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Editorial

An Introductory Overview of Bessel Polynomials, the Generalized Bessel Polynomials and the q-Bessel Polynomials

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Abstract: Named essentially after their close relationship with the modified Bessel function $K_\nu(z)$ of the second kind, which is known also as the Macdonald function (or, with a slightly different definition, the Basset function), the so-called Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ stemmed naturally in some systematic investigations of the classical wave equation in spherical polar coordinates. Our main purpose in this invited survey-cum-expository review article is to present an introductory overview of the Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ involving the asymmetric parameters α and β. Each of these polynomial systems, as well as their reversed forms $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$, has been widely and extensively investigated and applied in the existing literature on the subject. We also briefly consider some recent developments based upon the basic (or quantum or q-) extensions of the Bessel polynomials. Several general families of hypergeometric polynomials, which are actually the truncated or terminating forms of the series representing the generalized hypergeometric function ${}_rF_s$ with r symmetric numerator parameters and s symmetric denominator parameters, are also investigated, together with the corresponding basic (or quantum or q-) hypergeometric functions and the basic (or quantum or q-) hypergeometric polynomials associated with ${}_r\Phi_s$ which also involves r symmetric numerator parameters and s symmetric denominator parameters.

Keywords: Bessel and generalized polynomials; basic (or quantum or q-) Bessel polynomials; Bessel functions and the modified Bessel functions; orthogonality properties; generating functions; polynomial expansions; asymptotic expansions and location of zeros; hypergeometric functions and hypergeometric polynomials; q-hypergeometric functions and q-hypergeometric polynomials

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1. Introduction and Motivation

In the theory of the Bessel function $J_\nu(z)$, the so-called modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ of the first and the second kinds, respectively, are solutions of the modified Bessel's differential equation given by

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}-(z^2+\nu^2)w=0 \qquad (\nu\in\mathbb{C}). \tag{1}$$

In particular, in Macdonald's notation, the modified Bessel function $K_\nu(z)$ of the second kind is defined by (see, for example, [1] and Chapter 7 in [2])

$$K_\nu(z) = \frac{1}{2}\,\pi[I_{-\nu}(z) - I_\nu(z)]\csc(\nu\pi), \tag{2}$$

where in terms of the familiar and the most fundamental mathematical function, the (Euler's) Gamma function $\Gamma(z)$ $(z \in \mathbb{C} \setminus \mathbb{Z}_0^-)$ is given by

$$\Gamma(z) := \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\,\mathrm{d}t & (\Re(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1}(z+j)} & (z \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ n \in \mathbb{N}), \end{cases} \tag{3}$$

we have

$$I_\nu(z) := \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\,z\right)^{\nu+2n}}{n!\,\Gamma(\nu+m+1)}. \tag{4}$$

A slightly different definition, with $\cot(\nu\pi)$ instead of $\csc(\nu\pi)$ on the right-hand side of Equation (2), was used by Basset in 1889 (see, for details, p. 373 on [3]).

Here, and in what follows, we make use of such standard notations as those that are listed below:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\} \qquad (\mathbb{Z}_0^- := \{0, \pm-1, \pm-2, \cdots\}).$$

We also use $\mathbb{Z}$ to denote the set of integers, $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ to denote the set of complex numbers.

As early as 1949, a systematic study of a close relative of the modified Bessel function $K_\nu(z)$ of the second kind was initiated by Krall and Frink [4], who, in light of this specific relationship, called it the Bessel polynomials $y_n(x)$ defined by

$$\begin{aligned} y_n(x) &:= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\, k!\left(\frac{x}{2}\right)^k \\ &= \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,k!}\left(\frac{x}{2}\right)^k. \end{aligned} \tag{5}$$

More precisely, the above-mentioned relationship resulting essentially in the nomenclature of the Bessel polynomials $y_n(x)$ is given by (see, for example, p. 10, Equation 7.2.6 (40) in [2])

$$y_n(x) = \sqrt{\frac{2}{\pi x}}\,\exp\left(\frac{1}{x}\right) K_{n+\frac{1}{2}}\left(\frac{1}{x}\right). \tag{6}$$

The Bessel polynomials $y_n(x)$ provide the polynomial solution of the following differential equation:

$$\frac{\mathrm{d}^2\mathfrak{w}}{\mathrm{d}x^2} + 2(x+2)\frac{\mathrm{d}\mathfrak{w}}{\mathrm{d}x} - n(n+1)\mathfrak{w} = 0 \qquad (\mathfrak{w} = \mathfrak{w}(x) = y_n(x)), \tag{7}$$

which is equal to 1 when $x = 0$. These polynomials satisfy a recurrence relation given by

$$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x), \tag{8}$$

which would readily yield the following special values of the Bessel polynomials $y_n(x)$:

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1+x,\\
y_2(x) &= 1+3x+3x^2,\\
y_3(x) &= 1+6x+15x^2+15x^3,\\
y_4(x) &= 1+10x+45x^2+105x^3+105x^4,\\
y_5(x) &= 1+15x+105x^2+420x^3+945x^4+945x^5,\\
y_6(x) &= 1+21x+210x^2+1260x^3+4725x^4\\
&\qquad +10395x^5+10395x^6,
\end{aligned}$$

and so on.

A *two-parameter* extension $y_n(x;\alpha,\beta)$ of the Bessel polynomials $y_n(x)$ is referred to as the *generalized* Bessel polynomials. Following the work of Krall and Frink [4], we define $y_n(x;\alpha,\beta)$ as follows:

$$\begin{aligned}
y_n(x;\alpha,\beta) &:= \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k\\
&= \sum_{k=0}^{n} \binom{n}{k} (n+\alpha-1)_k \left(\frac{x}{\beta}\right)^k \qquad (9)\\
&\qquad \left(n\in\mathbb{N}_0;\ \alpha\notin\mathbb{Z}_0^-;\ \beta\neq 0\right),
\end{aligned}$$

so that, clearly, we have

$$y_n(x) = y_n(x;2,2) = y_n\left(\frac{\beta x}{2};2,\beta\right). \qquad (10)$$

We remark in passing that the parameter β in the definition (9) may be viewed as a mere scaling factor.

In the definition (9), and in the remainder of this paper, we make use of the general Pochhammer symbol or the *shifted* factorial $(\lambda)_\nu$, since

$$(1)_n = n! \qquad (n\in\mathbb{N}_0),$$

which is defined (for $\lambda,\nu\in\mathbb{C}$), in terms of the Gamma function in (3), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \qquad (11)$$

it is understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists.

The generalized Bessel polynomials $y_n(x;\alpha,\beta)$ satisfy the following differential equation:

$$\frac{d^2\mathfrak{w}}{dx^2} + (\alpha x+\beta)\frac{d\mathfrak{w}}{dx} - n(n+\alpha-1)\mathfrak{w} = 0 \qquad \left(\mathfrak{w}=\mathfrak{w}(x)=y_n(x;\alpha,\beta)\right), \qquad (12)$$

and their recurrence relation is given by (see, for example, p. 111, Equation (51) in [4])

$$
\begin{aligned}
&(n+\alpha-1)(2n+\alpha-2)y_{n+1}(x;\alpha,\beta) \\
&\qquad = \left[(2n+\alpha)(2n+\alpha-2)\left(\frac{x}{\beta}\right)+\alpha-2\right](2n+\alpha-1)y_n(x;\alpha,\beta) \\
&\qquad\qquad + n(2n+\alpha)y_{n-1}(x;\alpha,\beta),
\end{aligned}
\tag{13}
$$

which leads us easily to the following special values of the generalized Bessel polynomials $y_n(x;\alpha,\beta)$:

$$
\begin{aligned}
y_0(x;\alpha,\beta) &= 1, \\
y_1(x;\alpha,\beta) &= 1+\alpha\left(\frac{x}{\beta}\right), \\
y_2(x;\alpha,\beta) &= 1+2(\alpha+1)\left(\frac{x}{\beta}\right)+(\alpha+1)(\alpha+2)\left(\frac{x}{\beta}\right)^2, \\
y_3(x;\alpha,\beta) &= 1+3(\alpha+2)\left(\frac{x}{\beta}\right)+3(\alpha+2)(\alpha+3)\left(\frac{x}{\beta}\right)^2 \\
&\qquad + (\alpha+2)(\alpha+3)(\alpha+4)\left(\frac{x}{\beta}\right)^3, \\
y_4(x;\alpha,\beta) &= 1+4(a+3)\left(\frac{x}{\beta}\right)+6(\alpha+3)(\alpha+4)\left(\frac{x}{\beta}\right)^2 \\
&\qquad + 4(\alpha+3)(\alpha+4)(\alpha+5)\left(\frac{x}{\beta}\right)^3 \\
&\qquad + (\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6)\left(\frac{x}{\beta}\right)^4,
\end{aligned}
$$

and so on.

The Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ emerged in the investigation by Krall and Frink [4] of the classical wave equation in spherical polar coordinates. In fact, not only the Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ but also various different forms of the corresponding reversed Bessel polynomials $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$ have also found applications in many different scientific and engineering fields, for example, in the design of the so-called *Bessel electronic filters* (see, for details, [5]). Traditionally, these reversed Bessel polynomials $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$ are given by

$$
\theta_n(x) = x^n\, y_n\left(\frac{1}{x}\right) \qquad \text{and} \qquad y_n(x) = x^n\, \theta_n\left(\frac{1}{x}\right) \tag{14}
$$

and

$$
\theta_n(x) = x^n\, y_n\left(\frac{1}{x};\alpha,\beta\right) \qquad \text{and} \qquad y_n(x) = x^n\, \theta_n\left(\frac{1}{x};\alpha,\beta\right), \tag{15}
$$

respectively. This survey-cum-expository review article is mainly motivated by, and presents an introductory overview of, the theory and multi-disciplinary applications of Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ as well as those of their above-mentioned reversed forms $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$. We investigate and examine some recent developments based upon Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ and also their basic (or quantum or q-) extensions. A number of general families of hypergeometric polynomials, which are actually the appropriately truncated or terminating forms of the series representing the generalized hypergeometric functions ${}_rF_s$ with r symmetric numerator parameters and s symmetric denominator parameters, are also investigated, together with the corresponding basic (or quantum or q-) hypergeometric functions and polynomials associated with ${}_r\Phi_s$.

For the interest and information of the targeted reader of this survey-cum-expository review, it should be remarked once again that the nomenclature for the so-called Bessel polynomials $y_n(x)$ essentially stems from their close relationship with the modified Bessel function $K_\nu(z)$ of the second kind, which is known also as the Macdonald function (or, with a slightly different definition, the Basset function).

The plan of this review-cum-expository review is described next. In Section 2, we introduce the generalized hypergeometric functions ${}_rF_s$ $(r, s \in \mathbb{N}_0)$ with r symmetric numerator parameters $\alpha_1, \cdots, \beta_r$ and s symmetric denominator parameters $\beta_1, \cdots, \beta_s$. It is in Section 1 itself that we choose to introduce various families of hypergeometric generating functions as well as many of the classical orthogonal polynomials together with their generating functions and interrelationships. Section 3 provides a discussion of the orthogonality properties of the Bessel and generalized Bessel polynomials and a systematic exposition of some erroneous claims that have been made in the cited literature. The important and potentially useful review of the asymptotic expansions and location of zeros of the Bessel and generalized Bessel polynomials are presented in Section 4 in which we also include a presumably not-yet-settled conjecture, which is attributed to Yudell Leo Luke (1918–1983). Basic (or quantum of q-) analogs of the generalized Bessel polynomials are systematically investigated in Section 5. For the interest and use of the targeted reader of this survey-cum-expository review, a discussion of many other orthogonal q-polynomials is also presented. Finally, in Section 6, several concluding remarks and observations are provided, together with the potential directions for further investigations based on the subject matter of this survey-cum-expository review.

2. Hypergeometric Representations and the Associated Generating Functions

First, by using the general Pochhammer symbol $(\lambda)_\nu$ in (11), one of the most useful and fundamental special functions of applicable and applied mathematical sciences happens to be the generalized hypergeometric function ${}_rF_s$, with r symmetric numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, r)$ and s symmetric denominator parameters $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \cdots, s)$, which is defined here as follows (see, for example, [6–10]):

$$
\begin{aligned}
{}_rF_s\left[\begin{array}{c} \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} z\right] &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \\
&=: {}_rF_s(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_s; z)
\end{aligned} \tag{16}
$$

$$\big(r \leqq s+1;\ r < s+1 \ \text{ and } \ |z| < \infty;\ r = s+1 \ \text{ and } \ z \in \mathbb{U} := \{z : z \in \mathbb{C} \ \text{ and } \ |z| < 1\}\big).$$

Obviously, since

$$
(-n)_k = \begin{cases} \dfrac{(-1)^n\, n!}{(n-k)!} & (0 \leqq k \leqq n) \\ \\ 0 & (k \geqq n+1), \end{cases} \tag{17}
$$

whenever one of the numerator parameters is a negative integer or zero, the generalized hypergeometric series in (16) would terminate (or become automatically truncated), thereby leading us to a generalized hypergeometric polynomial of the following type:

$$\begin{aligned}
{}_{r+1}F_s\left[\begin{array}{c} -n,\alpha_1,\cdots,\alpha_r; \\ \\ \beta_1,\cdots,\beta_s; \end{array} z\right]
&:= \sum_{k=0}^{n} \frac{(-n)_k\,(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_s)_k}\,\frac{z^k}{k!} \\
&= \frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\,(-z)^n \\
&\quad \cdot\ {}_{s+1}F_r\left[\begin{array}{c} -n,1-\beta_1-n,\cdots,1-\beta_s-n; \\ \\ 1-\alpha_1-n,\cdots,1-\alpha_r-n; \end{array} \frac{(-1)^{r+s}}{z}\right] \qquad (n\in\mathbb{N}_0).
\end{aligned} \tag{18}$$

Most (if not all) of the familiar families of the classical orthogonal polynomials, including, for example, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the Hermite polynomials $H_n(x)$ (see, for details, [11]), Bessel polynomials $y_n(x)$, and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, , as well as a significantly large number of other polynomial systems, are essentially one form or other of the generalized hypergeometric polynomials, which are given by (18). In fact, in view of definitions (5) and (9), we have the following hypergeometric representations:

$$y_n(x) = {}_2F_0\left(-n,n+1;\underline{\quad};-\frac{x}{2}\right) \tag{19}$$

and

$$y_n(x;\alpha,\beta) = {}_2F_0\left(-n,\alpha+n-1;\underline{\quad};-\frac{x}{\beta}\right), \tag{20}$$

respectively. In particular, we can apply the hypergeometric representation (19) to derive the following interesting identities for the Bessel polynomials $y_n(x)$:

$$y_{-n}(x) = y_{n-1}(x) \qquad \text{and} \qquad y_{-1}(x) = y_0(x) = 1,$$

which would extend the Bessel polynomials $y_n(x)$ to negative integer values of n. Thus, remarkably, the recurrence relation (8) satisfied by the Bessel polynomials $y_n(x)$ holds true for all the integer values of n (that is, for $n\in\mathbb{Z}$).

The following three hypergeometric generating functions were extensively investigated by Chaundy p. 62, Equations (25) to (27) in [12] (see also pp. 138–139, Equations 2.6 (8) to 2.6 (10) in [13]):

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_{r+1}F_s\left[\begin{array}{c} -n,\alpha_1,\cdots,\alpha_r; \\ \\ \beta_1,\cdots,\beta_s; \end{array} z\right] t^n \\
&\qquad = (1-t)^{-\lambda}\ {}_{r+1}F_s\left[\begin{array}{c} \lambda,\alpha_1,\cdots,\alpha_r; \\ \\ \beta_1,\cdots,\beta_s; \end{array} -\frac{zt}{1-t}\right] \\
&\qquad\qquad (\lambda\in\mathbb{C};\ |t|<1),
\end{aligned} \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{r+2}F_s \left[\begin{array}{r} -n, \lambda+n, \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} z \right] t^n$$
$$= (1-t)^{-\lambda} \; {}_{r+2}F_s \left[\begin{array}{r} \Delta(2;\lambda), \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} -\frac{4zt}{(1-t)^2} \right] \tag{22}$$
$$(\lambda \in \mathbb{C};\ |t| < 1)$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{r+1}F_{s+1} \left[\begin{array}{r} -n, \alpha_1, \cdots, \alpha_r; \\ \\ 1-\lambda-n, \beta_1, \cdots, \beta_s; \end{array} z \right] t^n$$
$$= (1-t)^{-\lambda} \; {}_rF_s \left[\begin{array}{l} \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} zt \right] \tag{23}$$
$$(\lambda \in \mathbb{C};\ |t| < 1),$$

where it is tacitly assumed that each member of the hypergeometric generating functions (21) to (23) exists. In the hypergeometric generating function (22) and elsewhere in this article, it is convenient to use $\Delta(m;\lambda)$ to denote the following set of m parameters:

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \cdots, \frac{\lambda+m-1}{m} \qquad (m \in \mathbb{N};\ \lambda \in \mathbb{C}).$$

Since

$$\lim_{|\lambda|\to\infty} \left\{ (\lambda)_n \left(\frac{z}{\lambda}\right)^n \right\} = z^n \qquad (n \in \mathbb{N}_0;\ \lambda, z \in \mathbb{C}), \tag{24}$$

a limit case of the hypergeometric generating function (22) when $t \mapsto \frac{t}{\lambda}$ and $|\lambda| \to \infty$ yields the following result given by Brafman, p. 947, Equation (27) in [14]:

$$\sum_{n=0}^{\infty} \; {}_{r+1}F_s \left[\begin{array}{r} -n, \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} z \right] \frac{t^n}{n!}$$
$$= \mathrm{e}^t \; {}_rF_s \left[\begin{array}{l} \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} -zt \right] \qquad (|t| < \infty), \tag{25}$$

which is usually attributed to Rainville (see, for details, p. 267, Equation (25) in [15]). Rainville [16] also rediscovered Chaundy's hypergeometric generating function (22).

The hypergeometric generating function (22) can be applied to derive the generating functions of the celebrated Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+k+\alpha+\beta}{k}\left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \\ &= \binom{\alpha+n}{n}\, {}_2F_1\left(-n,\alpha+\beta+n+1;\alpha+1;\frac{1-x}{2}\right), \end{aligned} \tag{26}$$

as well as the generating functions of their such special or limit cases as, for example, the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, the Legendre (or spherical) polynomials $P_n(x)$, and the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ (see, for details, [11]). In the case of the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, we thus find from (22) (with $\lambda = \alpha - 1$) that (see p. 294, Equation (6) in [17] and p. 139, Equation 2.6 (11) in [13])

$$\begin{aligned} &(1-t)^{1-\alpha}\, {}_2F_0\left[\begin{array}{l} \Delta(2;\alpha-1); \\ \underline{\qquad\qquad}; \end{array} \frac{4xt}{\beta(1-t)^2}\right] \\ &\cong \sum_{n=0}^{\infty} \frac{(\alpha-1)_n}{n!}\, y_n(x;\alpha,\beta)\, t^n \qquad (|t|<1), \end{aligned} \tag{27}$$

which, in a further special case when $\alpha = \beta = 2$, yields the corresponding *divergent* generating function for the Bessel polynomials $y_n(x)$.

Each of the hypergeometric generating functions (21) to (23) and (25) is applicable in deriving generating functions for the Laguerre polynomials $L_n^{(\alpha)}(x)$, as well as for their relatives, for which we have (see, for details, [11] and [18,19])

$$\begin{aligned} L_n^{(\alpha)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \\ &= \binom{\alpha+n}{n}\, {}_1F_1(-n;\alpha+1;x) \\ &= \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}. \end{aligned} \tag{28}$$

In the case of the Hermite polynomials $H_n(x)$ for which we have

$$\begin{aligned} H_n(x) &:= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!}{k!\,(n-2k)!}\, x^{n-2k} \\ &= (2x)^n\, {}_2F_0\left[\begin{array}{l} \Delta(2;-n); \\ \underline{\qquad\qquad}; \end{array} -\frac{1}{x^2}\right], \end{aligned} \tag{29}$$

$[\kappa]$ being the largest integer in $\kappa \in \mathbb{R}$, so that

$$H_{2n}(x) = \lim_{|\epsilon|\to\infty} \left\{ (-1)^n\, n!\, 2^{2n}\, P_n^{(\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right)\right\}$$

and

$$H_{2n+1}(x) = \lim_{|\epsilon|\to\infty} \left\{ (-1)^n\, n!\, 2^{2n+1}\, x\, P_n^{(-\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right)\right\},$$

one can make use of these last limit relations in conjunction with the generating functions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in order to derive the corresponding generating functions for the Hermite polynomials $H_n(x)$. These obviously long and involved derivations can, in fact, be significantly simplified by means of Brafman's general form of the hypergeometric generating function (21) (see [20] and p. 136, Equation 2.6 (2) in [13]):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{m+r}F_s\left[\begin{array}{c}\Delta(m;-n),\alpha_1,\cdots,\alpha_r;\\ \\ \beta_1,\cdots,\beta_s;\end{array} x\right] t^n \\ = (1-t)^{-\lambda} \; {}_{m+r}F_s\left[\begin{array}{c}\Delta(m;\lambda),\alpha_1,\cdots,\alpha_r;\\ \\ \beta_1,\cdots,\beta_s;\end{array} x\left(-\frac{t}{1-t}\right)^m\right] \end{aligned} \tag{30}$$

$$(\lambda \in \mathbb{C};\ m \in \mathbb{N};\ |t| < 1),$$

where $\Delta(m;\lambda)$ denotes the m-parameter sequence:

$$\left\{\frac{\lambda+j-1}{m}\right\}_{j=1}^{m} \qquad (\lambda \in \mathbb{C};\ m \in \mathbb{N}).$$

This last hypergeometric generating function (30) not only reduces to (21) when $m = 1$, but it also applies to the Gould–Hopper generalization $g_n^m(x,h)$ of the Hermite polynomials $H_n(x)$, which is defined by (see, for details, [21])

$$\begin{aligned} g_n^m(x,h) &:= \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!}\, h^k\, x^{n-mk} \\ &= x^n \; {}_mF_0\left[\begin{array}{c}\Delta(m;-n);\\ \\ \underline{\qquad\qquad};\end{array} \left(-\frac{m}{x}\right)^m h\right], \end{aligned} \tag{31}$$

leading us to the following *divergent* generating function from (30):

$$\begin{aligned}(1-xt)^{-\lambda} \; {}_mF_q\left[\begin{array}{c}\Delta(m;\lambda);\\ \\ \underline{\qquad\qquad};\end{array} \left(\frac{mt}{1-xt}\right)^m h\right] \\ \cong \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, g_n^m(x,h)\, t^n. \end{aligned} \tag{32}$$

For the orthogonal family of the *two-parameter* Bessel polynomials $y_n(x,a,b)$, it is easily observed from the limit relationship in (28), in conjunction with

$$y_n(x;\alpha,\beta) = n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right), \tag{33}$$

that

$$y_n(x;\alpha,\beta) = n!\left(-\frac{x}{\beta}\right)^n \lim_{\epsilon\to\infty}\left\{P_n^{(1-\alpha-2n,\epsilon)}\left(1-\frac{2\beta}{\epsilon x}\right)\right\} \tag{34}$$

or, equivalently, that (see, for example, [22])

$$y_n(x;\alpha,\beta) = \lim_{\epsilon\to\infty}\left\{\frac{n!}{(\epsilon)_n}\, P_n^{(\epsilon-1,\alpha-\epsilon-1)}\left(\frac{1+2\epsilon x}{\beta}\right)\right\}, \tag{35}$$

together with similar relationships for the reversed Bessel polynomials $\theta_n(x;\alpha,\beta)$ defined by (14) and (15). Thus, obviously, the generating functions of the generalized Bessel polynomials can possibly be deduced from the (known or new) generating functions for the relatively more familiar Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and for Laguerre polynomials $L_n^{(\alpha)}(x)$.

In addition to the *divergent* generating function of Bessel polynomials $y_n(x)$, which corresponds to the special case of (27) when $\alpha = \beta = 2$, the following frequently-cited generating function of Bessel polynomials $y_n(x)$ was presented by Krall and Frink (see p. 106, Equation (25) in [4]):

$$\sum_{n=0}^{\infty} y_{n-1}(x)\, \frac{t^n}{n!} = \exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) \qquad \big(y_{-1}(x) = y_0(x) = 1\big). \tag{36}$$

In view of the success and usefulness of the hypergeometric generating functions (21) to (23) in the derivation of simpler generating functions for numerous families of hypergeometric polynomials, including Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, it may be of interest to recall the following general families of generating functions involving an appropriately bounded sequence $\{\Omega(n)\}_{n\in\mathbb{N}_0}$ of essentially arbitrary real or complex numbers (see, for details, [23]):

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{k=0}^{[\frac{n}{m}]} (-n)_{mk}\, \Omega(k)\, \frac{z^k}{k!}\right) t^n \\ &\qquad = (1-t)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_{mk}}{k!}\, \Omega(k) \left(\frac{z(-t)^m}{(1-t)^m}\right)^k \\ &\qquad\qquad (\lambda \in \mathbb{C};\ m \in \mathbb{N};\ |t| < 1),\end{aligned} \tag{37}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{k=0}^{[\frac{n}{m}]} (-n)_{mk}\, (\lambda+n)_{mk}\, \Omega(k)\, \frac{z^k}{k!}\right) t^n \\ &\qquad = (1-t)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_{2mk}}{k!}\, \Omega(k) \left(\frac{z(-t)^m}{(1-t)^{2m}}\right)^k \\ &\qquad\qquad (\lambda \in \mathbb{C};\ m \in \mathbb{N};\ |t| < 1)\end{aligned} \tag{38}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{k=0}^{[\frac{n}{m}]} \frac{(-n)_{mk}}{(1-\lambda-n)_{mk}}\, \Omega(k)\, \frac{z^k}{k!}\right) t^n \\ &\qquad = (1-t)^{-\lambda} \sum_{k=0}^{\infty} \Omega(k)\, \frac{(zt^m)^k}{k!} \\ &\qquad\qquad (\lambda \in \mathbb{C};\ m \in \mathbb{N};\ |t| < 1),\end{aligned} \tag{39}$$

where it is assumed that each member of the generating functions (37) to (39) exists.

By applying the limit relationship (24), a limit case of the hypergeometric generating function (38) when $t \mapsto \frac{t}{\lambda}$ and $|\lambda| \to \infty$ leads us to the following companion of the generating functions (37) to (39):

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n}{m}]} (-n)_{mk}\, \Omega(k)\, \frac{z^k}{k!}\right) \frac{t^n}{n!} = \mathrm{e}^t \sum_{k=0}^{\infty} \Omega(k)\, \frac{\big[z(-t)^m\big]^k}{k!} \tag{40}$$

$$(m \in \mathbb{N};\ |t| < \infty),$$

which analogously extends the hypergeometric generating function (25).

Remark 1. *In the assertions (37) to (40), and elsewhere in this paper, all of the parametric values that would render any member invalid or undefined are tacitly excluded.*

We turn now toward some further advances in the study of generating functions for Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$. First, we recall the following *corrected and modified* form of Burchnall's generating function for Bessel polynomials $y_n(x,\alpha,\beta)$ (see p. 67 in [24] and p. 84 in [13]):

$$\sum_{n=0}^{\infty} y_n(x,\alpha,\beta)\,\frac{t^n}{n!} = \frac{1}{\sqrt{1-\frac{4xt}{\beta}}}\left(\frac{2}{1+\sqrt{1-\frac{4xt}{\beta}}}\right)^{\alpha-2} \cdot \exp\left(\frac{2t}{1+\sqrt{1-\frac{4xt}{\beta}}}\right). \tag{41}$$

Some further developments emerging from Burchnall's work [24] can be found in [16,25–27], and other publications (see, for details, [5,13,28]). In particular, by appropriately applying the following hypergeometric reduction formula (see p. 101, Equation 2.8 (6) in [7]):

$$ {}_2F_1\left(\lambda,\lambda+\frac{1}{2};2\lambda;z\right) = \frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\lambda-1} \qquad (|z|<1), \tag{42}$$

Rainville [16] showed that

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(\alpha_j)_n}{\prod_{j=1}^{s}(\beta_j)_n}\,\psi_n(x;\lambda)\,\frac{t^n}{n!} = \frac{1}{\sqrt{1-4xt}}\left(\frac{2}{1+\sqrt{1-4xt}}\right)^{\lambda-1} \cdot {}_rF_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r; \\ \beta_1,\cdots,\beta_s;\end{matrix}\ \frac{2t}{1+\sqrt{1-4xt}}\right], \tag{43}$$

where $\sqrt{1-4xt} \to 1$ as $t \to 0$ and the hypergeometric polynomial sequence $\{\psi_n(x;\lambda)\}_{n\in\mathbb{N}_0}$ is given by

$$\psi_n(x;\lambda) := {}_{s+2}F_r\left[\begin{matrix}-n,\lambda+n,1-\beta_1-n,\cdots,1-\beta_s-n; \\ 1-\alpha_1-n,\cdots,1-\alpha_r-n;\end{matrix}\ (-1)^{r+s+1}\,x\right]. \tag{44}$$

Now, as in the case of the generalized hypergeometric polynomial identity (18), using a reversal of the order of terms, it is not difficult to show that

$$
\begin{aligned}
&{}_{s+2}F_r\left[\begin{array}{c}-n,\lambda+n,1-\beta_1-n,\cdots,1-\beta_s-n;\\ \\ 1-\alpha_1-n,\cdots,1-\alpha_r-n;\end{array}\ z\right]\\
&\quad=(-1)^{(r+s+1)n}\,(\lambda+n)_n\,\frac{(\beta_1)_n\cdots(\beta_s)_n}{(\alpha_1)_n\cdots(\alpha_r)_n}\,z^n\\
&\quad\quad\cdot\ {}_rF_{s+1}\left[\begin{array}{c}-n,\alpha_1,\cdots,\alpha_r;\\ \\ 1-\lambda-2n,\beta_1,\cdots,\beta_s;\end{array}\ \frac{(-1)^{r+s+1}}{z}\right],
\end{aligned}\tag{45}
$$

so that the definition (44) can be rewritten as follows:

$$
\begin{aligned}
\psi_n(x;\lambda)&=(\lambda+n)_n\,\frac{(\beta_1)_n\cdots(\beta_s)_n}{(\alpha_1)_n\cdots(\alpha_r)_n}\,x^n\\
&\quad\cdot\ {}_rF_{s+1}\left[\begin{array}{c}-n,\alpha_1,\cdots,\alpha_r;\\ \\ 1-\lambda-2n,\beta_1,\cdots,\beta_s;\end{array}\ \frac{1}{x}\right].
\end{aligned}\tag{46}
$$

Thus, clearly, Rainville's result (43) can be put in the following hypergeometric form:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(\lambda+n)_n\ {}_rF_{s+1}\left[\begin{array}{c}-n,\alpha_1,\cdots,\alpha_r;\\ \\ 1-\lambda-2n,\beta_1,\cdots,\beta_s;\end{array}\ \frac{1}{x}\right]\frac{(xt)^n}{n!}\\
&\quad=\frac{1}{\sqrt{1-4xt}}\left(\frac{2}{1+\sqrt{1-4xt}}\right)^{\lambda-1}{}_rF_s\left[\begin{array}{c}\alpha_1,\cdots,\alpha_r;\\ \\ \beta_1,\cdots,\beta_s;\end{array}\ \frac{2t}{1+\sqrt{1-4xt}}\right],
\end{aligned}\tag{47}
$$

An extension of this last hypergeometric generating function (47), which would belong to the family of the generating function relations (37) to (40) involving the suitably bounded sequence $\{\Omega(n)\}_{n\in\mathbb{N}_0}$ of essentially arbitrary real or complex numbers, is expressed below for the sake of completeness:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(\lambda+n)_n\left(\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-n)_{mk}}{(1-\lambda-2n)_{mk}}\,\Omega(k)\,\frac{z^k}{k!}\right)\frac{t^n}{n!}\\
&\quad=\frac{1}{\sqrt{1-4t}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\lambda-1}\sum_{k=0}^{\infty}\Omega(k)\left(\frac{2t}{1+\sqrt{1-4xt}}\right)^{mk}\frac{z^k}{k!}
\end{aligned}\tag{48}
$$

$$
\left(\lambda\in\mathbb{C};\ m\in\mathbb{N};\ |t|<\frac{1}{4}\right),
$$

which yields (47) when we set

$$
m=1,\ \ z=\frac{1}{x},\ \ t\mapsto xt\quad\text{and}\quad\Omega(k)=\frac{(\alpha_1)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta)_s}\qquad(k\in\mathbb{N}_0).
$$

Here, it is worth noting that, in contrast to Rainville's derivation of the generating function (43) or, equivalently, the generating function (47), which made use of the series rearrangement technique (see, for details, Chapter 2 in [13]) in conjunction with the hypergeometric reduction formula (42), Burchnall [24] applied several identities and differential equations involving the derivative operator $\delta=x\,\frac{\mathrm{d}}{\mathrm{d}x}$ in proving (41) (see also [29]). More recently, by means of the Lie algebraic (or group-theoretic) technique of Weisner [30] (see also Chapter 6 in [13] and the work by Miller [31]), several interesting and potentially useful generating functions and generating relations for Bessel polynomials $y_n(x)$ were derived by McBride pp. 47–50 in [28]. For example, we recall the following generating relation for Bessel polynomials $y_n(x)$ (see p. 50, Equation (12) in [28]):

$$\sum_{n=0}^{\infty} y_{m+n}(x)\,\frac{t^n}{n!} = (1-2xt)^{-\frac{1}{2}(m+1)}\,\exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) \cdot y_m\left(\frac{x}{\sqrt{1-2xt}}\right) \qquad \left(m \in \mathbb{N}_0;\ |t| < \frac{1}{2}|x|^{-1}\right), \tag{49}$$

which can be applied to show that, for identically nonvanishing function $\Omega_\mu(\xi_1, \cdots, \xi_s)$ of s real or complex variables $\xi_1, \cdots, \xi_s \quad (s \in \mathbb{N})$ and of order $\mu \in \mathbb{C}$, if we let

$$\Lambda_{m,\mathfrak{p},\mathfrak{q}}[x;\xi_1,\cdots,\xi_s : z] := \sum_{n=0}^{\infty} a_n\, y_{m+\mathfrak{q}n}(x)\,\Omega_{\mu+\mathfrak{p}n}(\xi_1,\cdots,\xi_s)\,\frac{z^n}{(\mathfrak{q}n)!} \tag{50}$$

$$(a_n \neq 0;\ m \in \mathbb{N}_0;\ \mathfrak{p},\mathfrak{q} \in \mathbb{N})$$

and

$$M_{n,\mathfrak{q}}^{\mathfrak{p},\mu}(\xi_1,\cdots,\xi_s;\eta) := \sum_{k=0}^{\left[\frac{n}{\mathfrak{q}}\right]} \binom{n}{\mathfrak{q}k}\, a_k\,\Omega_{\mu+\mathfrak{p}k}(\xi_1,\cdots,\xi_s)\,\eta^k, \tag{51}$$

then the following family of multilinear or mixed multilateral generating functions for Bessel polynomials $y_n(x)$ holds true (see, for details, Part I, p. 229, Corollary 2 in [32] and p. 421, Corollary 2 in [13]):

$$\begin{aligned}\sum_{n=0}^{\infty} & y_{m+n}(x)\, M_{n,\mathfrak{q}}^{\mathfrak{p},\mu}(\xi_1,\cdots,\xi_s;\eta)\,\frac{t^n}{n!} \\ &= (1-2xt)^{-\frac{1}{2}(m+1)}\,\exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) \\ &\quad \cdot \Lambda_{m,\mathfrak{p},\mathfrak{q}}\left[\frac{x}{\sqrt{1-2xt}};\xi_1,\cdots,\xi_s : \eta\left(\frac{t}{\sqrt{1-2xt}}\right)^{\mathfrak{q}}\right] \qquad \left(|t| < \frac{1}{2}|x|^{-1}\right).\end{aligned} \tag{52}$$

In the case of the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, one can similarly apply each of the following generating functions:

$$\sum_{n=0}^{\infty} y_{m+n}(x;\alpha-n,\beta)\,\frac{t^n}{n!} = \left(1-\frac{xt}{\beta}\right)^{1-\alpha-m} e^t\, y_m\left(\frac{\beta x}{\beta-xt};\alpha,\beta\right) \tag{53}$$

$$\left(m \in \mathbb{N}_0;\ |t| < \left|\frac{\beta}{x}\right|\right),$$

$$\sum_{n=0}^{\infty} \binom{\alpha+m+n-2}{n}\, y_m(x;\alpha+n;\beta)\, t^n = (1-t)^{1-\alpha-m}\, y_m\left(\frac{x}{1-t};\alpha,\beta\right) \tag{54}$$

$$(m \in \mathbb{N}_0;\ |t| < 1),$$

$$\sum_{n=0}^{\infty} y_m(x;\alpha-n,\beta)\,\frac{t^n}{n!} = \left(1-\frac{xt}{\beta}\right)^{m} e^t\, y_m\left(\frac{\beta x}{\beta-xt};\alpha,\beta\right) \qquad (m \in \mathbb{N}_0) \tag{55}$$

and

$$\sum_{n=0}^{\infty} y_{m+n}(x;\alpha-2n,\beta)\,\frac{(-\beta t)^n}{n!} = (1-xt)^{\alpha-2}\,\exp\left(-\frac{\beta t}{1-xt}\right)$$
$$\cdot\, y_m\big(x(1-xt);\alpha,\beta\big) \qquad \big(m\in\mathbb{N}_0;\ |t|<|x|^{-1}\big). \tag{56}$$

Such families of multilinear or mixed multilateral generating functions for the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ as those derivable from (53) to (56), which are analogous to (52), can be found in the works of Chen and Srivastava (see p. 154 in [33]); Chen et al. (see pp. 363–364 in [34]); and Srivastava (see p. 129 in [35]) (see also some related developments reported by Lin et al. [36]). Some rather obvious special cases of the general families of multilinear or mixed multilateral generating functions for the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ as those derivable from (53) to (56) were recently considered by Biswas and Chongdar [37].

Various other families of generating functions for Bessel polynomials $y_n(x)$ and $y_n(x;\alpha,\beta)$ involving the Stirling numbers $S(n,k)$ of the second kind, given by (see p. 90 et seq. in [38])

$$S(n,k) := \begin{cases} \dfrac{1}{k!}\displaystyle\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\,j^n & (k\in\mathbb{N}) \\[2ex] \delta_{n,0} & (n\in\mathbb{N}_0), \end{cases} \tag{57}$$

can be found in the aforementioned work (see p. 273 in [36]; see also [39,40]), $\delta_{m,n}$ being the Kronecker symbol.

Remark 2. *It is remarkable to observe that generating functions can be appropriately applied in the determination of the asymptotic behavior of the generated sequence* $\{\mathfrak{f}_n\}_{n=0}^{\infty}$ *by suitably adapting Darboux's method. Additionally, the existence of a generating function for a sequence* $\{\mathfrak{f}_n\}_{n=0}^{\infty}$ *of numbers or functions may be useful in determining* $\sum_{n=0}^{\infty}\mathfrak{f}_n$ *by means of such summability methods as those attributed to Abel and Cesàro.*

Remark 3. *For the convenience of the interested reader, we refer here to several other noteworthy contributions (see, for example, [41–79]) on the Bessel polynomials* $y_n(x)$ *and the generalized Bessel polynomials* $y_n(x;\alpha,\beta)$*, their reversed forms* $\theta_n(x)$ *and* $\theta_n(x;\alpha,\beta)$*, and other developments which are related to the work presented in this section.*

3. Orthogonality Relations and Polynomial Expansions

In their monumental work, Krall and Frink presented the following orthogonality property of the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ (see [4]):

$$\frac{1}{2\pi \mathrm{i}}\int_{|z|=1}\rho^{(\alpha,\beta)}(z)\;y_m(z;\alpha,\beta)\;y_n(z;\alpha,\beta)\;\mathrm{d}z$$
$$= (-1)^{n+1}\,\frac{n!}{\alpha+2n-1}\,\frac{\beta\Gamma(\alpha)}{\Gamma(\alpha+n-1)}\,\delta_{m,n} \qquad (m,n\in\mathbb{N}_0), \tag{58}$$

where $\delta_{m,n}$ denotes, as usual, the Kronecker symbol, and the weight function $\rho^{(\alpha,\beta)}(z)$ is given, in terms of the Kummer's confluent hypergeometric function ${}_1F_1(a;b;z)$, by

$$\begin{aligned}\rho^{(\alpha,\beta)}(z) &= (\alpha-1)\,{}_1F_1\left(1;\alpha-1;-\frac{\beta}{z}\right)\\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)}\left(-\frac{\beta}{z}\right)\end{aligned} \tag{59}$$

In light of the limit relationship (35) with the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the orthogonality property (58) is presumably not unexpected. However, each of the following observations concerning (58) is admittedly somewhat surprising:

(i) The path of integration is not a segment of the real axis but is rather a curve, that is, the unit circle $|z| = 1$ in the complex z-plane;
(ii) The integral in (58) is not the inner product with the kernel:

$$\rho^{(\alpha,\beta)}(z)\; y_m(z;\alpha,\beta)\; \overline{y_n(z;\alpha,\beta)},$$

which is expected for polynomials that are orthogonal on curves in the complex plane.

Such orthogonality properties as (58) are potentially useful in the numerical evaluation of inverse Laplace transforms (see, for details, [80,81]).

Remarkably, in the special case when $\alpha \in \mathbb{N}$, the following relatively simpler orthogonality property was given by Burchnall (see p. 68, Equation (29) in [24]). The interested reader should refer to Chapter 4 in the monograph by Grosswald [5] and also to the monograph by Luke (see [82] and p. 194, Equation 14.2 (26) in [83]):

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}} \int_{\mathfrak{C}} z^{\alpha-2}\, \exp\left(-\frac{\beta}{z}\right)\, y_m(z;\alpha,\beta)\; y_n(z;\alpha,\beta)\, \mathrm{d}z\\ &\qquad = (-1)^{\alpha+n-1}\, \beta^{\alpha-1}\, \frac{n!}{(\alpha+n-2)!\,(\alpha+2n-1)}\, \delta_{m,n}\end{aligned} \tag{60}$$

$$(m,n \in \mathbb{N}_0;\ \alpha \in \mathbb{N}),$$

where $\mathfrak{C}$ denotes any closed contour surrounding the origin $z = 0$ in the complex z-plane. In fact, the weight function $z^{\alpha-2}\ \exp\left(-\frac{\beta}{z}\right)$, which appears in the orthogonality property (60), was suggested by Krall and Frink (see p. 109, Equation (38) in [4]).

Next, we turn to the hypergeometric representation (20), which easily yields

$$y_n\left(-\frac{\beta}{x};\alpha,\beta\right) =: y_n(1;\alpha,x) \qquad (n \in \mathbb{N}_0). \tag{61}$$

The following orthogonality property of these special polynomials $y_n(1;\alpha,x)$ was asserted by Hamza [84] (see also [85]):

$$\begin{aligned}&\int_0^{\infty} x^{1-\alpha}\, \mathrm{e}^{-x}\, y_m(1;\alpha,x)\; y_n(1;\alpha,x)\, \mathrm{d}x\\ &\qquad = n!\,\Gamma(2-\alpha-n)\, \delta_{m,n}\end{aligned} \tag{62}$$

$$(m,n \in \mathbb{N}_0;\ \Re(2-\alpha-n) > 0).$$

It was shown by Srivastava [86] that the assertion (62) does not hold true except in the case when $m = n \;\; (m,n \in \mathbb{N}_0)$, in which case (62) happens to be a simple consequence of the following well-known orthogonality property of the Laguerre polynomials $L_n^{(\alpha)}(x)$ because of the known relationship (33) (see, for example, [11,13]):

$$\int_0^\infty x^\alpha\, e^{-x}\, L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, dx$$
$$= \frac{\Gamma(\alpha+n+1)}{n!} \qquad \big(m,n \in \mathbb{N}_0;\ \Re(\alpha) > -1\big). \tag{63}$$

Thus, clearly, (62) cannot, in any event, be viewed as an orthogonality property of the special polynomials $y_n(1;\alpha,x)$. As an important consequence of this observation by Srivastava [86], none of the so-claimed expansion formulas for the Meijer G-function (see [87]) hold true, nor does the Fox H-function or its multivariate extensions (see [88–90]) in the series of the special polynomials $y_n(1;\alpha,x)$, which were subsequently derived by Saxena and Hamza (see p. 84, Equation (3.1) in [91]); Mathai and Saxena (see pp. 80–81, Exercise 3.1 in [92]); Saxena and Kalla (see p. 74, Equation (3.1) in [93]); and Gokhroo and Saxena [94], using (62) in each work as an orthogonality property in every situation.

Remark 4. *As demonstrated by Srivastava [86], Hamza's claimed assertion (62), which has been blindly reproduced and erroneously used in many subsequent works, including those that are listed above, should be corrected to read as follows (see p. 212, Equation (21) in [86]):*

$$\int_0^\infty x^{1-\alpha}\, e^{-x}\, \big[y_n(1;\alpha,x)\big]^2\, dx = n!\, \Gamma(2-\alpha-n)\, \delta_{m,n} \tag{64}$$
$$\big(n \in \mathbb{N}_0;\ \Re(2-\alpha-2n) > 0\big).$$

Motivated by Srivastava's observation in [86] that the assertion in (62) is not valid as claimed and used in the literature, Exton [95] investigated the case of the differential Equation (12) when $\beta = 1$ in the following *self-adjoint* form:

$$\frac{d}{dx}\left\{x^\alpha\, \exp\left(-\frac{1}{x}\right) \frac{d\mathfrak{w}}{dx}\right\} - n(n+\alpha-1)x^{\alpha-2}\, \exp\left(-\frac{1}{x}\right) \mathfrak{w} = 0. \tag{65}$$
$$\big(\mathfrak{w} = \mathfrak{w}(x) = y_n(x;\alpha,1)\big).$$

Thus, by applying the general theory of Sturm-Liouville systems, Exton [95] was finally led to the following orthogonality relation (see, for details, p. 215, Equation (13) in [95]; see also [82] and p. 194, Equation 14.2 (26) in [83]):

$$\frac{1}{2\pi i} \int_0^{(\infty+)} z^{\alpha-2}\, \exp\left(-\frac{1}{z}\right) y_m(z;\alpha,1)\, y_n(z;\alpha,1)\, dz$$
$$= (-1)^{\alpha+n}\, \frac{n!}{(\alpha+2n-1)\Gamma(\alpha+n-1)}\, \delta_{m,n} \qquad (m,n \in \mathbb{N}_0), \tag{66}$$

in which the contour of integration is taken along a simple loop starting at the origin ($z = 0$), encircling the point at ∞ once in the positive (counter-clockwise) direction, and then returning to the origin (see, for example, p. 245 in [3]).

The existence of the following orthogonality relation was also pointed out by Exton, p. 215, Equation (14) in [95]:

$$\int_0^\infty x^{\alpha-2}\, \exp\left(-\frac{1}{x}\right) y_m(x;\alpha,1)\, y_n(x;\alpha,1)\, dx$$
$$= (-1)^n\, \frac{n!}{(\alpha+2n-1)\Gamma(\alpha+n-1)}\, \frac{\pi}{\sin(\pi\alpha)}\, \delta_{m,n} \tag{67}$$
$$\big(m,n \in \mathbb{N}_0;\ \Re(\alpha) < 1-m-n\big).$$

Since, according to the hypergeometric representation (20), we have

$$y_n\left(\frac{z}{\beta};\alpha,1\right)=y_n(z;\alpha,\beta)\qquad(n\in\mathbb{N}_0)$$

and, by means of the familiar Γ-function identity, we have

$$\frac{\pi}{\sin(\pi z)}=\Gamma(z)\,\Gamma(1-z)\qquad(z\in\mathbb{C}\setminus\mathbb{Z}),\tag{68}$$

we can easily rewrite Exton's results (66) and (67) in the following *equivalent* forms:

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_0^{(\infty+)}z^{\alpha-2}\exp\left(-\frac{1}{z}\right)y_m(z;\alpha,\beta)\,y_n(z;\alpha,\beta)\,\mathrm{d}z&\\=(-1)^{\alpha+n}\,\beta^{\alpha-1}\,\frac{n!}{(\alpha+2n-1)\Gamma(\alpha+n-1)}\,\delta_{m,n}&\qquad(m,n\in\mathbb{N}_0)\end{aligned}\tag{69}$$

and

$$\begin{aligned}\int_0^{\infty}x^{\alpha-2}\exp\left(-\frac{1}{x}\right)y_m(x;\alpha,\beta)\,y_n(x;\alpha,\beta)\,\mathrm{d}x&\\=\beta^{\alpha-1}\,\frac{n!}{(1-\alpha-2n)}\,\Gamma(2-\alpha-n)\,\delta_{m,n}&\end{aligned}\tag{70}$$

$$\big(m,n\in\mathbb{N}_0;\ \Re(\alpha)<1-m-n;\ \Re(\beta)>0\big),$$

respectively.

Remark 5. *The orthogonality relation (69) is worth comparing with Burchnall's result (60). Furthermore, Equation (70) may be viewed as another orthogonality property of the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ in the general case.*

Noting once again from the hypergeometric representation (20) that

$$\begin{aligned}y_n(\xi;\alpha,x)&=y_n\left(\frac{\xi}{x};\alpha,1\right)\\&=y_n\left(\frac{\beta\xi}{x};\alpha,\beta\right)\qquad(n\in\mathbb{N}_0),\end{aligned}\tag{71}$$

would enable us to rewrite the orthogonality property (70) in an equivalent form given by

$$\begin{aligned}\int_0^{\infty}x^{-\alpha}\exp\left(-\frac{x}{\xi}\right)y_m(\xi;\alpha,x)\,y_n(\xi;\alpha,x)\,\mathrm{d}x&\\=\xi^{1-\alpha}\,\frac{n!}{(1-\alpha-2n)}\,\Gamma(2-\alpha-n)\,\delta_{m,n}&\end{aligned}\tag{72}$$

$$\big(m,n\in\mathbb{N}_0;\ \Re(\alpha)<1-m-n;\ \xi>0\big).$$

Remark 6. *In the special case when $\xi=1$, this last result (72) provides the corrected version of Hamza's erroneous claim (62), as it was pointed out by Bajpai [96]. For some related developments, the interested reader may also refer to [35,97,98].*

Each of the orthogonality relations for Bessel polynomials and the generalized Bessel polynomials, which we presented in this section, is indeed capable of producing the corresponding polynomial expansions for suitably restricted functions in the series of these polynomials. The usual details involved in such derivations of polynomial expansions by means of known orthogonality relations are omitted here.

4. Asymptotic Expansions, Location of Zeros, and Luke's Conjecture

Since the second parameter β is merely a scaling factor, as we have already remarked above, essentially, there is no loss of generality if we define the modified Bessel polynomials $Y_n(z;\alpha)$ as follows:

$$Y_n(z;\alpha) := y_n(z;\alpha,2) = {}_2F_0\left(-n, n+\alpha-1; \underline{\quad\quad}; -\frac{z}{2}\right). \tag{73}$$

In the year 1951, Grosswald [99] showed that for Bessel polynomials $y_n(z)$,

$$y_n(z) \sim \frac{(2n)!}{2^n\, n!}\, z^n \, \exp\left(\frac{1}{z}\right) \qquad (n \to \infty;\ z \neq 0), \tag{74}$$

which, in light of Striling's formula for factorials (that is, for quotients of Gamma functions), yields the following equivalent form:

$$y_n(z) \sim \left(\frac{2nz}{e}\right)^n \sqrt{2}\, \exp\left(\frac{1}{z}\right) \qquad (n \to \infty;\ z \neq 0). \tag{75}$$

A generalization of the asymptotic formula (75) was proven by Obreshkov [100], which may be recalled in the following form:

$$Y_n(z;\alpha) \sim \left(\frac{2nz}{e}\right)^n 2^{\alpha-\frac{3}{2}}\, \exp\left(\frac{1}{z}\right) \qquad (n \to \infty;\ z \neq 0). \tag{76}$$

Subsequently, Dočev [101] further improved Obreshkov's asymptotic formula (76) as follows:

$$\begin{aligned} Y_n(z;\alpha) \sim &\left(\frac{2nz}{e}\right)^n 2^{\alpha-\frac{3}{2}}\, \exp\left(\frac{1}{z}\right) \\ &\cdot \left[1 - \frac{1+6(\alpha-2)\left(\alpha-1+\frac{2}{z}\right)+\frac{6}{z^2}}{24n} + O\left(\frac{1}{n^2}\right)\right] \end{aligned} \tag{77}$$

$$(n \to \infty;\ z \neq 0).$$

The location of the zeros of the Bessel polynomial $Y_n(z;\alpha)$ in the complex z-plane was also investigated by Krall and Frink (see, for details, Chapter 10 in [5]). Some of the known facts in this connection can be summarized as follows:

1. The zeros of $Y_n(z;\alpha)$ are simple;
2. None of the zeros of $Y_n(z;\alpha)$ is also a zero of $Y_{n+1}(z;\alpha)$;
3. In the case when $\alpha \geqq 2$, the zeros of $Y_n(z;\alpha)$ lie in the left half of the complex z-plane.

It was observed by Saff and Varga [102] (see also [103]) that, in the case when $\Re(\alpha) > 1-n$, the zeros of $\left|Y_n(z;\alpha)\right|$ lie inside the cardioid given, on the $(\mathfrak{r},)$-plane, by

$$\mathfrak{r} = \frac{1-\cos}{\alpha+n-1} \qquad (z = \mathfrak{r}e^{\mathrm{i}}) \tag{78}$$

and outside the circle given by

$$\mathfrak{r} = \frac{1}{n(\alpha+n-1)} \qquad (z = \mathfrak{r}e^{\mathrm{i}}). \tag{79}$$

In the special case when $\alpha = 2$, that is, for Bessel polynomials $y_n(z)$, asymptotic formulas for the zeros can be deduced from Olver's work [104] on uniform asymptotic expansions of the modified Bessel function $K_\nu(z)$ by means of relationships such as (6) (see also [105–108]).

We choose to conclude this section by stating a presumably not-yet-settled conjecture attributed to Yudell Leo Luke (1918–1983).

Luke's Conjecture. Prove or disprove that the root φ of the largest magnitude of the modified Bessel polynomials, which is defined above by (73), satisfies the following asymptotic expansion:

$$\varphi \sim -2\left(1.32548n + (\alpha - 1) - \frac{1}{\pi}\right)^{-1} \qquad (n \to \infty). \tag{80}$$

5. Basic (or Quantum or q-) Bessel Polynomials

Basic (or quantum or q-) series and basic (or quantum or q-) polynomials, especially the basic (or quantum or q-) hypergeometric functions and the basic (or quantum or q-) hypergeometric polynomials, are known to have widespread applications, particularly in many areas of number theory and combinatorial analysis such as the theory of partitions. Additionally, the q-theory and the q-calculus are also useful in a remarkably wide variety of fields, including finite vector spaces, lie theory and lie groups, particle physics, non-linear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology, and statistics (see also pp. 350–351 in [10] and the references cited therein).

In this section, we first present some definitions and notations that are needed in its development.

For $q, \lambda, \mu \in \mathbb{C}$ $\quad (|q| < 1)$, the basic (or quantum or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [8,10,109,110])

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty}\left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}}\right) \qquad (|q| < 1;\ \lambda, \mu \in \mathbb{C}), \tag{81}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \displaystyle\prod_{j=0}^{n-1}\left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{82}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty}\left(1 - \lambda q^j\right) \qquad (|q| < 1;\ \lambda \in \mathbb{C}), \tag{83}$$

where, as usual, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{N}$ denotes the set of positive integers *with*

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

In terms of the Pochhammer symbol, which we introduced in (11), by using the l'Hôpital's limit identity, it is not difficult to verify that

$$\lim_{q \to 1}\left\{\frac{(q^\lambda; q)_n}{(q^\mu; q)_n}\right\} = \begin{cases} 1 & (n = 0;\ \lambda, \mu \in \mathbb{C}) \\ \displaystyle\lim_{q \to 1}\left\{\prod_{j=0}^{n-1}\left(\frac{1 - q^{\lambda+j}}{1 - q^{\mu+j}}\right)\right\} & (n \in \mathbb{N};\ \lambda, \mu \in \mathbb{C}), \end{cases} \tag{84}$$

in other words,

$$\lim_{q \to 1}\left\{\frac{(q^\lambda; q)_n}{(q^\mu; q)_n}\right\} = \frac{(\lambda)_n}{(\mu)_n} \qquad (n \in \mathbb{N}_0;\ \lambda, \mu \in \mathbb{C}).$$

For convenience, we choose to write

$$(a_1, \cdots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n \tag{85}$$

and

$$(a_1, \cdots, a_r; q)_\infty = (a_1; q)_\infty \cdots (a_r; q)_\infty \tag{86}$$

whenever it is needed for the sake of brevity.

We next introduce the basic (or quantum or q-) hypergeometric function ${}_r\Phi_s$ with r symmetric numerator parameters and s symmetric denominator parameters, which is defined by (see, for example, p. 347, Equation 9.4 (272) in [10]; see also [8,109,111,112])

$$\begin{aligned} {}_r\Phi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, z) &= {}_r\Phi_s \left[\begin{matrix} a_1, \cdots, a_r; \\ \\ b_1, \cdots, b_s; \end{matrix} \; q, z \right] \\ &:= \sum_{k=0}^{\infty} (-1)^{(1-r+s)k} \, q^{\frac{1}{2}(1-r+s)k(k-1)} \\ &\quad \cdot \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k} \frac{z^k}{(q; q)_k}, \end{aligned} \tag{87}$$

whenever the generalized basic (or q-) series in (87) converges, that is, when

$$|q| < 1 \quad \text{and} \quad |z| < \infty \quad \text{when} \quad r \leqq s$$

or

$$\max\{|q|, |z|\} < 1 \quad \text{when} \quad r = s + 1,$$

it is tacitly assumed that no zeros appear in the denominator in (87). More precisely, we can write the following definition (see, for example, p. 347, Equation 9.4 (272) in [10]):

$$\begin{aligned} {}_{r+1}\Phi_{r+j} &\left[\begin{matrix} a_1, \cdots, a_{r+1}; \\ \\ b_1, \cdots, b_{r+j}; \end{matrix} \; q, z \right] \\ &= \sum_{k=0}^{\infty} (-1)^{jk} \, q^{\frac{1}{2}jk(k-1)} \frac{(a_1, \cdots, a_{r+1}; q)_k}{(b_1, \cdots, b_{r+j}; q)_k} \frac{z^k}{(q; q)_k} \qquad (j \in \mathbb{Z}), \end{aligned} \tag{88}$$

where, for convergence, $|q| < 1$ and $|z| < \infty$ when $j \in \mathbb{N}$, or $|q| < 1$ and $|z| < 1$ when $j = 0$, provided that no zeros appear in the denominator of (88).

Since (see, for example, [8,109])

$$(q^{-n}; q)_k = (-1)^k \, q^{\frac{1}{2}k(k-1)-nk} \frac{(q; q)_n}{(q; q)_{n-k}} \qquad (0 \leqq k \leqq n; \; n \in \mathbb{N}_0), \tag{89}$$

the q-hypergeometric series in (87) would terminate, and we are thus led to a generalized q-hypergeometric polynomial in argument z. In fact, most (if not all) of the known q-extensions of the Jacobi, Laguerre, and Hermite polynomials are defined in terms of the q-hypergeometric polynomials emerging from (87) and (88). For example, we have (see, for details, [113] and [109,114–116]; see also [117] and [118]).

I. Big q-Jacobi Polynomials:

$$P_n^{(\alpha,\beta)}(x; \alpha, \beta, \gamma; q) := \frac{(\alpha q; q)_n}{(q, q)_n} \, {}_3\Phi_2\Big(q^{-n}, \alpha\beta q^{n+1}, x; \alpha q, \gamma q; q, q\Big). \tag{90}$$

II. Little q-Jacobi Polynomials:

$$p_n^{(\alpha,\beta)}(x;q) := \frac{(\alpha q;q)_n}{(q,q)_n}\, {}_2\Phi_1\Big(q^{-n}, \alpha\beta q^{n+1}, x; \alpha q; q, qx\Big). \tag{91}$$

III. Continuous q-Jacobi Polynomials:

$$P_n^{(\alpha,\beta)}(x;q) := \frac{(q^{n+1};q)_n}{(q;q)_n}\, {}_4\Phi_3\left[\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}\mathrm{e}^{\mathrm{i}\vartheta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\vartheta}; \\ q^{n+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; \end{array} q, q\right] \tag{92}$$
$$(x = \cos\vartheta).$$

The existing literature on q-hypergeometric polynomials contains systematic investigations of various q-extensions of the modified Bessel polynomials $Y_n(z;\alpha)$, which are defined above by (73). We first recall here the q-Bessel polynomials $J(q;c,n;x)$ defined according to p. 210, Equation II. (1), in [119] (see also [116])

$$J(q;c,n;x) := \frac{(q^c;q)_n}{(q;q)_n}\, {}_2\Phi_1\big(q^{-n}, q^{c+n}; 0; q, x\big). \tag{93}$$

More recently, the following q-extension of the modified Bessel polynomials $Y_n(z;\alpha)$ was considered by Ismail, p. 455, Equation (1.11) in [120]:

$$y_n(x;\alpha|q^2) := q^{\frac{1}{2}q(q-1)}\, {}_2\Phi_1\Big(q^{-n}, q^{\alpha+n-1}; -q; q, -2qx\Big). \tag{94}$$

Indeed, in their limit case when $q \to 1$, each of the above-defined q-Bessel polynomials would essentially lead us to the modified Bessel polynomials $Y_n(z;\alpha)$ defined with (73).

The q-Bessel polynomials $y_n(x;c;x)$, which were systematically investigated by Exton and Srivastava [121], are substantially the same as those studied by Abdi [119] and, subsequently, by Khan and Khan [122] (as well as, more recently, by Koekoek et al. [116]). In fact, we have

$$\begin{aligned} y_n(x;c;x) &:= \frac{(q^c;q)_n}{(q;q)_n}\, {}_2\Phi_1\big(q^{-n}, q^{c+n}; 0; q, -x\big) \\ &= J(q;c,n;x). \end{aligned} \tag{95}$$

By appropriately applying the q-binomial theorem for ${}_1\Phi_0$ (also known as *Heine's Theorem*):

$${}_1\Phi_0\left[\begin{array}{c} \lambda; \\ \text{—}; \end{array} q, z\right] := \sum_{k=0}^{\infty} \frac{(\lambda;q)_k}{(q;q)_k} z^k = \frac{(\lambda z;q)_\infty}{(z;q)_\infty} \qquad (|q|<1;\ |z|<1), \tag{96}$$

which considerably simplifies to the following form when we set $\lambda = q^{-n}$ $(n \in \mathbb{N}_0)$:

$$\begin{aligned} {}_1\Phi_0\left[\begin{array}{c} q^{-n}; \\ \text{——}; \end{array} q, z\right] &:= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} z^k = (zq^{-n};q)_n \\ &= \left(\frac{q}{z}\right)_n \left(-\frac{z}{q}\right)^n q^{-\binom{n}{2}} \qquad (|q|<1;\ n \in \mathbb{N}_0); \end{aligned} \tag{97}$$

the q-exponential functions $\mathrm{e}_q(z)$ and $\mathrm{E}_q(z)$ defined, respectively, by

$$\mathrm{e}_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} =: {}_1\Phi_0\left[\begin{array}{c} 0; \\ \text{—}; \end{array} q, z\right] = \frac{1}{(z;q)_\infty}, \tag{98}$$

where we use the *special* case of the q-binomial theorem (96) when $\lambda = 0$, and

$$\mathrm{E}_q(z) := \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{z^k}{(q;q)_k} =: {}_0\Phi_0 \left[\begin{array}{c} \text{—;} \\ \\ \text{—;} \end{array} q, -z \right] = (-z;q)_\infty, \tag{99}$$

where we use the *limit* case of the q-binomial theorem (96) when z is replaced by $\frac{z}{\lambda}$ and $\lambda \to \infty$; the q-derivative operator $\mathfrak{D}_{z,q}$:

$$\mathfrak{D}_{z,q} f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \in \mathbb{C} \setminus \{0\}) \\ \\ f'(0) & (z = 0), \end{cases} \tag{100}$$

provided that $f'(0)$ exists; the q-Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z} \qquad (|q| < 1;\ z \in \mathbb{C}), \tag{101}$$

so that

$$\lim_{q \to 1} \{\Gamma_q(z)\} = \Gamma(z)$$

in terms of the familiar (Euler's) Gamma function $\Gamma(z)$ given by (3); the basic (or q-) integrals of Jackson [123] given by

$$\int_0^a f(t)\, \mathrm{d}_q t = (1-q)a \sum_{n=0}^{\infty} f(aq^n)\, q^n \qquad (a > 0), \tag{102}$$

so that

$$\int_a^b f(t)\, \mathrm{d}_q t = \int_0^b f(t)\, \mathrm{d}_q t - \int_0^a f(t)\, \mathrm{d}_q t \qquad (0 \leqq a < b), \tag{103}$$

$$\int_a^\infty f(t)\, \mathrm{d}_q t = (1-q)a \sum_{n=0}^{\infty} f(aq^{-n})\, q^{-n} \tag{104}$$

and

$$\int_0^\infty f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)\, q^n, \tag{105}$$

provided that each of the series in (102) to (105) is convergent, Exton and Srivastava [121] presented a homogeneous second-order q-differential equation, together with its *self-adjoint* form, as well as the orthogonality properties of the q-Bessel polynomials $y_n(x;c;x)$ defined by (95).

Some recent developments concerning the q-Bessel polynomials include those by Riyasat and Khan [124] on their determinantal representations and by Riyasat et al. [125] on the study of their analogs in two dimensions.

6. Concluding Remarks and Observations

Related rather closely to the modified Bessel function $K_\nu(z)$ of the second kind, which is known also as the Macdonald function (or, with a slightly different definition, the Basset function), the so-called Bessel polynomials $y_n(x)$ and their two-parameter version $y_n(x;\alpha,\beta)$, together with their reversed forms $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$, are widely and extensively investigated and applied in the existing literature on the subject. In this article, we presented an introductory overview of some of the important and potentially useful developments in the theory and applications of Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, which involve the asymmetric parameters α and

β. We briefly considered some recent developments based on the basic (or quantum or q-) extensions of the Bessel polynomials. We also investigated several related families of generalized hypergeometric polynomials, which are essentially the truncated or terminating forms of the series representing the generalized hypergeometric function ${}_rF_s$ with r symmetric numerator parameters and s symmetric denominator parameters, together with the corresponding basic (or quantum or q-) hypergeometric functions and the basic (or quantum or q-) hypergeometric polynomials associated with ${}_r\Phi_s$ which also involves r symmetric numerator parameters and s symmetric denominator parameters.

For further reading and research, those who are interested in pursuing this subject may refer to the various useful monographs and textbooks cited in this review that focus on the theory and applications of Bessel polynomials $y_n(x)$ and the generalized Bessel polynomials $y_n(x;\alpha,\beta)$, their reversed forms $\theta_n(x)$ and $\theta_n(x;\alpha,\beta)$, respectively, as well as their quantum (or basic or q-) extensions.

One can find a considerable number of studies in the literature investigating and applying the basic (or quantum or q-) calculus not only in the area of higher transcendental functions and geometric function theory of complex analysis (see, for a detailed historical and introductory overview, each of the recently-published survey-cum-expository review articles [126–129]), but also in the modeling and analysis of applied problems as well as in extending the well-established theory and applications of various rather classical mathematical functions, mathematical inequalities, and generating functions (see, for example, [130–137]). It is regretful, however, to see that a large number of mostly amateurish-type researchers on these and other related topics continue to produce and publish obvious and inconsequential variations and straightforward translations of the known q-results in terms of the so-called (p,q)-calculus by unnecessarily forcing-in an obviously superfluous (or redundant) parameter p into the classical q-calculus and thereby falsely claiming "generalization" (see p. 340 in [126] and Section 5, pp. 1511–1512 in [127]). Such tendencies to produce and flood the literature with trivialities should be discouraged by all means.

The importance of Bessel polynomials can be appreciated by the fact that they arise rather naturally in several seemingly diverse contexts, for example, in connection with the solution of the wave equation in spherical polar coordinates (see [4]), network synthesis and design (see [49]), the analysis of the student t-distribution (see, for example, [41,42]), in a representation of the energy spectral functions for a family of isotropic turbulence fields (see [138,139]), in developing a matrix technique applicable in solving some multi-order pantograph differential equations of fractional order (see [140]), etc. (see also [141–144] for several families of operators of fractional integrals and fractional derivatives, together with their usages in the mathematical modeling and analysis of various applied problems involving fractional differential equations and fractional integrodifferential equations). Indeed, as is revealed in the recent publications [145–154], Bessel polynomials and the reversed Bessel polynomials continue to be useful in developing numerical and approximation techniques, and other collocation and quasilinearization approaches, in successfully handling a wide variety of problems stemming from the mathematical, physical, chemical, biological and engineering sciences. The interested reader's familiarization with these and other recent publications will surely lead to further studies applying Bessel polynomials and the reversed Bessel polynomials.

In this survey-cum-expository review, the targeted reader also finds a systematic introduction and description of many other families of orthogonal polynomials and orthogonal q-polynomials, together with the potentially useful inter-relationships between them. Furthermore, with a view to making this article as comprehensively informative as possible, the reader has access to an up-to-date listing and citation of the available literature on the subject.

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