphic and satisfies (46), i.e., it is univalent starlike. Hence, for the function $h(z) := \theta(\phi_c(z)) + Q(z) = \phi_c(z) + Q(z)$, $z \in \mathbb{D}$, by using (46) we obtain

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\frac{z\phi_c'(z)}{Q(z)} + \operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\phi_c(z) + \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

Thus, for any function $p \in \mathcal{H}$ with $p(0) = \phi_c(0) = 1$ such that $p(\mathbb{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathbb{D}$, from Lemma 4 it follows that the subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec \phi_c(z) + \frac{z\phi_c'(z)}{\phi_c(z)} = z + \sqrt{1+z^2} + \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{D}, \tag{54}$$

yields the subordination $p \prec \phi_c$.

II. Let now take $g \in \mathcal{H}$ with $g(0) = 1$ and $g(z) \neq 0$ for $z \in \mathbb{D}$ satisfying (43). Define the function p as in (53). We see that

$$\begin{aligned} p(0) &= \lim_{z\to 0} z\left(\frac{g(z)}{1-z}\right)^2 \left(\int_0^z \left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} \\ &= (g(0))^2 \lim_{z\to 0} z\left(\int_0^z \left(\frac{g(\zeta)}{1-\zeta}\right)^2 d\zeta\right)^{-1} = 1 = \phi_c(0), \end{aligned}$$

$p(z) \neq 0$ for $z \in \mathbb{D}$ and p is holomorphic. Since

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

from (54), (52) follows, which completes the proof. □

Theorem 10. *If $g \in \mathcal{H}$ with $g(0) = 1$ and satisfies*

$$\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z} \prec \sqrt{1+z} + \frac{z}{2(1+z)}, \quad z \in \mathbb{D},$$

then $p \prec \phi_L$, i.e., $p \in \mathcal{P}_L$, where p is defined in (53).

Proof. Let $\theta(w) := w$, $w \in \mathbb{C}$, and $\varphi(w) := 2/w$, $w \in \mathbb{C} \setminus \{0\}$. Note that $\phi_L(\mathbb{D}) \subset D := \mathbb{C} \setminus \{0\}$ and θ and φ are holomorphic in D. Thus, the function Q defined by (49) is holomorphic and satisfies (50), i.e., it is univalent starlike. Hence, for the function $h(z) := \theta(\phi_L(z)) + Q(z) = \phi_L(z) + Q(z)$, $z \in \mathbb{D}$, by using (50) we obtain

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\frac{z\phi_L'(z)}{Q(z)} + \operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\phi_L(z) + \operatorname{Re}\frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

Thus, for any function $p \in \mathcal{H}$ with $p(0) = \phi_L(0) = 1$ such that $p(\mathbb{D}) \subset D$, i.e., $p(z) \neq 0$ for $z \in \mathbb{D}$, from Lemma 4 it follows that the subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec \phi_L(z) + \frac{z\phi_L'(z)}{\phi_L(z)} = \sqrt{1+z} + \frac{z}{2(1+z)}, \quad z \in \mathbb{D},$$

yields the subordination $p \prec \phi_L$.

Further argumentation is as in Part II of the proof of Theorem 9. $\square$

5. Conclusions

In this paper, two ideas were combined, namely the class $\mathcal{P}^*(1)$, which contains normalized holomorphic functions ϕ having positive real part mapping univalently the unit disk $\mathbb{D}$ onto a set $\phi(\mathbb{D})$ symmetric with respect to the real axis and starlike with respect to 1, and the class of starlike functions with respect to the boundary point. Our research concerns the case when $\phi := \phi_c$ and $\phi := \phi_L$, of which they map the unit disk onto either a crescent shaped domain or a domain bounded by lemniscate of Bernoulli, respectively, and both domains are symmetric with respect to the real axis. This property of symmetry is the basis for finding the analytical and geometrical properties of the studied classes. The use of the functions ϕ_c and ϕ_L is reasonable and makes sense as these functions have been studied by other authors before, and have been used to construct subclasses of starlike functions with respect to the interior point. Using them for starlikeness with respect to the boundary point is a new original idea. Research on the introduced classes can be developed both for further geometrical and analytical properties, with particular study on the coefficient problems as Hankel or Hermitian–Toeplitz matrices.

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