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# Comparative Analysis of Advection–Dispersion Equations with Atangana–Baleanu Fractional Derivative

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**Abstract:** In this study, we solve the fractional advection–dispersion equation (FADE) by applying the Laplace transform decomposition method (LTDM) and the variational iteration transform method (VITM). The Atangana–Baleanu (AB) sense is used to describe the fractional derivative. This equation is utilized to determine solute transport in groundwater and soils. The FADE is converted into a system of non-linear algebraic equations whose solution leads to the approximate solution for this equation using the techniques presented. The proposed approximate method's convergence is examined. The suggested method's applicability is demonstrated by testing it on several illustrative examples. The series solutions to the specified issues are obtained, and they contain components that converge more quickly to the precise solutions. The actual and estimated results are demonstrated in graphs and tables to be quite similar, demonstrating the usefulness of the proposed strategy. The innovation of the current work is in the application of an effective method that requires less calculation and achieves a greater level of accuracy. Furthermore, the proposed approaches may be implemented to prove their utility in tackling fractional-order problems in science and engineering.

**Keywords:** fractional advection–dispersion equations; variational iteration transform method; Laplace transform decomposition method; fractional advection–dispersion equations; Atangana–Baleanu fractional derivative



**Citation:** Alshehry, A.S.; Yasmin, H.; Ghani, F.; Shah, R.; Nonlaopon, K. Comparative Analysis of Advection–Dispersion Equations with Atangana–Baleanu Fractional Derivative. *Symmetry* **2023**, *15*, 819. https://doi.org/10.3390/sym15040819

Academic Editor: Dongfang Li

Received: 27 December 2022
Revised: 19 January 2023
Accepted: 20 January 2023
Published: 29 March 2023



## 1. Introduction

Fractional calculus (FC) has recently received a great deal of interest from scholars, with many distinct topics being researched. This is because fractional calculus is a valuable tool for explaining the dynamical behavior of numerous physical systems. The benefit of fractional differential operators is that they have nonlocal properties absent in integer-order differential operators. Fractional differential equations (FDEs) are notable for providing memory and transmission qualities for numerous mathematical models. Fractional-order models are, in reality, more realistic and practical than integer-order models. In these models, the derivative possessing fractional order gives a higher degree of flexibility. The global properties of these models, which do not exist in classical-order models, are their most iterating feature. FC has received a great deal of interest in many areas, including fluid dynamics, solid mechanics, ecology, financial mathematics, biological diseases, and many other fields [1–8]. Since finding the solution to FDEs can be difficult, affective computing approaches for solving FDEs may be required [9–12]. Many authors, including Baleanu et al. [13], Miller and Ross [14], Kilbas et al. [15], and Podlubny [16] have published books on FC in the last few years. For such challenges, several approximate and analytical approaches have been developed [17–22]. The fractional differential equation is a useful tool for expressing nonlinear events in scientific and engineering models. Partial differential equations, especially nonlinear ones, have been used in applied mathematics and engineering to model a variety of scientific phenomena. Parallel to their work in the physical sciences, researchers were able to identify and model a wide variety of significant and practical physical difficulties thanks to fractional-order partial differential equations (FPDEs). It has long been argued that it is crucial for scientists to use analytical or numerical approaches to obtain approximations. As a result, symmetry analysis is an excellent tool for understanding partial differential equations, particularly in the case of equations derived from mathematical ideas related to accounting. Contrary to popular belief, symmetry is not the fundamental principle of nature.

The study of Brownian motion of particles in a fluid results in the simultaneous occurrence of particle dispersion and advection, which gives rise to the advection–dispersion equation (ADE). The phenomenon of anomalous diffusion of particles in the transport process is better described by the FADE, as in anomalous diffusion, the solute transport is faster or quicker than the inferred square root of time given by Baeumer et al. [23]. The equation has been applied to research a variety of environmental issues, including smoke and dust pollution of the atmosphere, groundwater pollution, pollutant discharges, and the spread of chemical solutes, among others [24]. Thus, FADE has gained the interest of several scholars. Our main focus in this work is to solve the FADE of the form [25]:

$$D_{\eta}^{\lambda}\psi(\nu,\eta) = \kappa\ell D_{\nu}^{2}\psi(\nu,\eta) - \jmath D_{\nu}\psi(\nu,\eta) \qquad \eta > 0, \nu > 0, 0 < \lambda \leq 1 \tag{1}$$

where $\psi$ is the solute concentration, $\kappa, \jmath$ are the average dispersion coefficient and fluid velocity, $\nu$ is the spatial domain, $\eta$ is time, and $\lambda$ is the parameter determining the order of the time- and space-fractional derivatives, respectively. In the Atangana–Baleanu sense, the fractional derivative is examined. The order of fractional derivatives is described by parameters in the general response expression, which can be changed to obtain numerous results. The fractional equation is reduced to the standard ADE when $\lambda = 1$. Gaussian densities with variances and means dependent on the values of the macroscopic transport coefficients $\kappa$ and $\jmath$ will constitute the fundamental solutions of the ADE over time. A number of authors have already studied the space–time FADE.

With the aid of the Laplace transform (LT) and fractional Attangana–Baleanu derivative operator, we used two analytical approaches to solve the FADE in this study [26,27]. To approach problem (1), the Adomian decomposition method [28,29] and the variational iteration method [30–35] were employed successfully. Both techniques are novel approaches in providing an analytical approximation to linear and nonlinear problems. They are particularly useful tools for scientists and applied mathematicians because they give extremely fast and visible symbolic terms of analytic solutions and approximate numerical solutions to linear and nonlinear differential equations. The literature has utilized the decomposition method to provide approximate solutions to a wide range of linear and nonlinear differential equations [36]. The methods used for FDEs have recently been extended [37–41]. He et al. [30–35] presented the variational iteration approach, which has been used for autonomous ordinary and partial differential equations [42–46] and other areas. He introduced the variational iteration method to solve FDEs. Momani and Odibat [47] recently used the variational iteration approach to solve fractional-order nonlinear ordinary differential equations. Furthermore, they provide a numerical comparison of the two approaches for solving fractional-order linear differential equations [48].

## 2. Preliminaries

Here, we examine some fundamental fractional calculus definitions that are relevant to our current research.

**Definition 1.** *The Caputo operator is defined as [15,16]*

$$^{C}D_{\eta}^{\lambda}\{y(\eta)\} = \frac{1}{(n-\lambda)}\int_{0}^{\eta}(\eta-k)^{n-\lambda-1}y^{n}(k)dk, \ \ where \ \ n<\lambda\leq n+1.$$

**Definition 2.** *In addition to the Caputo derivative, the Laplace transformation $^{L}D_{\eta}^{\lambda}\{y(\eta)\}$ is defined as [15,16]*

$$L\{^{LC}D_{\eta}^{\lambda}\{y(\eta)\}\}(\varpi) = \frac{1}{\varpi^{n-\lambda}}\Big[\varpi^{n}L\{y(\nu,\eta)\}(\varpi) - \varpi^{n-1}y(\nu,0) - \cdots - y^{n-1}(\nu,0)\Big].$$

**Definition 3.** *The Atangana–Baleanu (AB) derivative is defined as [49]*

$$^{AB}D_{\eta}^{\lambda}\{y(\eta)\} = \frac{A(\lambda)}{1-\lambda}\int_{a}^{\eta}y^{'}(k)E_{\lambda}\Big[-\frac{\lambda}{1-\lambda}(1-k)^{\lambda}\Big]dk.$$

*Here, $A(\gamma)$ represents the normalization function with $A(0)=A(1)=1, y\in H^{1}(a,b), b> a, \lambda \ \in [0,1]$, and $E_{\lambda}$ illustrates the Mittag–Leffler function.*

**Definition 4.** *The Riemann–Liouville derivative of AB is given as [49]*

$$^{AB}D_{\eta}^{\lambda}\{y(\eta)\} = \frac{A(\lambda)}{1-\lambda}\frac{d}{d\eta}\int_{a}^{\eta}y(k)E_{\lambda}\Big[-\frac{\gamma}{1-\lambda}(1-k)^{\lambda}\Big]dk$$

**Definition 5.** *The AB operator is used in association with the Laplace transformation as [50]*

$$^{AB}D_{\eta}^{\lambda}\{y(\eta)\}(\varpi) = \frac{A(\gamma)\varpi^{\lambda}L\{y(\eta)\}(\varpi) - \varpi^{\lambda-1}y(0)}{(1-\lambda)\Big(\varpi^{\lambda}+\frac{\lambda}{1-\gamma}\Big)}$$

**Definition 6.** *If $0<y<1$ is a function of $\eta$, the integral operator with fractional order is defined as [50]*

$$^{AB}I_{\eta}^{\lambda}\{y(\eta)\} = \frac{1-\lambda}{A(\lambda)}y(\eta) + \frac{\lambda}{A(\lambda)\Gamma(\lambda)}\int_{a}^{\eta}y(k)(\eta-k)^{\lambda-1}dk.$$

## 3. Idea of LTDM

The general form of the proposed technique for solving a nonlinear partial differential equation is defined as

$$^{AB}D_{\eta}^{\lambda}\psi(\nu,\eta) + \bar{\mathcal{G}}_{1}(\nu,\eta) + \mathcal{N}_{1}(\nu,\eta) = \mathcal{F}(\nu,\eta), 0<\lambda\leq 1, \tag{2}$$

with the initial source

$$\psi(\nu,0) = \Phi(\nu).$$

where $^{AB}D_{\eta}^{\lambda} = \frac{\partial^{\lambda}}{\partial\eta^{\lambda}}$ is the time-fractional derivative in an AB manner, $\bar{\mathcal{G}}_{1}, \mathcal{N}_{1}$ represent the linear and non-linear parts, and $\mathcal{F}(\nu,\eta)$ is a known function.

By employing LT in Equation (2), we obtain

$$L[^{AB}D_{\eta}^{\lambda}\psi(\nu,\eta) + \bar{\mathcal{G}}_{1}(\nu,\eta) + \mathcal{N}_{1}(\nu,\eta)] = L[\mathcal{F}(\nu,\eta)]. \tag{3}$$

Using Laplace differentiation property yields

$$L[\psi(\nu,\eta)] = \Theta(\nu,\varpi) - \frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}L[\bar{\mathcal{G}}_{1}(\nu,\eta) + \mathcal{N}_{1}(\nu,\eta)], \tag{4}$$

where $\Theta(\nu,\varpi) = \frac{1}{\varpi^{\lambda+1}}[\varpi^{\lambda}g_0(\nu) + \varpi^{\lambda-1}g_1(\nu) + \cdots + g_1(\nu)] + \frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}\mathcal{F}(\nu,\eta)$. Applying the inverse LT, we obtain

$$\psi(\nu,\eta) = \Theta(\nu,\varpi) - L^{-1}\left\{\frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}L[\bar{\mathcal{G}}_1(\nu,\eta) + \mathcal{N}_1(\nu,\eta)]\right\}, \tag{5}$$

Here, $\Theta(\nu,\varpi)$ is the term obtained from the result of the initial and nonhomogeneous terms. Thus, the infinite-series form solution is shown as

$$\psi(\nu,\eta) = \sum_{m=0}^{\infty}\psi_m(\nu,\eta). \tag{6}$$

$\mathcal{N}_1$ indicates the nonlinear term as

$$\mathcal{N}_1(\nu,\eta) = \sum_{m=0}^{\infty}\mathcal{A}_m(\nu,\eta). \tag{7}$$

Here, $\mathcal{A}_m$ is the Adomian polynomial:

$$\mathcal{A}_m(\nu,\eta) = \frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{N}_1\left(\sum_{k=0}^{\infty}\ell^k\nu_k, \sum_{k=0}^{\infty}\ell^k\eta_k\right)\right\}\right]_{\ell=0}. \tag{8}$$

On incorporating Equations (6) and (7) into (5), we obtain

$$\sum_{m=0}^{\infty}\psi_m(\nu,\eta) = \Theta(\nu,\varpi) - L^{-1}\left\{\frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}L[\bar{\mathcal{G}}_1(\sum_{m=0}^{\infty}\nu_m, \sum_{m=0}^{\infty}\eta_m) + \sum_{m=0}^{\infty}\mathcal{A}_m]\right\}, \tag{9}$$

The rest of the components are derived as

$$\begin{cases}\psi_0(\nu,\eta) = \Theta(\nu,\varpi),\\ \psi_1(\nu,\eta) = L^{-1}\left\{\frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}L[\bar{\mathcal{G}}_1(\nu_0,\eta_0) + \mathcal{A}_0]\right\},\end{cases} \tag{10}$$

On continuing the same process, all the components for $m \geq 1$ are calculated as

$$\psi_{m+1}(\nu,\eta) = L^{-1}\left\{\frac{\varpi^{\lambda}+\lambda(1-\lambda)}{\varpi^{\lambda}}L[\bar{\mathcal{G}}_1(\nu_m,\eta_m) + \mathcal{A}_m]\right\}.$$

## 4. VITM Formulation

The general form of the proposed technique for solving a nonlinear partial differential equation is defined as

$${}^{AB}D_{\eta}^{\lambda}\psi(\nu,\eta) + \mathcal{M}\psi(\nu,\eta) + \mathcal{N}\psi(\nu,\eta) - \mathcal{P}(\nu,\eta) = 0, \quad m-1 < \delta \leq m, \tag{11}$$

with the initial source

$$\psi(\nu,0) = g_1(\nu). \tag{12}$$

where ${}^{AB}D^{\lambda} = \frac{\partial^{\lambda}}{\partial^{\lambda}}$ is the time-fractional derivative in an AB manner, $\mathcal{M}, \mathcal{N}$ represent the linear and non-linear parts, and $\mathcal{P}$ is a known function.

By employing LT in Equation (11), we obtain

$$L[{}^{AB}D_{\eta}^{\lambda}\psi(\nu,\eta)] + L[\mathcal{M}\psi(\nu,\eta) + \mathcal{N}\psi(\nu,\eta) - \mathcal{P}(\nu,\eta)] = 0. \tag{13}$$

Using Laplace differentiation property yields

$$L[\psi(\nu,\eta)] = \frac{\varpi^{\lambda}}{\varpi^{\lambda}+\lambda(1-\lambda)}L[\mathcal{M}\psi(\nu,\eta) + \mathcal{N}\psi(\nu,\eta) - \mathcal{P}(\nu,\eta)]. \tag{14}$$

The iterative scheme for Equation (13) is given as

$$\psi_{m+1}(\nu,\eta) = \psi_m(\nu,\eta) + \lambda(\varpi)\left[\frac{\varpi^\lambda}{\varpi^\lambda + \lambda(1-\lambda)} L[\mathcal{M}\psi(\nu,\eta) + \mathcal{N}\psi(\nu,\eta) - \mathcal{P}(\nu,\eta)]\right]. \quad (15)$$

and $\lambda(\varpi)$ is the Lagrange multiplier

$$\lambda(\varpi) = -\frac{\varpi^\lambda + \lambda(1-\lambda)}{\varpi^\lambda}, \quad (16)$$

The series form solution for Equation (14) is obtained by taking the inverse LT as

$$\begin{aligned}
\psi_0(\nu,\eta) &= \psi(0) + L^{-1}[\lambda(\varpi)L[-\mathcal{P}(\nu,\eta)]],\\
\psi_1(\nu,\eta) &= L^{-1}[\lambda(\varpi)L[\mathcal{M}\psi(\nu,\eta) + \mathcal{N}\psi(\nu,\eta)]],\\
&\vdots
\end{aligned}$$

$$\psi_{n+1}(\nu,\eta) = L^{-1}[\lambda(\varpi)L[\mathcal{M}[\psi_0(\nu,\eta) + \psi_1(\nu,\eta) + \cdots, \psi_n(\nu,\eta)]] + \mathcal{N}[\psi_0(\nu,\eta) + \psi_1(\nu,\eta), \cdots, \psi_n(\nu,\eta)]].$$

## 5. Applications

Here, we extract the solutions of FADE by implementing LTDM and VITM.

### *5.1. Example*

Consider the following ADE:

$$D_\eta^\lambda \psi(\nu,\eta) = \ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta) \qquad 0 < \lambda \leq 1, \eta > 0, \quad (17)$$

with the initial source

$$\psi(\nu,0) = e^{-\nu}.$$

By employing LT, we obtain

$$\frac{\varpi^\lambda L[\psi(\nu,\eta)] - \varpi^{-1}\psi(\nu,0)}{\varpi^\lambda + \lambda(1-\varpi^\lambda)} = L\left[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\right]. \quad (18)$$

By applying inverse LT, we obtain

$$\psi(\nu,\eta) = e^{-\nu} + L^{-1}\left[\frac{\varpi^\lambda + \lambda(1-\varpi^\lambda)}{\varpi^\lambda} L\left[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\right]\right]. \quad (19)$$

The $\psi(\nu,\eta)$ series solution is given as

$$\begin{aligned}
\psi(\nu,\eta) &= \sum_{m=0}^{\infty} \psi_m(\nu,\eta),\\
\sum_{m=0}^{\infty} \psi_m(\nu,\eta) &= e^{-\nu} + L^{-1}\left[\frac{\varpi^\lambda + \lambda(1-\varpi^\lambda)}{\varpi^\lambda} L\left[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\right]\right].
\end{aligned} \quad (20)$$

By comparing both sides of Equation (20), we obtain

$$\psi_0(\nu,\eta) = e^{-\nu},$$

For $m = 0$:

$$\psi_1(\nu,\eta) = e^{-\nu}(\ell+1)\left[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)} + (1-\lambda)\right],$$

For $m = 1$:

$$\psi_2(\nu,\eta) = e^{-\nu}(\ell+1)^2\left[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\right],$$

For $m = 2$:

$$\psi_3(\nu,\eta) = e^{-\nu}(\ell+1)^3\left[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\right],$$

Consequently, we determine the solution in series form as

$$\psi(\nu,\eta) = \sum_{m=0}^{\infty}\psi_m(\nu,\eta) = \psi_0(\nu,\eta) + \psi_1(\nu,\eta) + \psi_2(\nu,\eta) + \psi_3(\nu,\eta) + \cdots$$

$$\psi(\nu,\eta) = e^{-\nu} + e^{-\nu}(\ell+1)\left[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\right] + e^{-\nu}(\ell+1)^2\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big] + e^{-\nu}(\ell+1)^3\left[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\right] + \cdots$$

If we set $\lambda = 1$, we obtain $\psi(\nu,\eta) = e^{(1+\ell)\eta-\nu}$.

**VITM Solution**:

Applying the iterative formula to Equation (17), we obtain

$$\psi_{m+1}(\nu,\eta) = \psi_m(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}\ell D_{\nu}^{2}\psi_m(\nu,\eta) - D_{\nu}\psi_m(\nu,\eta)\Big\}\Big], \tag{21}$$

where

$$\psi_0(\nu,\eta) = e^{-\nu}.$$

For $m = 0, 1, 2, \cdots$

$$\begin{aligned}\psi_1(\nu,\eta) &= \psi_0(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}\ell D_{\nu}^{2}\psi_0(\nu,\eta) - D_{\nu}\psi_0(\nu,\eta)\Big\}\Big],\\ \psi_1(\nu,\eta) &= e^{-\nu} + e^{-\nu}(\ell+1)\left[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\right],\end{aligned} \tag{22}$$

$$\begin{aligned}\psi_2(\nu,\eta) &= \psi_1(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}\ell D_{\nu}^{2}\psi_1(\nu,\eta) - D_{\nu}\psi_1(\nu,\eta)\Big\}\Big],\\ \psi_2(\nu,\eta) &= e^{-\nu} + e^{-\nu}(\ell+1)\left[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\right] + e^{-\nu}(\ell+1)^2\left[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\right],\end{aligned} \tag{23}$$

$$\begin{aligned}\psi_3(\nu,\eta) &= \psi_2(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda}+\lambda(1-\varpi^{\lambda})}\ell D_{\nu}^{2}\psi_2(\nu,\eta) - D_{\nu}\psi_2(\nu,\eta)\Big\}\Big],\\ \psi_3(\nu,\eta) &= e^{-\nu} + e^{-\nu}(\ell+1)\left[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\right] + e^{-\nu}(\ell+1)^2\left[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\right] +\\ &\quad e^{-\nu}(\ell+1)^3\left[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\right],\end{aligned} \tag{24}$$

$$\psi(\nu,\eta) = \sum_{m=0}^{\infty}\psi_m(\nu,\eta) = e^{-\nu} + e^{-\nu}(\ell+1)\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big] + e^{-\nu}(\ell+1)^2\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big] + e^{-\nu}(\ell+1)^3\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\Big] + \cdots \tag{25}$$

If we set $\lambda = 1$, we obtain $\psi(\nu,\eta) = e^{(1+\ell)\eta-\nu}$.

### *5.2. Example*

Consider the following ADE:

$$D_{\eta}^{\lambda}\psi(\nu,\eta) = \ell D_{\nu}^{2}\psi(\nu,\eta) - D_{\nu}\psi(\nu,\eta) \qquad 0<\lambda\le 1, \eta>0, \tag{26}$$

subject to the initial condition

$$\psi(\nu,0) = \nu^3 - \nu^2.$$

By employing LT, we obtain

$$\frac{\varpi^{\lambda}L[\psi(\nu,\eta)] - \varpi^{-1}\psi(\nu,0)}{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})} = L\Big[\ell D_{\nu}^{2}\psi(\nu,\eta) - D_{\nu}\psi(\nu,\eta)\Big]. \tag{27}$$

By applying inverse LT, we obtain

$$\psi(\nu,\eta) = \nu^3 - \nu^2 + L^{-1}\bigg[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big[\ell D_{\nu}^{2}\psi(\nu,\eta) - D_{\nu}\psi(\nu,\eta)\Big]\bigg]. \tag{28}$$

The $\psi(\nu,\ell,\eta)$ series solution is defined as

$$\begin{aligned}\psi(\nu,\eta) &= \sum_{m=0}^{\infty}\psi_m(\nu,\eta),\\ \sum_{m=0}^{\infty}\psi_m(\nu,\eta) &= \nu^3 - \nu^2 + L^{-1}\bigg[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}}L\Big[\ell D_{\nu}^{2}\psi(\nu,\eta) - D_{\nu}\psi(\nu,\eta)\Big]\bigg].\end{aligned} \tag{29}$$

By comparing both sides of Equation (29), we obtain

$$\psi_0(\nu,\eta) = \nu^3 - \nu^2,$$

For $m = 0$:

$$\psi_1(\nu,\eta) = \{-3\nu^2 + 2\nu(1+3\ell) - 2\ell\}\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big],$$

For $m = 1$:

$$\psi_2(\nu,\eta) = \{6\nu - 2 - 12\ell\}\bigg[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\bigg],$$

For $m = 2$:

$$\psi_3(\nu,\eta) = -6\bigg[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\bigg],$$

Consequently, we determine the solution in series form as

$$\psi(\nu,\eta)=\sum_{m=0}^{\infty}\psi_m(\nu,\eta)=\psi_0(\nu,\eta)+\psi_1(\nu,\eta)+\psi_2(\nu,\eta)+\psi_3(\nu,\eta)+\cdots$$

$$\psi(\nu,\eta)=\nu^3-\nu^2+\{-3\nu^2+2\nu(1+3\ell)-2\ell\}\Big[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)\Big]+\{6\nu-2-12\ell\}\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)}+2\lambda(1-\lambda)\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^2\Big]-6\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)}+3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)}+3\lambda(1-\lambda)^2\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^3\Big]+\cdots$$

**VITM Solution**:
Applying the iterative formula to Equation (27), we obtain

$$\psi_{m+1}(\nu,\eta)=\psi_m(\nu,\eta)+L^{-1}\Big[\frac{\varpi^\lambda+\lambda(1-\varpi^\lambda)}{\varpi^\lambda}L\Big\{\frac{\varpi^\lambda}{\varpi^\lambda+\lambda(1-\varpi^\lambda)}\ell D_\nu^2\psi_m(\nu,\eta)-D_\nu\psi_m(\nu,\eta)\Big\}\Big], \tag{30}$$

where

$$\psi_0(\nu,\eta)=\nu^3-\nu^2.$$

For $m=0,1,2,\cdots$

$$\begin{aligned}\psi_1(\nu,\eta)&=\psi_0(\nu,\eta)+L^{-1}\Big[\frac{\varpi^\lambda+\lambda(1-\varpi^\lambda)}{\varpi^\lambda}L\Big\{\frac{\varpi^\lambda}{\varpi^\lambda+\lambda(1-\varpi^\lambda)}\ell D_\nu^2\psi_0(\nu,\eta)-D_\nu\psi_0(\nu,\eta)\Big\}\Big],\\ \psi_1(\nu,\eta)&=\nu^3-\nu^2+\{-3\nu^2+2\nu(1+3\ell)-2\ell\}\Big[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)\Big],\end{aligned} \tag{31}$$

$$\begin{aligned}\psi_2(\nu,\eta)&=\psi_1(\nu,\eta)+L^{-1}\Big[\frac{\varpi^\lambda+\lambda(1-\varpi^\lambda)}{\varpi^\lambda}L\Big\{\frac{\varpi^\lambda}{\varpi^\lambda+\lambda(1-\varpi^\lambda)}\ell D_\nu^2\psi_1(\nu,\eta)-D_\nu\psi_1(\nu,\eta)\Big\}\Big],\\ \psi_2(\nu,\eta)&=\nu^3-\nu^2+\{-3\nu^2+2\nu(1+3\ell)-2\ell\}\Big[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)\Big]+\{6\nu-2-12\ell\}\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)}+2\lambda(1-\lambda)\\ &\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^2\Big],\end{aligned} \tag{32}$$

$$\begin{aligned}\psi_3(\nu,\eta)&=\psi_2(\nu,\eta)+L^{-1}\Big[\frac{\varpi^\lambda+\lambda(1-\varpi^\lambda)}{\varpi^\lambda}L\Big\{\frac{\varpi^\lambda}{\varpi^\lambda+\lambda(1-\varpi^\lambda)}\ell D_\nu^2\psi_2(\nu,\eta)-D_\nu\psi_2(\nu,\eta)\Big\}\Big],\\ \psi_3(\nu,\eta)&=\nu^3-\nu^2+\{-3\nu^2+2\nu(1+3\ell)-2\ell\}\Big[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)\Big]+\{6\nu-2-12\ell\}\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)}+2\lambda(1-\lambda)\\ &\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^2\Big]-6\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)}+3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)}+3\lambda(1-\lambda)^2\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^3\Big],\end{aligned} \tag{33}$$

$$\begin{aligned}\psi(\nu,\eta)=\sum_{m=0}^{\infty}\psi_m(\nu,\eta)&=\nu^3-\nu^2+\{-3\nu^2+2\nu(1+3\ell)-2\ell\}\Big[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)\Big]+\{6\nu-2-12\ell\}\\ &\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)}+2\lambda(1-\lambda)\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^2\Big]-6\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)}+3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)}+\\ &3\lambda(1-\lambda)^2\frac{\eta^\lambda}{\Gamma(\lambda+1)}+(1-\lambda)^3\Big]+\cdots\end{aligned} \tag{34}$$

### 5.3. Example

Consider the following ADE:

$$D_\eta^\lambda \psi(\nu,\eta) = \ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta) \quad 0 < \lambda \leq 1, \eta > 0, \tag{35}$$

subject to the initial condition

$$\psi(\nu,0) = \cos(\nu).$$

By employing LT, we obtain

$$\frac{\varpi^\lambda L[\psi(\nu,\eta)] - \varpi^{-1}\psi(\nu,0)}{\varpi^\lambda + \lambda(1-\varpi^\lambda)} = L\Big[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\Big]. \tag{36}$$

By applying inverse LT, we obtain

$$\psi(\nu,\eta) = \cos(\nu) + L^{-1}\left[\frac{\varpi^\lambda + \lambda(1-\varpi^\lambda)}{\varpi^\lambda} L\Big[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\Big]\right]. \tag{37}$$

The $\psi(\nu,\eta)$ series solution is given as

$$\begin{aligned} \psi(\nu,\eta) &= \sum_{m=0}^{\infty} \psi_m(\nu,\eta), \\ \sum_{m=0}^{\infty} \psi_m(\nu,\eta) &= \cos(\nu) + L^{-1}\left[\frac{\varpi^\lambda + \lambda(1-\varpi^\lambda)}{\varpi^\lambda} L\Big[\ell D_\nu^2 \psi(\nu,\eta) - D_\nu \psi(\nu,\eta)\Big]\right] \end{aligned} \tag{38}$$

By comparing both sides of Equation (38), we obtain

$$\psi_0(\nu,\eta) = \cos(\nu),$$

For $m = 0$:

$$\psi_1(\nu,\eta) = (\sin(\nu) - \ell\cos(\nu))\left[\frac{\lambda\eta^\lambda}{\Gamma(\lambda+1)} + (1-\lambda)\right],$$

For $m = 1$:

$$\psi_2(\nu,\eta) = (-\cos(\nu) - 2\ell\sin(\nu) + \ell^2\cos(\nu))\left[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^\lambda}{\Gamma(\lambda+1)} + (1-\lambda)^2\right],$$

For $m = 2$:

$$\psi_3(\nu,\eta) = (-\sin(\nu) + 3\ell\cos(\nu) + 3\ell^2\sin(\nu) - \ell^3\cos(\nu))\left[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2 \frac{\eta^\lambda}{\Gamma(\lambda+1)} + (1-\lambda)^3\right],$$

Consequently, we determine the solution in series form as

$$\psi(\nu,\eta) = \sum_{m=0}^{\infty} \psi_m(\nu,\eta) = \psi_0(\nu,\eta) + \psi_1(\nu,\eta) + \psi_2(\nu,\eta) + \psi_3(\nu,\eta) + \cdots$$

$$\psi(\nu,\eta) = \cos(\nu) + (\sin(\nu) - \ell\cos(\nu))\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big] + (-\cos(\nu) - 2\ell\sin(\nu) + \ell^2\cos(\nu))\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big] + (-\sin(\nu) + 3\ell\cos(\nu) + 3\ell^2\sin(\nu) - \ell^3\cos(\nu))\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\Big] + \cdots$$

**VITM Solution**:

Applying the iterative formula to Equation (27), we obtain

$$\psi_{m+1}(\nu,\eta) = \psi_m(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}} L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}\ell D_{\nu}^2\psi_m(\nu,\eta) - D_{\nu}\psi_m(\nu,\eta)\Big\}\Big], \tag{39}$$

where

$$\psi_0(\nu,\eta) = \cos(\nu).$$

For $m = 0,1,2,\cdots$

$$\begin{aligned}\psi_1(\nu,\eta) &= \psi_0(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}} L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}\ell D_{\nu}^2\psi_0(\nu,\eta) - D_{\nu}\psi_0(\nu,\eta)\Big\}\Big],\\ \psi_1(\nu,\eta) &= \cos(\nu) + (\sin(\nu) - \ell\cos(\nu))\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big],\end{aligned} \tag{40}$$

$$\begin{aligned}\psi_2(\nu,\eta) &= \psi_1(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}} L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}\ell D_{\nu}^2\psi_1(\nu,\eta) - D_{\nu}\psi_1(\nu,\eta)\Big\}\Big],\\ \psi_2(\nu,\eta) &= \cos(\nu) + (\sin(\nu) - \ell\cos(\nu))\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big] + (-\cos(\nu) - 2\ell\sin(\nu) + \ell^2\cos(\nu))\\ &\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big],\end{aligned} \tag{41}$$

$$\begin{aligned}\psi_3(\nu,\eta) &= \psi_2(\nu,\eta) + L^{-1}\Big[\frac{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}{\varpi^{\lambda}} L\Big\{\frac{\varpi^{\lambda}}{\varpi^{\lambda} + \lambda(1-\varpi^{\lambda})}\ell D_{\nu}^2\psi_2(\nu,\eta) - D_{\nu}\psi_2(\nu,\eta)\Big\}\Big],\\ \psi_3(\nu,\eta) &= \cos(\nu) + (\sin(\nu) - \ell\cos(\nu))\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big] + (-\cos(\nu) - 2\ell\sin(\nu) + \ell^2\cos(\nu))\\ &\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big] + (-\sin(\nu) + 3\ell\cos(\nu) + 3\ell^2\sin(\nu) - \ell^3\cos(\nu))\\ &\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\Big],\end{aligned} \tag{42}$$

$$\begin{aligned}\psi(\nu,\eta) = \sum_{m=0}^{\infty}\psi_m(\nu,\eta) &= \cos(\nu) + (\sin(\nu) - \ell\cos(\nu))\Big[\frac{\lambda\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)\Big] + (-\cos(\nu) - 2\ell\sin(\nu) +\\ &\ell^2\cos(\nu))\Big[\frac{\lambda^2\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 2\lambda(1-\lambda)\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^2\Big] + (-\sin(\nu) + 3\ell\cos(\nu) + 3\ell^2\sin(\nu) - \ell^3\cos(\nu))\\ &\Big[\frac{\lambda^3\eta^{3\lambda}}{\Gamma(3\lambda+1)} + 3\lambda^2(1-\lambda)\frac{\eta^{2\lambda}}{\Gamma(2\lambda+1)} + 3\lambda(1-\lambda)^2\frac{\eta^{\lambda}}{\Gamma(\lambda+1)} + (1-\lambda)^3\Big] + \cdots\end{aligned} \tag{43}$$

## 6. Results and Discussion

The numerical investigation of space–time FADE using LTDM and VITM is presented in this section. The solutions are graphically illustrated in the Figures and Table using Maple. Table 1 shows the error analysis of fractional advection–dispersion equations achieved using the proposed approaches for various $\nu, \eta$, and $\lambda$ values, whereas Table 2 demonstrates the error analysis of the proposed methods and results obtained by HPTM in terms of absolute error. Figure 1a,b depict the nature of the proposed methods' accurate and analytical solutions, respectively, while Figure 1c,d depict the proposed methods' nature at various fractional orders within the domain $0 \leq \nu, \eta \leq 1$. Figures 2 and 3a–d exhibit the suggested methods' solutions at different fractional orders, respectively, within the domain $0 \leq \nu, \eta \leq 1$ and within the domain $0 \leq \nu, \eta \leq 5$. In comparison to other approaches, the proposed methods were quite successful and accurate, as shown in Table 2. Furthermore, the fractional-order solution shows that when the value of $\lambda$ approaches the integer order, the solution gets closer to the precise solution.

**Table 1.** LTDM and VITM absolute error comparison of Example 1.

| $\eta$ | $\nu$ | $\|Exact - LTDM\|$ | $\|Exact - LTDM\|$ | $\|Exact - VITM\|$ | $\|Exact - VITM\|$ |
|---|---|---|---|---|---|
| | | $\lambda = 0.6$ | $\lambda = 1$ | $\lambda = 0.8$ | $\lambda = 1$ |
| 0.1 | 0.5 | $3.94307831 \times 10^{-02}$ | $2.6838000 \times 10^{-06}$ | $1.06407390 \times 10^{-02}$ | $2.6838000 \times 10^{-06}$ |
| | 1 | $2.39159790 \times 10^{-02}$ | $1.6278000 \times 10^{-06}$ | $6.45393460 \times 10^{-03}$ | $1.6278000 \times 10^{-06}$ |
| | 1.5 | $1.45057744 \times 10^{-02}$ | $9.8730000 \times 10^{-07}$ | $3.91450910 \times 10^{-03}$ | $9.8730000 \times 10^{-07}$ |
| | 2 | $8.79819690 \times 10^{-03}$ | $5.9890000 \times 10^{-07}$ | $2.37426970 \times 10^{-03}$ | $5.9890000 \times 10^{-07}$ |
| | 2.5 | $5.33637619 \times 10^{-03}$ | $3.6321000 \times 10^{-07}$ | $1.44006743 \times 10^{-03}$ | $3.6321000 \times 10^{-07}$ |
| | 3 | $3.23667578 \times 10^{-03}$ | $2.2030000 \times 10^{-07}$ | $8.73445050 \times 10^{-04}$ | $2.2030000 \times 10^{-07}$ |
| | 3.5 | $1.96314309 \times 10^{-03}$ | $1.3362000 \times 10^{-07}$ | $5.29771200 \times 10^{-04}$ | $1.3362000 \times 10^{-07}$ |
| | 4 | $1.19070648 \times 10^{-03}$ | $8.1040000 \times 10^{-08}$ | $3.21322480 \times 10^{-04}$ | $8.1040000 \times 10^{-08}$ |
| | 4.5 | $7.22199990 \times 10^{-04}$ | $4.9150000 \times 10^{-08}$ | $1.94891940 \times 10^{-04}$ | $4.9150000 \times 10^{-08}$ |
| | 5 | $4.38036433 \times 10^{-04}$ | $2.9814000 \times 10^{-08}$ | $1.18207933 \times 10^{-04}$ | $2.9814000 \times 10^{-08}$ |
| 0.2 | 0.5 | $5.86532951 \times 10^{-02}$ | $4.3835800 \times 10^{-05}$ | $1.72341631 \times 10^{-02}$ | $4.3835800 \times 10^{-05}$ |
| | 1 | $3.55750218 \times 10^{-02}$ | $2.6587800 \times 10^{-05}$ | $1.04530483 \times 10^{-02}$ | $2.6587800 \times 10^{-05}$ |
| | 1.5 | $2.15773414 \times 10^{-02}$ | $1.6126300 \times 10^{-05}$ | $6.34009420 \times 10^{-03}$ | $1.6126300 \times 10^{-05}$ |
| | 2 | $1.30873191 \times 10^{-02}$ | $9.7811000 \times 10^{-06}$ | $3.84546160 \times 10^{-03}$ | $9.7811000 \times 10^{-06}$ |
| | 2.5 | $7.93786031 \times 10^{-03}$ | $5.9326000 \times 10^{-06}$ | $2.33239034 \times 10^{-03}$ | $5.9326000 \times 10^{-06}$ |
| | 3 | $4.81455565 \times 10^{-03}$ | $3.5982600 \times 10^{-06}$ | $1.41466625 \times 10^{-03}$ | $3.5982600 \times 10^{-06}$ |
| | 3.5 | $2.92017561 \times 10^{-03}$ | $2.1824600 \times 10^{-06}$ | $8.58038460 \times 10^{-04}$ | $2.1824600 \times 10^{-06}$ |
| | 4 | $1.77117604 \times 10^{-03}$ | $1.3237300 \times 10^{-06}$ | $5.20426630 \times 10^{-04}$ | $1.3237300 \times 10^{-06}$ |
| | 4.5 | $1.07427258 \times 10^{-03}$ | $8.0288000 \times 10^{-07}$ | $3.15654710 \times 10^{-04}$ | $8.0288000 \times 10^{-07}$ |
| | 5 | $6.51579253 \times 10^{-04}$ | $4.8697300 \times 10^{-07}$ | $1.91454259 \times 10^{-04}$ | $4.8697300 \times 10^{-07}$ |
| 0.3 | 0.5 | $7.39988156 \times 10^{-02}$ | $2.2660620 \times 10^{-04}$ | $2.27580592 \times 10^{-02}$ | $2.2660620 \times 10^{-04}$ |
| | 1 | $4.48825505 \times 10^{-02}$ | $1.3744360 \times 10^{-04}$ | $1.38034607 \times 10^{-02}$ | $1.3744360 \times 10^{-04}$ |
| | 1.5 | $2.72226429 \times 10^{-02}$ | $8.3363800 \times 10^{-05}$ | $8.37222200 \times 10^{-03}$ | $8.3363800 \times 10^{-05}$ |
| | 2 | $1.65113676 \times 10^{-02}$ | $5.0562700 \times 10^{-05}$ | $5.07800940 \times 10^{-03}$ | $5.0562700 \times 10^{-05}$ |
| | 2.5 | $1.00146506 \times 10^{-02}$ | $3.0667900 \times 10^{-05}$ | $3.07996838 \times 10^{-03}$ | $3.0667900 \times 10^{-05}$ |
| | 3 | $6.07419268 \times 10^{-03}$ | $1.8600970 \times 10^{-05}$ | $1.86809527 \times 10^{-03}$ | $1.8600970 \times 10^{-05}$ |
| | 3.5 | $3.68418409 \times 10^{-03}$ | $1.1282060 \times 10^{-05}$ | $1.13305705 \times 10^{-03}$ | $1.1282060 \times 10^{-05}$ |
| | 4 | $2.23457061 \times 10^{-03}$ | $6.8429200 \times 10^{-06}$ | $6.87233850 \times 10^{-04}$ | $6.8429200 \times 10^{-06}$ |
| | 4.5 | $1.35533559 \times 10^{-03}$ | $4.1504400 \times 10^{-06}$ | $4.16828400 \times 10^{-04}$ | $4.1504400 \times 10^{-06}$ |
| | 5 | $8.22052586 \times 10^{-04}$ | $2.5173680 \times 10^{-06}$ | $2.52819201 \times 10^{-04}$ | $2.5173680 \times 10^{-06}$ |

**Table 2.** Absolute error comparison of the HPTM and proposed methods.

| $\eta$ | $\nu$ | *HPTM* | *Our Methods* |
|---|---|---|---|
| | | $\lambda = 1$ | $\lambda = 1$ |
| | 1 | $2.0000000 \times 10^{-10}$ | $1.42429233 \times 10^{-12}$ |
| | 1.5 | $1.0000000 \times 10^{-10}$ | $8.63876960 \times 10^{-12}$ |
| | 2 | $1.0000000 \times 10^{-10}$ | $5.23967870 \times 10^{-12}$ |
| | 2.5 | $4.0000000 \times 10^{-11}$ | $3.17802575 \times 10^{-12}$ |
| 0.1 | 3 | $2.0000000 \times 10^{-11}$ | $1.92757006 \times 10^{-12}$ |
| | 3.5 | $2.0000000 \times 10^{-11}$ | $1.16913033 \times 10^{-12}$ |
| | 4 | $1.0000000 \times 10^{-11}$ | $7.09113390 \times 10^{-12}$ |
| | 1 | $3.5000000 \times 10^{-09}$ | $2.21784191 \times 10^{-11}$ |
| | 1.5 | $2.1000000 \times 10^{-09}$ | $1.34518911 \times 10^{-11}$ |
| | 2 | $1.3000000 \times 10^{-09}$ | $8.15898440 \times 10^{-11}$ |
| | 2.5 | $7.8000000 \times 10^{-10}$ | $4.94867420 \times 10^{-11}$ |
| 0.2 | 3 | $4.7000000 \times 10^{-10}$ | $3.00152262 \times 10^{-11}$ |
| | 3.5 | $2.9000000 \times 10^{-10}$ | $1.82051550 \times 10^{-11}$ |
| | 4 | $1.7000000 \times 10^{-10}$ | $1.10419847 \times 10^{-11}$ |
| | 1 | $1.8100000 \times 10^{-08}$ | $2.86980942 \times 10^{-10}$ |
| | 1.5 | $1.1000000 \times 10^{-08}$ | $1.74062739 \times 10^{-10}$ |
| | 2 | $6.7000000 \times 10^{-09}$ | $1.05574388 \times 10^{-10}$ |
| | 2.5 | $4.0400000 \times 10^{-09}$ | $6.40341033 \times 10^{-10}$ |
| 0.3 | 3 | $2.4500000 \times 10^{-09}$ | $3.88386470 \times 10^{-10}$ |
| | 3.5 | $1.4900000 \times 10^{-09}$ | $2.35568301 \times 10^{-10}$ |
| | 4 | $9.0000000 \times 10^{-10}$ | $1.42879398 \times 10^{-10}$ |

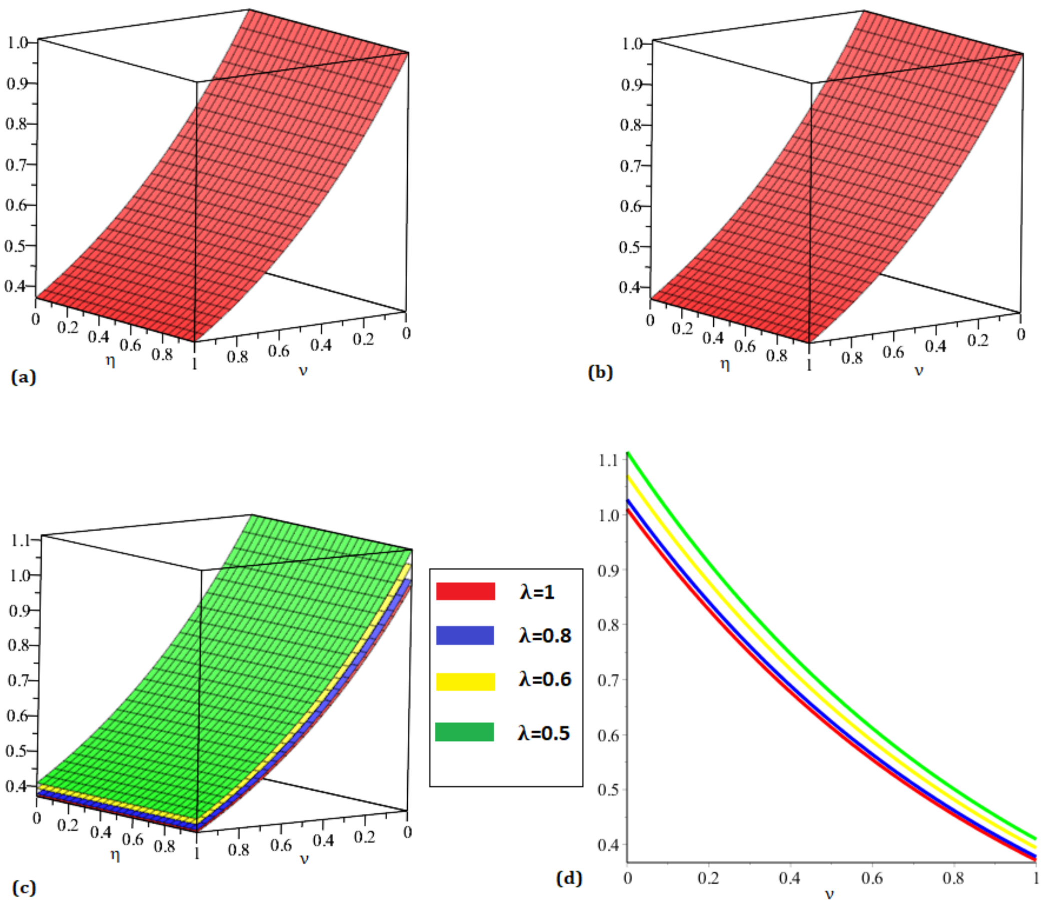


**Figure 1.** The solution plot of Example 1.

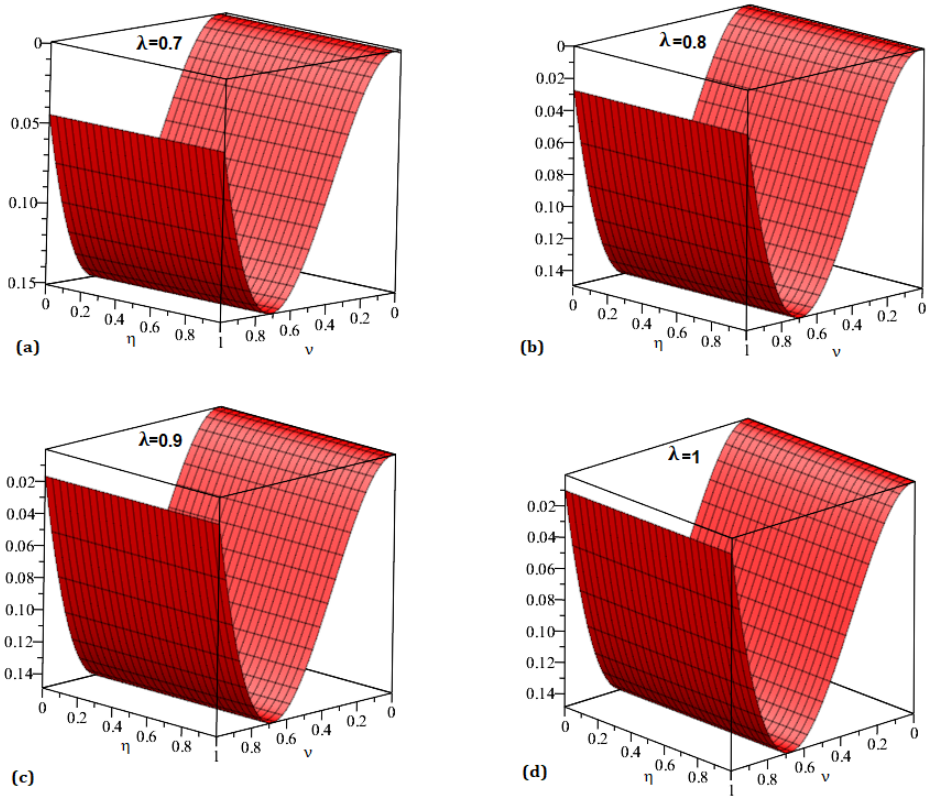


**Figure 2.** The solution plot of Example 2.

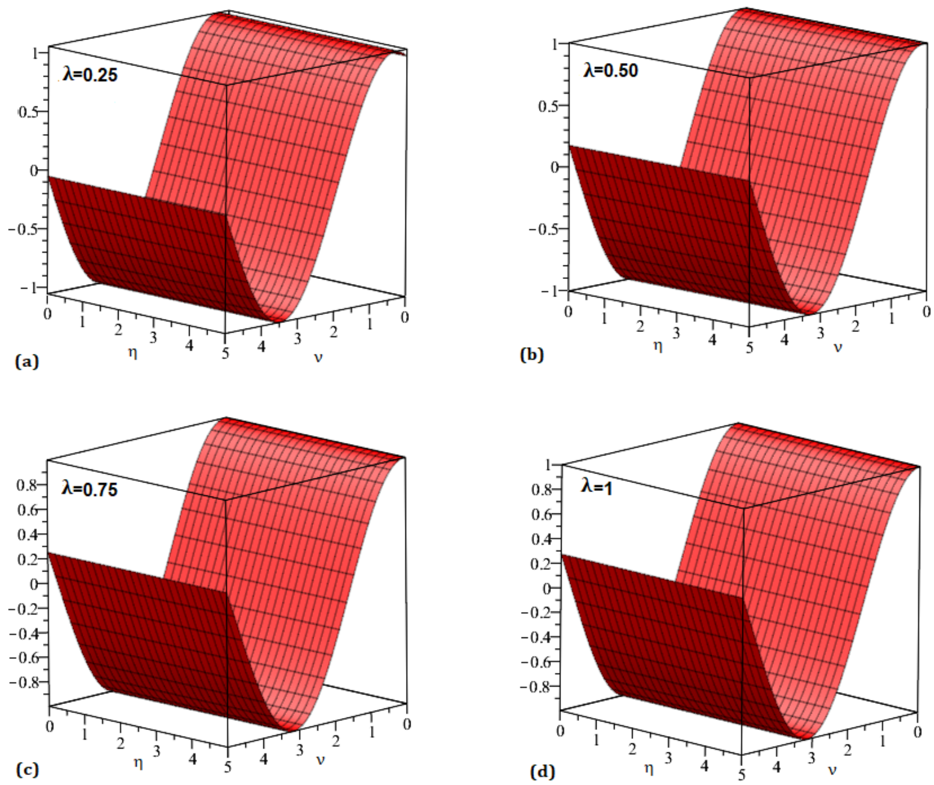


**Figure 3.** The solution plot of Example 3.

## 7. Conclusions

The use of the Adomian decomposition approach and the variational iteration method to achieve explicit and numerical solutions to the space–time fractional advection–dispersion problem has been expanded in this research. In obtaining the solutions to the provided equations, both of the approaches were clearly very effective and powerful; while providing quantitatively accurate results, the Adomian decomposition method and variational iteration method demand less computational effort than existing techniques. The obtained results indicate that the methods are reliable and that they may be utilized for solving fractional evolution problems. The current methods have shown to be an effective and straightforward approach when compared to other analytical and numerical techniques. In addition, the proposed solutions required fewer calculations and can thus be applied to other fractional-order problems.

**Author Contributions:** Methodology, A.S.A. and H.Y.; Software, F.G. and R.S.; Investigation, K.N.; Writing—original draft, A.S.A.; Funding, K.N. All authors have read and agreed to the published version of the manuscript.

**Funding:** Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R183), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This work was supported by the Deanship of Scientific Research, the Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Grant No. 3159).

**Data Availability Statement:** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Acknowledgments:** Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R183), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia. This work was supported by the Deanship of Scientific Research, the Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Grant No. 3159).

**Conflicts of Interest:** The authors declare no conflict of interest.

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