

Article

Picture Fuzzy Soft Prioritized Aggregation Operators and Their Applications in Medical Diagnosis

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Abstract: A medical diagnosis is one the most efficient processes of determining a disease based on a person's symptoms and signs. In recent days, due to the complexities of the same type of diseases, it is very difficult to diagnose a disease by using old methods and techniques. In this way, new and efficient medical diagnosis methods can help a lot in reaching an accurate conclusion, depending upon the timing and sequences of symptoms and medical history. The physician relies on other clues like medical tests and imaging tests. So, in this way, a list of possible diagnoses can be determined, which are referred to as different diagnoses. To handle these types of issues in this manuscript, additional information is identified, and possible disease is confirmed. Under the consideration of classical data, it is a very difficult task to deal with complex and asymmetric sorts of data. Fuzzy set theory has a wide range of applications, from engineering to the medical field. Different methods and techniques have been proposed to support the decision-making process in medical fields. Picture fuzzy soft sets are more generalized structures and efficient tools to formalize the information more decently and accurately. So, devoted from this notion, in this article based on picture fuzzy soft settings, we firstly have established some basic operational laws for picture fuzzy soft number; then based on these operational laws, we have developed some aggregation operators named as picture fuzzy soft prioritized average and geometric aggregation operators. In real-world problems, these operators can be useful in analyzing uncomfortable and asymmetric information. Furthermore, some basic properties of the introduced operators have been initiated and discussed briefly. Moreover, to show the effective use of this developed approach to medical diagnoses, we have proposed an algorithm, along with a descriptive example. Additionally, a comparative analysis of the proposed work shows the superiority and effectiveness of the introduced approach.

Keywords: picture fuzzy soft set; prioritized aggregation operators; medical diagnosis



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1. Introduction

Before the development of modern technologies, techniques for the diagnosis and treatment in medicine were rudimentary, inconsistent, and unresponsive to a wide range of medical symptoms. Medical diagnosis before treatment is very important. Medical diagnosis is a challenging procedure that includes patient history, physical examinations, and tests. Keep in mind, a patient may describe a minor illness, such as a headache or an injured joint, using medical symptoms. However, in many cases medical symptoms are non-specific. For instance, two people with different illnesses—one with the heart, and the other with the lungs—might both experience chest pain. To make an accurate determination, a medical expert can create a list of symptoms and then compare it to other data; this means it is a very difficult task for the medical professional to decide the ailment of the patient suffering from a certain disease. In this regard, FST helps researchers to use it in medical fields. Many researchers have developed different methods and techniques to conform to the uses of FST in medical diagnoses. Since its inception, FST has been regarded as a suitable formalism to deal with many medical problems. Zadeh [1] initiated the idea of a fuzzy set that uses the membership grades belonging to $[0, 1]$. Many researchers find the

uses of FS in medical fields, Steimann [2] uses FST in medicine, and Adlassnig [3] proposed a medical diagnosis based on FS settings. Moreover, based on the interval-valued fuzzy set, Choi et al. [4] proposed a medical diagnosis.

Note that FS is a limited notion, and it cannot describe the two types of aspects like membership grade (MG) and non-membership grade (NMG) in one structure. So, the idea of the intuitionistic fuzzy set (IFS) [5] has been developed to generalize the FS notion. IFS uses MG and NMG in one structure belonging to $[0, 1]$. After the successful invention of IFS, De et al. [6] utilizes IFS in medical diagnoses. Moreover, Yu [7] developed a group decision-making approach based on generalized IF geometric aggregation operators.

Additionally, Luo and Zhao [8] established distance measures between IFSs, and provided its application to medical diagnoses; this means that IFS also has applications in medical diagnoses. IFS is also a limited notion because when decision-makers come up with 0.5 as MG and 0.7 as NMG, then IFS fails to tackle this data because $\text{sum}(0.5, 0.7) \notin [0, 1]$. To cover this issue, the notion of the Pythagorean fuzzy set (PyFS) [9] has been developed. PyFS also has applications in medical diagnoses, because Xiao and Ding [10] established divergence measures based on PyFS settings and proved its application to medical diagnoses. Likewise, Khan et al. [11] proposed prioritized aggregation operators based on PyF settings and provide their applications to multi-attribute group decision-making (MAGDM). Moreover, Ejegwa [12] proposed an improved composite relations for PyFS, and initiated its applications to medical diagnoses. Furthermore, Lin et al. [13] provided directional correlation coefficient measures for PyFS and their applications to medical diagnoses and cluster analysis. Additionally, Molla et al. [14] provided an extended PROMETHEE method under the environment of PyFS, and used this method for medical diagnoses. Note that a q-rung orthopair fuzzy set (q-ROFS) [15] can generalize both IFS and PyFS, and this idea also has applications in medical fields. So, Liu et al. [16] introduced the definition of the complex q-rung orthopair fuzzy variation coefficient similarity measures and used this approach for medical diagnoses and pattern recognition. Similarly, Jan et al. [17] defined generalized dice similarity measures for q-ROFS with applications. Furthermore, Riaz et al. [18] introduced prioritized aggregation operators under the environment of q-ROF settings.

Note that all of the above ideas have their applications in medical diagnoses, but these structures are limited. We observe that in our real-life problems, we cannot restrict ourselves to two-dimensional structures. Nevertheless, in any situation, we have to discuss three types of aspects like MG, abstinence grade (AG), or NMG. To cover these problems, Cuong [19] introduced the idea of a picture fuzzy set (PFS). PFS is a more advanced structure, and many researchers utilize this structure for medical diagnoses. We can see the applications and uses of PFS in medical diagnosis forms [20–22]. Additionally, Qiyas et al. [23] established linguistic PF Dombi aggregation operators and provided their application in MAGDM problems.

Molodtsov's concept of a soft set [24] $(S_{ft}S)$ is a fundamental mathematical parameterization structure that can generalize FS. $S_{ft}S$ is a function from a parameter to a crisp subset of the universe for modeling complex systems involving uncertain and unclear objects. Ali et al. [25] initiated basic algebraic structure for $S_{ft}S$. After the notions of $S_{ft}S$, many new developments have been made to show the importance of this idea. Many hybrid notions, like a fuzzy soft set $(FS_{ft}S)$ [26], IF soft set $(IFS_{ft}S)$ [27], PyF soft set $(PyFS_{ft}S)$ [28], and q-ROF soft set $(q-ROFS_{ft}S)$ [29], have been developed. Moreover, Akram and Nawaz [30] proposed fuzzy soft graphs with applications, and Arora and Garg [31] proposed prioritized average aggregation operators under the environment of $IFS_{ft}S$ settings. Furthermore, Hayat et al. [32] introduced applications of generalized $IFS_{ft}S$.

In the real world, there is a great deal of ambiguity, imprecision, and uncertainty. Dealing with uncertainty is a serious challenge in many disciplines, including economics, engineering, environmental research, medical science, and social science. Recently, model-

ing ambiguity has become a growing area of study for researchers. When there are several human answers, such as “no”, “yes”, “abstain”, and “refusal”, $PFS_{ft}S$ models are typically used. A departmental employee of a corporation, for instance, may make a suitable example of a $PFS_{ft}S$. Some employees want to travel to two locations: one in Germany, and the other in Australia. However, some employees prefer to travel to Germany (MG) over Australia (NMG), while others prefer to travel to Australia (MG) over Germany (NMG), and still, others prefer to travel to both Germany and Australia—i.e., neutral employees. However, there are some employees who do not want to visit both places—i.e., refusal grades. The information aggregation step of the decision-making process is primarily responsible for the accuracy of the overall conclusion. So, in this regard, $PFS_{ft}S$ is regarded as a more suitable structure to cover such kind of situations. Similarly, when we consider the phenomenon of voting, someone can vote in favor of someone, vote against someone, abstain to vote, or refuse to vote. So, picture fuzzy soft structure is the best tool where the three types of grades are utilized to model the human opinion—like yes, abstain, or no. We observe that all of the above structures are very useful and have their respective applications to FST, but the limitations of these existing theories are obvious. So, in this regard, the idea of $P_cPFS_{ft}S$, introduced by Yang et al. [33], is a stronger structure, because note that

1. $P_cPFS_{ft}S$ can discuss the parameterization tool.
2. $P_cPFS_{ft}S$ involves three aspects—i.e., MG, NMG, and AG in one structure.
3. When we ignore the abstinence grade then the $PFS_{ft}S$ reduces to $IFS_{ft}S$.

Furthermore, note that when we use only one parameter in the basic structure of $PFS_{ft}S$, then $PFS_{ft}S$ reduces to a simple picture fuzzy set.

Asymmetric information is frequently used since choosing the best alternative information is not always symmetric because the alternatives do not have all the information. Nowadays, as medical diagnoses have become a hot topic in FST, we can observe from the above analysis that many scholars have used their respective fuzzy structures for medical diagnoses. So, due to the dominant features of picture fuzzy soft sets, in this article we have established some aggregation operators for picture fuzzy soft set settings. Additionally, based on these developed structures, we have introduced an application of these structures for medical diagnoses. All of the results show the effective use of this developed structure in the medical field.

The rest of the article is organized as follows: We have provided some fundamental definitions in the second part like FS, $S_{ft}S$, $FS_{ft}S$, $IFS_{ft}S$, $P_yPFS_{ft}S$, $q-ROFS_{ft}S$, PFS, and $PFS_{ft}S$. We have presented certain $PFS_{ft}S$ operating laws in Section 3. Section 4 deals with $PFS_{ft}PWA$ aggregation operators and their properties are elaborated precisely. In Section 5, we have proposed $PFS_{ft}PWG$ aggregation operators. Section 6 discusses an algorithm and an example to demonstrate how these operators operate. In Section 7, we have given a comparative analysis of established work. Finally, Section 8 deals with concluding remarks.

2. Preliminaries

A medical diagnosis refers to the process of determining a disease, condition, or manifestation of its signs and symptoms. A health history physical exam and biopsies may be used to help make a diagnosis. The process of a medical diagnosis helps medical professionals to reach an appropriate decision. The use of different FS structures in medical diagnoses can be seen in [3,6,7].

In the following, we will propose some basic definitions of FS, $S_{ft}S$, $FS_{ft}S$, $IFS_{ft}S$, $P_yPFS_{ft}S$, $q-ROFS_{ft}S$, PFS, and $PFS_{ft}S$.

Definition 1. [1]: Considering Z as a general set, the notion of the form

$$\zeta = \{\mathfrak{L}(\tau) : \tau \in Z\}$$

is called an FS, where $\mathfrak{L}(\tau) : Z \rightarrow [0, 1]$ denotes the MG and $0 \leq \mathfrak{L}(\tau) \leq 1$.

Definition 2. [24]: Let Z be a general set and ρ denote a parameter set (PS), $\mathfrak{P} \subseteq \rho$, a $S_{ft}S$ is a pair $(\mathcal{I}, \mathfrak{P})$, where $\mathcal{I} : \mathfrak{P} \rightarrow P(Z)$, where $P(Z)$ is the power set of Z .

Definition 3. [26]: Consider the universal set Z , ρ as a PS, and let $\mathfrak{P} \subseteq \rho$. The notion of $FS_{ft}S$ is a pair $(\mathcal{I}, \mathfrak{P})$, where $\mathcal{I} : \mathfrak{P} \rightarrow P(FS)$ and $P(FS)$ denotes the power set of FSs.

Definition 4. [27]: Consider the universal set Z , ρ as a PS, and let $\mathfrak{P} \subseteq \rho$. An $IFS_{ft}S$ is the pair $(\mathcal{I}, \mathfrak{P})$, where $\mathcal{I} : \mathfrak{P} \rightarrow IFS^Z$, and IFS^Z denotes the power set of IFS given by

$$\zeta_{\rho_{rf}} = (\tau, \mathfrak{L}_{rf}(\tau), \mathfrak{T}_{rf}(\tau) \mid \tau \in Z)$$

where $\mathfrak{L}_{rf} : Z \rightarrow [0, 1]$ denotes the MG and $\mathfrak{T}_{rf} : Z \rightarrow [0, 1]$, denotes the NMG with the condition that $0 \leq \mathfrak{L}_{rf}(\tau) + \mathfrak{T}_{rf}(\tau) \leq 1$.

Definition 5. [27]: Consider the universal set Z , ρ as a set of parameters, and let $\mathfrak{P} \subseteq \rho$. A $PyFS_{ft}S$ is the pair $(\mathcal{I}, \mathfrak{P})$, where $\mathcal{I} : \mathfrak{P} \rightarrow PyFS^Z$, and $PyFS^Z$ denotes the power set of $PyFS$ given by

$$\zeta_{\rho_{rf}} = (\mathfrak{L}_{rf}, \mathfrak{T}_{rf})$$

where $\mathfrak{L}_{rf} : Z \rightarrow [0, 1]$ represents the MG, and $\mathfrak{T}_{rf} : Z \rightarrow [0, 1]$ denotes the NMG with $0 \leq \mathfrak{L}_{rf}^2 + \mathfrak{T}_{rf}^2 \leq 1$.

Definition 6. [29]: Consider the universal set Z , ρ as a PS, and let $\mathfrak{P} \subseteq \rho$. A $q-ROFS_{ft}S$ is the pair $(\mathcal{I}, \mathfrak{P})$, where $\mathcal{I} : \mathfrak{P} \rightarrow q-ROFS^Z$, and $q-ROFS^Z$ denotes the power set of $q-ROFS$ given by

$$\zeta_{\rho_{rf}} = (\mathfrak{L}_{rf}, \mathfrak{T}_{rf})$$

where $\mathfrak{L}_{rf} : Z \rightarrow [0, 1]$ represents the MG, and $\mathfrak{T}_{rf} : Z \rightarrow [0, 1]$ denotes the NMG with $0 \leq \mathfrak{L}_{rf}^q + \mathfrak{T}_{rf}^q \leq 1$ for $q \geq 1$.

Definition 7. [19]: A set of the form

$$\zeta = \{\mathfrak{L}(\tau), \mathfrak{D}(\tau), \mathfrak{T}(\tau) : \tau \in Z\}$$

is called a PFS on a universal set Z , where $\mathfrak{L}(\tau) : Z \rightarrow [0, 1]$ denotes the MG, $\mathfrak{D}(\tau) : Z \rightarrow [0, 1]$ denotes the AG, and $\mathfrak{T}(\tau) : Z \rightarrow [0, 1]$ denotes the NMG with $su\mathfrak{K}(\mathfrak{L}(\tau), \mathfrak{D}(\tau), \mathfrak{T}(\tau)) \in [0, 1]$.

Definition 8. [33]: For general set Z , ρ as a PS, and $\mathfrak{P} \subseteq \rho$. A pair $(\mathcal{I}, \mathfrak{P})$ is called $PFS_{ft}S$ where $\mathcal{I} : \mathfrak{P} \rightarrow PFS^Z$, is defined by

$$\zeta_{\rho_{rf}} = \{\mathfrak{L}_{rf}(\tau), \mathfrak{D}_{rf}(\tau), \mathfrak{T}_{rf}(\tau) : \tau \in Z\}$$

where PFS^Z denotes the collection of the PFS. Here, $\mathfrak{L}_{rf}(\tau)$, $\mathfrak{D}_{rf}(\tau)$ and $\mathfrak{T}_{rf}(\tau)$ denotes the MG, AG, and NMG, respectively, with $0 \leq (\mathfrak{L}_{rf}(\tau)) + (\mathfrak{D}_{rf}(\tau)) + (\mathfrak{T}_{rf}(\tau)) \leq 1$.

3. Operational Laws for Picture Fuzzy Soft Numbers

In this section, we will discuss the fundamental operating rules for $PFS_{ft}Ns$.

Definition 9. Assume that $\zeta_{\rho_{11}} = \{(\mathfrak{L}_{11}(\tau), \mathfrak{D}_{11}(\tau), \mathfrak{T}_{11}(\tau)) : \tau \in Z\}$, $\zeta_{\rho_{12}} = \{(\mathfrak{L}_{12}(\tau), \mathfrak{D}_{12}(\tau), \mathfrak{T}_{12}(\tau)) : \tau \in Z\}$ represents two $PFS_{ft}Ns$, and for any real number $\mathcal{R} > 0$, the rules are given by

$$1. \quad \zeta_{\rho_{11}} \oplus \zeta_{\rho_{12}} = \{(\mathfrak{L}_{11}(\tau) + \mathfrak{L}_{12}(\tau) - \mathfrak{L}_{11}(\tau)\mathfrak{L}_{12}(\tau)), (\mathfrak{D}_{11}(\tau)\mathfrak{D}_{12}(\tau)), (\mathfrak{T}_{11}(\tau)\mathfrak{T}_{12}(\tau))\}.$$

2. $\zeta_{\rho_{11}} \otimes \zeta_{\rho_{12}} = \left\{ (\mathfrak{L}_{11}(\tau)\mathfrak{L}_{12}(\tau)), (\mathfrak{D}_{11}(\tau)\mathfrak{D}_{12}(\tau)), \left(\mathfrak{T}_{11}(\tau) + \mathfrak{T}_{12}(\tau) - \mathfrak{T}_{11}(\tau)\mathfrak{T}_{12}(\tau) \right) \right\}.$
3. $\mathcal{R}\zeta_{\rho_{11}} = \left\{ \left(1 - (1 - \mathfrak{L}_{11}(\tau))^{\mathcal{R}} \right), (\mathfrak{D}_{11}(\tau))^{\mathcal{R}}, (\mathfrak{T}_{11}(\tau))^{\mathcal{R}} \right\}.$
4. $\zeta_{\rho_{11}}^{\mathcal{R}} = \left\{ (\mathfrak{L}_{11}(\tau))^{\mathcal{R}}, \left(1 - (1 - \mathfrak{D}_{11}(\tau))^{\mathcal{R}} \right), \left(1 - (1 - \mathfrak{T}_{11}(\tau))^{\mathcal{R}} \right) \right\}$

Definition 10. For a PFS_{ft}N $\zeta_{\rho} = \{(\mathfrak{L}(\tau), \mathfrak{D}(\tau), \mathfrak{T}(\tau)) : \tau \in Z\}$, the score function (SF) and accuracy function (AF) are respectively given by

$$Sc(\zeta_{\rho}) = \frac{(1 + \mathfrak{L}(\tau) - \mathfrak{T}(\tau))}{2}, Sc(\zeta_{\rho}) \in [0, 1]$$

$$Ac(\zeta_{\rho}) = \mathfrak{L}(\tau) - \mathfrak{T}(\tau), Ac(\zeta_{\rho}) \in [-1, 1]$$

Definition 11. Suppose $\zeta_{\rho_{11}} = \{(\mathfrak{L}_{11}(\tau), \mathfrak{D}_{11}(\tau), \mathfrak{T}_{11}(\tau)) : \tau \in Z\}$, $\zeta_{\rho_{12}} = \{(\mathfrak{L}_{12}(\tau), \mathfrak{D}_{12}(\tau), \mathfrak{T}_{12}(\tau)) : \tau \in Z\}$ denotes two PFS_{ft}Ns, then we have

1. If $Sc(\zeta_{\rho_{11}}) < Sc(\zeta_{\rho_{12}})$ then $\zeta_{\rho_{11}} < \zeta_{\rho_{12}}$.
2. If $Sc(\zeta_{\rho_{11}}) = Sc(\zeta_{\rho_{12}})$ then
 - (1) If $Ac(\zeta_{\rho_{11}}) < Ac(\zeta_{\rho_{12}})$ then $\zeta_{\rho_{11}} < \zeta_{\rho_{12}}$.
 - (2) If $Ac(\zeta_{\rho_{11}}) = Ac(\zeta_{\rho_{12}})$ then $\zeta_{\rho_{11}} = \zeta_{\rho_{12}}$.

Theorem 1. (Commutative law) For two PFS_{ft}Ns, $\zeta_{\rho_{11}} = \{(\mathfrak{L}_{11}(\tau), \mathfrak{D}_{11}(\tau), \mathfrak{T}_{11}(\tau)) : \tau \in Z\}$, $\zeta_{\rho_{12}} = \{(\mathfrak{L}_{12}(\tau), \mathfrak{D}_{12}(\tau), \mathfrak{T}_{12}(\tau)) : \tau \in Z\}$, we have

- (1) $\zeta_{\rho_{11}} \oplus \zeta_{\rho_{12}} = \zeta_{\rho_{12}} \oplus \zeta_{\rho_{11}}$
- (2) $\zeta_{\rho_{11}} \otimes \zeta_{\rho_{12}} = \zeta_{\rho_{12}} \otimes \zeta_{\rho_{11}}$.

Theorem 2. (Associative law) For three PFS_{ft}Ns, $\zeta_{\rho_{11}} = \{(\mathfrak{L}_{11}(\tau), \mathfrak{D}_{11}(\tau), \mathfrak{T}_{11}(\tau)) : \tau \in Z\}$, $\zeta_{\rho_{12}} = \{(\mathfrak{L}_{12}(\tau), \mathfrak{D}_{12}(\tau), \mathfrak{T}_{12}(\tau)) : \tau \in Z\}$ and $\zeta_{\rho_{13}} = \{(\mathfrak{L}_{13}(\tau), \mathfrak{D}_{13}(\tau), \mathfrak{T}_{13}(\tau)) : \tau \in Z\}$,

- (1) $(\zeta_{\rho_{11}} \oplus \zeta_{\rho_{12}}) \oplus \zeta_{\rho_{13}} = \zeta_{\rho_{11}} \oplus (\zeta_{\rho_{12}} \oplus \zeta_{\rho_{13}})$
- (2) $(\zeta_{\rho_{11}} \otimes \zeta_{\rho_{12}}) \otimes \zeta_{\rho_{13}} = \zeta_{\rho_{11}} \otimes (\zeta_{\rho_{12}} \otimes \zeta_{\rho_{13}}).$

Theorem 3. For three PFS_{ft}Ns, $\zeta_{\rho_{11}} = \{(\mathfrak{L}_{11}(\tau), \mathfrak{D}_{11}(\tau), \mathfrak{T}_{11}(\tau)) : \tau \in Z\}$, $\zeta_{\rho_{12}} = \{(\mathfrak{L}_{12}(\tau), \mathfrak{D}_{12}(\tau), \mathfrak{T}_{12}(\tau)) : \tau \in Z\}$, and $\zeta_{\rho_{13}} = \{(\mathfrak{L}_{13}(\tau), \mathfrak{D}_{13}(\tau), \mathfrak{T}_{13}(\tau)) : \tau \in Z\}$, the following properties hold

- (1) $\mathcal{R}(\zeta_{\rho_{11}} \oplus \zeta_{\rho_{12}}) = \mathcal{R}\zeta_{\rho_{11}} \oplus \mathcal{R}\zeta_{\rho_{12}}$
- (2) $(\zeta_{\rho_{11}} \otimes \zeta_{\rho_{12}})^{\mathcal{R}} = \zeta_{\rho_{11}}^{\mathcal{R}} \otimes \zeta_{\rho_{12}}^{\mathcal{R}}$
- (3) $\mathcal{R}_1\zeta_{\rho} \oplus \mathcal{R}_2\zeta_{\rho} = (\mathcal{R}_1 + \mathcal{R}_2)\zeta_{\rho}$
- (4) $\zeta_{\rho}^{\mathcal{R}_1} \otimes \zeta_{\rho}^{\mathcal{R}_2} = \zeta_{\rho}^{(\mathcal{R}_1 + \mathcal{R}_2)}.$

Proofs of Theorems 1, 2, and 3 are straightforward, so we will omit their proofs.

4. Picture Fuzzy Soft Prioritized Average and Geometric Aggregation Operators

The fundamental description of PFS_{ft} prioritized average (PFS_{ft}PA) aggregation operators is in this section. Additionally, the characteristics of these newly introduced operators have been described.

4.1. Picture Fuzzy Soft Prioritized Average (PFS_{ft}PA) Aggregation Operators

In this subsection, we have to present PFS_{ft}PA aggregation operators. Furthermore, the properties of these operators are described.

Definition 12. Let $\zeta_{\rho_{r\mathfrak{f}}} = (\mathfrak{L}_{r\mathfrak{f}}, \mathfrak{D}_{r\mathfrak{f}}, \mathfrak{T}_{r\mathfrak{f}})$ denote the family of PFS_{ft}Ns, ($r = 1, 2, \dots, \mathfrak{K}$; $\mathfrak{f} = 1, 2, \dots, \mathfrak{n}$). Then, picture fuzzy soft prioritized weighted average aggregation operator is defined by

$$PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \oplus_{r=1}^{\mathfrak{K}} \frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r} \left(\oplus_{\mathfrak{f}=1}^{\mathfrak{n}} \frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}} \zeta_{\rho_{r\mathfrak{f}}} \right) \quad (1)$$

where $\mathcal{F}_1 = 1$, $\mathfrak{h}_1 = 1$, $\mathcal{F}_{\mathfrak{f}} = \prod_{t=1}^{\mathfrak{f}-1} Sc(\zeta_{\rho_{rt}})$, $\mathfrak{f} = 2, 3, \dots, \mathfrak{n}$, $\mathfrak{h}_r = \prod_{o=1}^{r-1} Sc(\zeta_{\rho_o})$ ($r = 2, 3, \dots, \mathfrak{K}$), and $Sc(\zeta_{\rho_{rt}})$ represents the SF of PFS_{ft}Ns.

Now, by using the above-given Definition 12, we can define the following results:

Theorem 4. For a collection of PFS_{ft}Ns and $\zeta_{\rho_{r\mathfrak{f}}} = (\mathfrak{L}_{r\mathfrak{f}}, \mathfrak{D}_{r\mathfrak{f}}, \mathfrak{T}_{r\mathfrak{f}})$, the aggregated result for PFS_{ft}PWA operator is again a PFS_{ft}N given by

$$PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \left(\begin{array}{l} 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (1 - \mathfrak{L}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (\mathfrak{D}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (\mathfrak{T}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \end{array} \right) \quad (2)$$

Proof. For $\mathfrak{n} = 1$, we get

$$\begin{aligned} PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{21}}, \dots, \zeta_{\rho_{\mathfrak{K}1}}) &= \oplus_{r=1}^{\mathfrak{K}} \frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r} \zeta_{\rho_{r1}} \\ &= \left(\begin{array}{l} 1 - \prod_{r=1}^{\mathfrak{K}} \left((1 - \mathfrak{L}_{r1})^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \right), \prod_{r=1}^{\mathfrak{K}} (\mathfrak{D}_{r1})^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} (\mathfrak{T}_{r1})^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \end{array} \right) \\ &= \left(\begin{array}{l} 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^1 (1 - \mathfrak{L}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^1 \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^1 (\mathfrak{D}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^1 \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^1 (\mathfrak{T}_{r\mathfrak{f}})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^1 \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \end{array} \right) \end{aligned}$$

and for $\mathfrak{K} = 1$,

$$\begin{aligned}
 PFS_{ftPWA}(\zeta_{\rho_{11}}, \zeta_{\rho_{21}}, \dots, \zeta_{\rho_{\mathfrak{K}1}}) &= \oplus_{f=1}^{\mathfrak{H}} \frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f} \zeta_{\rho_{1f}} \\
 &= \left(1 - \prod_{f=1}^{\mathfrak{H}} \left((1 - \mathfrak{L}_{1f})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right), \prod_{j=1}^{\mathfrak{n}} (\mathfrak{D}_{1j})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}}, \right. \\
 &\quad \left. \prod_{j=1}^{\mathfrak{n}} (\mathfrak{T}_{1j})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right) \\
 &= \left(1 - \prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{H}} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^1 \mathfrak{H}_r}}, \right. \\
 &\quad \prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{H}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^1 \mathfrak{H}_r}}, \\
 &\quad \left. \prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{H}} (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^1 \mathfrak{H}_r}} \right)
 \end{aligned}$$

Hence, Equation (2) holds for $\mathfrak{H} = 1$ and $\mathfrak{K} = 1$. Assuming that Equation (2) holds for $\mathfrak{K} = \alpha_1 + 1$, $\mathfrak{H} = \alpha_2$, $\mathfrak{K} = \alpha_1$, and $\mathfrak{H} = \alpha_2 + 1$, we get

$$\begin{aligned}
 PFS_{ftPWA}(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) &= \oplus_{r=1}^{\alpha_1+1} \frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r} \left(\oplus_{f=1}^{\alpha_2} \frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f} \zeta_{\rho_{rf}} \right) \\
 &= \left(1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}}, \right. \\
 &\quad \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}}, \\
 &\quad \left. \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 &\oplus_{r=1}^{\alpha_1} \frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r} \left(\oplus_{f=1}^{\alpha_2+1} \frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f} \zeta_{\rho_{rf}} \right) \\
 &= \left(1 - \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}}, \right. \\
 &\quad \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}}, \\
 &\quad \left. \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{H}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{H}_r}} \right)
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \oplus_{r=1}^{\alpha_1+1} \frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r} \left(\oplus_{f=1}^{\alpha_2+1} \frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f} \zeta_{\rho_{rf}} \right) \\
 &= \oplus_{r=1}^{\alpha_1+1} \frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r} \left(\oplus_{f=1}^{\alpha_2} \frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f} \zeta_{\rho_{rf}} \oplus \frac{\mathcal{F}_{\alpha_2+1}}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f} \zeta_{\rho_{(\alpha_2+1)f}} \right) \\
 &= \left(\oplus_{r=1}^{\alpha_1+1} \frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r} \oplus_{f=1}^{\alpha_2} \left(\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f} \zeta_{\rho_{rf}} \right) \right) \oplus_{r=1}^{\alpha_1+1} \frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r} \left(\frac{\mathcal{F}_{\alpha_2+1}}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f} \zeta_{\rho_{(\alpha_2+1)f}} \right) \\
 &= \left(1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \oplus 1 - \prod_{r=1}^{\alpha_1+1} \left(\left(1 - \mathfrak{L}_{(\alpha_2+1)f} \right)^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \right), \\
 & \quad \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \oplus \prod_{r=1}^{\alpha_1+1} \left((\mathfrak{D}_{(\alpha_2+1)f})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \\
 & \quad \oplus \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \oplus \prod_{r=1}^{\alpha_1+1} \left((\mathfrak{T}_{(\alpha_2+1)f})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \\
 &= \left(1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \right), \\
 & \quad \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\
 & \quad \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}
 \end{aligned}$$

Hence, Equation (2) holds for $\mathfrak{x} = \alpha_1 + 1$ and $\mathfrak{u} = \alpha_2 + 1$. So, the result holds for all positive integers $\mathfrak{x}$, $\mathfrak{u} \geq 1$, by mathematical induction. $\square$

4.2. Properties of Picture Fuzzy Soft Prioritized Weighted Average Aggregation Operators

Here in this section, we will discuss some basic properties of $PFS_{ft}PWA$ aggregation operators.

1. (Idempotency): If $\zeta_{\rho_{rf}} = \zeta_{\rho} = (\mathfrak{L}, \mathfrak{D}, \mathfrak{T})$ for all r, f , then

$$PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \zeta_{\rho}.$$

Proof. Proof is trivial. $\square$

2. (Boundedness): If $\zeta_{\rho_{rf}}^- = (\min_r \min_f \{\mathfrak{L}_{rf}\}, \max_r \max_f \{\mathfrak{D}_{rf}\}, \max_r \max_f \{\mathfrak{T}_{rf}\})$ and $\zeta_{\rho_{rf}}^+ = (\max_r \max_f \{\mathfrak{L}_{rf}\}, \min_r \min_f \{\mathfrak{D}_{rf}\}, \min_r \min_f \{\mathfrak{T}_{rf}\})$, then $\zeta_{\rho_{rf}}^- \leq PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq \zeta_{\rho_{rf}}^+$.

Proof. As $\zeta_{\rho_{rf}}$ denote $PFS_{ft}Ns$, then for all r, f

$$\min_r \min_f \{\mathfrak{L}_{rf}\} \leq \{\mathfrak{L}_{rf}\} \leq \max_r \max_f \{\mathfrak{L}_{rf}\} \Leftrightarrow 1 - \max_r \max_f \{\mathfrak{L}_{rf}\} \leq 1 - \mathfrak{L}_{rf} \leq 1 - \min_r \min_f \{\mathfrak{L}_{rf}\}$$

$$\begin{aligned}
&\Leftrightarrow (1 - \max_r \max_f (\mathfrak{L}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \leq (1 - (\mathfrak{L}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \leq (1 - \min_r \min_f (\mathfrak{L}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \\
&\Leftrightarrow (1 - \max_r \max_f (\mathfrak{L}_{rf})) \leq \left(\prod_{f=1}^{\mathfrak{H}} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right) \leq (1 - \max_r \max_f (\mathfrak{L}_{rf})) \\
&\Leftrightarrow (1 - \max_r \max_f (\mathfrak{L}_{rf})) \leq \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathfrak{H}} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (1 - \min_r \min_f (\mathfrak{L}_{rf}))
\end{aligned}$$

Hence,

$$(\min_r \min_f (\mathfrak{L}_{rf})) \leq 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathfrak{H}} (1 - \mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{L}_{rf})) \quad (3)$$

Furthermore,

$$\begin{aligned}
&\min_r \min_f (\mathfrak{D}_{rf}) \leq (\mathfrak{D}_{rf}) \leq \max_r \max_f (\mathfrak{D}_{rf}) \\
&\Leftrightarrow (\min_r \min_f (\mathfrak{D}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \leq (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \leq (\max_r \max_f (\mathfrak{D}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \\
&\Leftrightarrow (\min_r \min_f (\mathfrak{D}_{rf})) \leq \prod_{f=1}^{\mathfrak{H}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \leq (\max_r \max_f (\mathfrak{D}_{rf})) \\
&\Leftrightarrow (\min_r \min_f (\mathfrak{D}_{rf}))^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq \left(\prod_{f=1}^{\mathfrak{H}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{D}_{rf}))^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \\
&\Leftrightarrow (\min_r \min_f (\mathfrak{D}_{rf}))^{\frac{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathfrak{H}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{D}_{rf}))^{\frac{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}
\end{aligned}$$

Hence,

$$(\min_r \min_f (\mathfrak{D}_{rf})) \leq \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathfrak{H}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\mathfrak{H}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{D}_{rf})) \quad (4)$$

Additionally,

$$\begin{aligned}
 \min_r \min_f (\mathfrak{T}_{rf}) &\leq (\mathfrak{T}_{rf}) \leq \max_r \max_f (\mathfrak{T}_{rf}) \\
 \Leftrightarrow (\min_r \min_f (\mathfrak{T}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} &\leq (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \leq (\max_r \max_f (\mathfrak{T}_{rf}))^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \\
 \Leftrightarrow (\min_r \min_f (\mathfrak{T}_{rf})) &\leq \prod_{f=1}^n (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \leq (\max_r \max_f (\mathfrak{T}_{rf})) \\
 \Leftrightarrow (\min_r \min_f (\mathfrak{T}_{rf}))^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} &\leq \left(\prod_{f=1}^n (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{T}_{rf}))^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \\
 \Leftrightarrow (\min_r \min_f (\mathfrak{T}_{rf}))^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} &\leq \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^n (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \\
 &\leq \left(\max_r \max_f (\mathfrak{T}_{rf}) \right)^{\frac{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}
 \end{aligned}$$

Hence,

$$(\min_r \min_f (\mathfrak{T}_{rf})) \leq \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^n (\mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^n \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \leq (\max_r \max_f (\mathfrak{T}_{rf})) \quad (5)$$

Let $\xi = PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = (\mathfrak{L}_{\xi}, \mathfrak{D}_{\xi}, \mathfrak{T}_{\xi})$, then by Equations (3)–(5), we have $\min_r \min_f (\mathfrak{L}_{rf}) \leq \mathfrak{L}_{\xi} \leq \max_r \max_f (\mathfrak{L}_{rf})$, $\min_r \min_f (\mathfrak{D}_{rf}) \leq \mathfrak{D}_{\xi} \leq \max_r \max_f (\mathfrak{D}_{rf})$, and $\min_r \min_f (\mathfrak{T}_{rf}) \leq \mathfrak{T}_{\xi} \leq \max_r \max_f (\mathfrak{T}_{rf})$. Now

$$Sc(\xi) = \frac{(1 + \mathfrak{L}_{\xi} - \mathfrak{T}_{\xi})}{2} \leq \left(\frac{1 + \max_r \max_f (\mathfrak{L}_{rf}) - \min_r \min_f (\mathfrak{T}_{rf})}{2} \right) = Sc(\zeta^+_{\rho_{rf}}),$$

and

$$\begin{aligned}
 Sc(\xi) &= \frac{(1 + \mathfrak{L}_{\xi} - \mathfrak{T}_{\xi})}{2} \geq \left(\frac{1 + \min_r \min_f (\mathfrak{L}_{rf}) - \max_r \max_f (\mathfrak{T}_{rf})}{2} \right) = Sc(\zeta^-_{\rho_{rf}}) \\
 &\Rightarrow Sc(\xi) \geq Sc(\zeta^-_{\rho_{rf}}). \square
 \end{aligned}$$

Now, the following cases arise

Case 1: If $Sc(\zeta_{\rho_{rf}}) < Sc(\zeta^+_{\rho_{rf}})$ and $Sc(\zeta_{\rho_{rf}}) > Sc(\zeta^-_{\rho_{rf}})$, then by the comparison law for the two $PFS_{ft}Ns$, we have

$$\zeta^-_{\rho_{rf}} \leq PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq \zeta^+_{\rho_{rf}}$$

Case 2: If $Sc(\zeta_{\rho_{rf}}) = Sc(\zeta^+_{\rho_{rf}})$, that is $\frac{(1 + \mathfrak{L}_{\xi} - \mathfrak{T}_{\xi})}{2} = \left(\frac{1 + \max_r \max_f (\mathfrak{L}_{rf}) - \min_r \min_f (\mathfrak{T}_{rf})}{2} \right)$, then by using the above inequalities, we get $\mathfrak{L}_{\xi} = \max_r \max_f (\mathfrak{L}_{rf})$, $\mathfrak{D}_{\xi} = \min_r \min_f (\mathfrak{D}_{rf})$, and $\mathfrak{T}_{\xi} = \min_r \min_f (\mathfrak{T}_{rf})$.

Thus, $Ac(\xi) = \mathfrak{L}_{\xi} - \mathfrak{T}_{\xi} = \max_r \max_f (\mathfrak{L}_{rf}) + \min_r \min_f (\mathfrak{D}_{rf}) + \min_r \min_f (\mathfrak{T}_{rf}) = Ac(\zeta^+_{\rho_{rf}})$; this implies that

$$PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = (\zeta^+_{\rho_{rf}}).$$

Case 3: If $Sc(\zeta_{\rho_{rf}}) = Sc(\zeta^-_{\rho_{rf}})$, that is, $\mathfrak{L}_{\xi} + \mathfrak{D}_{\xi} + \mathfrak{T}_{\xi} = \left(\frac{1 + \min_r \min_f (\mathfrak{L}_{rf}) - \max_r \max_f (\mathfrak{T}_{rf})}{2} \right)$, then by using the above inequalities, we get $\mathfrak{L}_{\xi} = \min_r \min_f (\mathfrak{L}_{rf})$, $\mathfrak{D}_{\xi} = \max_r \max_f (\mathfrak{D}_{rf})$ and $\mathfrak{T}_{\xi} = \max_r \max_f (\mathfrak{T}_{rf})$.

Thus, $Ac(\xi) = \mathfrak{L}_\xi - \mathfrak{T}_\xi = \min_{\mathfrak{f}} \min_{\mathfrak{r}} (\mathfrak{L}_{\mathfrak{r}\mathfrak{f}}) + \max_{\mathfrak{f}} \max_{\mathfrak{r}} (\mathfrak{D}_{\mathfrak{r}\mathfrak{f}}) + \max_{\mathfrak{f}} \max_{\mathfrak{r}} (\mathfrak{T}_{\mathfrak{r}\mathfrak{f}}) = Ac(\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}}^-)$; this implies that

$$PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = (\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}}^-).$$

3. Monotonicity: Let $\hat{\zeta}_{\rho_{\mathfrak{r}\mathfrak{f}}} = (\hat{\mathfrak{L}}_{\mathfrak{r}\mathfrak{f}}, \hat{\mathfrak{D}}_{\mathfrak{r}\mathfrak{f}}, \hat{\mathfrak{T}}_{\mathfrak{r}\mathfrak{f}})$ be any other family of the $PFS_{ft}Ns$, such that $\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}} \leq \hat{\zeta}_{\rho_{\mathfrak{r}\mathfrak{f}}}$ for all $\mathfrak{r}, \mathfrak{f}$, then the $PFS_{ft}PWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq PFS_{ft}PWA(\hat{\zeta}_{\rho_{11}}, \hat{\zeta}_{\rho_{12}}, \dots, \hat{\zeta}_{\rho_{mn}})$.

Proof. The proof is similar to the proof of property 2. $\square$

4.3. Picture Fuzzy Soft Prioritized Ordered Weighted Average ($PFS_{ft}POWA$) Aggregation Operator

The fundamental definition of $PFS_{ft}POWA$ aggregation operators and their characteristics are covered in this subsection.

Definition 13. Let $\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}} = (\mathfrak{L}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{D}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{T}_{\mathfrak{r}\mathfrak{f}})$ denote the family of the $PFS_{ft}Ns$, where $(\mathfrak{r} = 1, 2, \dots, \mathfrak{n}; \mathfrak{f} = 1, 2, \dots, \mathfrak{K})$. Then, picture fuzzy soft prioritized ordered weighted average aggregation operator is defined by

$$PFS_{ft}POWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \oplus_{\mathfrak{r}=1}^{\mathfrak{K}} \frac{\mathfrak{h}_{\mathfrak{r}}}{\sum_{\mathfrak{r}=1}^{\mathfrak{K}} \mathfrak{h}_{\mathfrak{r}}} \left(\oplus_{\mathfrak{f}=1}^{\mathfrak{n}} \frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}} \zeta_{\rho_{d(\mathfrak{r})\mathfrak{t}(\mathfrak{f})}} \right) \quad (6)$$

where $\mathcal{F}_1 = 1; \mathfrak{h}_1 = 1; \mathcal{F}_{\mathfrak{f}} = \prod_{\mathfrak{t}=1}^{\mathfrak{f}-1} Sc(\zeta_{\rho_{\mathfrak{r}\mathfrak{t}(\mathfrak{t})}})$, where $\mathfrak{f} = 2, 3, \dots, \mathfrak{n}$; and $\mathfrak{h}_{\mathfrak{r}} = \prod_{\mathfrak{o}=1}^{\mathfrak{r}-1} Sc(\zeta_{\rho_{d(\mathfrak{o})}})$ ($\mathfrak{r} = 2, 3, \dots, \mathfrak{K}$), where $Sc(\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}})$ represents the SF of $PFS_{ft}Ns$. Furthermore, d and $\mathfrak{t}$ are the permutations of $(1, 2, \dots, \mathfrak{K})$ and $(1, 2, 3, \dots, \mathfrak{n})$, such that $\rho_{d(\mathfrak{r})\mathfrak{f}} \geq \rho_{d(\mathfrak{r}-1)\mathfrak{f}}$ and $\rho_{\mathfrak{r}\mathfrak{t}(\mathfrak{f}1)} \geq \rho_{\mathfrak{r}\mathfrak{t}(\mathfrak{f}-1)}$ for $\mathfrak{r} = 2, 3, \dots, \mathfrak{K}$ and $\mathfrak{j} = 2, 3, \dots, \mathfrak{n}$.

Theorem 5. For a family of $PFS_{ft}Ns$, $\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}} = (\mathfrak{L}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{D}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{T}_{\mathfrak{r}\mathfrak{f}})$, the aggregated result for the $PFS_{ft}POWA$ operator is again a $PFS_{ft}N$, given by

$$PFS_{ft}POWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{\mathfrak{n}\mathfrak{K}}}) = \left(\begin{array}{l} 1 - \prod_{\mathfrak{r}=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (1 - \mathfrak{L}_{d(\mathfrak{r})\mathfrak{t}(\mathfrak{f})})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_{\mathfrak{r}}}{\sum_{\mathfrak{r}=1}^{\mathfrak{K}} \mathfrak{h}_{\mathfrak{r}}}}, \\ \prod_{\mathfrak{r}=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (\mathfrak{D}_{d(\mathfrak{r})\mathfrak{t}(\mathfrak{f})})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_{\mathfrak{r}}}{\sum_{\mathfrak{r}=1}^{\mathfrak{K}} \mathfrak{h}_{\mathfrak{r}}}}, \\ \prod_{\mathfrak{r}=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{f}=1}^{\mathfrak{n}} (\mathfrak{T}_{d(\mathfrak{r})\mathfrak{t}(\mathfrak{f})})^{\frac{\mathcal{F}_{\mathfrak{f}}}{\sum_{\mathfrak{f}=1}^{\mathfrak{n}} \mathcal{F}_{\mathfrak{f}}}} \right)^{\frac{\mathfrak{h}_{\mathfrak{r}}}{\sum_{\mathfrak{r}=1}^{\mathfrak{K}} \mathfrak{h}_{\mathfrak{r}}}} \end{array} \right) \quad (7)$$

where $\mathcal{F}_1 = 1; \mathfrak{h}_1 = 1; \mathcal{F}_{\mathfrak{f}} = \prod_{\mathfrak{t}=1}^{\mathfrak{f}-1} Sc(\zeta_{\rho_{\mathfrak{r}\mathfrak{t}(\mathfrak{t})}})$, where $\mathfrak{f} = 2, 3, \dots, \mathfrak{n}$; and $\mathfrak{h}_{\mathfrak{r}} = \prod_{\mathfrak{o}=1}^{\mathfrak{r}-1} Sc(\zeta_{\rho_{d(\mathfrak{o})}})$ ($\mathfrak{r} = 2, 3, \dots, \mathfrak{K}$), where $Sc(\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}})$ represents the SF of $PFS_{ft}Ns$.

Proof. The proof is similar to the proof of Theorem 4. $\square$

Similar to the $PFS_{ft}PWA$ aggregation operators, the $PFS_{ft}POWA$ aggregation operator also satisfies some properties given by

1. Idempotency: If $\zeta_{\rho_{r\mathfrak{F}}} = \zeta_{\rho} = (\mathfrak{L}, \mathfrak{D}, \mathfrak{T})$ for all $r, \mathfrak{F}$, then

$$PFS_{ft}POWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \zeta_{\rho}.$$

2. Boundedness: If $\zeta_{\rho_{r\mathfrak{F}}}^- = (\min_r \min_{\mathfrak{F}} \{\mathfrak{L}_{r\mathfrak{F}}\}, \max_r \max_{\mathfrak{F}} \{\mathfrak{D}_{r\mathfrak{F}}\}, \max_r \max_{\mathfrak{F}} \{\mathfrak{T}_{r\mathfrak{F}}\})$ and $\zeta_{\rho_{r\mathfrak{F}}}^+ = (\max_r \max_{\mathfrak{F}} \{\mathfrak{L}_{r\mathfrak{F}}\}, \min_r \min_{\mathfrak{F}} \{\mathfrak{D}_{r\mathfrak{F}}\}, \min_r \min_{\mathfrak{F}} \{\mathfrak{T}_{r\mathfrak{F}}\})$, then $\zeta_{\rho_{r\mathfrak{F}}}^- \leq PFS_{ft}POWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq \zeta_{\rho_{r\mathfrak{F}}}^+$.
3. Monotonicity: Let $\tilde{\zeta}_{\rho_{r\mathfrak{F}}} = (\tilde{\mathfrak{L}}_{r\mathfrak{F}}, \tilde{\mathfrak{D}}_{r\mathfrak{F}}, \tilde{\mathfrak{T}}_{r\mathfrak{F}})$ be any other family of the $PFS_{ft}Ns$, such that $\zeta_{\rho_{r\mathfrak{F}}} \leq \tilde{\zeta}_{\rho_{r\mathfrak{F}}}$ for all $r, \mathfrak{F}$, then $PFS_{ft}POWA(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq PFS_{ft}POWA(\tilde{\zeta}_{\rho_{11}}, \tilde{\zeta}_{\rho_{12}}, \dots, \tilde{\zeta}_{\rho_{mn}})$.

5. Picture Fuzzy Prioritized Weighted Geometric ($PFS_{ft}PWG$) Aggregation Operators

In this part of the article, we will discuss the basic definition and properties of $PFS_{ft}PWG$ aggregation operators.

Definition 14. Let $\zeta_{\rho_{r\mathfrak{F}}} = (\mathfrak{L}_{r\mathfrak{F}}, \mathfrak{D}_{r\mathfrak{F}}, \mathfrak{T}_{r\mathfrak{F}})$ denote the family of $S_{ft}Ns$ ($r = 1, 2, \dots, \mathfrak{u}$; $\mathfrak{F} = 1, 2, \dots, \mathfrak{K}$). Then, picture fuzzy soft prioritized weighted geometric aggregation operator is defined by

$$PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \otimes_{r=1}^{\mathfrak{K}} \frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r} \left(\otimes_{\mathfrak{F}=1}^{\mathfrak{u}} \frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{u}} \mathcal{F}_{\mathfrak{F}}} \zeta_{\rho_{r\mathfrak{F}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \quad (8)$$

where $\mathcal{F}_1 = 1$; $\mathfrak{h}_1 = 1$; $\mathcal{F}_{\mathfrak{F}} = \prod_{t=1}^{\mathfrak{F}-1} Sc(\zeta_{\rho_{rt}})$, where $\mathfrak{F} = 2, 3, \dots, \mathfrak{u}$; and $\mathfrak{h}_r = \prod_{o=1}^{r-1} Sc(\zeta_{\rho_o})$ ($r = 2, 3, \dots, \mathfrak{K}$), where $Sc(\zeta_{\rho_{r\mathfrak{F}}})$ represents the SF of $PFS_{ft}Ns$.

Theorem 6. For a family of $PFS_{ft}Ns$, $\zeta_{\rho_{r\mathfrak{F}}} = (\mathfrak{L}_{r\mathfrak{F}}, \mathfrak{D}_{r\mathfrak{F}}, \mathfrak{T}_{r\mathfrak{F}})$, the aggregated result for $PFS_{ft}PWG$ operator is again a $PFS_{ft}N$, given by

$$PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{\mathfrak{u}\mathfrak{K}}}) = \left(\begin{array}{l} \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{u}} (\mathfrak{L}_{r\mathfrak{F}})^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{u}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{u}} (\mathfrak{D}_{r\mathfrak{F}})^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{u}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{u}} (1 - \mathfrak{T}_{r\mathfrak{F}})^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{u}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \end{array} \right) \quad (9)$$

Proof. For $\mathfrak{u} = 1$, we get

$$\begin{aligned}
PFS_{ft}PWG\left(\zeta_{\rho_{11}}, \zeta_{\rho_{21}}, \dots, \zeta_{\rho_{\mathfrak{K}1}}\right) &= \otimes_{r=1}^{\mathfrak{K}} \zeta_{\rho_{r1}}^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \\
&= \left(\prod_{r=1}^{\mathfrak{K}} (\mathfrak{L}_{r1})^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \prod_{r=1}^{\mathfrak{K}} (\mathfrak{D}_{r1})^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \right. \\
&\quad \left. 1 - \prod_{r=1}^{\mathfrak{K}} \left((1 - \mathfrak{T}_{r1})^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \right) \right) \\
&= \left(\prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathcal{F}_f} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^1 \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \right. \\
&\quad \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathcal{F}_f} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^1 \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\
&\quad \left. 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{f=1}^{\mathcal{F}_f} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^1 \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \right)
\end{aligned}$$

and for $\mathfrak{K} = 1$,

$$\begin{aligned}
PFS_{ft}PWG\left(\zeta_{\rho_{11}}, \zeta_{\rho_{21}}, \dots, \zeta_{\rho_{1n}}\right) &= \otimes_{f=1}^{\mathfrak{n}} \zeta_{\rho_{1f}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}} \\
&= \left(\prod_{j=1}^{\mathfrak{n}} (\mathfrak{L}_{1j})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}}, \prod_{j=1}^{\mathfrak{n}} (\mathfrak{D}_{1j})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}}, \right. \\
&\quad \left. 1 - \prod_{f=1}^{\mathfrak{n}} \left((1 - \mathfrak{T}_{1f})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}} \right) \right) \\
&= \left(\prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{n}} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^1 \mathfrak{h}_r}}, \right. \\
&\quad \prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{n}} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^1 \mathfrak{h}_r}}, \\
&\quad \left. 1 - \prod_{r=1}^1 \left(\prod_{f=1}^{\mathfrak{n}} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\mathfrak{n}} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^1 \mathfrak{h}_r}} \right)
\end{aligned}$$

Hence, Equation (2) holds for $\mathfrak{n} = 1$ and $\mathfrak{K} = 1$. Assume that Equation (2) holds for $\mathfrak{K} = \alpha_1 + 1$, $\mathfrak{n} = \alpha_2$, $\mathfrak{K} = \alpha_1$, and $\mathfrak{n} = \alpha_2 + 1$, we get

$$\begin{aligned}
PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) &= \otimes_{r=1}^{\alpha_1+1} \left(\otimes_{f=1}^{\alpha_2} \zeta_{\rho_{rf}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \\
&= \left(\begin{array}{c} \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ 1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \end{array} \right)
\end{aligned}$$

and

$$\otimes_{r=1}^{\alpha_1} \left(\otimes_{f=1}^{\alpha_2+1} \zeta_{\rho_{rf}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1} \mathfrak{h}_r}} = \left(\begin{array}{c} \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1} \mathfrak{h}_r}}, \\ 1 - \prod_{r=1}^{\alpha_1} \left(\prod_{f=1}^{\alpha_2+1} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1} \mathfrak{h}_r}} \end{array} \right)$$

Now,

$$\begin{aligned}
&\otimes_{r=1}^{\alpha_1+1} \left(\otimes_{f=1}^{\alpha_2+1} \zeta_{\rho_{rf}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \\
&= \otimes_{r=1}^{\alpha_1+1} \left(\otimes_{f=1}^{\alpha_2} \zeta_{\rho_{rf}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \otimes \zeta_{\rho_{(\alpha_2+1)f}}^{\frac{\mathcal{F}_{\alpha_2+1}}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \\
&= \left(\otimes_{r=1}^{\alpha_1+1} \left(\otimes_{f=1}^{\alpha_2} \left(\zeta_{\rho_{rf}}^{\frac{\mathcal{F}_f}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right) \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \right) \otimes_{r=1}^{\alpha_1+1} \left(\zeta_{\rho_{(\alpha_2+1)f}}^{\frac{\mathcal{F}_{\alpha_2+1}}{\Sigma_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\Sigma_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}
\end{aligned}$$

$$\begin{aligned}
&= \left(\begin{array}{l} \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \otimes \prod_{r=1}^{\alpha_1+1} \left((\mathfrak{L}_{(\alpha_2+1)f})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \otimes \prod_{r=1}^{\alpha_1+1} \left((\mathfrak{D}_{(\alpha_2+1)f})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ 1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \otimes \prod_{r=1}^{\alpha_1+1} \left((1 - \mathfrak{T}_{(\alpha_2+1)f})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \end{array} \right) \\
&= \left(\begin{array}{l} \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{L}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (\mathfrak{D}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}}, \\ 1 - \prod_{r=1}^{\alpha_1+1} \left(\prod_{f=1}^{\alpha_2+1} (1 - \mathfrak{T}_{rf})^{\frac{\mathcal{F}_f}{\sum_{f=1}^{\alpha_2+1} \mathcal{F}_f}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\alpha_1+1} \mathfrak{h}_r}} \end{array} \right)
\end{aligned}$$

Hence, Equation (2) holds for $\mathfrak{K} = \alpha_1 + 1$ and $\mathfrak{H} = \alpha_2 + 1$. So, the result holds for all positive integers of $\mathfrak{K}$, $\mathfrak{H} \geq 1$, by mathematical induction. $\square$

5.1. Properties of Picture Fuzzy Soft Prioritized Weighted Geometric Aggregation Operators

The characteristics of $PFS_{ft}PWG$ aggregation operators are covered in this subsection.

1. Idempotency: If $\zeta_{\rho_{rf}} = \zeta_{\rho} = (\mathfrak{L}, \mathfrak{D}, \mathfrak{T})$ for all r, f , then

$$PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{\mathfrak{H}\mathfrak{K}}}) = \zeta_{\rho}.$$

2. Boundedness: If $\zeta_{\rho_{rf}}^- = (\min_r \min_f \{\mathfrak{L}_{rf}\}, \min_r \min_f \{\mathfrak{D}_{rf}\}, \max_r \max_f \{\mathfrak{T}_{rf}\})$ and $\zeta_{\rho_{rf}}^+ = (\max_r \max_f \{\mathfrak{L}_{rf}\}, \max_r \max_f \{\mathfrak{D}_{rf}\}, \min_r \min_f \{\mathfrak{T}_{rf}\})$, then $\zeta_{\rho_{rf}}^- \leq PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq \zeta_{\rho_{rf}}^+$.
3. Monotonicity: Let $\zeta_{\rho_{rf}} = (\mathfrak{L}_{rf}, \mathfrak{D}_{rf}, \mathfrak{T}_{rf})$ be any other family of $PFS_{ft}Ns$, such that $\zeta_{\rho_{rf}} \leq \zeta_{\rho_{rf}}$ for all r, f , then $PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq PFS_{ft}PWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}})$.

Proof. The proof is similar to the proof in Section 4.2. $\square$

5.2. Picture Fuzzy Soft Prioritized Ordered Weighted Geometric ($PFS_{ft}POWG$) Aggregation Operator

The notions of $PFS_{ft}POWG$ aggregation operators are defined here, and their characteristics are covered in this subsection.

Definition 15. Let $\zeta_{\rho_{r\mathfrak{F}}} = (\mathfrak{L}_{r\mathfrak{F}}, \mathfrak{D}_{r\mathfrak{F}}, \mathfrak{T}_{r\mathfrak{F}})$ denote the family of $S_{ft}Ns$ ($r = 1, 2, \dots, \mathfrak{H}$; $\mathfrak{F} = 1, 2, \dots, \mathfrak{K}$). Then, picture fuzzy soft prioritized ordered weighted geometric aggregation operator is defined by

$$PFS_{ft}POWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \otimes_{r=1}^{\mathfrak{K}} \left(\otimes_{\mathfrak{F}=1}^{\mathfrak{H}} \zeta_{\rho_{d(r)\mathfrak{t}(\mathfrak{F})}}^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{H}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \quad (10)$$

where $\mathcal{F}_1 = 1$; $\mathfrak{h}_1 = 1$; $\mathcal{F}_{\mathfrak{F}} = \prod_{t=1}^{\mathfrak{F}-1} Sc(\zeta_{\rho_{r\mathfrak{t}(t)}})$, where $\mathfrak{F} = 2, 3, \dots, \mathfrak{H}$; and $\mathfrak{h}_r = \prod_{o=1}^{r-1} Sc(\zeta_{\rho_{d(o)}})$ ($r = 2, 3, \dots, \mathfrak{K}$), where $Sc(\zeta_{\rho_{r\mathfrak{F}}})$ represents the SF of $PFS_{ft}Ns$. Additionally, d and $\mathfrak{t}$ are the permutations of $(1, 2, \dots, \mathfrak{K})$ and $(1, 2, 3, \dots, \mathfrak{H})$, such that $\rho_{d(r)\mathfrak{F}} \geq \rho_{d(r-1)\mathfrak{F}}$ and $\rho_{r\mathfrak{t}(\mathfrak{F})} \geq \rho_{r\mathfrak{t}(\mathfrak{F}-1)}$ for $r = 2, 3, \dots, \mathfrak{K}$ and $\mathfrak{F} = 2, 3, \dots, \mathfrak{H}$.

Theorem 7. For a family of $PFS_{ft}Ns$, $\zeta_{\rho_{r\mathfrak{F}}} = (\mathfrak{L}_{r\mathfrak{F}}, \mathfrak{D}_{r\mathfrak{F}}, \mathfrak{T}_{r\mathfrak{F}})$, the aggregated result for $PFS_{ft}POWG$ operator is again a $PFS_{ft}N$, given by

$$PFS_{ft}POWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) = \left(\begin{array}{c} \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{H}} \left(\mathfrak{L}_{d(r)\mathfrak{t}(\mathfrak{F})}^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{H}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \right. \\ \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{H}} \left(\mathfrak{D}_{d(r)\mathfrak{t}(\mathfrak{F})}^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{H}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}}, \\ \left. 1 - \prod_{r=1}^{\mathfrak{K}} \left(\prod_{\mathfrak{F}=1}^{\mathfrak{H}} \left(1 - \mathfrak{T}_{d(r)\mathfrak{t}(\mathfrak{F})}^{\frac{\mathcal{F}_{\mathfrak{F}}}{\sum_{\mathfrak{F}=1}^{\mathfrak{H}} \mathcal{F}_{\mathfrak{F}}}} \right)^{\frac{\mathfrak{h}_r}{\sum_{r=1}^{\mathfrak{K}} \mathfrak{h}_r}} \right) \right) \end{array} \right) \quad (11)$$

Proof. The proof is similar to the proof of Theorem 4. $\square$

5.3. Properties of Picture Fuzzy Soft Prioritized Ordered Weighted Geometric Aggregation Operators

Similar to the $PFS_{ft}PWG$ aggregation operators, the $PFS_{ft}POWG$ aggregation operator also satisfies some properties given by

1. Idempotency: If $\zeta_{\rho_{r\mathfrak{F}}} = \zeta_{\rho} = (\mathfrak{L}, \mathfrak{D}, \mathfrak{T})$ for all $r, \mathfrak{F}$, then

$$PFS_{ft}POWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{n\mathfrak{K}}}) = \zeta_{\rho}.$$

2. Boundedness: If $\zeta_{\rho_{r\mathfrak{F}}}^- = (\min_r \min_{\mathfrak{F}} \{\mathfrak{L}_{r\mathfrak{F}}\}, \min_r \min_{\mathfrak{F}} \{\mathfrak{D}_{r\mathfrak{F}}\}, \max_r \max_{\mathfrak{F}} \{\mathfrak{T}_{r\mathfrak{F}}\})$ and $\zeta_{\rho_{r\mathfrak{F}}}^+ = (\max_r \max_{\mathfrak{F}} \{\mathfrak{L}_{r\mathfrak{F}}\}, \max_r \max_{\mathfrak{F}} \{\mathfrak{D}_{r\mathfrak{F}}\}, \min_r \min_{\mathfrak{F}} \{\mathfrak{T}_{r\mathfrak{F}}\})$, then $\zeta_{\rho_{r\mathfrak{F}}}^- \leq PFS_{ft}POWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq \zeta_{\rho_{r\mathfrak{F}}}^+$.
3. Monotonicity: Let $\tilde{\zeta}_{\rho_{r\mathfrak{F}}} = (\tilde{\mathfrak{L}}_{r\mathfrak{F}}, \tilde{\mathfrak{D}}_{r\mathfrak{F}}, \tilde{\mathfrak{T}}_{r\mathfrak{F}})$ be any other family of $PFS_{ft}Ns$, such that $\zeta_{\rho_{r\mathfrak{F}}} \leq \tilde{\zeta}_{\rho_{r\mathfrak{F}}}$ for all $r, \mathfrak{F}$, then $PFS_{ft}POWG(\zeta_{\rho_{11}}, \zeta_{\rho_{12}}, \dots, \zeta_{\rho_{mn}}) \leq PFS_{ft}POWG(\tilde{\zeta}_{\rho_{11}}, \tilde{\zeta}_{\rho_{12}}, \dots, \tilde{\zeta}_{\rho_{mn}})$.

6. Algorithm for Proposed Prioritized Aggregation Operators

In this part of the article, we will introduce a stepwise algorithm for the proposed work.

Let $\mathfrak{A} = \{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \dots, \mathfrak{A}_{\mathfrak{h}}\}$ denote the set of different alternatives. Additionally, suppose that $\rho = \{\rho_1, \rho_2, \rho_3, \dots, \rho_n\}$ denote the set of parameters, and $Y = \{Y_1, Y_2, Y_3, \dots, Y_m\}$ denote the set of experts. Suppose experts provide their assessment of alternatives $\mathfrak{A}_c (c = 1, 2, 3, \dots, \mathfrak{h})$ corresponding to each parameter

$\rho_{\mathfrak{f}}$ ($\mathfrak{f} = 1, 2, 3, \dots, \mathfrak{n}$) in the form of $PFS_{ft}Ns$ $\zeta_{\rho_{\mathfrak{f}}} = (\mathfrak{L}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{D}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{T}_{\mathfrak{r}\mathfrak{f}})$. Now, the step-wise algorithm is given as follows

Step 1: Collect the information about each alternative provided by the decision analyst in the form of $PFS_{ft}Ns$ corresponding to each parameter.

Step 2: Normalize the information given in step 1 by using the formula given below

$$\mathcal{W}_{\mathfrak{r}\mathfrak{f}} = \begin{cases} \zeta_{\rho_{\mathfrak{r}\mathfrak{f}}} ; \text{ For cost type parameter} \\ (\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}})^c ; \text{ For benefit type parameter} \end{cases}$$

where $(\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}})^c$ is the complement of $\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}} = (\mathfrak{L}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{D}_{\mathfrak{r}\mathfrak{f}}, \mathfrak{T}_{\mathfrak{r}\mathfrak{f}})$.

Step 3: Calculate $\mathcal{F}_{\mathfrak{f}}$ ($\mathfrak{f} = 1, 2, 3, \dots, \mathfrak{n}$) and $\mathfrak{h}_{\mathfrak{r}}$ ($\mathfrak{r} = 1, 2, 3, \dots, \mathfrak{k}$) by using $\mathcal{F}_1 = 1$; $\mathfrak{h}_1 = 1$; $\mathcal{F}_{\mathfrak{f}} = \prod_{t=1}^{\mathfrak{f}-1} Sc(\zeta_{\rho_{\mathfrak{r}t}})$, where $\mathfrak{f} = 2, 3, \dots, \mathfrak{n}$; and $\mathfrak{h}_{\mathfrak{r}} = \prod_{o=1}^{\mathfrak{r}-1} Sc(\zeta_{\rho_o})$ ($\mathfrak{r} = 2, 3, \dots, \mathfrak{k}$), where $Sc(\zeta_{\rho_{\mathfrak{r}\mathfrak{f}}})$ represents the SF of $PFS_{ft}Ns$.

Step 4: Aggregate $PFS_{ft}Ns$ for each alternative $\mathfrak{A}_c$ ($c = 1, 2, 3, \dots, \mathfrak{h}$) into a collective preference value $\mathcal{W}^c$ by using $PFS_{ft}PWA$, $PFS_{ft}POWA$, $PFS_{ft}PWG$, or $PFS_{ft}POWG$ aggregation operators.

Step 5: By using the definition of the score function, find out the score values for each alternative $\mathcal{W}^c$.

Step 6: Rank all alternatives and choose the best one.

6.1. Numerical Example

The medical field has become complicated due to the complexities of diseases. Furthermore, the decision analyst takes difficulty in making their decision about the diagnosis of some diseases. So, there is a need to ensure some valuable techniques that can handle the issues of decision analysts working in the medical field. As fuzzy set theory has a wide range of applications in medical fields, it assists—to some extent—the decision-makers working in this field. So, devoted to this idea, in this section, we will provide a numerical example of a medical diagnosis that shows that our work can be used in the medical field more effectively.

6.2. Medical Diagnosis

Example 1: Suppose a team of five senior doctors— $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$, $\mathbb{D}_4$, and $\mathbb{D}_5$ —will provide their assessment of four different patients based on the parameter set given as $\rho = \{\rho_1 = \text{chest pain}, \rho_2 = \text{vomiting}, \rho_3 = \text{fever}, \rho_4 = \text{cough}, \rho_5 = \text{fatigue}\}$. Suppose doctors give their assessment in the form of $PFS_{ft}Ns$ corresponding to each parameter. Now we can use a stepwise algorithm to solve the problem.

By using $PFS_{ft}PWA$ aggregation operators.

Step 1: Doctors provide their evaluation information in the form of $PFS_{ft}Ns$, according to each parameter given in Tables 1–4.

Table 1. PFS_{ft} data for patient $\mathfrak{U}_1$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
$\mathbb{D}_1$	$\begin{pmatrix} 0.13, 0.23, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.11, 0.28, \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} 0.17, 0.16, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.17, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.17, \\ 0.16 \end{pmatrix}$
$\mathbb{D}_2$	$\begin{pmatrix} 0.16, 0.17, \\ 0.18 \end{pmatrix}$	$(0.1, 0.2, 0.3)$	$\begin{pmatrix} 0.29, 0.27, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.23, 0.18, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.34, \\ 0.33 \end{pmatrix}$
$\mathbb{D}_3$	$\begin{pmatrix} 0.21, 0.22, \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.3, 0.31, \\ 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.14, \\ 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.19, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.36, 0.41, \\ 0.1 \end{pmatrix}$
$\mathbb{D}_4$	$\begin{pmatrix} 0.19, 0.34, \\ 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.26, \\ 0.41 \end{pmatrix}$	$\begin{pmatrix} 0.24, 0.17, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.17, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.19, \\ 0.31 \end{pmatrix}$
$\mathbb{D}_5$	$\begin{pmatrix} 0.12, 0.27, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.18, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.13, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.21, \\ 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.32, 0.15, \\ 0.16 \end{pmatrix}$

Table 2. PFS_{ft} data for patient $\mathfrak{U}_2$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
$\mathbb{D}_1$	$\begin{pmatrix} 0.11, 0.28, \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.17, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.17, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.17, 0.16, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.23, \\ 0.25 \end{pmatrix}$
$\mathbb{D}_2$	$\begin{pmatrix} 0.1, 0.2, \\ 0.3 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.34, \\ 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.23, 0.18, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.29, 0.27, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.17, \\ 0.18 \end{pmatrix}$
$\mathbb{D}_3$	$\begin{pmatrix} 0.3, 0.31, \\ 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.36, 0.41, \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.19, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.14, \\ 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.22, \\ 0.28 \end{pmatrix}$
$\mathbb{D}_4$	$\begin{pmatrix} 0.25, 0.26, \\ 0.41 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.19, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.17, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.24, 0.17, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.34, \\ 0.33 \end{pmatrix}$
$\mathbb{D}_5$	$\begin{pmatrix} 0.22, 0.18, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.32, 0.15, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.21, \\ 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.13, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.27, \\ 0.21 \end{pmatrix}$

Table 3. PFS_{ft} data for patient $\mathfrak{U}_3$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
$\mathbb{D}_1$	$\begin{pmatrix} 0.12, 0.33, \\ 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.31, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.12, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.15, \\ 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.25, \\ 0.41 \end{pmatrix}$
$\mathbb{D}_2$	$\begin{pmatrix} 0.20, 0.21, \\ 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.12, \\ 0.30 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.24, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.33, \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.23, \\ 0.24 \end{pmatrix}$
$\mathbb{D}_3$	$\begin{pmatrix} 0.21, 0.24, \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.34, \\ 0.35 \end{pmatrix}$	$\begin{pmatrix} 0.17, 0.28, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.18, \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.35, \\ 0.41 \end{pmatrix}$
$\mathbb{D}_4$	$\begin{pmatrix} 0.27, 0.21, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.21, \\ 0.39 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.22, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.23, \\ 0.43 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.13, \\ 0.14 \end{pmatrix}$
$\mathbb{D}_5$	$\begin{pmatrix} 0.11, 0.42, \\ 0.39 \end{pmatrix}$	$\begin{pmatrix} 0.10, 0.49, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.10, 0.29, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.32, 0.21, \\ 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.25, \\ 0.27 \end{pmatrix}$

Table 4. PFS_{ft} data for patient $\mathfrak{U}_4$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
$\mathbb{D}_1$	$\begin{pmatrix} 0.12, 0.33, \\ 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.17, \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.23, 0.19, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.34, \\ 0.35 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.17, \\ 0.16 \end{pmatrix}$
$\mathbb{D}_2$	$\begin{pmatrix} 0.13, 0.25, \\ 0.41 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.22, \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.31, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.21, \\ 0.39 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.19, \\ 0.17 \end{pmatrix}$
$\mathbb{D}_3$	$\begin{pmatrix} 0.21, 0.25, \\ 0.27 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.24, \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.12, \\ 0.30 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.23, \\ 0.43 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.17, \\ 0.19 \end{pmatrix}$
$\mathbb{D}_4$	$\begin{pmatrix} 0.12, 0.35, \\ 0.41 \end{pmatrix}$	$\begin{pmatrix} 0.27, 0.21, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.1, 0.2, \\ 0.3 \end{pmatrix}$	$\begin{pmatrix} 0.32, 0.21, \\ 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.27, 0.21, \\ 0.25 \end{pmatrix}$
$\mathbb{D}_5$	$\begin{pmatrix} 0.22, 0.23, \\ 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.10, 0.29, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.3, 0.31, \\ 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.19, \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} 0.11, 0.42, \\ 0.39 \end{pmatrix}$

Step 2: There is no need for normalization because all parameters are of the same type.

Step 3: Calculate $\mathcal{F}_f$ ($f = 1, 2, 3, \dots, n$) and $\mathfrak{h}_r$ ($r = 1, 2, 3, \dots, \mathfrak{x}$) by using $\mathcal{F}_1 = 1$; $\mathfrak{h}_1 = 1$; $\mathcal{F}_f = \prod_{t=1}^{f-1} Sc(\zeta_{\rho_{rt}})$, where $f = 2, 3, \dots, n$; and $\mathfrak{h}_r = \prod_{o=1}^{r-1} Sc(\zeta_{\rho_o})(r = 2, 3, \dots, \mathfrak{x})$, where $Sc(\zeta_{\rho_{rf}})$ represents the SF of $PFS_{ft}Ns$.

$$\mathcal{F}_f^{(1)} = \begin{bmatrix} 1 & 0.44 & 0.41 & 0.49 & 0.53 \\ 1 & 0.49 & 0.40 & 0.59 & 0.55 \\ 1 & 0.465 & 0.485 & 0.49 & 0.49 \\ 1 & 0.43 & 0.42 & 0.54 & 0.54 \\ 1 & 0.455 & 0.515 & 0.49 & 0.49 \end{bmatrix}, \mathcal{F}_f^{(2)} = \begin{bmatrix} 1 & 0.41 & 0.485 & 0.53 & 0.49 \\ 1 & 0.40 & 0.405 & 0.55 & 0.59 \\ 1 & 0.485 & 0.63 & 0.49 & 0.49 \\ 1 & 0.42 & 0.435 & 0.495 & 0.54 \\ 1 & 0.515 & 0.58 & 0.53 & 0.49 \end{bmatrix}$$

$$\mathcal{F}_f^{(3)} = \begin{bmatrix} 1 & 0.44 & 0.465 & 0.505 & 0.45 \\ 1 & 0.49 & 0.41 & 0.445 & 0.445 \\ 1 & 0.49 & 0.405 & 0.49 & 0.535 \\ 1 & 0.51 & 0.415 & 0.475 & 0.38 \\ 1 & 0.36 & 0.455 & 0.465 & 0.50 \end{bmatrix}, \mathcal{F}_f^{(4)} = \begin{bmatrix} 1 & 0.44 & 0.49 & 0.51 & 0.405 \\ 1 & 0.36 & 0.465 & 0.465 & 0.415 \\ 1 & 0.47 & 0.49 & 0.41 & 0.38 \\ 1 & 0.355 & 0.51 & 0.40 & 0.50 \\ 1 & 0.49 & 0.465 & 0.485 & 0.425 \end{bmatrix}$$

and

$$\mathfrak{h}_r^1 = \begin{bmatrix} 1 \\ 0.44 \\ 0.2156 \\ 0.1002 \\ 0.0431 \end{bmatrix}, \mathfrak{h}_r^2 = \begin{bmatrix} 1 \\ 0.41 \\ 0.164 \\ 0.0795 \\ 0.0334 \end{bmatrix}$$

$$\mathfrak{h}_r^3 = \begin{bmatrix} 1 \\ 0.44 \\ 0.2156 \\ 0.1056 \\ 0.0538 \end{bmatrix}, \mathfrak{h}_r^4 = \begin{bmatrix} 1 \\ 0.44 \\ 0.1584 \\ 0.0744 \\ 0.0264 \end{bmatrix}$$

Step 4: Now we use the $PFS_{ft}PWA$ aggregation operator to find the aggregated values for each patient, given as

$$\mathcal{W}^1 = (0.1740, 0.2235, 0.2100), \mathcal{W}^2 = (0.1709, 0.2227, 0.2188),$$

$$\mathcal{W}^3 = (0.1513, 0.2521, 0.2378), \mathcal{W}^4 = (0.1660, 0.2538, 0.2514)$$

Step 5: Calculate the score value for each alternative by using the definition of the score function.

Now

$$Sc(\mathcal{W}^1) = 0.482, Sc(\mathcal{W}^2) = 0.4760, Sc(\mathcal{W}^3) = 0.4567, Sc(\mathcal{W}^4) = 0.4573$$

Step 6: Thus, the ranking is $Sc(\mathcal{W}^1) > Sc(\mathcal{W}^2) > Sc(\mathcal{W}^4) > Sc(\mathcal{W}^3)$.

Therefore, from the analyses of the experts, we can note that patient $\mathfrak{A}_1$ has more serious illnesses than other patients.

By using $PFS_{ft}PWG$ aggregation operators

Step 1: Doctors provide their evaluation information in the form of $PFS_{ft}Ns$, according to each parameter given in Tables 1–4.

Step 2: There is no need for normalization because all parameters are of the same type.

Step 3: Same as above.

Step 4: Now we use the $PFS_{ft}PWG$ aggregation operator to find the aggregated values for each patient, given as

$$\mathcal{W}^1 = (0.1624, 0.2235, 0.2229), \mathcal{W}^2 = (0.1571, 0.2227, 0.233),$$

$$\mathcal{W}^3 = (0.1460, 0.2521, 0.2528), \mathcal{W}^4 = (0.11594, 0.2538, 0.2689)$$

Step 5: Calculate the score value for each alternative by using the definition of the score function.

Now,

$$Sc(\mathcal{W}^1) = 0.4697, Sc(\mathcal{W}^2) = 0.4620, Sc(\mathcal{W}^3) = 0.4446, Sc(\mathcal{W}^4) = 0.4235$$

Step 6: Thus, the ranking is $Sc(\mathcal{W}^1) > Sc(\mathcal{W}^2) > Sc(\mathcal{W}^3) > Sc(\mathcal{W}^4)$.

Therefore, from the analyses of the experts, we can note that patient $\mathfrak{A}_1$ has more serious illnesses than other patients.

7. Comparative Analysis

In this part of the article, we will discuss the comparative analysis of the introduced work by comparing our work with some existing notions to establish the reliability and superiority of the introduced work. We will compare our work with the Yu method [7], Khan et al. method [11], Riaz et al. method [18], and Arora and Garg method [31]. The overall discussion is given below.

Example 2: Suppose we have a set of three alternatives $\mathring{A}^{\top} = \{\mathring{A}_1^{\top}, \mathring{A}_2^{\top}, \mathring{A}_3^{\top}\}$ and five parameters $\rho = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$. Suppose a team of five experts $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5 —is invited to give their assessment of each alternative, and they provide their evaluation corresponding to their parameters in the form of a picture fuzzy soft number, as given in Tables 5–7.

Table 5. PFS_{ft} data for alternative $\hat{A}_1^7$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ε_1	$\begin{pmatrix} 0.10, 0.13, \\ 0.15 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.18, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.17, 0.16, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.27, \\ 0.9 \end{pmatrix}$	$\begin{pmatrix} 0.23, 0.27, \\ 0.26 \end{pmatrix}$
ε_2	$\begin{pmatrix} 0.26, 0.27, \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.21, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.17, \\ 0.22 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.28, \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.24, 0.33, \\ 0.37 \end{pmatrix}$
ε_3	$\begin{pmatrix} 0.11, 0.12, \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.2, 0.21, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.23, 0.24, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.29, \\ 0.27 \end{pmatrix}$	$\begin{pmatrix} 0.26, 0.40, \\ 0.12 \end{pmatrix}$
ε_4	$\begin{pmatrix} 0.29, 0.30, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.16, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.14, 0.27, \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.27, \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.29, \\ 0.31 \end{pmatrix}$
ε_5	$\begin{pmatrix} 0.22, 0.17, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.28, \\ 0.29 \end{pmatrix}$	$\begin{pmatrix} 0.27, 0.26, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.11, \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.25, \\ 0.26 \end{pmatrix}$

Table 6. PFS_{ft} data for alternative $\hat{A}_2^7$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ε_1	$\begin{pmatrix} 0.19, 0.21, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.17, 0.22, \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.19, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.14, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.13, \\ 0.14 \end{pmatrix}$
ε_2	$\begin{pmatrix} 0.18, 0.13, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.11, 0.21, \\ 0.23 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.23, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.22, 0.11, \\ 0.11 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.36, \\ 0.35 \end{pmatrix}$
ε_3	$\begin{pmatrix} 0.26, 0.26, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.31, 0.32, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.16, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.12, 0.13, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.32, 0.43, \\ 0.14 \end{pmatrix}$
ε_4	$\begin{pmatrix} 0.14, 0.30, \\ 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.23, \\ 0.44 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.19, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.16, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.15, 0.16, \\ 0.37 \end{pmatrix}$
ε_5	$\begin{pmatrix} 0.13, 0.29, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.16, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.17, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} 0.28, 0.29, \\ 0.21 \end{pmatrix}$	$\begin{pmatrix} 0.38, 0.19, \\ 0.10 \end{pmatrix}$

Table 7. PFS_{ft} data for alternative $\hat{A}_3^7$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ε_1	$\begin{pmatrix} 0.10, 0.29, \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.24, \\ 0.25 \end{pmatrix}$	$\begin{pmatrix} 0.10, 0.11, \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.14, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.13, 0.15, \\ 0.19 \end{pmatrix}$
ε_2	$\begin{pmatrix} 0.15, 0.16, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.10, 0.21, \\ 0.32 \end{pmatrix}$	$\begin{pmatrix} 0.25, 0.24, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.20, 0.11, \\ 0.12 \end{pmatrix}$	$\begin{pmatrix} 0.24, 0.34, \\ 0.30 \end{pmatrix}$
ε_3	$\begin{pmatrix} 0.22, 0.23, \\ 0.24 \end{pmatrix}$	$\begin{pmatrix} 0.33, 0.32, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.17, \\ 0.18 \end{pmatrix}$	$\begin{pmatrix} 0.11, 0.12, \\ 0.13 \end{pmatrix}$	$\begin{pmatrix} 0.26, 0.47, \\ 0.15 \end{pmatrix}$
ε_4	$\begin{pmatrix} 0.19, 0.30, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} 0.26, 0.25, \\ 0.44 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.10, \\ 0.19 \end{pmatrix}$	$\begin{pmatrix} 0.16, 0.15, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.18, 0.11, \\ 0.33 \end{pmatrix}$
ε_5	$\begin{pmatrix} 0.16, 0.27, \\ 0.28 \end{pmatrix}$	$\begin{pmatrix} 0.29, 0.18, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.19, 0.18, \\ 0.17 \end{pmatrix}$	$\begin{pmatrix} 0.21, 0.28, \\ 0.27 \end{pmatrix}$	$\begin{pmatrix} 0.36, 0.14, \\ 0.11 \end{pmatrix}$

From the above analyses, we can note that

1. When decision-makers provide their assessment in the form of picture fuzzy soft numbers, in which we see that three types of aspects like MG, AG, and NMG are given, then the Yu method [7], Khan et al. method [11], Riaz method [18], and Arora and Garg method [31] fail to handle these types of situations, because all of the above-given methods cannot discuss three types of aspects in their structures. Meanwhile, the proposed aggregation operators can handle this situation. So, our work is superior.
2. When we discuss the parameterization tool that makes the soft set theory more valuable and fruitful than the fuzzy set theory, note that the Yu method [7], Khan et al.

- method [11], and Riaz method [18] cannot consider the parameterizations tool, while our proposed work can. Due to this reason, our work is more effective.
3. Additionally, note that although Arora and Garg's method [31] can deal with parameterization tools, this structure consists of intuitionistic fuzzy soft prioritized average and geometric aggregation operators that can consider two types of aspects—i.e., MG and NMG—in their structure, while data given in Tables 5–7 consist of three types of aspects—i.e., MG, AG, and NMG. Hence, in this regard, all data given in Tables 5–7 cannot be handled by the Arora and Garg method [31]. Meanwhile, our initiated work can handle all issues faced by the existing literature. So, our introduced work is more efficient. The overall results for data given in Tables 5–7 are given in Table 8.

Table 8. Overall results for above-given Tables 5–7.

Methods	Score Values	Ranking
Yu Method [7]	No result	No result
Kahn et al. Method [11]	No result	No result
Riaz et al. method [18]	No result	No result
Arora and Garg Method [31]	No result	No result
$PFS_{ft}PWA$ aggregation operators (Proposed)	$Sc(\hat{A}_1^{\neg}) = 0.4843,$ $Sc(\hat{A}_2^{\neg}) = 0.5049,$ $Sc(\hat{A}_3^{\neg}) = 0.4820,$	$\hat{A}_2^{\neg} > \hat{A}_1^{\neg} > \hat{A}_3^{\neg}$
$PFS_{ft}PWG$ aggregation operators (Proposed)	$Sc(\hat{A}_1^{\neg}) = 0.4719,$ $Sc(\hat{A}_2^{\neg}) = 0.4929,$ $Sc(\hat{A}_3^{\neg}) = 0.4671,$	$\hat{A}_2^{\neg} > \hat{A}_1^{\neg} > \hat{A}_3^{\neg}$

Correspondingly, a characteristic analysis of the introduced work along with existing literature has been given in Table 9.

Table 9. Characteristic analysis of introduced work and existing literature.

Methods	Parameterization Toll
Yu Method [7]	NO
Kahn et al. Method [11]	NO
Riaz et al. method [18]	NO
Arora and Garg Method [31]	Yes
$PFS_{ft}PWA$ aggregation operators (Proposed)	Yes
$PFS_{ft}PWG$ aggregation operators (Proposed)	Yes

Moreover, the abbreviation used throughout the article is given in Table 10.

Table 10. Useful abbreviations used throughout the article.

Full Name	Abbreviations
Picture fuzzy soft set	$PFS_{ft}S$
Picture fuzzy soft Prioritized weighted average aggregation operators	$PFS_{ft}PWA$
Picture fuzzy soft Prioritized weighted geometric aggregation operators	$PFS_{ft}PWG$
Multi-attribute group decision making	MAGDM
Fuzzy Set	FS
Fuzzy set theory	FST
Soft set	$S_{ft}S$
Intuitionistic fuzzy soft set	$IFS_{ft}S$
Pythagorean fuzzy soft set	$P_yFS_{ft}S$
q-rung orthopair fuzzy soft set	$q - ROFS_{ft}S$

8. Conclusions

Although medical diagnoses have become more complex in recent days, recent techniques and tools are available to tackle this issue. FS theory has played its role in this regard, and many developments have been made to cover this field. Picture fuzzy soft set is a decent and valuable tool to discuss the three grades to model human opinion in the form of MG, AG, and NMD. Additionally, aggregation operators are familiar structures in FST to convert the overall information into a single value that can further help in many decision-making scenarios. So, due to the strong structure of $PFS_{ft}S$ and characteristics of aggregation operators, here, in this article, we have first proposed basic operational laws for picture fuzzy soft numbers. Moreover, based on these developed operations, we have introduced the PFS_{ft} prioritized average and geometric aggregation operators. Furthermore, we have used this defined approach to medical diagnoses to show the effective use of this developed technique. The use of elaborated work in medical diagnoses shows that this technique has a fruitful influence in the medical field and plays a vital role. It also helps the experts working in the medical field to make their decisions accurately. In this regard, medical diagnoses become easier and more reliable. Moreover, the cooperative analysis shows the effectiveness and superiority of the initiated work.

The proposed aggregation operators are also limited notions because they use the condition that $sum(MG, AG, NMG) \in [0, 1]$. However, when decision-makers come up with 0.3 as MG, 0.6 as AG, and 0.4 as NMG, then note that $sum(0.3, 0.6, 0.4) \notin [0, 1]$. So, the main condition for picture fuzzy soft data is violated, and we can say that these notions are also limited.

In future work, we can propose some other aggregation operators based on developed operations as given in [34]. This work can be extended to bipolar complex fuzzy sets for generalized similarity measures, as given in [35], and for Aczel-Alsina aggregation operators, as given in [36]. Additionally, some new methods, like TOPSIS, can be defined based on the developed approach, as given in [37]. Furthermore, we can also use the developed notions for bipolar soft sets [38]. Moreover, we can extend these notions to complex fuzzy set theory, as given in [39]. Likewise, we can extend these notions to rough set theory [40], interval-valued picture fuzzy set theory [41], and N-soft set theory [42]. Some new developments based on developed aggregation operators can be made, as given in [43]. Moreover, we can extend this work to a spherical fuzzy environment and develop a new approach, as given in [44,45].

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