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A Class of Rough Generalized Marcinkiewicz Integrals on Product Domains

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Abstract: In this article, suitable estimates for a class of rough generalized Marcinkiewicz integrals on product spaces are established. By these estimates, together with employing Yano's extrapolation technique, we obtain the boundedness of the aforementioned integral operators under weak conditions on singular kernels. A number of known previous results on Marcinkiewicz as well as generalized Marcinkiewicz operators over a symmetric space are essentially improved or extended.

Keywords: rough integral operators; Marcinkiewicz integrals; product domains; extrapolation

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1. Introduction

Throughout this article, we let $d \geq 2$ ($d = n$ or m) and $\mathbb{R}^d$ be a Euclidean space of dimensions d. Furthermore, we let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the normalized Lebesgue surface measure $d\mu_d(\cdot) \equiv d\mu$.

For $\lambda_1 = \tau_1 + i\nu_1, \lambda_2 = \tau_2 + i\nu_2$ ($\tau_1, \tau_2, \nu_1, \nu_2 \in \mathbb{R}$ *with* $\tau_1, \tau_2 > 0$), we assume that

$$K_{\Omega,h}(\omega, v) = \frac{\Omega(\omega, v)h(|\omega|, |v|)}{|\omega|^{n-\lambda_1}|v|^{m-\lambda_2}},$$

where h is a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and Ω is a measurable function defined on $\mathbb{R}^n \times \mathbb{R}^m$ which satisfies the following properties:

$$\Omega(r\omega, sv) = \Omega(\omega, v), \ \ \forall r, s > 0, \tag{1}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(\omega, .)d\mu(\omega) = \int_{\mathbb{S}^{m-1}} \Omega(., v)d\mu(v) = 0, \tag{2}$$

and

$$\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}). \tag{3}$$

For $\alpha > 1$ and $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we consider the generalized parametric Marcinkiewicz integral over the symmetric space $\mathbb{R}^n \times \mathbb{R}^m$

$$\mathfrak{M}^{(\alpha)}_{\Omega,h}(f)(x, y) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |F_{r,s}(f)(x, y)|^{\alpha} \frac{drds}{rs} \right)^{1/\alpha}, \tag{4}$$

where

$$F_{r,s}(f)(x, y) = \frac{1}{r^{\lambda_1} s^{\lambda_2}} \int_{|\omega| \leq r} \int_{|v| \leq s} K_{\Omega,h}(\omega, v) f(x - \omega, y - v) d\omega dv.$$

When $\alpha = 2$, $h \equiv 1$, and $\lambda_1 = 1 = \lambda_2$, we denote the operator $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ by $\mathcal{M}_\Omega$. In this case, $\mathcal{M}_\Omega$ is essentially the classical Marcinkiewicz integral on product spaces. The study of the L^p boundedness of the operator $\mathcal{M}_\Omega$ was started by Ding in [1], in which he established the L^2 boundedness of $\mathcal{M}_\Omega$ whenever Ω lies in the space $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Thereafter, the boundedness of $\mathcal{M}_\Omega$ has been studied by many researchers. For example, Choi in [2] proved the L^2 boundeness of $\mathcal{M}_\Omega$ if Ω satisfies the weaker condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. In [3], the authors proved that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$ if $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Later on, the authors of [4] improved and extended the above results. In fact, they showed that the operator $\mathcal{M}_\Omega$ is of type (p, p) for all $1 < p < \infty$, provided that $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Furthermore, they found that by adapting the technique employed in [5] to the product space setting, the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is optimal in the sense that it cannot be replaced by a weaker condition $\Omega \in L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $\varepsilon \in (0, 1)$. On the other hand, Al-Qassem in [6] showed that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $1 < p < \infty$ if $\Omega \in B^{(0,0)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $q > 1$. Moreover, he showed that the condition $\Omega \in B^{(0,0)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is optimal in the sense that we cannot replace it by $\Omega \in B^{(0,\varepsilon)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for any $\varepsilon \in (-1, 0)$. Here, $B^{(0,\nu)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is a special class of block spaces introduced in [7].

By using an extrapolation argument, the authors of [8] proved that the L^p boundedness of $\mathfrak{M}^{(2)}_{\Omega,h}$ for all $|1/2 - 1/p| < \min\{1/\gamma', 1/2\}$ whenever $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma > 1$ and Ω lies in either the space $L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or in the space $B^{(0,0)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$. Here, $\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\gamma > 1$) indicates the class of measurable functions h which are defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and satisfy

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)} = \sup_{j,k \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(r, s)|^\gamma \frac{drds}{rs} \right)^{1/\gamma} < \infty.$$

Recently, the authors of [9] established that if $h \equiv 1$ and $\Omega \in L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or $\Omega \in B^{(0,\frac{2}{\alpha}-1)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, then

$$\left\| \mathfrak{M}^{(\alpha)}_{\Omega,1}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \|f\|_{\dot{F}^{\vec{0},\alpha}_p(\mathbb{R}^n \times \mathbb{R}^m)} \tag{5}$$

for all $p \in (1, \infty)$.

It is well known that the Marcinkiewicz integral, $\mathcal{M}_\Omega$, on product spaces naturally generalizes the Marcinkiewicz integral in one parameter setting which was introduced by E. Stein in [10]. The singularity of $\mathcal{M}_\Omega$ is along the diagonals $\{x = \omega\}$ and $\{y = v\}$. The study of singular integrals on product spaces and the study of $\mathcal{M}_\Omega$ as well as its generalizations, which may have singularities along subvarities, has attracted the attention of many authors in recent years. One of the principal motivations for the study of such operators is the requirements of several complex variables and large classes of "subelliptic" equations. For more background information, readers may refer to Stein's survey articles [11,12].

Let us recall the definition of Triebel–Lizorkin spaces, $\dot{F}^{\vec{m},\alpha}_p(\mathbb{R}^n \times \mathbb{R}^m)$. Assume that $\vec{m} = (\beta, \epsilon) \in \mathbb{R} \times \mathbb{R}$ and $\alpha, p \in (1, \infty)$. The homogeneous Triebel–Lizorkin space $\dot{F}^{\vec{m},\alpha}_p(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the class of all tempered distributions f on $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$\|f\|_{\dot{F}^{\vec{m},\alpha}_p(\mathbb{R}^n \times \mathbb{R}^m)} = \left\| \left(\sum_{j,k \in \mathbb{Z}} 2^{k\beta\alpha} 2^{j\epsilon\alpha} \left| (\phi_k \otimes \psi_j) * f \right|^\alpha \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} < \infty,$$

where $\widehat{\phi}_k(x) = 2^{-kn}E(2^{-k}x)$ for $k \in \mathbb{Z}$, $\widehat{\psi}_j(y) = 2^{-jm}J(2^{-j}y)$ for $j \in \mathbb{Z}$, and the functions $E \in C_0^\infty(\mathbb{R}^n)$ and $J \in C_0^\infty(\mathbb{R}^m)$ are radial functions satisfying the following proprieties:

(i) $E, J \in [0,1]$;

(ii) $supp\,(E) \subset \left\{x : |x| \in [\frac{1}{2}, 2]\right\}, \;\; supp\,(J) \subset \left\{y : |y| \in [\frac{1}{2}, 2]\right\}$;

(iii) $E(x), J(y) \geq T > 0$ if $|x|, |y| \in [\frac{3}{5}, \frac{5}{3}]$ for some constant T;

(iv) $\sum\limits_{k\in\mathbb{Z}} E(2^{-k}x) = \sum\limits_{j\in\mathbb{Z}} J(2^{-j}y) = 1$ with $x \neq 0 \neq y$.

It was shown in [13] that the space $\dot{F}_p^{\vec{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)$ satisfies the following:

(a) The Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $\dot{F}_p^{\vec{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)$;

(b) $\dot{F}_p^{0,\vec{2}}(\mathbb{R}^n \times \mathbb{R}^m) = L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$;

(c) $\dot{F}_p^{\vec{m},\alpha_1}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq \dot{F}_p^{\vec{m},\alpha_2}(\mathbb{R}^n \times \mathbb{R}^m)$ if $\alpha_1 \leq \alpha_2$;

(d) $\left(\dot{F}_p^{\vec{m},\alpha}(\mathbb{R}^n \times \mathbb{R}^m)\right)^* = \dot{F}_{p'}^{-\vec{m},\alpha'}(\mathbb{R}^n \times \mathbb{R}^m)$,

where p' denotes the exponent conjugate to p, that is, $1/p + 1/p' = 1$ whenever $1 < p < \infty$ and $p' := 1$ or $p' := +\infty$ for $p := +\infty$ or $p := 1$, respectively.

In light of the results in [8] concerning the boundedness of the operator $\mathfrak{M}_{\Omega,h}^{(2)}$ and of the results in [9] concerning the boundedness of the generalized operator $\mathfrak{M}_{\Omega,1}^{(\alpha)}$, a natural questions arises in the following:

Question: Is the operator $\mathfrak{M}_{\Omega,h}^{(\alpha)}$ bounded under the same assumptions in [8] with replacing $\alpha = 2$ by $\alpha > 1$?

The main purpose of this work is to answer the above question affirmatively. Precisely, we have the following:

Theorem 1. *Let $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma \in (1,2]$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q \in (1,2]$. Then, there is a constant $C_{p,\Omega,h}$ such that*

$$\left\|\mathfrak{M}_{\Omega,h}^{(\alpha)}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{p,\Omega,h}\left(\frac{1}{(q-1)(\gamma-1)}\right)^{2/\alpha}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

for all $p \in (\frac{\alpha\gamma'}{\alpha+\gamma'-1}, \frac{\alpha'\gamma}{\alpha'-\gamma})$ if $\alpha \leq \gamma'$, and

$$\left\|\mathfrak{M}_{\Omega,h}^{(\alpha)}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{p,\Omega,h}\left(\frac{1}{(\gamma-1)(q-1)}\right)^{2/\alpha}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

for all $\gamma' < p < \infty$ if $\alpha \geq \gamma'$, where $C_{p,\Omega,h} = C_p\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|\Omega\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}$.

Theorem 2. *Assume that $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\gamma \in (2,\infty)$ and that Ω lies in $L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $q \in (1,2]$. Then, we have*

$$\left\|\mathfrak{M}_{\Omega,h}^{(\alpha)}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{p,\Omega,h}\left(\frac{\gamma}{q-1}\right)^{2/\alpha}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

for all $p \in (1,\alpha)$ if $\alpha \leq \gamma'$, and

$$\left\|\mathfrak{M}_{\Omega,h}^{(\alpha)}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{p,\Omega,h}\left(\frac{\gamma}{q-1}\right)^{2/\alpha}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

for all $p \in (\gamma',\infty)$ if $\alpha \geq \gamma'$.

By employing the estimates in Theorems 1 and 2 and employing an extrapolation argument as in [14] (see also [15,16]), we obtain the following:

Theorem 3. *Let h be given as in Theorem 1.*
(i) If $\Omega \in L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, then the inequality

$$\left\|\mathfrak{M}^{(\alpha)}_{\Omega,h}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C_p\left(1+\|\Omega\|_{L(logL)^{2/\alpha}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\right)\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

holds for $p \in (\frac{\alpha\gamma'}{\alpha+\gamma'-1}, \frac{\alpha'\gamma}{\alpha'-\gamma})$ if $\alpha \le \gamma'$, and for $\gamma' < p < \infty$ if $\alpha \ge \gamma'$.
(ii) If $\Omega \in B_q^{(0,\frac{2}{\alpha}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$, then the inequality

$$\left\|\mathfrak{M}^{(\alpha)}_{\Omega,h}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C_p\left(1+\|\Omega\|_{B_q^{(0,\frac{2}{\alpha}-1)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\right)\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}$$

holds for $p \in (\frac{\alpha\gamma'}{\alpha+\gamma'-1}, \frac{\alpha'\gamma}{\alpha'-\gamma})$ if $\alpha \le \gamma'$, and for $\gamma' < p < \infty$ if $\alpha \ge \gamma'$.

Theorem 4. *Let $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma \in (2,\infty)$ and $\Omega \in L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B_q^{(0,\frac{2}{\alpha}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then, the operator $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p \in (1,\alpha)$ if $\alpha \le \gamma'$, and for $p \in (\gamma',\infty)$ if $\alpha \ge \gamma'$.*

Remark 1.
(1) The conditions assumed for Ω in Theorems 3 and 4 are the weakest conditions in their respective classes for the case $\alpha = 2$ and $h \equiv 1$ (see [4,6]).
(2) For the special case $h \equiv 1$, Theorem 4 gives that $\mathfrak{M}^{(\alpha)}_{\Omega,1}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1,\infty)$, provided that Ω belongs to $L(\log L)^{2/\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or to $B_q^{(0,\frac{2}{\alpha}-1)}$ $(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, which is Theorem 2.7 in [9].
(3) The result in Theorem 3 in the case $\alpha = 2$ and $1 < \gamma \le 2$ essentially improves Theorem 2 in [8], in which the authors proved the L^p boundedness of $\mathfrak{M}^{(2)}_{\Omega,h}$ for $p \in (\frac{2\gamma'}{\gamma'-2}, \frac{2\gamma}{2-\gamma})$. Hence, the range of p in Theorem 3 is better than the range of that obtained in [8].
(4) The authors of [17] proved the L^p ($\gamma' < p < \infty$) boundedness of $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ only for the special case $1 < \gamma \le 2$ and $\alpha = \gamma'$. Therefore, the results in Theorem 3 essentially improve the main results in [17].
(5) For the special case $\alpha = \gamma'$ with $2 < \gamma < \infty$, Theorem 4 leads to the boundedness of $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ for all $p \in (1,\infty)$.

Henceforward, the constant C signifies a positive real number that could be different at each occurrence but is independent of all essential variables.

2. Auxiliary Lemmas

This section is devoted to introducing some notation and establishing some lemmas that will be needed to prove the main results of this paper. For $\theta \ge 2$, consider the family of measures $\{\mu_{K_{\Omega,h},r,s} := \mu_{r,s} : r,s \in \mathbb{R}_+\}$ and its corresponding maximal operators μ^*_h and $S_{h,\theta}$ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\iint_{\mathbb{R}^n\times\mathbb{R}^m} f d\mu_{r,s} = \frac{1}{r^{\lambda_1}s^{\lambda_2}}\int_{1/2r\le|\omega|\le r}\int_{1/2s\le|v|\le s} f(\omega,v)K_{\Omega,h}(\omega,v)d\omega dv,$$

$$\mu^*_h(f)(\omega,v) = \sup_{r,s\in\mathbb{R}_+}||\mu_{r,s}|*f(\omega,v)|,$$

and

$$\mathrm{S}_{h,\theta}(f)(\omega,v) = \sup_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} ||\mu_{r,s}| * f(\omega,v)| \frac{drds}{rs},$$

where $|\mu_{r,s}|$ is defined in the same way as $\mu_{r,s}$ but with Ωh replaced by $|\Omega h|$.

Lemma 1. *Assume that $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$, with $\gamma > 1$ and $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Then, for any $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$ with $p \in (\gamma', \infty)$, we have*

$$\|\mu_h^*(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq \widetilde{C}_{p,h,\Omega}\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \tag{6}$$

and

$$\|\mathrm{S}_{h,\theta}(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq \widetilde{C}_{p,h,\Omega}\ln^2(\theta)\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \tag{7}$$

where $\widetilde{C}_{p,h,\Omega} = \|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}$.

Proof. Thanks to Hölder's inequality, we obtain that

$$\begin{aligned}||\mu_{r,s}| * f(x,y)| \leq{}& C\|\Omega\|^{1/\gamma}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\frac{1}{rs}\int_{\frac{s}{2}}^{s}\int_{\frac{r}{2}}^{r}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(\omega,v)|\right.\\ \times{}& \left.|f(x-r\omega,y-sv)|^{\gamma'}d\mu(\omega)d\mu(v)drds\right)^{1/\gamma'}.\end{aligned}$$

Therefore, Minkowski's inequality for the integrals and Corollary 5 in [18] lead to

$$\begin{aligned}&\|\mu_h^*(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C\|\Omega\|^{1/\gamma}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\\ &\times \left(\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}|\Omega(\omega,v)|\|\mu^*(|f|^{\gamma'})\|_{L^{(p/\gamma')}(\mathbb{R}^n\times\mathbb{R}^m)}d\mu(\omega)d\mu(v)\right)^{1/\gamma'}\\ &\leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|\mu^*(|f|)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\leq C\widetilde{C}_{p,h,\Omega}\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)},\end{aligned}$$

where

$$\mu^*(f)(x,y) = \sup_{r,s>0}\frac{1}{rs}\int_0^s\int_0^r|f(x-r\omega,y-sv)|drds.$$

Inequality (7) is easily deduced from Inequality (6). □

The next lemma is found in [8] with very minor modifications. We omit the proof.

Lemma 2. *Let $\theta \geq 2$, $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then, the following estimates hold:*

$$\|\mu_{r,s}\| \leq C_{\Omega,h}, \tag{8}$$

$$\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}|\hat{\mu}_{r,s}(\zeta,\xi)|^2\frac{dsdt}{st} \leq C^2_{\Omega,h}\ln^2(\theta)\left|\theta^k\zeta\right|^{\pm\frac{2\delta}{\ln(\theta)}}\left|\theta^j\xi\right|^{\pm\frac{2\delta}{\ln(\theta)}}, \tag{9}$$

where $2\delta q' < 1$ and $\|\mu_{r,s}\|$ is the total variation of $\mu_{r,s}$.

In order to prove our main results, we need to prove the following lemmas.

Lemma 3. *Suppose that $\theta \geq 2$, $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$ with $1 < \gamma \leq 2$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $1 < q \leq 2$. Let $\alpha \in (1, \gamma']$ and $\{\mathcal{G}_{j,k}(\cdot,\cdot),\ j,k \in \mathbb{Z}\}$ be arbitrary functions defined on $\mathbb{R}^n \times \mathbb{R}^m$. Then, there exists a positive constant $C_{\Omega,h}$ such that the inequality*

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int\limits_{\theta^j}^{\theta^{j+1}} \int\limits_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * \mathcal{G}_{j,k}\right|^\alpha \frac{drds}{rs} \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{\Omega,h} \ln^{2/\alpha}(\theta) \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{G}_{j,k}\right|^\alpha \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \tag{10}$$

holds for all $p \in \left(\frac{\alpha\gamma'}{\alpha+\gamma'-1}, \frac{\alpha'\gamma}{\alpha'-\gamma}\right)$.

Proof. We employ a similar argument used in [19]. First, let us consider the case $p \in \left(\alpha, \frac{\alpha'\gamma}{\alpha'-\gamma}\right)$. By duality, there is a non-negative function $\vartheta \in L^{(p/\alpha)'}(\mathbb{R}^n \times \mathbb{R}^m)$ such that $\|\vartheta\|_{L^{(p/\alpha)'}(\mathbb{R}^n\times\mathbb{R}^m)} \leq 1$ and

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * \mathcal{G}_{j,k}\right|^\alpha \frac{drds}{rs} \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}^\alpha \\ = &\iint_{\mathbb{R}^n\times\mathbb{R}^m} \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * \mathcal{G}_{j,k}(\omega,v)\right|^\alpha \frac{drds}{rs} \vartheta(\omega,v) d\omega dv. \end{aligned} \tag{11}$$

By Hölder's inequality, it is easy to obtain that

$$\begin{aligned} &\left|\mu_{r,s} * \mathcal{G}_{j,k}(\omega,v)\right|^\alpha \leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha/\alpha')} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^{(\alpha/\alpha')} \\ \times &\int\limits_{s/2}^{s} \int\limits_{r/2}^{r} \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \left|\mathcal{G}_{j,k}(\omega - \kappa x, v - \eta y)\right|^\alpha |\Omega(x,y)| d\mu(x) d\mu(y) |h(\kappa,\eta)|^{\alpha - \frac{\alpha\gamma}{\alpha'}} \frac{d\kappa d\eta}{\kappa\eta}. \end{aligned} \tag{12}$$

Again, by using Hölder's inequality and Inequalities (11) and (12), we have

$$\begin{aligned} &\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * \mathcal{G}_{j,k}\right|^\alpha \frac{drds}{rs} \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}^\alpha \leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha/\alpha')} \|h\|_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\alpha/\alpha')} \\ \times &\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{G}_{j,k}(\omega,v)\right|^\alpha \right) \mathrm{S}_{|h|^{\alpha - \frac{\alpha\gamma}{\alpha'}},\theta}(\overline{\vartheta})(-\omega,-v) d\omega dv \\ \leq &C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha/\alpha')} \|h\|_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\alpha/\alpha')} \left\| \sum_{j,k\in\mathbb{Z}} \left|\mathcal{G}_{j,k}\right|^\alpha \right\|_{L^{(p/\alpha)}(\mathbb{R}^n\times\mathbb{R}^m)} \left\| \mathrm{S}_{|h|^{\frac{\alpha(\alpha'-\gamma)}{\alpha'}},\theta}(\overline{\vartheta}) \right\|_{L^{(p/\alpha)'}(\mathbb{R}^n\times\mathbb{R}^m)}, \end{aligned}$$

where $\overline{\vartheta}(\omega,v) = \vartheta(-\omega,-v)$. Therefore, since $|h|^{\frac{\alpha(\alpha'-\gamma)}{\alpha'}} \in \Delta_{\frac{\alpha'\gamma}{\alpha(\alpha'-\gamma)}}(\mathbb{R}_+\times\mathbb{R}_+)$, then we have

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * \mathcal{G}_{j,k}\right|^\alpha \frac{drds}{rs} \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{\Omega,h} \ln^{2/\alpha}(\theta) \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{G}_{j,k}\right|^\alpha \right)^{1/\alpha} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \tag{13}$$

for all $p \in \left(\alpha, \frac{\alpha'\gamma}{\alpha'-\gamma}\right)$. For the case $p = \alpha$, we use (12) and Hölder's inequality to obtain that

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*\mathcal{G}_{j,k}\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\right\|^{\alpha}_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C\|\Omega\|^{(\alpha/\alpha')}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|^{(\alpha/\alpha')}_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}$$

$$\times \sum_{j,k\in\mathbb{Z}}\int_{\mathbb{R}^n\times\mathbb{R}^m}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\int_{s/2}^{s}\int_{r/2}^{r}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\left|\mathcal{G}_{j,k}(\omega-\kappa x,v-\eta y)\right|^{\alpha}$$

$$\times |\Omega(x,y)||h(\kappa,\eta)|^{\frac{\alpha(\alpha'-\gamma)}{\alpha'}}d\mu(x)d\mu(y)\frac{d\kappa d\eta}{\kappa\eta}\frac{drds}{rs}d\omega dv$$

$$\leq C\ln^2(\theta)\|\Omega\|^{(\alpha/\alpha')+1}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|^{(\alpha/\alpha')+1}_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}\int_{\mathbb{R}^n\times\mathbb{R}^m}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}(\omega,v)\right|^{\alpha}\right)d\omega dv. \tag{14}$$

Finally, we consider the case $p\in(\frac{\alpha\gamma'}{\alpha+\gamma'-1},\alpha)$. Define the linear operator $\mathcal{T}$ on any function $\mathcal{G}=\mathcal{G}_{j,k}(x,y)$ by $\mathcal{T}(\mathcal{G})=\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}(x,y)$. Then, we have

$$\left\|\left\|\|\mathcal{T}(\mathcal{G})\|_{L^1([1,\theta)\times[1,\theta)),\frac{drds}{rs}}\right\|_{l^1(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)} \leq C\ln^2(\theta)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|\right)\right\|_{L^1(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{15}$$

On the other hand, by using (6), we obtain

$$\begin{aligned}\left\|\sup_{j,k\in\mathbb{Z}}\sup_{(r,s)\in[1,\theta]\times[1,\theta]}\left|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq \left\|\mu_h^*\left(\sup_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|\right)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\ &\leq C_{\Omega,h}\left\|\sup_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\end{aligned}$$

for all $\gamma'<p<\infty$, which in turn implies

$$\left\|\left\|\|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\|_{L^\infty([1,\theta]\times[1,\theta],\frac{drds}{rs})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{\Omega,h}\left\|\left\|\mathcal{G}_{j,k}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{16}$$

Therefore, by interpolating between (15) and (16) we get (10) for any $p\in(\frac{\alpha\gamma'}{\alpha+\gamma'-1},\alpha)$. □

Lemma 4. *Assume that $\theta\geq 2$, $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $2<\gamma<\infty$ and $\Omega\in L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ for some $1<q\leq 2$. Let $\alpha\leq\gamma'$ and $\{\mathcal{G}_{j,k}(\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ be arbitrary functions defined on $\mathbb{R}^n\times\mathbb{R}^m$. Then, there exists a positive constant $C_{\Omega,h}$ such that*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*\mathcal{G}_{j,k}\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{\Omega,h}\ln^{2/\alpha}(\theta)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|^{\alpha}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \tag{17}$$

for all $p\in(1,\alpha)$.

Proof. By duality, there is a set of functions $\{M_{j,k}(\omega,v,r,s)\}$ defined on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}_+\times\mathbb{R}_+$ with $\left\|\left\|\|M_{j,k}\|_{L^{\alpha'}([\theta^k,\theta^{k+1}]\times[\theta^j,\theta^{j+1}],\frac{drds}{rs})}\right\|_{l^{\alpha'}(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}\leq 1$ and

$$
\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*\mathcal{G}_{j,k}\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
=\;&\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left(\mu_{r,s}*\mathcal{G}_{j,k}(\omega,v)\right)M_{j,k}(\omega,v,r,s)\frac{drds}{rs}d\omega dv\\
\leq\;& C(\ln\theta)^{2/\alpha}\left\|(\mathcal{N}(M))^{1/\alpha'}\right\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|^{\alpha}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \qquad (18)
\end{aligned}
$$

where

$$\mathcal{N}(M)(\omega,v)=\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*M_{j,k}(\omega,v,r,s)\right|^{\alpha'}\frac{drds}{rs}.$$

Since $\gamma\geq 2\geq\gamma'\geq\alpha$, then by Hölder's inequality we obtain

$$
\begin{aligned}
&\left|\mu_{r,s}*M_{j,k}(\omega,v)\right|^{\alpha'}\leq C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha'/\alpha)}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^{(\alpha'/\alpha)}\\
\times&\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}}\left|M_{j,k}(\omega-\kappa x,v-\eta y,r,s)\right|^{\alpha'}|\Omega(x,y)|d\mu(x)d\mu(y)\frac{d\kappa d\eta}{\kappa\eta}. \qquad (19)
\end{aligned}
$$

Since $p'>\alpha'$, there exists a function $\rho\in L^{(p'/\alpha')'}(\mathbb{R}^n\times\mathbb{R}^m)$ such that

$$\|\mathcal{N}(M)\|_{L^{(p'/\alpha')}(\mathbb{R}^n\times\mathbb{R}^m)}=\sum_{j,k\in\mathbb{Z}}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*M_{j,k}(\omega,v,r,s)\right|^{\alpha'}\frac{drds}{rs}\rho(\omega,v)d\omega dv.$$

Therefore, a simple change in variable together with Lemmas 1 and (19) give

$$
\begin{aligned}
\|\mathcal{N}(M)\|_{L^{(p'/\alpha')}(\mathbb{R}^n\times\mathbb{R}^m)}\;&\leq\; C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha'/\alpha)}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^{(\alpha')}\|\mu^*(\rho)\|_{L^{(p'/\alpha')'}(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\times\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|M_{j,k}(\cdot,\cdot,r,s)\right|^{\alpha'}\frac{drds}{rs}\right)\right\|_{L^{(p'/\alpha')}(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\leq\; C\|\Omega\|_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}^{(\alpha'/\alpha)+1}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^{\alpha'}\|(\rho)\|_{L^{(p'/\alpha')'}(\mathbb{R}^n\times\mathbb{R}^m)}. \qquad (20)
\end{aligned}
$$

Therefore, by (18) and (20), Inequality (17) is proved. Consequently, the proof of Lemma 4 is complete. □

Lemma 5. *Assume that θ, Ω, and $\{\mathcal{G}_{j,k}(\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ are given as in Lemma 3. Suppose that $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $1<\gamma<\infty$ and $\alpha\geq\gamma'$. Then, there exists a constant $C_{\Omega,h}>0$ such that*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*\mathcal{G}_{j,k}\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\leq C_{\Omega,h}\ln^{2/\alpha}(\theta)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|^{\alpha}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \qquad (21)$$

for all $\gamma'<p<\infty$.

Proof. By (6), we have

$$\left\|\sup_{j,k\in\mathbb{Z}}\sup_{(r,s)\in[1,\theta]\times[1,\theta]}\left|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le \left\|\mu_h^*\left(\sup_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|\right)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C_{\Omega,h}\left\|\sup_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \quad (22)$$

for all $\gamma' < p < \infty$. Hence,

$$\left\|\left\|\left\|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right\|_{L^\infty([1,\theta]\times[1,\theta],\frac{drds}{rs})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \le C_{\Omega,h}\left\|\left\|\mathcal{G}_{j,k}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}. \quad (23)$$

By the duality, there exists $\psi\in L^{(p/\gamma')'}(\mathbb{R}^n\times\mathbb{R}^m)$ such that $\|\psi\|_{L^{(p/\gamma')'}(\mathbb{R}^n\times\mathbb{R}^m)}\le 1$ and

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right|^{\gamma'}\frac{drds}{rs}\right)^{1/\gamma'}\right\|^{\gamma'}_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
=&\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right|^{\gamma'}\frac{drds}{rs}\psi(\omega,v)d\omega dv\\
\le& C\|\Omega\|^{(\gamma'/\gamma)}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|^{\gamma'}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\\
\times&\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}(\omega,v)\right|^{\gamma'}\right)\mu^*(\overline{\psi})(-\omega,-v)d\omega dv\\
\le& C\ln^2(\theta)\|\Omega\|^{(\gamma'/\gamma)}_{L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\|h\|^{(\gamma')}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|^{\gamma'}\right\|_{L^{(p/\gamma')}(\mathbb{R}^n\times\mathbb{R}^m)}\left\|\mu^*(\overline{\psi})\right\|_{L^{(p/\gamma')'}(\mathbb{R}^n\times\mathbb{R}^m)}, \quad (24)
\end{aligned}$$

where $\overline{\psi}(\omega,v)=\psi(-\omega,-v)$. Define the linear operator $\mathcal{L}$ on any function $\mathcal{G}_{j,k}(\omega,v)$ by $\mathcal{L}(\mathcal{G}_{j,k}(\omega,v))=\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}(\omega,v)$. Hence, by interpolating between (23) and (24), we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\mu_{r,s}*\mathcal{G}_{j,k}\right|^\alpha\frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\le C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\mu_{\theta^k r,\theta^j s}*\mathcal{G}_{j,k}\right|^\alpha\frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\le C_{\Omega,h}\ln^{2/\alpha}(\theta)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{G}_{j,k}\right|^\alpha\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}
\end{aligned}$$

for all $\gamma' < p < \infty$ with $\gamma' < \alpha$. The proof of this lemma is complete. □

3. Proof of the Main Results

Proof of Theorem 1. We employ similar arguments as those in [19,20]. Assume that $\alpha > 1$, $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $1<\gamma\le 2$ and $\Omega\in L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ with $q\in(1,2]$. By Minkowski's inequality, we obtain

$$\begin{aligned}
\mathfrak{M}^{(\alpha)}_{\Omega,h}(f)(x,y) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\sum_{j,k=0}^{\infty}\frac{1}{r^{\lambda_1}s^{\lambda_2}}\int_{2^{-j-1}s<|\omega|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r} K_{\Omega,h}(\omega,v)\right.\right.\\
&\times \left.\left. f(x-\omega,y-v)d\omega dv\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\\
&\leq \sum_{j,k=0}^{\infty}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\frac{1}{r^{\lambda_1}s^{\lambda_2}}\int_{2^{-j-1}s<|\omega|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r}\right.\right.\\
&\times \left.\left. K_{\Omega,h}(\omega,v)f(x-\omega,y-v)d\omega dv\right|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}\\
&\leq \frac{2^{\tau_1+\tau_2}}{(2^{\tau_1}-1)(2^{\tau_2}-1)}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\mu_{r,s}*f(x,y)|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}.
\end{aligned} \tag{25}$$

Take $\theta = 2^{\gamma' q'}$, then $\ln(\theta) \leq \frac{C}{(\gamma-1)(q-1)}$. Choose a set of functions $\{\varphi_k\}_{-\infty}^{\infty}$ defined on $(0,\infty)$ with the following properties:

$$\begin{aligned}
\varphi_k &\in C^{\infty},\ 0\leq\varphi_k\leq 1,\ \sum_{k\in\mathbb{Z}}\varphi_k(r)=1,\\
supp\ (\varphi_k) &\subseteq \mathcal{I}_k \equiv [\theta^{-1-k},\theta^{1-k}]\ \ and\ \ \left|\frac{d^{\beta}\varphi_k(r)}{dr^{\beta}}\right| \leq \frac{C_{\beta}}{r^{\beta}},
\end{aligned}$$

where C_β is independent of θ. Define the operators $(\widehat{\mho_k}(\zeta)) = \varphi_k(|\zeta|)$ and $(\widehat{\mho_j}(\xi)) = \varphi_j(|\xi|)$ for $(\zeta,\xi)\in\mathbb{R}^n\times\mathbb{R}^m$. Therefore, we obtain that for any $f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\mu_{r,s}*f(x,y)|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha} \leq C\sum_{t,i\in\mathbb{Z}}\mathcal{A}_{t,i}(f)(x,y), \tag{26}$$

where

$$\mathcal{A}_{t,i}(f)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\mathcal{B}_{t,i}(f)(x,y,r,s)|^{\alpha}\frac{drds}{rs}\right)^{1/\alpha}$$

and

$$\mathcal{B}_{t,i}(f)(x,y,r,s) = \sum_{j,k\in\mathbb{Z}}\mu_{r,s}*\left(\mho_{k+i}\otimes\mho_{j+t}\right)*f(x,y)\chi_{[\theta^k,\theta^{k+1})\times[\theta^j,\theta^{j+1})}(r,s).$$

Therefore, to prove Theorem 1, it is sufficient to prove that there exists a positive constant ε such that

$$\|\mathcal{A}_{t,i}(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_p C_{\Omega,h}2^{-\frac{\varepsilon}{2}(|t|+|i|)}(\ln\theta)^{2/\alpha}\|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)} \tag{27}$$

for all $p\in(\frac{\alpha\gamma'}{\alpha+\gamma'-1},\frac{\alpha'\gamma}{\alpha'-\gamma})$ with $\gamma'\geq\alpha$, and also for all $\gamma'<p<\infty$ with $\gamma'\leq\alpha$.

Let us first estimate the norm of $\mathcal{A}_{t,i}(f)$ for the case $p=\alpha=2$. Indeed, by Plancherel's theorem, Fubini's theorem, and Lemma 2, we obtain

$$
\begin{aligned}
&\|\mathcal{A}_{t,i}(f)\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & \sum_{j,k\in\mathbb{Z}} \iint_{E_{j+t,k+i}} \left(\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} |\hat{\mu}_{r,s}(\zeta,\xi)|^2 \frac{drds}{rs}\right) \left|\hat{f}(\zeta,\xi)\right|^2 d\zeta d\xi \\
\leq\ & C_p \ln^2(\theta) C^2_{\Omega,h} \sum_{j,k\in\mathbb{Z}} \iint_{E_{j+t,k+i}} \left|\theta^k\zeta\right|^{\pm\frac{2\delta}{\ln(\theta)}} \left|\theta^j\xi\right|^{\pm\frac{2\delta}{\ln(\theta)}} \left|\hat{f}(\zeta,\xi)\right|^2 d\zeta d\xi \\
\leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|t|+|i|)}\, C^2_{\Omega,h} \sum_{j,k\in\mathbb{Z}} \iint_{E_{j+t,k+i}} \left|\hat{f}(\zeta,\xi)\right|^2 d\zeta d\xi \\
\leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|t|+|i|)}\, C^2_{\Omega,h} \|f\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^m)},
\end{aligned} \tag{28}
$$

where $E_{j,k} = \left\{(\zeta,\xi) \in \mathbb{R}^n \times \mathbb{R}^m : (|\zeta|,|\xi|) \in \mathcal{I}_k \times \mathcal{I}_j\right\}$ and $\varepsilon \in (0,1)$.

However, we estimate the L^p-norm of $\mathcal{A}_{t,i}(f)$ in the following. By Lemmas 3 and 5, together with the Littlewood–Paley theory and invoking Lemma 2.3 in [9], we obtain

$$
\begin{aligned}
&\|\mathcal{A}_{t,i}(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & C\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} \left|\mu_{r,s} * (\mho_{k+i}\otimes\mho_{j+t}) * f\right|^\alpha \frac{drds}{rs}\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & C_{\Omega,h} \ln^{2/\alpha}(\theta) \left\| \left(\sum_{j,k\in\mathbb{Z}} \left|(\mho_{k+i}\otimes\mho_{j+t}) * f\right|^\alpha\right)^{1/\alpha}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & C_p \frac{1}{[(q-1)(\gamma-1)]^{2/\alpha}} C_{\Omega,h} \|f\|_{\dot{F}_p^{\vec{0},\alpha}(\mathbb{R}^n\times\mathbb{R}^m)}
\end{aligned} \tag{29}
$$

for all $p \in \left(\frac{\alpha\gamma'}{\alpha+\gamma'-1}, \frac{\alpha'\gamma}{\alpha'-\gamma}\right)$ with $\alpha \leq \gamma'$, and also for all $\gamma' < p < \infty$ with $\alpha \geq \gamma'$. Therefore, by interpolating (28) with (29), we immediately obtain (27). This ends the proof of Theorem 1.

Proof of Theorem 2. To prove this theorem, we follow the exact procedure that was used in the proof of Theorem 1, employing Lemma 4 instead of Lemma 3.

4. Conclusions

In this article, we established appropriate L^p bounds for the generalized parametric Marcinkiewicz integral operator $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ under the assumption that $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then, we used these bounds, along with Yano's extrapolation argument, to prove the boundedness of the operator $\mathfrak{M}^{(\alpha)}_{\Omega,h}$ under very weak conditions on the kernel function Ω. Such conditions on Ω are considered to be the best possible among their respective classes. The results in this article improve and extend several known results in the field of Marcinkiewicz and generalized Marcinkiewicz operators. In fact, our results improve and extend the results in [1–4,6,8,9,17].

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References

1. Ding, Y. L^2-boundedness of Marcinkiewicz integral with rough kernel. *Hokk. Math. J.* **1998**, *27*, 105–115.
2. Choi, Y. Marcinkiewicz integrals with rough homogeneous kernel of degree zero in product domains. *J. Math. Anal. Appl.* **2001**, *261*, 53–60. [CrossRef]
3. Chen, J.; Fan, D.; Ying, Y. Rough Marcinkiewicz integrals with $L(\log L)^2$ kernels. *Adv. Math.* **2001**, *30*, 179–181.
4. Al-Qassem, A.; Al-Salman, A.; Cheng, L.; Pan, Y. Marcinkiewicz integrals on product spaces. *Studia Math.* **2005**, *167*, 227–234. [CrossRef]
5. Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, *44*, 203–217. [CrossRef]
6. Al-Qassem, H. Rough Marcinkiewicz integral operators on product spaces. *Collec. Math.* **2005**, *36*, 275–297.
7. Jiang, Y.; Lu, S. A class of singular integral operators with rough kernel on product domains. *Hokkaido Math. J.* **1995**, *24*, 1–7. [CrossRef]
8. Ali, M.; Al-Senjlawi, A. Boundedness of Marcinkiewicz integrals on product spaces and extrapolation. *Inter. J. Pure Appl. Math.* **2014**, *97*, 49–66. [CrossRef]
9. Al-Qassem, H.; Cheng, L.; Pan, Y. Generalized Littlewood-Paley functions on product spaces. *Turk. J. Math.* **2021**, *45*, 319–345. [CrossRef]
10. Stein, E. On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. *Trans. Amer. Math. Soc.* **1958**, *88*, 430–466. [CrossRef]
11. Stein, E. Problems in harmonic analysis related to curvature and oscillatory integrals. In Proceedings of the International Congress of Mathematicians, Berkeley, CA, USA, 3–11 August 1986; pp. 196–221.
12. Stein, E. Some geometrical concepts arising in harmonic analysis. In *Visions in Mathematics*; Springer: Berlin/Heidelberg, Germany, 2010; pp. 434–453.
13. Fan, D.; Wu, H. On the generalized Marcinkiewicz integral operators with rough kernels. *Canad. Math. Bull.* **2011**, *54*, 100–112. [CrossRef]
14. Yano, S. Notes on Fourier analysis. XXIX. An extrapolation theorem. *J. Math. Soc. Jpn.* **1951**, *3*, 296–305. [CrossRef]
15. Sato, S. Estimates for singular integrals and extrapolation. *Stud. Math.* **2009**, *192*, 219–233. [CrossRef]
16. Ali, M.; Al-Mohammed, O. Boundedness of a class of rough maximal functions. *J. Ineq. Appl.* **2018**, *2018*, 305. [CrossRef] [PubMed]
17. Ali, M.; Reyyashi, M. L^p estimates for maximal functions along surfaces of revolution on product spaces. *Open Math.* **2019**, *17*, 1361–1373. [CrossRef]
18. Duoandikoetxea, J.; Rubio de Francia, J. Maximal and singular integral operators via Fourier transform estimates. *Invent Math.* **1986**, *84*, 541–561. [CrossRef]
19. Al-Qassem, H.; Cheng, L.; Pan, Y. On rough generalized parametric Marcinkiewicz integrals. *J. Math. Ineq.* **2017**, *11*, 763–780. [CrossRef]
20. Ali, M.; Al-Refai, O. Boundedness of Generalized Parametric Marcinkiewicz Integrals Associated to Surfaces. *Mathematics* **2019**, *7*, 886. [CrossRef]