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Clifford Odd and Even Objects in Even and Odd Dimensional Spaces Describing Internal Spaces of Fermion and Boson Fields

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Abstract: In a long series of works, it has been demonstrated that the *spin-charge-family* theory, assuming a simple starting action in even dimensional spaces with $d \geq (13+1)$, with massless fermions interacting with gravity only, offers the explanation for all assumed properties of the second quantized fermion and boson fields in the *standard model*, as well as offering predictions and explanations for several of the observed phenomena. The description of the internal spaces of the fermion and boson fields by the Clifford odd and even objects, respectively, justifies the choice of the simple starting action of the *spin-charge-family* theory. The main topic of the present article is the analysis of the properties of the internal spaces of the fermion and boson fields in odd dimensional spaces, $d = (2n+1)$, which can again be described by the Clifford odd and even objects, respectively. It turns out that the properties of fermion and boson fields differ essentially from their properties in even dimensional spaces, resembling the ghosts needed when looking for final solutions with Feynman diagrams.

Keywords: second quantization of fermion and boson fields with Clifford algebra; beyond the standard model; Kaluza–Klein-like theories in higher dimensional spaces; Clifford algebra in odd dimensional spaces; ghosts in quantum field theories

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1. Introduction

Thirty years ago, I recognized that there are two kinds of Clifford algebra objects, γ^as and $\tilde{\gamma}^a$s [1–5], originating from Grassmann algebra. The Clifford and Grassmann algebras can be used to describe the internal spaces of fermions in even-dimensional spaces (while the Clifford odd algebras describe fermions with a half-integer spin, the "Grassmann's" fermions carry integer spins [1,3]). The superposition of odd products of either γ^a or $\tilde{\gamma}^a$ or θ^a anti-commute, fulfilling the anti-commutation relations on the vacuum states [4] of the second quantization postulates for fermion fields [6–10]. The superpositions of odd products of either γ^a or $\tilde{\gamma}^a$ carry half integer spins and appear in irreducible representations, offering the description of families of fermions [3].

Only one kind of fermion has been observed so far and this appears in several families. If we use one of the two kinds of Clifford algebra objects, say γ^a, to describe the internal space of fermions and the second kind of Clifford algebra objects, $\tilde{\gamma}^a$, to describe the family quantum numbers of each irreducible representation determined by γ^as, we are left with one kind of fermion [11,12], Section 3.2.3 of [3].

In a long series of works [5,11,13] I have found, together with my collaborators ([3,4,12,14] and the references therein), phenomenological success with the model named the *spin-charge-family* theory with the following properties:

a. The internal space of fermions is described by the "basis vectors", which are superpositions of odd products of anticommuting objects γ^a in $d = (13+1)$ [3,4]. Correspondingly, the "basis vectors" of one Lorentz irreducible representation in the internal space of fermions, together with their Hermitian conjugated partners, anticommute, fulfilling (on the vacuum state) all the requirements for the second quantized fermion fields [3,12] (In even dimensional spaces, the Clifford odd "basis vectors" have only left or right handedness, depending on the definition ($\Gamma = \prod_a^d (\sqrt{\eta^{aa}}\gamma^a) \cdot (i)^{\frac{d}{2}}$) and the choice of the "basis vectors". The reader can find one irreducible representation of the Clifford odd "basis vectors" for $d = (13+1)$ in Appendix D, which is analyzed from the point of view of the subgroups $SO(3,1) \times SO(4)$ (included in $SO(7,1)$) and $SO(7,1) \times SO(6)$ (included in $SO(13,1)$, while $SO(6)$ breaks into $SU(3) \times U(1)$), containing quarks, leptons, antiquarks, and antileptons with the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ groups and $Y = \tau^{23} + \tau^4$, one irreducible representation includes the right handed neutrinos and left handed antineutrinos, which are not found in the *standard model* scheme. Table A1 of Appendix D shows that either quarks or leptons or antiquarks and antileptons manifest in $d = (3+1)$ left and right handedness [3].).

b. The second kind of the anticommuting object, $\tilde{\gamma}^a$s, equips each irreducible representation of odd "basis vectors" with the family quantum number [3,4,12].

c. The creation operators for single fermion states, which are tensor products, $*_T$, of a finite number of odd "basis vectors" appearing in $2^{\frac{d}{2}-1}$ families, with each family having $2^{\frac{d}{2}-1}$ members, and the (continuously) infinite momentum/coordinate basis applied on the vacuum state [3,4], inherit the anticommutativity of "basis vectors". Creation operators and their Hermitian conjugated partners correspondingly anticommute, explaining the second quantization postulates of Dirac (Two fermion states with the orthogonal basis part in ordinary space "do not meet". Correspondingly, each can carry the same "basis vector". They must be distinguished in terms of their internal basis if they have an identical ordinary part of the basis. Otherwise, the tensor product, $*_{T_H}$, of the two fermion states would be zero. Illustration: Let us consider an atom with many electrons. Each electron has a spin of either $1/2$ or $-1/2$. The orthogonal basis of the electrons in ordinary space allows them to have an internal spin of $\pm 1/2$ (leading to a total angular momentum of $\pm 1/2$ or greater due to the angular momentum in ordinary space)).

d. The Hilbert space of second quantized fermions is represented by the tensor products, $*_{T_H}$, of all possible numbers of creation operators, from zero to infinity [4], applied on a vacuum state.

e. In a simple starting action, Equation (1), massless fermions carry only spins and interact with only gravity with the vielbeins and the two kinds of spin connection fields (the gauge fields of momenta, of $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and of $\tilde{S}^{ab} = \frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, respectively.

A few years ago, I recognized that Clifford algebra offers a description of the internal spaces for not only fermion fields but also for boson fields [15]:

i. The Clifford odd "basis vectors", the superpositions of the odd products of γ^a in $d \geq (13+1)$-dimensional space, manifest in $d = (3+1)$ families of quarks and leptons and antiquarks and antileptons, explaining all of the assumptions of the *standard model* for fermions. The Clifford odd "basis vectors" anticommute, transferring the anticommuting properties to the corresponding creation and annihilation operators, which are tensor products of the $2^{\frac{d}{2}-1}$ families of the Clifford odd "basis vectors", each family with $2^{\frac{d}{2}-1}$ family members, and the infinite basis in ordinary space.

The Hermitian conjugated partners of the Clifford odd "basis vectors" appear in a separate group, determining the annihilation operators ([3] and references therein).

ii. The Clifford even "basis vectors" in $d \geq (13+1)$-dimensional space manifest in $d = (3+1)$ all the properties of the vector gauge fields of the corresponding fermion fields, and the scalar gauge fields, explaining the appearance of the scalar Higgs and Yukawa couplings.

They appear in two orthogonal groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Each member of the two groups has their Hermitian conjugated partner within the same group. The members of one of the two groups transform the Clifford odd "basis vector" of a particular family into other members of the same family (keeping the family quantum number unchanged).

The members of the second group transfer the Clifford odd "basis vector" of a particular family into the Clifford odd "basis vector" of another family, keeping the family member's quantum number unchanged. The members of one of the two groups therefore cause changes in the Clifford odd "basis vector" of a particular family and a particular family member quantum number, as do the operators of the Lorentz transformations in the internal space of fermions S^{ab}. The other group causes changes in the Clifford odd "basis vector" as do the operators $\tilde{S}^{ab}$. These properties of the Clifford even "basis vectors" obviously dictate the interaction of the boson fields with the fermion fields, justifying the choice of the two spin connection fields in Equation (1) [15].

The Clifford even "basis vectors" commute, transferring the commuting properties to the corresponding creation operators, which are tensor products of the two times $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" and the infinite basis in ordinary space.

iii. The properties of the Clifford odd and the Clifford even "basis vectors" in odd dimensional spaces, $d = (2n+1)$, differ essentially from their properties in even dimensional spaces. Although anticommuting, the Clifford odd "basis vectors" manifest the properties of the Clifford even "basis vectors" in even dimensional spaces. The Clifford even "basis vectors" do not manifest the properties of the second quantized boson fields in even dimensional spaces. Although commuting, they manifest properties of the Clifford odd "basis vectors" in even dimensional spaces, resembling the ghosts needed when looking for finite solutions with Feynman diagrams.

In addition, since the operator of handedness has the Clifford odd character ($\Gamma = \prod_a^d (\sqrt{\eta^{aa}}\gamma^a) \cdot (i)^{\frac{d-1}{2}}$) in odd dimensional spaces, it transforms Clifford odd "basis vectors" into Clifford even "basis vectors" [16]. The eigenstates of the operator of handedness are in odd dimensional spaces corresponding to the superpositions of the Clifford odd and Clifford even "basis vectors".

We present the simple starting action of the *spin-charge-family* theory in which fermions interact in $d = (13+1)$-dimensional space with the gravitational fields only. As discussed above in points **i., ii., iii.** and in Ref. [16]; the assumption that "nature has made a choice" to use the Clifford algebra to describe the internal spaces of fermion fields (by using Clifford odd "basis vectors") and boson fields (by using Clifford even "basis vectors") requires the starting action of Equation (1) to have two kinds of spin connection field, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$). Ref. [3], and references therein, demonstrates that this action offers the description of all properties of fermion and boson fields—vector and scalar—as observed so far in $d = (3+1)$, offering several predictions for the observed phenomena.

$$
\begin{aligned}
\mathcal{A} &= \int d^d x \; E \; \tfrac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
&\quad \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \\
p_{0a} &= f^\alpha{}_a p_{0\alpha} + \tfrac{1}{2E} \, \{ p_\alpha, E f^\alpha{}_a \}_- \,, \\
p_{0\alpha} &= p_\alpha - \tfrac{1}{2} S^{ab} \omega_{ab\alpha} - \tfrac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \,, \\
R &= \tfrac{1}{2} \, \{ f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^c{}_{b\beta}) \} + h.c. \,, \\
\tilde{R} &= \tfrac{1}{2} \, \{ f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^c{}_{b\beta}) \} + h.c. \,.
\end{aligned}
\tag{1}
$$

Here, $f^\alpha{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, ..., m, n, ..., s, t, ...$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, ..., \mu, \nu, ..., \sigma, \tau, ...$ denote an Einstein index (a curved index). Letters from the beginning of both alphabets indicate a general index ($a, b, c, ...$ and $\alpha, \beta, \gamma, ...$). From the middle of both alphabets are the observed dimensions $0, 1, 2, 3$ ($m, n, ...$ and $\mu, \nu, ...$). Indexes from the bottom of the alphabets indicate compactified dimensions (s, t, ... and σ, τ, ...). We assume the signature is $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.) $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$.

A short overview of the properties of the Clifford odd and Clifford even "basis vectors" in even dimensional spaces is given in Section 2.1, showing that the Clifford odd "basis vectors", when applied on the appropriate vacuum states, manifest the postulates of the second quantized fermion fields, while the Clifford even "basis vectors" manifest the postulates for their gauge fields—the second quantized boson fields.

The properties of the Clifford odd and the Clifford even "basis vectors" in odd dimensional spaces are discussed in Section 2.2, demonstrating that, in odd-dimensional space, the Clifford odd and Clifford even objects have drastically different properties than in even dimensional spaces, offering an explanation for the postulated ghost fields that appear in several theories by taking care of the singular contributions when evaluating Feynman graphs.

In Section 2, an appropriate definition of the eigenstates of the Cartan subalgebra members is presented for even dimensional spaces, and this is extended to odd dimensional spaces.

In Section 3, the internal spaces for fermion and boson fields in even and odd dimensional spaces are discussed for simple cases: in Section 3.1, for the choices $d = (1+1)$, $d = (3+1)$, and in Section 3.2, for $d = (2+1)$ and $d = (4+1)$ (In Ref. [17], produced 20 years ago, discuss the question of q time and $d-q$ dimensions in odd and even dimensional spaces for any q. Using the requirement that the inner product of two fermions must be unitary and invariant under Lorentz transformations, the authors conclude that odd dimensional spaces are not appropriate due to the existence of fermions of both handedness and, correspondingly, are not mass protected. The recognition of this paper might further clarify the "effective" choice of nature for one time and three space dimensions.).

In Section 4, an overview of the main ideas of this paper is given.

In Appendix A, some helpful relations of the Clifford algebra can be found.

In Appendix B, the Grassmann algebra, expressible with the two Clifford subalgebras, γ^a and $\tilde{\gamma}^a$, is reviewed.

In Appendix C the algebra of $2^{d=(3+1)}$ products of Dirac γ^a in $d = (3+1)$ shows that the Dirac vectors and the Clifford odd "basis vectors" used to describe the spins and handedness of quarks, leptons, antiquarks, and antileptons are related.

In Appendix D one irreducible representation of $SO(13,1)$, analyzed with respect to $SO(3,1)$, $SU(2)_I$, $SU(2)_{II}$, $SU(3)$, and $U(1)$ is presented, demonstrating "basis vectors" of quarks and leptons and antiquarks and antileptons in the *spin-charge-family* theory. The relations with the corresponding vector and scalar gauge fields with respect to $d = (3+1)$ are described.

Appendix E presents the creation operators (and annihilation of their Hermitian conjugated partners) as the tensor products of the "basis vectors" and the basis in ordinary momentum or coordinate space, explaining the Dirac second quantized postulates. The Hilbert space of fermions, formed from the tensor products of creation operators is discussed.

The last four Appendixes are (indirectly) suggested by the referees.

Let me repeat, the recognition that the Clifford even "basis vectors" in even dimensional spaces offer an explanation for the properties of the boson gauge fields of the corresponding fermion fields, described by the Clifford odd "basis vectors", together with the recognition that, in odd dimensional spaces, the Clifford odd and even "basis vectors" demonstrate the properties of the ghosts, has introduced a new step beyond the *standard model* that can be used in cosmology.

In this article, I do not confront the achievements of the *spin-charge-family* theory but offer a simple action (Equation (1), the choice of which is supported by the description of the internal spaces of fermion and boson fields with the Clifford odd and even algebra, respectively, in $d \geq (13+1)$-dimensional space, which treats quarks and leptons and antiquarks and antileprons in an equivalent way, as manifested in Appendix D) with suggestions in the literature for the next step beyond the *standard model*. There are many suggestions for unifying charges in larger groups by adding additional groups for describing families [18–32], going to higher dimensional spaces [33–43], and looking for anomalies in gravity [44–46]. An explanation of what and how the *spin-charge-family* theory elegantly shows the way beyond the *standard model* is provided in Ref. [3]. Although there are several steps still to take, such as the quantization of gravity, the method suggested in this paper is promising and will hopefully lead to new findings, such as understanding the second quantized fields in black holes [47].

I do not know any literature that explains the internal spaces for fermion *and boson fields* in an equivalent way: not for $d = (3+1)$, and not for general d—either even or odd (Let me repeat again, the Clifford odd "basis vectors" transfer their anticommutativity to creation and annihilation operators, and the Clifford even "basis vectors" transfer their commutativity to creation and annihilation operators. The main point is that the Clifford odd "basis vectors" appear in families and have their Hermitian conjugated partners in a separate group, while the Clifford even "basis vectors" have their Hermitian conjugated partners in the same group and appear in two groups. The member of one group, when applied on the Clifford odd "basis vector" transforms it to one of the member of the same family, while the member of another group, when applied on the Clifford odd "basis vector" transforms it to the same member of another family. This article and also Ref. [15] clearly demonstrate these properties, empowering Clifford odd and even "basis vectors" and offering an explanation for the second quantized fermion and boson fields, respectively.).

Other references used a different approach by trying to make the next step with Clifford algebra to the second quantized fermion, which might also be a boson field [48–50].

2. Eigenstates of Cartan Subalgebra Members of Lorentz Algebra for Clifford Odd and Clifford Even "Basis Vectors"

In this section, the properties of the two kinds of Clifford algebra objects, γ^as and $\tilde{\gamma}^a$s, are repeated in accordance with several papers [1,4,11,15,51], in particular, the reference ([3], and the references therein).

In Appendix B, the starting Grassmann algebra is introduced and the corresponding Clifford subalgebras are discussed.

The two kinds of Clifford algebra objects, γ^a and $\tilde{\gamma}^a$, each offering 2^d superposition of products of either γ^a or $\tilde{\gamma}^a$, fulfill the relation [1,12]

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,, \\
(\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,.
\end{aligned} \tag{2}$$

Each of these two kinds of Clifford algebra objects can be used to describe the internal spaces of fermion and boson fields.

We can reduce the two possibilities to only one by deciding to describe the internal spaces of fermion and boson fields with the superpositions of the Clifford odd (for fermion fields) and Clifford even (for boson fields) products of γ^as, while using $\tilde{\gamma}^a$'s to equip the irreducible representations of the Lorentz group in the internal space of fermions with the family quantum numbers by assuming

$$\{\tilde{\gamma}^a B \;=\; (-)^B\, i\, B\gamma^a\}\,|\psi_{oc}>\,, \tag{3}$$

with $(-)^B = -1$ if B is (a function of) an odd product of γ^as; otherwise, $(-)^B = 1$, $|\psi_{oc}>$, as defined in Equations (8) and (A8). It is proven in [3] (Appendix I, Statement 3, 3.a, 3.b) that all relations of Equation (2) remain valid after the assumption of Equation (3).

The "basis vectors" describing the internal spaces of fermion and boson fields are chosen to be eigenstates of all Cartan subalgebra members. There are $\frac{d}{2}$ commuting operators of the Lorentz algebra in the even dimensional spaces, as described in Equation (A3), and $\frac{d-1}{2}$ in odd dimensional spaces, as described in Equation (A4).

If S^{ab}, $a \neq b$, (or $\tilde{S}^{ab}$ or $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$) are members of the Cartan subalgebra group of the Lorentz algebra in the internal space of fermion and boson fields, then it is not difficult to find the eigenstate of each of the members by taking into account the relations shown in Equation (2): $S^{ab}\, \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{k}{2}\, \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$ and $S^{ab}\, \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\, \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, with $k^2 = \eta^{aa}\eta^{bb}$. The first eigenstate is nilpotent, $(\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b))^2 = 0$, and the second eigenstate is projector, $(\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b))^2 = \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$.

Let us introduce the graphic notation in accordance with Refs. [1,12,15].

$$
\begin{aligned}
\stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,\\
\stackrel{ab}{(\tilde{k})}: &= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad \stackrel{ab}{[\tilde{k}]}: \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,,\\
(\stackrel{ab}{(k)})^\dagger &= \stackrel{ab}{(-k)}\,, \quad (\stackrel{ab}{(k)})^2 = 0\,, \quad (\stackrel{ab}{[k]})^\dagger = \stackrel{ab}{[k]}\,, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\;.
\end{aligned}
\tag{4}
$$

After taking into account Equations (2) and (3), the relations follow

$$
\begin{aligned}
\gamma^a \stackrel{ab}{(k)} &= \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}\,,\\
\tilde{\gamma}^a \stackrel{ab}{(k)} &= -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma}^b \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \quad \tilde{\gamma}^a \stackrel{ab}{[k]} = i \stackrel{ab}{(k)}, \quad \tilde{\gamma}^b \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}\,,
\end{aligned}
\tag{5}
$$

More relations can be found in Appendix A.

2.1. *Properties of Clifford Odd and Clifford Even "Basis Vectors" in Even Dimensional Spaces*

In each even dimensional space, there are $2^{\frac{d}{2}-1}$ members of the Clifford odd "basis vectors" appearing in $2^{\frac{d}{2}-1}$ families, and the same number of $2^{\frac{d}{2}-1}$ Hermitian conjugated partners appearing in $2^{\frac{d}{2}-1}$ families.

There are two orthogonal groups of the Clifford even "basis vectors". The members of each group have their Hermitian conjugated partners within the same group.

2.1.1. **Clifford Odd "Basis Vectors"**

The Clifford odd "basis vectors" describing the internal space of fermion fields are products of odd numbers of nilpotents and the rest of the projectors. Each nilpotent and each projector represent the eigenstate of one of the Cartan subalgebra members.

Let us call the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_f$, if it is the m^{th} member of the family f.

Let us choose the first member $\hat{b}^{1\dagger}_1$ in $d = 2(2n+1)$ as the product of nilpotents only.

$$
\begin{aligned}
&d = 2(2n+1)\,,\\
&\hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-1\,d}{(+)}\,,\\
&\hat{b}^{2\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \cdots \stackrel{d-1\,d}{(+)}\,,\\
&\cdots\\
&\hat{b}^{2^{\frac{d}{2}-1}\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \ldots \stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[-]}\,,\\
&\cdots\,.
\end{aligned}
\tag{6}
$$

In the case that $d = 4n, n = 1, 2, ...$, the first member must have one projector.

$$
\begin{aligned}
& d = 4n\,, \\
& \hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-1\,d}{[+]}\,, \\
& \cdots\,.
\end{aligned} \tag{7}
$$

All the rest of the members of the same family, $2^{\frac{d}{2}-1} - 1$, follow through the application of all possible S^{ab} on $\hat{b}^{1\dagger}_{1}$, while all the rest of the $2^{\frac{d}{2}-1} - 1$ families follow through the application of all possible $\tilde{S}^{ab}$ on all members of the starting family.

The Hermitian conjugated partners $(\hat{b}^{m\dagger}_{f})^{\dagger}$ of the "basis vectors" $\hat{b}^{m\dagger}_{f}$ follow from these $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ "basis vectors" by replacing each nilpotent $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(-k)}$.

Let us recognize that all Clifford odd "basis vectors" are orthogonal: $\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0$. The Hermitian conjugated partners are also orthogonal $\hat{b}^{m}_{f} *_A \hat{b}^{m'}_{f'} = 0$.

When we choose a vacuum state equal to

$$
|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*A} \hat{b}^{m\dagger}_{f}\,|\,1\;>\,, \tag{8}
$$

for one of the members m, whcih can be any one of the odd irreducible representations f with $|\,1\;>$, which is the vacuum without any structure (the identity) it follows that $\hat{b}^{m}_{f}|\psi_{oc}>= 0$.

Each Clifford odd "basis vector" carries the family quantum number, and so does its Hermitian conjugated partner. One correspondingly finds that the "basis vectors" and their Hermitian conjugated partners fulfill the postulates for the second quantized fermion fields.

$$
\begin{aligned}
\hat{b}^{m}_{f}{}_{*A}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,, \\
\hat{b}^{m\dagger}_{f}{}_{*A}|\psi_{oc}> &= |\psi^{m}_{f}>\,, \\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,, \\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,, \\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*A+}|\psi_{oc}> &= \delta^{mm'}_{ff'}\,|\psi_{oc}>\,,
\end{aligned} \tag{9}
$$

where $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_{f}$ and $\hat{b}^{m'}_{f'}$ among themselves and with the vacuum state $|\psi_{oc}>$ of Equation (8). Equation (9) follows by taking into account Equations (2) and (3).

These "basis vectors" are not yet the representatives of the creation and annihilation operators: the tensor, $*_T$, products of the "basis vectors" m and the basis in ordinary momentum or coordinate space [3] (In even dimensional spaces with $d = 4n$, one can proceed, as we did for the $d = 2(2n + 1)$ dimensional case after taking into account the requirement that the odd number of nilpotents forms the anticommuting "basis vectors" describing the internal space of fermions. The starting "basis vector" $\hat{b}^{1\dagger}_{1}$ must have one projector, while all the rest are nilpotents. S^{ab}s then generate all the members of one family, while $\tilde{S}^{ab}$s generate all of the families. The "basis vectors" and their Hermitian conjugated partners fulfill the requirements on the vacuum state, as shown in Equation (A8). The anti-commuting properties of Equation (9)) represent the creation operators and the corresponding Hermitian conjugated partners, the annihilation operators.

2.1.2. **Clifford Even "Basis Vectors"**

We can define the Clifford even "basis vectors" describing the internal space of the boson fields as products of even numbers of nilpotents and the rest as projectors if each nilpotent and each projector represent an eigenstate of one of the Cartan subalgebra members [15].

Let us call the Clifford even "basis vectors" ${}^{i}\mathcal{A}^{m\dagger}_{f}, i = I, II$. There are two groups of Clifford even "basis vectors". Each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.

Let us choose the starting Clifford even "basis vector", ${}^{i=I}\mathcal{A}^{1\dagger}_{1}$, to be the product of projectors $\stackrel{ab}{[k]}$ with $k = i$ for S^{03}, and $k = 1$ for the rest $2^{\frac{d}{2}-1} - 1$ members of the Cartan subalgebra.

$$
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[+]} \cdots \stackrel{d-1\,d}{[+]} \,. \tag{10}
$$

The starting Clifford even "basis vector" of the second group ${}^{i=II}\mathcal{A}^{1\dagger}_{1}$ can again be the product of projectors only, but in this case, with $\stackrel{03}{[-i]}$ instead of $\stackrel{03}{[+i]}$, and for all the rest of the $2^{\frac{d}{2}-1} - 1$ members of the Cartan subalgebra, with $k = +1$. (This starting member can not be obtained from ${}^{I}\mathcal{A}^{1\dagger}_{1}$ through the application of S^{ab}s or $\tilde{S}^{ab}$s, since these operators always change the eigenvalues of two Cartan subalgebra members.)

$$
{}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[+]} \cdots \stackrel{d-1\,d}{[+]} \,. \tag{11}
$$

The rest of the members of each group follow on from the starting member through the application of either S^{ab}s or $\tilde{S}^{ab}$s.

Since S^{01} transforms $\stackrel{03}{[+i]}\stackrel{12}{[+]}$ into $\stackrel{03}{(-i)}\stackrel{12}{(-1)}$, while $\tilde{S}^{01}$ transforms $\stackrel{03}{[+i]}\stackrel{12}{[+]}$ into $\stackrel{03}{(+i)}\stackrel{ab}{(+)}$, we can immediately see that the Clifford even "basis vectors" have Hermitian conjugated partners within the same group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.

2.1.3. **Clifford Even "Basis Vectors" Applied on Clifford Odd "Basis Vectors"**

Let us apply ${}^{I}\mathcal{A}^{1\dagger}_{1}$, which is made from the projectors $\stackrel{ab}{[k]}$ only, with $k = i$ for S^{03} and $k = 1$ for the rest of the members of the Cartan subalgebra, on $\hat{b}^{1\dagger}_{1}$, which is the product of the nilpotents only, with an eigenvalue of S^{03} equal to $k = \frac{i}{2}$ and that of the rest of the Cartan subalgebra members equal to $k = \frac{1}{2}$.

Taking into account Equations (A9) and (A10), one sees that, in this application, ${}^{I}\mathcal{A}^{1\dagger}_{1} *_A \hat{b}^{1\dagger}_{1}$, leaving $\hat{b}^{1\dagger}_{1}$ unchanged. When applying ${}^{I}\mathcal{A}^{2\dagger}_{1}$, with the first two projectors transformed into two nilpotents, $\stackrel{03}{(-i)}\stackrel{12}{(-1)}$, and all of the rest remain the same. We see that this application transforms $\hat{b}^{1\dagger}_{1}$ into $\hat{b}^{2\dagger}_{1}$ ($=\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{(+)}\stackrel{78}{(+)}$ .... (all the rest remain the same). The application of ${}^{I}\mathcal{A}^{2\dagger}_{1}$ on $\hat{b}^{1\dagger}_{1}$ obviously changes the eigenvalues of S^{03} and S^{12} to $\hat{b}^{1\dagger}_{1}$ for the integer values of $-i$ and -1, respectively.

We conclude that the algebraic application $*_A$ of the Clifford even "basis vectors" on the Clifford odd "basis vectors", describing the internal space of fermion fields, changes the eigenvalues of the Cartan subalgebra members to 0 or for the integer values $\pm i$ or ± 1, leading to

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f'} *_A \hat{b}^{m'\dagger}_{f} \to \left\{ \begin{array}{c} \hat{b}^{m\dagger}_{f}\,, \\ \text{or zero}\,. \end{array} \right. \tag{12}
$$

For each m', there exists one f, so that the Equation (12) is fulfilled for all f' and all m [15].

2.1.4. **Clifford Even "Basis Vectors" Applied on Clifford Even "Basis Vectors" [15]**

It is not difficult to see, by taking Equations (A9) and (A10) into account, that the algebraic applications of ${}^{I}\mathcal{A}^{f\dagger}_{1} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f'} = 0 = {}^{II}\mathcal{A}^{m'\dagger}_{f'} *_A {}^{I}\mathcal{A}^{m\dagger}_{f}$, for all (m, m', f, f').

The algebraic application $*_A$, of ${}^{i}\mathcal{A}^{m\dagger}_{f} *_A {}^{i}\mathcal{A}^{m'\dagger}_{f'}$, $i = (I, II)$ within each of the two groups general has a nonzero contribution, demonstrating the properties of the internal spaces of the gauge fields to the corresponding fermion fields, the internal space of which is described by the Clifford odd "basis vectors".

In each of the two groups, there are $2^{\frac{d}{2}-1}$ members, which are products of projectors only. They are self adjoint and have the eigenvalues of all the Cartan subalgebra members equal to zero: $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.

All the rest ${}^{i}\mathcal{A}^{m\dagger}_{f}$ (there are $2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1)$ members) appear in pairs that are Hermitian-conjugated to each other. Their mutual algebraic products form one of $2^{\frac{d}{2}-1}$ self adjoint members.

The algebraic multiplication of the Clifford even "basis vectors" on the Clifford even "basis vectors" leads to

$$
{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'} \rightarrow \left\{ \begin{array}{c} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f'} \,, \\ \text{or zero}\,. \end{array} \right. \; i = (I, II)\,. \tag{13}
$$

Special cases for Clifford odd and even "basis vectors" are discussed in Section 3.1 for $d = (1+1)$ and $d = (3+1)$ and in Appendixes C and D. In Appendix D, one irreducible representation in $d = (13+1)$ is presented to describe the internal space of quarks and leptons and antiquarks and antileptons.

In Ref. [15], the reader can find the Clifford odd and Clifford even "basis vectors" for the case where the dimensions of the space are $d = (5+1)$, describing the internal space of fermion and boson fields, respectively, as illustrated by the figures.

2.2. Properties of Clifford Odd and Clifford Even "Basis Vectors" in Odd Dimensional Spaces

In this Section 2.2 the Clifford odd and Clifford even "basis vectors" in odd dimensional spaces [15] are discussed.

While in even dimensional spaces the Clifford odd "basis vectors" fulfill the postulates for the second quantized fermion fields, as shown in Equation (9), and Clifford even "basis vectors" have all the properties of the internal spaces of their corresponding gauge fields, as shown in Equations (12) and (13), in odd dimensional spaces, the Clifford odd and even "basis vectors" have unusual properties resembling properties of the internal spaces of the Faddeev–Popov ghosts, as we describe in the following text.

Looking at $d = (2n+1)$-dimensional cases, $n = 1, 2, \ldots$, for the Clifford odd and Clifford even "basis vectors" in the $2n$-dimensional part of space, we find half of the "basis vectors" with properties presented in Equations (6), (7), (10) and (11). In Equations (14) and (15), they are presented on the left-hand side.

The rest of the "basis vectors" in odd dimensional spaces follow if $S^{0\,2n+1}$ is applied on the obtained half of the Clifford odd and the Clifford even "basis vectors". Since $S^{0\,2n+1}$ are Clifford even operators, they do not change the oddness or evenness of the "basis vectors".

For the Clifford odd "basis vectors", correspondingly, the additional $2^{\frac{d-1}{2}-1}$ members appear in $2^{\frac{d-1}{2}-1}$ families and the same number of their Hermitian conjugated partners are present on the right-hand side of Equation (14).

$$
\begin{array}{llll}
 & d = & 2(2n+1)+1 & \\
\hat{b}^{1\dagger}_{1} = & \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-2\,d-1}{(+)}\,, & \hat{b}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = & \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-2\,d-1}{(+)}\;\gamma^{d}\,, \\
\hat{b}^{2\dagger}_{1} = & \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \cdots \stackrel{d-2\,d-1}{(+)}\,, & \hat{b}^{2\dagger}_{2^{\frac{d-1}{2}-1}+1} = & \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(+)} \cdots \stackrel{d-2\,d-1}{(+)}\;\gamma^{d}\,, \\
 & \cdots & \cdots & \\
\hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = & \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} \ldots \stackrel{d-2\,d-1}{[-]}\,, & \hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{d-12-1}+1} = & \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(+)} \ldots \stackrel{d-2\,d-1}{[-]}\;\gamma^{d}\,, \\
 & \cdots & \cdots\,. &
\end{array} \tag{14}
$$

The half of the "basis vectors" or their Hermitian conjugated partners appearing on the right-hand side follow on from those appearing on the left-hand side through the application of S^{0d} or $\tilde{S}^{0d}$ on the left-hand side. The application of S^{0d} or $\tilde{S}^{0d}$ on the left-hand side of the "basis vectors" and their Hermitian conjugated partners generates the whole set of two 2^{d-2} members of the Clifford odd "basis vectors" and their Hermitian conjugated partners in $d = (2n+1)$- dimensional space appearing on the left- and right-hand sides of Equation (14).

When applied on the Clifford even "basis vectors" that appear on the left-hand side of Equation (15), the operators $S^{0\,2n+1}$ and the additional two groups of $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ "basis vectors" follow, as presented in Equation (15), on the right-hand side.

For the 2^{d-2} Clifford odd objects present on the right-hand side of Equation (14), and for the special cases in Equations (23) and (25), although they are the superpositions of the Clifford odd products of γ^as, do not manifest the properties of "basis vectors" and their Hermitian conjugated partners, presented on the left-hand side of Equation (14) and in the special cases of Equations (23) and (25).

The eigenstates appearing on the right-hand side of Equation (14) can be divided into two groups that are orthogonal to each other, having their Hermitian conjugated partners within the same group or being self adjoint. Although they are Clifford odd objects, they resemble the properties of the Clifford even partners of the "basis vectors" that appear on the left-hand side of Equation (15).

Let us see the application of the operators S^{0d} and $\tilde{S}^{0d}$ on the Clifford even "basis vectors" on the even dimensional part of the $d = (2(2n+1)+1)$ space. The Clifford even "basis vectors" must have an even number of nilpotents, which means that, in $d = 2(2n+1)$, we must have at least one projector. To obtain all of the Clifford even "basis vectors", we must apply the operators S^{0d} or $\tilde{S}^{0d}$ on these starting Clifford even "basis vectors", as presented in Equation (15) on the left-hand side.

$$
\begin{aligned}
&d = && 2(2n+1)+1 \\
&{}^{I}\mathcal{A}^{1\dagger}_{1} = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-2\,d-1}{[+]}\,, && {}^{I}\mathcal{A}^{1\dagger}_{2^{d-1}2-1+1} = \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-2\,d-1}{[+]}\;\gamma^{d}\,, \\
&{}^{I}\mathcal{A}^{2\dagger}_{1} = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-2\,d-1}{[+]}\,, && {}^{I}\mathcal{A}^{2\dagger}_{2^{d-1}2-1+1} = \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-2\,d-1}{[+]}\;\gamma^{d}\,, \\
&\cdots && \cdots \\
&{}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\ldots\overset{d-2\,d-1}{[+]}\,, && {}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{d-1}2-1+1} = \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}\ldots\overset{d-2\,d-1}{[+]}\;\gamma^{d}\,, \\
&\cdots && \cdots\,.
\end{aligned}
\tag{15}
$$

The right-hand side of Equation (15) and the special cases of the Clifford even part of Equations (23) and (25) show the Clifford even "basis vectors" as the left-handed partners. However, they resemble the properties of the Clifford odd "basis vectors", as presented in Equation (14) and for the special cases of the Clifford odd part of Equations (23) and (25). These Clifford even objects can be arranged into $2^{\frac{d-1}{2}-1}$ members in $2^{\frac{d-1}{2}-1}$ families of "basis vectors" and into a separate group of Hermitian conjugated partners. However, they are the Clifford even "basis vectors".

Let us point out that the Lorentz transformations in the internal spaces of fermion and boson fields transform the left-hand sides of Equation (14) and (15) into the corresponding right-hand sides vice versa.

If we algebraically apply the Clifford even "basis vectors" appearing on the right-hand side of Equation (15) to the Clifford odd "basis vectors" appearing on the right-hand side of Equation (14), we end up with the Clifford odd "basis vector" appearing on the left-hand side of Equation (14) or on one of their Hermitian conjugated partners. Otherwise, we obtain a value of zero.

In the next section, we discuss concrete cases to make the discussion more transparent.

Let us conclude this section with what we have learned:

a. In $d = 2n+1$ dimensional spaces, $n = 1, 2, \ldots$, there are two kinds of Clifford odd "basis vectors":

a.i. The "basis vectors" are the products of an odd number of nilpotents and the rest of the projectors. These "basis vectors" appear in $2^{\frac{d-1}{2}-1}$ families, where each family has $2^{\frac{d-1}{2}-1}$ members. They anticommute, fulfilling together with their Hermitian conjugated partners the postulates for the second quantized fermion fields. Their Hermitian conjugated partners appear in a separate group.

a.ii. When the operators S^{0d} or $\tilde{S}^{0d}$ are applied on these Clifford odd "basis vectors", the additional two times $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ of the Clifford odd "basis vectors" follow. These Clifford odd "basis vectors" resemble the properties of the Clifford even "basis vectors" from the case **b.i.** presented below. They form two orthogonal groups. The members of each group have their Hermitian conjugated partners within the same group, or they are self adjoint.

b. In $d = 2n + 1$ dimensional spaces, $n = 1, 2, \ldots$, there are two kinds of Clifford even "basis vectors":

b.i. The "basis vectors" are products of even numbers of nilpotents and the rest of the projectors. These "basis vectors" appear in two orthogonal groups with $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ members. Each group has its Hermitian conjugated members within the group or is self adjoint. They commute, fulfilling the postulates for the second quantized boson fields or the gauge fields of the corresponding fermion fields for the case **a.i.**.

b.ii. When the operators S^{0d} or $\tilde{S}^{0d}$ are applied to these "basis vectors" the additional two times $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ Clifford even "basis vectors" follow. These Clifford even "basis vectors" resemble the properties of the Clifford odd "basis vectors" for the case **a.i.**. They form two groups with $2^{\frac{d-1}{2}-1}$ members in each of the $2^{\frac{d-1}{2}-1}$ families. Their Hermitian conjugated partners appear in a separate group. However, they commute.

c. When Clifford even "basis vectors" of the kind b:

c.i. When Clifford even "basis vectors" of the **b.i.** type are algebraically applied on the Clifford odd "basis vectors" of the **a.i.** type, they transfer to the Clifford odd "basis vectors" the integer values of the Cartan subalgebra members ($\pm i$, ± 1 or 0) or they have a value of zero.

c.ii. When Clifford even basis vectors" of the **b.ii.** type are algebraically applied to the Clifford odd "basis vectors" of the **a.ii.** type, they transfer to the Clifford odd "basis vectors" the integer values of the Cartan subalgebra members, ($\pm i$, ± 1 or 0) or they have a value of zero, as in the case **c.i.**.

d. While the Clifford odd "basis vectors" in even dimensional spaces have well-defined handedness, since the operator of handedness is the Clifford even operator, Equation (A1), the eigenvectors of the operator of handedness in odd dimensional spaces are the superpositions of the "basis vectors" of the **a.i.** and **a.ii.** types.

3. "Basis Vectors" in Even, $d = 2n$ for $n = 1, 2$, and Odd, $d = 2n + 1$ for $n = 1, 2$, Dimensional Spaces

The internal spaces for fermion and boson fields in even and odd dimensional spaces are discussed for simple cases: In Section 3.1 for the choices $d = (1 + 1)$, $d = (3 + 1)$ and in Section 3.2 for $d = (2 + 1)$ and $d = (4 + 1)$. This section is meant to be an illustration of Section 2.

The case with $d = (3 + 1)$ is also discussed in Appendix C, relating $2^{d=4}$ products of Dirac's matrices γ^a to the Clifford odd and even "basis vectors" and their Hermitian conjugated partners. In Appendix D, a description of the internal spaces of quarks, leptons, antiquarks, and antileptons with the "basis vectors" in $d = (13 + 1)$ is presented.

The reader can also find the definition of the "basis vectors" as the eigenstates of the Cartan subalgebra of the Lorentz algebra describing the internal spaces of fermion and boson fields in Refs. [3,4,15,51]. The "basis vectors" are written as superpositions of the Clifford odd (for fermions) and Clifford even (for bosons) products of γ^as.

The "basis vectors" for fermions have either left- or righthandedness, Γ^d (the handedness is defined in Equation (A1)), and they appear in families (the family quantum numbers are determined by $\tilde{\gamma}^a$s, with $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$). The Clifford odd "basis vectors" have their Hermitian conjugated partners in a separate group.

"Basis vectors" for bosons have no families, and they have their Hermitian conjugated partners within the same group, as described in Section 2.

The "basis vectors" in odd dimensional spaces differ in properties from the "basis vectors" in even dimensional spaces, as we concluded in the previous Section 2. Let us repeat:

Half of the Clifford odd "basis vectors" have the same properties in odd dimensional spaces as in even dimensional spaces (The same choice of Cartan subalgebra members is made in the case of $d = (2n+1)$ and in the case of $d = 2n$. The Lorentz transformations take place from the left-hand side into the right-hand side and vice versa in the internal spaces of the fermion and boson fields, as shown in Equations (14) and (15)). The remaining half of the Clifford odd "basis vectors", although anticommuting, gain the properties of the Clifford even "basis vectors".

Half of the Clifford even "basis vectors" have the properties of the Clifford even "basis vectors" in even dimensional spaces. The remaining half of the Clifford even "basis vectors", although commuting, gain the properties of the Clifford odd "basis vectors".

Since the operator of handedness is the Clifford odd object (it is the product of the odd number of γ^as), only the superpositions of the Clifford odd and the Clifford even "basis vectors" have definite handedness (Correspondingly, the eigenvectors of the Cartan subalgebra members have both handednesses, $\Gamma^{(2n+1)} = \pm 1$.).

3.1. "Basis Vectors" in Even Dimensional Spaces: $d = (1+1), (3+1)$

To illustrate the differences in the properties of the internal spaces of fermion and boson fields in even and odd dimensional spaces, a few simple cases are discussed. The definitions of nilpotents and projectors and the relations among them can be found in Equation (4) and Appendix A.

3.1.1. $d = (1+1)$

There are $4\,(2^{d=2})$ "eigenvectors" of the Cartan subalgebra members, as shown in Equation (A3), S^{01} and $\mathcal{S}^{01}$ of the Lorentz algebra S^{ab} and $\mathcal{S}^{ab} = S^{01} + \tilde{S}^{01}$ ($S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_-$, $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$), representing one Clifford odd "basis vector" $\hat{b}^{1\dagger}_1 = \stackrel{01}{(+i)}$ (m = 1), appearing in one family (f = 1) and correspondingly one Hermitian conjugated partner $\hat{b}^1_1 = \stackrel{01}{(-i)}$ (It is our choice as to whether $\stackrel{01}{(+i)}$ or $\stackrel{01}{(-i)}$ is chosen as the "basis vector" $\hat{b}^{1\dagger}_1$; the remainder is its Hermitian conjugated partner. The choice of "basis vector" determines the vacuum state $|\psi_{oc}>$, Equation (8). For $\hat{b}^{1\dagger}_1 = \stackrel{01}{(+i)}$, the vacuum state is $|\psi_{oc}> = \stackrel{01}{[-i]}$ (due to the requirement that $\hat{b}^{1\dagger}_1|\psi_{oc}>$ must be nonzero, while $\hat{b}^1_1|\psi_{oc}>$ is zero). $\stackrel{01}{[-i]}$ is the Clifford even object.) and two Clifford even "basis vectors" ${}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[+i]}$ and ${}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[-i]}$, which are both self adjoint.

Correspondingly, after using Equations (A14) and (A7), we have two Clifford odd and two Clifford even eigenvectors of the Cartan subalgebra members

$$
\begin{aligned}
& & &\text{Clifford odd} \\
\hat{b}^{1\dagger}_1 &= & &\stackrel{01}{(+i)}\,, \quad \hat{b}^1_1 = \stackrel{01}{(-i)}\,, \\
& & &\text{Clifford even} \\
{}^{I}\mathcal{A}^{1\dagger}_1 &= & &\stackrel{01}{[+i]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{01}{[-i]}\;.
\end{aligned}
\tag{16}
$$

The two Clifford odd "basis vectors" are Hermitian conjugated to each other. $\hat{b}_1^{1\dagger}$ is chosen to be the "basis vector", and the second Clifford odd object is its Hermitian conjugated partner. After defining the handedness as $\Gamma^{(1+1)} = \gamma^0\gamma^1$, Equation (A1), it follows, using Equation (A5), that $\Gamma^{(1+1)}\, \hat{b}_1^{1\dagger} = \hat{b}_1^{1\dagger}$. $\hat{b}_1^{1\dagger}$ is the right-handed "basis vector" (We could choose left-handed "basis vectors" if choosing $\hat{b}_1^{1\dagger} = \stackrel{01}{(-i)}$, but the choice of handedness would remain at one.).

Each of the two Clifford even "basis vectors" is self adjoint $(({}^{I,II}\mathcal{A}_1^{1\dagger})^\dagger = {}^{I,II}\mathcal{A}_1^{1\dagger})$.

Taking into account Equations (A5) and (A9), we can observe that

$$
\begin{aligned}
\{\hat{b}_1^{1}(\equiv\stackrel{01}{(-i)}) *_A \hat{b}_1^{1\dagger}(\equiv\stackrel{01}{(+i)})\}|\psi_{oc}> &= {}^{II}\mathcal{A}_1^{1\dagger}(\equiv\stackrel{01}{[-i]})|\psi_{oc}>=|\psi_{oc}>\,,\\
\{\hat{b}_1^{1\dagger}(\equiv\stackrel{01}{(+i)}) *_A \hat{b}_1^{1}(\equiv\stackrel{01}{(-i)})\}|\psi_{oc}> &= 0\,,\\
{}^{I}\mathcal{A}_1^{1\dagger}(\equiv\stackrel{01}{[+i]}) *_A \hat{b}_1^{1\dagger}(\equiv\stackrel{01}{(+i)})|\psi_{oc}> &= \hat{b}_1^{1\dagger}(\equiv\stackrel{01}{(+i)})|\psi_{oc}>\,,\\
{}^{I}\mathcal{A}_1^{1\dagger}(\equiv\stackrel{01}{[+i]})\, \hat{b}_1^{1}(\equiv\stackrel{01}{(-i)})|\psi_{oc}> &= 0\,,\\
{}^{I}\mathcal{A}_1^{1\dagger} *_A {}^{II}\mathcal{A}_1^{1\dagger} &= 0 = {}^{II}\mathcal{A}_1^{1\dagger} *_A {}^{I}\mathcal{A}_1^{1\dagger}\,.
\end{aligned}
\tag{17}
$$

The case with $d = (3+1)$ is more informative:

3.1.2. $d = (3+1)$

In $d = (3+1)$ there are 16 ($2^{d=4}$) "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, as shown in Equation (A3).

Half of them are Clifford odd "basis vectors", which appear in two families $2^{\frac{4}{2}-1}$, $f = (1,2)$), each with two ($2^{\frac{4}{2}-1}$, $m = (1,2)$), members, $\hat{b}_f^{m\dagger}$.

There are $2^{\frac{4}{2}-1}\times\, 2^{\frac{4}{2}-1}$ Hermitian conjugated partners $\hat{b}_f^{m}$ that appear in a separate group (not reachable by S^{ab}).

There are $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{I}\mathcal{A}_f^{m\dagger}$, which are self adjoint or have their Hermitian conjugated partners within the same group.

All members of this group are reachable by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ from any starting "basis vector" ${}^{I}\mathcal{A}_1^{1\dagger}$.

There is another group of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors". They are the members of ${}^{II}\mathcal{A}_f^{m\dagger}$, which again are either Hermitian conjugated to each other or are self adjoint. All are reachable from the starting vector ${}^{II}\mathcal{A}_1^{1\dagger}$ by the application of $\mathcal{S}^{ab}$.

Choosing the right-handed Clifford odd "basis vectors" as

$$
\begin{array}{cccc}
f=1 & f=2 & & \\
\tilde{S}^{03}=\frac{i}{2},\tilde{S}^{12}=-\frac{1}{2} & \tilde{S}^{03}=-\frac{i}{2},\tilde{S}^{12}=\frac{1}{2} & S^{03} & S^{12}\\
\hat{b}_1^{1\dagger}=\stackrel{03}{(+i)}\stackrel{12}{[+]} & \hat{b}_2^{1\dagger}=\stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2}\\
\hat{b}_1^{2\dagger}=\stackrel{03}{[-i]}\stackrel{12}{(-)} & \hat{b}_2^{2\dagger}=\stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2}\,,
\end{array}
\tag{18}
$$

We find, for their Hermitian conjugated partners,

$$
\begin{array}{cccc}
S^{03}=-\frac{i}{2},S^{12}=\frac{1}{2} & S^{03}=\frac{i}{2},S^{12}=-\frac{1}{2} & \tilde{S}^{03} & \tilde{S}^{12}\\
\hat{b}_1^{1}=\stackrel{03}{(-i)}\stackrel{12}{[+]} & \hat{b}_2^{1}=\stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2}\\
\hat{b}_1^{2}=\stackrel{03}{[-i]}\stackrel{12}{(+)} & \hat{b}_2^{2}=\stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2}\,.
\end{array}
\tag{19}
$$

The vacuum state on which the Clifford odd "basis vectors" are applied is equal to: $|\psi_{oc}>= \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]})$ (The case $SO(1,1)$ can be viewed as a subspace of the case $SO(3,1)$, recognizing the "basis vectors" $\stackrel{03}{(+i)}\stackrel{12}{[+]}$ and $\stackrel{03}{(-)}\stackrel{12}{[-]}$ with $\stackrel{03}{(+i)}$ and $\stackrel{03}{(-i)}$, respectively, as carrying two different types of handedness in $d = (1+1)$, but each of them also carries a different "charge" S^{12}. In the whole internal space, all of the Clifford odd "basis vectors" have only one kind of handedness.).

Let us recognize that all of the Clifford odd "basis vectors" are orthogonal: $\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0$.

Let us present $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{I}\mathcal{A}^{m\dagger}_{f}$, which are Hermitian conjugated to each other or are self adjoint (Let it be repeated that $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ [15].)

$$\begin{array}{lcc lcc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0 & ,\ {}^{I}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1 \\
{}^{I}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1 & ,\ {}^{I}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0\,,
\end{array} \tag{20}$$

and $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{II}\mathcal{A}^{m\dagger}_{f}$, $m = (1,2)$, $f = (1,2)$, which are again Hermitian conjugated to each other or are self adjoint

$$\begin{array}{lcc lcc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0 & ,\ {}^{II}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(-)} & i & -1 \\
{}^{II}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1 & ,\ {}^{II}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0\,.
\end{array} \tag{21}$$

The Clifford even "basis vectors" have no families. The two groups ${}^{I}\mathcal{A}^{m\dagger}_{f}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}$ (they are not reachable by $\mathcal{S}^{ab}$) are orthogonal.

$$ {}^{I}\mathcal{A}^{m\dagger}_{f} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f'} = 0, \quad \text{for any } (m, m', f, f')\,. \tag{22}$$

Even dimensional spaces have the properties of the fermion and boson second quantized fields [15].

In Appendix C, the 16 members of the Dirac's products of γ^a, arranged into the 16 Clifford odd and even "basis vectors" presented in Equations (18)–(21), are presented.

The reader can find discussions about the $d = (5+1)$- dimensional case in [3,15] and the references therein.

3.2. *"Basis Vectors" in Odd Dimensional Spaces with $d = (2+1)$, and $d = (4+1)$*

Half of the Clifford odd and Clifford even Clifford objects in the $(2n+1)$-dimensional cases can be found by treating the Clifford odd "basis vectors" and their Hermitian conjugated partners and the Clifford even "basis vectors" in the $2(2n'+1)$ (or $4n'$) dimensional part of space. The properties of these "basis vectors" are presented in Equations (6), (7), (10) and (11).

The rest of the "basis vectors" follow by the application of S^{0d} on the "basis vectors" determining the internal spaces of the fermion and boson fields in the $2(2n'+1)$ (or $4n'$)-dimensional part of space. Since S^{0d} represents the Clifford even operators, they do not change oddness or evenness of the "basis vectors" or their Hermitian conjugated partners. However, they do change their properties:

i. In even dimensional subspace, $2(2n+1)$ of $d = 2(2n+1)+1$ (or $4n$ of $d = 4n+1$), the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, have $2^{\frac{d-1}{2}-1}$ members, m, in $2^{\frac{d-1}{2}-1}$ families, f, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d-1}{2}-1}$ members in $2^{\frac{d-1}{2}-1}$ families. The Clifford even "basis vectors" appear in two mutually orthogonal groups, each with $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ members.

ii. The second part of the "basis vectors" and their Hermitian conjugated partners, obtained from the first part through the application of S^{0d} with the same number of either Clifford odd or Clifford even objects as the first part manifests as follows: The Clifford odd "basis vectors" appear in two mutually orthogonal groups, each with $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ members, self adjoint, or with the Hermitian conjugated partners within the same group. The Clifford even "basis vectors" appear with $2^{\frac{d-1}{2}-1}$ members, m, in $2^{\frac{d-1}{2}-1}$ families, f, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d-1}{2}-1}$ members in $2^{\frac{d-1}{2}-1}$ families.

iii. While $\hat{b}^{m\dagger}_f$ has one handedness only (either right or left, depending on the definition of handedness) in even dimensional spaces, in odd dimensional spaces, the operator of handedness is a Clifford odd object—the product of an odd number of γ^as, as shown in Equation (A1), (still commuting with S^{ab}). This transforms the Clifford odd "basis vectors" into Clifford even "basis vectors" and vice versa. Correspondingly, the eigenvectors of the operator of handedness are the superpositions of the Clifford odd and the Clifford even basis vectors", offering the right- and left-handed eigenvectors of the operator of handedness in odd dimensional spaces.

Let us illustrate the abovementioned properties of the "basis vectors" in odd dimensional spaces, starting with the simplest case:

3.2.1. **d = (2 + 1)**

In $d = (2+1)$, there are 8 $(2^{d=3})$ "eigenvectors" of the Cartan subalgebra members $(S^{01}, \mathcal{S}^{01})$ for the Lorentz algebras, S^{ab} and $\mathcal{S}^{ab}$, Equation (A4).

Half of the Clifford odd and Clifford even "basis vectors" and their Hermitian conjugated partners can be taken from Equation (16); the rest are obtained by the application of S^{02} or $\tilde{S}^{02}$ on the first half. One obtains

$$
\begin{array}{rl}
d = & 2+1 \\
 & \text{Clifford odd} \\
\hat{b}^{1\dagger}_1 = \overset{01}{(+i)}\,, & \hat{b}^{1\dagger}_2 = \overset{01}{[-i]}\,\gamma^2\,, \\
\hat{b}^{1}_1 = \overset{01}{(-i)}\,, & \hat{b}^{1}_2 = \overset{01}{[+i]}\,\gamma^2\,, \\
 & \\
 & \text{Clifford even} \\
{}^{I}\mathcal{A}^{1\dagger}_1 = \overset{01}{[+i]}\,, & {}^{I}\mathcal{A}^{1\dagger}_2 = \overset{01}{(-i)}\,\gamma^2\,, \\
{}^{II}\mathcal{A}^{1\dagger}_1 = \overset{01}{[-i]}\,, & {}^{II}\mathcal{A}^{1\dagger}_2 = \overset{01}{(+i)}\,\gamma^2\,.
\end{array}
\tag{23}
$$

It can clearly be seen that the left-hand side of the Clifford odd "basis vectors" and the right-hand side of the Clifford even "basis vectors", although the former are the Clifford odd objects and the latter are Clifford even objects, have similar properties.

For example,

$$
\hat{b}^{1}_1 *_A \hat{b}^{1\dagger}_1 = {}^{I}\mathcal{A}^{1\dagger}_2 *_A {}^{II}\mathcal{A}^{1\dagger}_2 = \overset{01}{(-i)}\overset{01}{(+i)} = \overset{01}{[-i]} = |\psi_{oc}> \;.
$$

The right-hand side of the Clifford odd "basis vectors" contains two self adjoint "basis vectors" that are orthogonal to each other, as does the left-hand side of the two Clifford even "basis vectors".

Let us find the eigenvectors of the operator of handedness $\Gamma^{(2+1)} = i\gamma^0\gamma^1\gamma^2$. Since it is the Clifford odd object, its eigenvectors are the superpositions of the Clifford odd and Clifford even "basis vectors".

$$
\begin{aligned}
\Gamma^{(2+1)}\{\overset{01}{[-i]} \pm i\,\overset{01}{[-i]}\,\gamma^2\} &= \mp\{\overset{01}{[-i]} \pm i\,\overset{01}{[-i]}\,\gamma^2\}\,, \\
\Gamma^{(2+1)}\{\overset{01}{(+i)} \pm i\,\overset{01}{(+i)}\,\gamma^2\} &= \mp\{\overset{01}{(+i)} \pm i\,\overset{01}{(+i)}\,\gamma^2\}\,, \\
\Gamma^{(2+1)}\{\overset{01}{[+i]} \pm i\,\overset{01}{[+i]}\,\gamma^2\} &= \pm\{\overset{01}{[+i]} \pm i\,\overset{01}{[+i]}\,\gamma^2\}\,, \\
\Gamma^{(2+1)}\{\overset{01}{(-i)}\,\gamma^2 \pm i\,\overset{01}{(-i)}\} &= \pm\{\overset{01}{(-i)}\,\gamma^2 \pm i\,\overset{01}{(-i)}\}\,.
\end{aligned}
\tag{24}
$$

3.2.2. ***d* = (4 + 1)**

In $d=(4+1)$ there are 32 $(2^{d=5})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ for the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, as shown in Equation (A4).

Half of the Clifford odd and Clifford even "basis vectors" and their Hermitian conjugated partners can be taken from Equations (18)–(21); the other half follows by the application of S^{05} or $\tilde{S}^{05}$ on the first half.

$$
\begin{array}{rl}
d= & 4+1\\
 & \text{Clifford odd}\\
\hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,, & \hat{b}^{1\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{[+i]}\,\gamma^5\,,\; \hat{b}^{1\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,\gamma^5\,,\\
\hat{b}^{2\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,,\; \hat{b}^{2\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,, & \hat{b}^{2\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,\gamma^5\,,\; \hat{b}^{2\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,\gamma^5\,,\\
\hat{b}^{1}_1 = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1}_2 = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,, & \hat{b}^{1}_3 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,\gamma^5\,,\; \hat{b}^{1}_4 = \stackrel{03}{(-i)}\stackrel{12}{(-)}\,\gamma^5\,,\\
\hat{b}^{2}_1 = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,,\; \hat{b}^{2}_2 = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,, & \hat{b}^{2}_3 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,\gamma^5\,,\; \hat{b}^{2}_4 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,\gamma^5\,,\\
 & \text{Clifford even}\\
{}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,,\; {}^{I}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,, & {}^{I}\mathcal{A}^{1}_3 = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,\gamma^5\,,\; {}^{I}\mathcal{A}^{1}_4 = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,\gamma^5\,,\\
{}^{I}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(-i)}\stackrel{12}{(-i)}\,,\; {}^{I}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,, & {}^{I}\mathcal{A}^{2}_3 = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,\gamma^5\,,\; {}^{I}\mathcal{A}^{2}_4 = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,\gamma^5\,,\\
 & \\
{}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,,\; {}^{II}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,, & {}^{II}\mathcal{A}^{1\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{1\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,\gamma^5\,,\\
{}^{II}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,,\; {}^{II}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,, & {}^{II}\mathcal{A}^{2\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{2\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,\gamma^5\,.
\end{array}
\tag{25}
$$

It can be observed that the right-hand side of the Clifford odd "basis vectors" appears to have two mutually orthogonal groups, each with either self adjoint members or with the Hermitian conjugated partners within the same group.

The members of one group

$$
\hat{b}^{1\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{[+i]}\,\gamma^5\,,\; \hat{b}^{1\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,\gamma^5\,,\; \hat{b}^{2\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,\gamma^5\,,\; \hat{b}^{2\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,\gamma^5
$$

have the same properties, except for commutativity (they are, namely, the Clifford odd objects), as the group of Clifford even objects

$$
{}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,,\; {}^{II}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,,\; {}^{II}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,,\; {}^{II}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\;.
$$

The Clifford odd members of the group also have comparable properties

$$
\hat{b}^{1}_3 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,\gamma^5\,,\; \hat{b}^{1}_4 = \stackrel{03}{(-i)}\stackrel{12}{(-)}\,\gamma^5\,,\; \hat{b}^{2}_3 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,\gamma^5\,,\; \hat{b}^{2}_4 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,\gamma^5\,,
$$

as do the Clifford even members of the group

$$
{}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,,\; {}^{I}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,,\; {}^{I}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(-i)}\stackrel{12}{(-i)}\,,\; {}^{I}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\;.
$$

The members of both groups have Hermitian conjugated partners within the same group or are self adjoint.

On the other side, the members of the Clifford even group

$$
{}^{II}\mathcal{A}^{1\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{1\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{2\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{2\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,\gamma^5\,,
$$

have their Hermitian conjugated partners in a separate group

$$ {}^{I}\mathcal{A}^{1}_{3} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,\gamma^5 \quad {}^{I}\mathcal{A}^{1}_{4} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,\gamma^5\,,\, {}^{I}\mathcal{A}^{2}_{3} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,\gamma^5\,,\, {}^{I}\mathcal{A}^{2}_{4} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,\gamma^5\,, $$

just like the Clifford odd objects on the left-hand side

$$ \hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,,\; \hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,,\; \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,, $$

which have their Hermitian conjugated partners in a separate group

$$ \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,,\; \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,,\; \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\;. $$

The "basis vectors" of the right-hand side keep their oddness if they are partners of the Clifford odd "basis vectors" on the left-hand side, but they demonstrate the properties of Clifford even objects on the left-hand side.

The "basis vectors" of the right-hand side keep their evenness if they are partners of the Clifford even "basis vectors" on the left-hand side, but they demonstrate the properties of Clifford odd objects on the left-hand side.

After the algebraic application of, for example, ${}^{II}\mathcal{A}^{1\dagger}_{3}(= \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5$ on $\hat{b}^{1\dagger}_{4} = \stackrel{03}{(-i)}\stackrel{12}{(+)}$ γ^5, we are left with $\hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)}$.

The eigenvectors of the operator of handedness in $d = (4+1)$, $\Gamma^{(4+1)} = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^5$, are the superpositions of the Clifford odd and Clifford even "basis vectors", for example, $\Gamma^{(4+1)}(\hat{b}^{1\dagger}_{1}[= \stackrel{03}{(+i)}\stackrel{12}{[+]}] \pm {}^{II}\mathcal{A}^{1\dagger}_{3}[= \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5]) = \mp((\hat{b}^{1\dagger}_{1} \pm {}^{II}\mathcal{A}^{1\dagger}_{3})$.

We can conclude that, in odd dimensional spaces, neither the Clifford odd nor the Clifford even "basis vectors" have the properties that they demonstrate in even dimensional spaces, such as the properties that empower the Clifford odd "basis vectors" to describe the internal space of fermion fields and the Clifford even "basis vectors" to describe the internal space of the corresponding gauge fields. After enlarging the "basis vectors" in a tensor product, $*_T$, with the basis in ordinary space [15], the corresponding creation and annihilation operators manifest the properties required by the postulates for the second quantized field, either fermion or boson.

In odd dimensional spaces, half of the Clifford odd "basis vectors" demonstrate the properties of the Clifford even "basis vectors" and half of the Clifford even "basis vectors" demonstrate the properties of the Clifford odd "basis vectors". Arbitrary Lorentz transformations transform the left-hand side into the right side and vice versa.

These are the properties of the internal spaces of the ghost scalar fields, used in the quantum field theory to make the contributions of the Feynman diagrams finite.

4. Discussion

This article briefly repeated the properties of the fermion and boson fields in even dimensional spaces [3,15] and discussed the properties of the fermion and boson fields in *odd* [16] dimensional spaces if their internal spaces are described by the superpositions of the Clifford odd products of γ^a (for fermions) and by the superpositions of the Clifford even products of γ^a (for bosons).

The discussion on the properties of the fermion and boson gauge fields in odd dimensional spaces is the main new contribution of this article.

The recognition that the internal spaces of both fermion and boson field can be described by the Clifford algebra objects offers a new step forward in the understanding of the laws of nature. It is a new step beyond the *standard model* of elementary fields, and it is a new step for cosmology, since this method of describing the internal spaces of fields not only explains the second quantization postulates for all fermion and boson fields, offering, in addition, the understanding of the assumption of Fadeev and Popov to use ghosts to ensure the final solutions are obtained when using Feynman diagrams [52], but also because the relations among fermions and bosons described by the Clifford objects influence the choice of starting action, as shown in Equation (1), which unifies all of the boson gauge fields:

In the starting action in $d = (13+1)$-dimensional space, as shown in Equation (1) and explained in $d = (3+1)$ for all of the properties of the quarks and leptons and antiquarks and antileptons of the corresponding vector gauge fields and the scalar fields, it is assumed in the *standard model* before the electroweak phase transition ([3] and the references therein) that the covariant momenta, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$, includes two spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$.

The analysis of the properties of admissible even Clifford "basis vectors" shows that there are two kinds of even Clifford "basis vectors": one kind transforms the Clifford odd "basis vectors" of a given family into other members of the same family, while the family quantum number remains unchanged; the second kind changes a family member of a particular family into the same family member of another family [15].

Let us repeat what we have learned in Sections 2.2 and 3.2 and Appendixes D and E about the properties of the Clifford even and Clifford odd objects in odd dimensional spaces:

In odd dimensional spaces, neither Clifford odd nor Clifford even "basis vectors" have the properties that they demonstrate in even dimensional spaces. These are the properties that empower the Clifford odd "basis vectors" to describe the internal space of fermion fields and the Clifford even "basis vectors" to describe the internal space of the corresponding gauge fields.

In odd dimensional spaces, half of the Clifford odd "basis vectors", although anticommuting, demonstrate the properties of the Clifford even "basis vectors" in even dimensional spaces, and half of the Clifford even "basis vectors", although commuting, demonstrate properties of the Clifford odd "basis vectors" in even dimensional spaces.

These "basis vectors" obviously resemble the properties of the internal spaces of the ghost scalar fields, used in quantum field theory to make the contributions of the Feynman diagrams finite (Arbitrary Lorentz transformations in odd dimensional spaces transform the left-hand sides of Equations (14), (15), (23) and (25) into the right-hand sides and vice versa.).

Further studies on the properties of the Clifford even "basis vectors" in even dimensional spaces are required to realise the benefits of this description (In particular, the contribution of the replacement of vielbeins, $f^{a}{}_{\alpha}$, and the two kinds of spin connection fields, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$) in the covariant derivative $p_{0\alpha}$, ($p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$,) with ($p_{0\alpha} = p_\alpha - \sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger I}_{f} \mathcal{C}^{m}_{f\alpha} - \sum_{mf} {}^{II}\hat{\mathcal{A}}^{m\dagger\, I}_{f} \mathcal{C}^{m}_{f\alpha}$.) in a simple starting action in $d = 2(2n+1), n \geq 7$, as shown in Equation (1), must be understood. Additionally, the properties of the Clifford odd and the Clifford even "basis vectors" in odd dimensional spaces need further study. Both recognitions are new and have not yet been studied enough.

We again observe that the simple starting actions, as presented in Equation (1), assuming massless fermion and boson fields, interact with only gravity in $d = (13+1)$-dimensional space, offering an explanation for *all assumptions* of the *standard model*, such as the *families* of fermions and the vector gauge fields and scalar gauge fields (higgs and Yukawa coupling included). They also offer explanations for the appearance of dark matter and matter/antimatter asymmetry. Several other observations [1,3–5,11–14,51,53–55] indicate that the postulation for the second quantization of the fermion and boson fields is too promising not to be presented to all who are looking for the next step beyond the *standard model*.

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Appendix A. Some Useful Formulas

This appendix contains the helpful relations needed for the reader of this paper. For more detailed explanations and for proofs, the reader is kindly asked to read [3] and the references therein.

For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}}\gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for } d \text{ even}\,,\\ (i)^{\frac{d-1}{2}}\,, & \text{for } d \text{ odd}\,, \end{cases} \tag{A1}$$

The Clifford objects γ^as and $\tilde{\gamma}^a$s fulfill the relations

$$\begin{aligned} \{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\ (\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,. \end{aligned} \tag{A2}$$

In the paper, the signature $\eta^{aa} = diag(1,-1,-1,\ldots,-1)$ is used.

The choice of Cartan subalgebra members for d even is

$$\begin{aligned} &\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\,d}\,,\\ &S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\,d}\,,\\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\,d}\,,\\ &\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,, \end{aligned} \tag{A3}$$

and for d odd, it is

$$\begin{aligned} &\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-2\,d-1}\,,\\ &S^{03}, S^{12}, S^{56}, \cdots, S^{d-2\,d-1}\,,\\ &\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-2\,d-1}\,,\\ &\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,. \end{aligned} \tag{A4}$$

Nilpotents and projectors are defined as follows: [1,12]

$$\stackrel{ab}{(k)}: \;=\; \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \tag{A5}$$

with $k^2 = \eta^{aa}\eta^{bb}$.

Taking Equation (A14) into account and assuming

$$\{\tilde{\gamma}^a B \;=\; (-)^B\, i\, B\gamma^a\}\,|\psi_{oc}>\,, \tag{A6}$$

with $(-)^B = -1$, if B is (a function of) an odd product of γ^as, otherwise $(-)^B = 1$, $|\psi_{oc}>$ is defined in Equation (A8), the eigenvalues of the Cartan subalgebra operators are

$$\begin{aligned} S^{ab}\,\stackrel{ab}{(k)} &= \frac{k}{2}\,\stackrel{ab}{(k)}\,, &\quad \tilde{S}^{ab}\,\stackrel{ab}{(k)} &= \frac{k}{2}\,\stackrel{ab}{(k)}\,,\\ S^{ab}\,\stackrel{ab}{[k]} &= \frac{k}{2}\,\stackrel{ab}{[k]}\,, &\quad \tilde{S}^{ab}\,\stackrel{ab}{[k]} &= -\frac{k}{2}\,\stackrel{ab}{[k]}\,. \end{aligned} \tag{A7}$$

The vacuum state for the Clifford odd "basis vectors", $|\psi_{oc}>$, is defined as

$$|\psi_{oc}>=\sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m\dagger}_{f}\,|\,1\,>\,. \tag{A8}$$

Taking Equation (A14) into account, it follows that

$$\begin{aligned}
\gamma^a \overset{ab}{(k)} &= \eta^{aa}\overset{ab}{[-k]}, \quad \gamma^b \overset{ab}{(k)}= -ik \overset{ab}{[-k]}, \quad \gamma^a \overset{ab}{[k]}=\overset{ab}{(-k)}, \quad \gamma^b \overset{ab}{[k]}= -ik\eta^{aa} \overset{ab}{(-k)}\,,\\
\tilde{\gamma}^a \overset{ab}{(k)} &= -i\eta^{aa}\overset{ab}{[k]}, \quad \tilde{\gamma}^b \overset{ab}{(k)}= -k \overset{ab}{[k]}, \quad \tilde{\gamma}^a \overset{ab}{[k]}= i\overset{ab}{(k)}, \quad \tilde{\gamma}^b \overset{ab}{[k]}= -k\eta^{aa} \overset{ab}{(k)}\,,\\
\overset{ab}{(k)}{}^{\dagger} &= \eta^{aa}\overset{ab}{(-k)}\,, \quad (\overset{ab}{(k)})^2=0\,, \quad \overset{ab}{(k)}\overset{ab}{(-k)}=\eta^{aa}\overset{ab}{[k]}\,,\\
\overset{ab}{[k]}{}^{\dagger} &= \overset{ab}{[k]}\,, \quad (\overset{ab}{[k]})^2=\overset{ab}{[k]}\,, \quad \overset{ab}{[k]}\overset{ab}{[-k]}=0\,,\\
\overset{ab}{(k)}\overset{ab}{[k]} &= 0\,, \quad \overset{ab}{[k]}\overset{ab}{(k)}=\overset{ab}{(k)}\,, \quad \overset{ab}{(k)}\overset{ab}{[-k]}=\overset{ab}{(k)}\,, \quad \overset{ab}{[k]}\overset{ab}{(-k)}=0\,,\\
\overset{ab}{(\tilde{k})}{}^{\dagger} &= \eta^{aa}\overset{ab}{(\widetilde{-k})}\,, \quad (\overset{ab}{(\tilde{k})})^2=0\,, \quad \overset{ab}{(\tilde{k})}\overset{ab}{(\widetilde{-k})}=\eta^{aa}\overset{ab}{[\tilde{k}]}\,,\\
\overset{ab}{[\tilde{k}]}{}^{\dagger} &= \overset{ab}{[\tilde{k}]}\,, \quad (\overset{ab}{[\tilde{k}]})^2=\overset{ab}{[\tilde{k}]}\,, \quad \overset{ab}{[\tilde{k}]}\overset{ab}{[\widetilde{-k}]}=0\,,\\
\overset{ab}{(\tilde{k})}\overset{ab}{[\tilde{k}]} &= 0\,, \quad \overset{ab}{[\tilde{k}]}\overset{ab}{(\tilde{k})}=\overset{ab}{(\tilde{k})}\,, \quad \overset{ab}{(\tilde{k})}\overset{ab}{[\widetilde{-k}]}=\overset{ab}{(\tilde{k})}\,, \quad \overset{ab}{[\tilde{k}]}\overset{ab}{(\widetilde{-k})}=0\,,\\
\overset{ab}{\widetilde{(-k)}}\overset{ab}{(k)} &= -i\eta^{aa}\overset{ab}{[k]}\,, \quad \overset{ab}{\widetilde{[k]}}\overset{ab}{(k)}=\overset{ab}{(k)}\,, \quad \overset{ab}{\widetilde{(k)}}\overset{ab}{[k]}=i\overset{ab}{(k)}\,, \quad \overset{ab}{\widetilde{[-k]}}\overset{ab}{[k]}=\overset{ab}{[k]}\,,\\
\overset{ab}{\widetilde{(k)}}\overset{ab}{(k)} &= \mathbf{0}\,, \quad \overset{ab}{\widetilde{[-k]}}\overset{ab}{(k)}=0\,, \quad \overset{ab}{\widetilde{(k)}}\overset{ab}{[-k]}=0\,, \quad \overset{ab}{\widetilde{[k]}}\overset{ab}{[k]}=0\,.
\end{aligned} \tag{A9}$$

One can further find that

$$\begin{aligned}
S^{ac}\overset{ab}{(k)}\overset{cd}{(k)} &= -\tfrac{i}{2}\eta^{aa}\eta^{cc}\overset{ab}{[-k]}\overset{cd}{[-k]}\,, \quad S^{ac}\overset{ab}{[k]}\overset{cd}{[k]}=\tfrac{i}{2}\overset{ab}{(-k)}\overset{cd}{(-k)}\,,\\
S^{ac}\overset{ab}{(k)}\overset{cd}{[k]} &= -\tfrac{i}{2}\eta^{aa}\overset{ab}{[-k]}\overset{cd}{(-k)}\,, \quad S^{ac}\overset{ab}{[k]}\overset{cd}{(k)}=\tfrac{i}{2}\eta^{cc}\overset{ab}{(-k)}\overset{cd}{[-k]}\,.
\end{aligned} \tag{A10}$$

Appendix B. Grassmann and Clifford Algebras

This part of the appendix presents a short overview of Section 3.2 of Ref. [3].

In Grassmann d-dimensional space, there are d anticommuting operators θ^a and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$,

$$\{\theta^a,\theta^b\}_+=0\,, \qquad \{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+=0\,, \quad \{\theta^a,\frac{\partial}{\partial\theta^b}\}_+=\delta ab\,,(a,b)=(0,1,2,3,5,\cdots,d)\,. \tag{A11}$$

θ^a and $\frac{\partial}{\partial\theta_a}$ are, up to the sign, Hermitian conjugated to each other.

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial\theta_a})^\dagger=\eta^{aa}\theta^a\,, \tag{A12}$$

with $\eta^{ab}=diag\{1,-1,-1,\cdots,-1\}$.

The identity is the self adjoint member of the algebra.

The operators θ^a offer 2^d superpositions of the products of θ^a, the Hermitian conjugated partners of which are the corresponding superpositions of the products of $\frac{\partial}{\partial\theta_a}$.

One can define two kinds of Clifford algebra elements—γ^a and $\tilde{\gamma}^a$—which are the superpositions of θ^a and their conjugate momenta $p^{\theta a}=i\,\frac{\partial}{\partial\theta_a}$ [1].

$$\begin{aligned}\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,, \\ \theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,\end{aligned} \tag{A13}$$

offering together $2\cdot 2^d$ operators which are the superpositions of the products of either γ^a, 2^d or $\tilde{\gamma}^a$, 2^d.

Taking Equations (A12) and (A13) into account, it is easy to prove that they form two anticommuting Clifford subalgebras, $\{\gamma^a and \tilde{\gamma}^b\}_+ = 0$, as shown in Refs. ([3] and the references therein)

$$\begin{aligned}\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,, \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,, \\ (\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,.\end{aligned} \tag{A14}$$

In each of the two subalgebras, half of the products of the operators (γ^a or $\tilde{\gamma}^a$) have an odd number of operators, and the rest have an even number of operators. The superposition of an odd number of operators can be arranged to describe the internal space of fermions [1,3], and the superposition of an even number of operators can be arranged to describe the internal space of bosons, the gauge fields of the corresponding fermions [15].

In even dimensional spaces, the superposition of an odd number of operators, either γ^a or $\tilde{\gamma}^a$, forms $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding generators of the Lorentz transformations (either S^{ab} or $\tilde{S}^{ab}$) with $2^{\frac{d}{2}-1}$ members each. Their Hermitian conjugated partners appear in a different group.

The superposition of an even number of operators of either γ^a or $\tilde{\gamma}^a$ forms two orthogonal groups with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, with the Hermitian conjugated partners appearing in the same group.

In odd dimensional spaces, the superposition of an odd number of operators either γ^a or $\tilde{\gamma}^a$, forms $2\times 2^{\frac{d-1}{2}-1}$ irreducible representations with $2\times 2^{\frac{d-1}{2}-1}$ members each. Their Hermitian conjugated partners appear in a different group.

The superposition of an even number of operators of either γ^a or $\tilde{\gamma}^a$ each forms two groups of $2\times 2^{\frac{d-1}{2}-1} \times 2\times 2^{\frac{d-1}{2}-1}$ members, which are no longer orthogonal.

Two Clifford spaces of γ^a and $\tilde{\gamma}^a$ can be reduced to only one by the assumption (A15). Let $\tilde{\gamma}^a$ operate on γ^a as follows: [1,11]

$$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}\, |\psi_{oc}>\,, \tag{A15}$$

where $(-)^B = -1$, if B is (a function of) an odd product of γ^a; otherwise, $(-)^B = 1$, $|\psi_{oc}>$ is defined in Equation (8).

After this postulate, the vector space of $\tilde{\gamma}^a$s is “frozen out”. No vector space of $\tilde{\gamma}^a$s needs to be taken into account any longer, in agreement with the observed properties of fermions.

One can check that all relations of Equation (A14) remain valid ([3], Appendix I , Statement 3, 3a, 3b) after the postulate of Equation (A15), and $\tilde{S}^{ab}$ is used to determine the family quantum numbers of the irreducible representations of S^{ab}.

Appendix C. Dirac γ^a, Spin States and Clifford Even and Odd “Basis Vectors”

This appendix relates the algebra of the products of Dirac γ^a, $a = (0,1,2,3)$, and the method of describing states of fermions and bosons with the Clifford algebra.

The Dirac algebra of γ^a with the commutation relations of Equation (A14) offers $2^{(3+1)} = 16$ products of γ^a. Half of them are odd products of γ^a, and the other half are even products of γ^a.

$$\begin{aligned}
&\text{odd products of} \gamma^a\,,\\
&\gamma^0\,,\gamma^1\,,\gamma^2\,,\gamma^3\,,\\
&\gamma^0\gamma^1\gamma^2\,,\gamma^0\gamma^1\gamma^3\,,\gamma^0\gamma^2\gamma^3\,,\gamma^1\gamma^2\gamma^3\,,\\
&\text{even products of} \gamma^a\,,\\
&1\,,\gamma^0\gamma^1\,,\gamma^0\gamma^2\,,\gamma^0\gamma^3\,,\gamma^1\gamma^2\,,\gamma^1\gamma^3\,,\gamma^2\gamma^3\,,\\
&\gamma^0\gamma^1\gamma^2\gamma^3\,.
\end{aligned} \tag{A16}$$

Let us arrange these 16 elements into four Clifford odd "basis vectors" which are eigenstates of S^{03} and S^{12} (or $\Gamma^{(3+1)}$) made up of the members presented in Equation (A16). The corresponding Dirac vectors are presented as $\binom{1}{0}$ and $\binom{0}{1}$.

$$\begin{aligned}
&\hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,, \quad &&\hat{b}^{1\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,, \quad &&\begin{pmatrix} 1 \\ 0 \end{pmatrix}_R\,, \quad S^{03} = +\frac{i}{2}\,, S^{12} = +\frac{1}{2}\,,\\
&\hat{b}^{2\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,, \quad &&\hat{b}^{2\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,. \quad &&\begin{pmatrix} 0 \\ 1 \end{pmatrix}_R\,, \quad S^{03} = -\frac{i}{2}\,, S^{12} = -\frac{1}{2}\,,
\end{aligned}$$

The Hermitian conjugated partners of the four "basis vectors" and the Dirac Hermitian conjugated vectors presented as $\binom{1}{0}^\dagger (= (1\,0)$ and $\binom{0}{1}^\dagger (= (0\,1)$, are

$$\begin{aligned}
&\hat{b}^{1}_1 = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,, \quad &&\hat{b}^{1}_2 = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,, \quad &&\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger_R\,, \quad \tilde{S}^{03} = -\frac{i}{2}\,, \tilde{S}^{12} = -\frac{1}{2}\,,\,,\\
&\hat{b}^{2}_1 = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,, \quad &&\hat{b}^{2}_2 = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,, \quad &&\begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger_R\,, \quad \tilde{S}^{03} = +\frac{i}{2}\,, \tilde{S}^{12} = +\frac{1}{2}\,,
\end{aligned}$$

The "basis vectors" $\hat{b}^{m\dagger}_f$ in the first column represent the family with $f = 1$. The two members carry $\tilde{S}^{03} = \frac{i}{2}$ and $\tilde{S}^{12} = -\frac{1}{2}$ (according to Equation (A9)). There are $(S^{01}, S^{02}, S^{31}, S^{32})$, which rotate the two members of the first family among themselves. The second irreducible representation has $\tilde{S}^{03} = -\frac{i}{2}$ and $\tilde{S}^{12} = \frac{1}{2}$.

The Dirac vectors do not pay attention to (in the *spin-charge-family* theory existing) irreducible representations which, in the *spin-charge-family* case, are equipped with the family quantum numbers.

The four "basis vectors", together with their Hermitian conjugated partners, exhaust the odd products of γ^a, as presented in Equation (A16).

The even products of γ^a can be arranged into two orthogonal groups of Clifford even "basis vectors", as presented in Equations (20) and (21). The members of each group are either self adjoint or have their Hermitian conjugated partners within the same group.

Dirac vectors do not pay attention to either irreducible representations or to the Clifford even "basis vectors".

Let me add that, in even dimensional spaces, the Clifford odd "basis vectors" describe "basis vectors" of only one handedness, either R or L, depending on the definition of handedness. However, we need "basis vectors" of both types of handedness if we want to describe quarks and leptons and antiquarks and antileptons. However, in any even dimensional subspace of an even dimensional space with $d > (3+1)$, there are always both types of handedness.

To see this, let us look at Table A1, presented in Appendix D. The whole one family representation (with 64 members) is right-handed. The right-handedness in $d = (3+1)$ carries the four "basis vectors" appearing in the first four lines in this table, representing the internal space of u and d quarks with the spin $S^{12} = \pm\frac{1}{2}$ and with the color charge $(\tau^{33} = \frac{1}{2}, \tau^{38} = \frac{1}{2\sqrt{3}})$. However, the Clifford odd "basis vectors", presented in the next four lines of this table, describe the "basis vectors" again of the u and d quarks of spin $S^{12} = \pm\frac{1}{2}$ and color charge ($\tau^{33} = \frac{1}{2}, \tau^{38} = \frac{1}{2\sqrt{3}}$), but for left-handedness. They follow by rotating their right-handed partners with, for example, S^{07}.

Appendix D. One Family Representation of Clifford Odd "Basis Vectors" in $d = (13 + 1)$

This appendix presents an overview of Appendix A of Ref. [15]. Short comments on the corresponding gauge vector and scalar fields and fermion and boson representations in $d = (14 + 1)$-dimensional space are also included.

In even dimensional space $d = (13 + 1)$, one irreducible representation of the Clifford odd "basis vectors", analyzed from the point of view of the subgroups $SO(3,1) \times SO(4)$ (included in $SO(7,1)$) and $SO(7,1) \times SO(6)$ (included in $SO(13,1)$, while $SO(6)$ breaks into $SU(3) \times U(1)$), contains the Clifford odd "basis vectors" describing the internal spaces of quarks and leptons and antiquarks and antileptons with the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ groups, $Y = \tau^{23} + \tau^{4}$, one irreducible representation includes the right-handed neutrinos and the left-handed antineutrinos, which are not in the *standard model* scheme.

The Clifford even "basis vectors", analyzed with respect to the same subgroups, offer a description of the internal spaces of the corresponding vector and scalar fields, appearing in the *standard model* before the electroweak break [15].

For an overview of the properties of the vector and scalar gauge fields in the *spin-charge-family* theory, the reader is invited to read Refs. ([3,14] and the references therein). The vector gauge fields, expressed as the superpositions of spin connections and vielbeins, carrying the space index $m = (0,1,2,3)$, manifest the properties of the observed boson fields. The scalar gauge fields, which cause the electroweak break, carry the space index $s = (7,8)$ and determine the symmetry of the mass matrices of quarks and leptons.

In Table A1, one can check the quantum numbers of the Clifford odd "basis vectors" representing quarks and leptons and antiquarks and antileptons if it is taken into account that all nilpotents and projectors are eigenvectors of one of the Cartan subalgebra members, $(S^{03}, S^{12}, S^{56}, \dots, S^{13\,14})$ with the eigenvalues $\pm\frac{i}{2}$ for $\overset{ab}{(\pm i)}$ and $\overset{ab}{[\pm i]}$, and with the eigenvalues $\pm\frac{1}{2}$ for $\overset{ab}{(\pm 1)}$ and $\overset{ab}{[\pm 1]}$.

Taking into account that the third component of the weak charge is $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, the second $SU(2)$ charge is $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, the color charge is $\tau^{33} = \frac{1}{2}(S^{9\,10} - S^{11\,12})$ and $\tau^{38} = \frac{1}{2\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})$, the "fermion charge" is $\tau^{4} = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, $Y = \tau^{23} + \tau^{4}$, and $Q = Y + \tau^{13}$, one can reproduce all of the quantum numbers of the quarks, leptons, antiquarks, and antileptons. It can be observed that the $SO(7,1)$ part is the same for quarks and leptons and the same for antiquarks and antileptons. Quarks are distinguished from leptons only in terms of their color and "fermion" quantum numbers and antiquarks are distinguished from antileptons only in terms of their anticolor and "antifermion" quantum numbers.

In odd dimensional space $d = (14 + 1)$ the eigenstates of handedness are the superposition of one irreducible representation of $SO(13,1)$, presented in Table A1, and the one obtained if on each "basis vector" appearing in $SO(13,1)$ the operator $S^{0\,(14+1)}$ applies. The comment on differences in odd dimensional space with respect to even dimensional space are discussed in Section 3.2.

Let me point out that, in addition to the electroweak break of the *standard model*, the break at $\geq 10^{16}$ GeV is needed.

This break is caused by the condensate of the two right-handed neutrinos, Ref. ([3]), Table 6, which interact with all of the scalar and vector gauge fields, except for the weak fields, $U(1)$, $SU(3)$ and the gravitational field in $d = (3 + 1)$, leaving these gauge fields massless up to the electroweak break. The scalar fields only leave the electromagnetic, color, and gravitational fields massless.

The theory predicts two groups of four families. So far, it is known that three of the four families contribute to the lower group. The theory predicts the symmetry of both groups to be $SU(2) \times SU(2) \times U(1)$, Ref. ([3], Section 7.3) , which enables the mixing matrices of quarks and leptons to be calculated for the 3×3 sub matrix of the 4×4 unitary matrix. No sterile neutrinos are needed, and no symmetries of the mass matrices must be guessed. In the literature, many authors have tried to reproduce mass matrices and measured mixing matrices for quarks and leptons [56–76].

The stable parts of the upper four families predicted by the *spin-charge-family* theory are candidates for dark matter, as discussed in Ref. [3]. In the literature, there are several works that suggest candidates for dark matter and also for matter/antimatter asymmetry [76–78].

Table A1. The left-handed ($\Gamma^{(13,1)} = -1$ [3]) irreducible representation of one family of spinors—the products of the odd number of nilpotents and projectors, which are eigenvectors of the Cartan subalgebra of the $SO(13,1)$ group [3,11–13], resulting in the subgroup $SO(7,1)$ of the color charged quarks and antiquarks and the colourless leptons and antileptons—is presented. It contains the left-handed ($\Gamma^{(3,1)} = -1$) weakly ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, and the $SU(2)_{II}$ chargeless ($\tau^{23} = 0$ quarks and leptons and the right-handed ($\Gamma^{(3,1)} = 1$) weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). Quarks are distinguished from leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colors ($c^i = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})$, carrying the "fermion charge" ($\tau^4 = \frac{1}{6}$. The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). The same multiplet contains also the left-handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right-handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks are distinguished from antileptons again only in the $SU(3) \times U(1)$ part: Antiquarks are antitriplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anticolourless antileptons carry the "fermion charge" ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hypercharge, and the electromagnetic charge is $Q = (\tau^{13} + Y)$.

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (Anti)Quarks and (Anti)leptons									
1	u_R^{c1}	$\stackrel{03}{(+i)}\,\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\,\stackrel{12}{(-)} \mid \stackrel{56}{[+]}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\,\stackrel{12}{[+]} \mid \stackrel{56}{(-)}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	d_R^{c1}	$\stackrel{03}{[-i]}\,\stackrel{12}{(-)} \mid \stackrel{56}{(-)}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\,\stackrel{12}{[+]} \mid \stackrel{56}{(-)}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	d_L^{c1}	$-\stackrel{03}{(+i)}\,\stackrel{12}{(-)} \mid \stackrel{56}{(-)}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	u_L^{c1}	$-\stackrel{03}{[-i]}\,\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\,\stackrel{12}{(-)} \mid \stackrel{56}{[+]}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	u_R^{c2}	$\stackrel{03}{(+i)}\,\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	u_R^{c2}	$\stackrel{03}{[-i]}\,\stackrel{12}{(-)} \mid \stackrel{56}{[+]}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	d_R^{c2}	$\stackrel{03}{(+i)}\,\stackrel{12}{[+]} \mid \stackrel{56}{(-)}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	d_R^{c2}	$\stackrel{03}{[-i]}\,\stackrel{12}{(-)} \mid \stackrel{56}{(-)}\,\stackrel{78}{[-]} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	d_L^{c2}	$\stackrel{03}{[-i]}\,\stackrel{12}{[+]} \mid \stackrel{56}{(-)}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	d_L^{c2}	$-\stackrel{03}{(+i)}\,\stackrel{12}{(-)} \mid \stackrel{56}{(-)}\,\stackrel{78}{(+)} \parallel \stackrel{9\,10}{[-]}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$

Table A1. *Cont.*

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)}=(-1)\,1$, $\Gamma^{(6)}=(1)\,-1$ of (Anti)Quarks and (Anti)leptons									
15	u_L^{c2}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	u_L^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	u_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	u_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	d_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	d_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	d_L^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	d_L^{c3}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	u_L^{c3}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	u_L^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}_R^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$

Table A1. *Cont.*

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
		(Anti)octet, $\mathbf{\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1}$ **of (Anti)Quarks and (Anti)leptons**									
47	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}_R^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\bar{e}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\bar{e}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\bar{\nu}_L$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\bar{\nu}_L$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\bar{\nu}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\bar{\nu}_R$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\bar{e}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\bar{e}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Appendix E. Second Quantization with Clifford Algebra

This appendix discusses the second quantization using the Clifford odd "basis vectors" to describe the internal space of fermion fields. It is a short overview of Section 3.3.2 of Ref. [3].

The second quantization of boson fields, using the Clifford even "basis vectors" to describe their internal space, can be found in Section 3 in [15].

Creation and annihilation operators are defined as the tensor product, $*_T$, of each "basis vector" with the basis in momentum (or coordinate) ordinary space, which infinitely generate many creation operators and, when applied on the vacuum state, infinitely many states for each "basis vector".

The Clifford odd "basis vectors" are anticommuting objects that transfer their anticommutativity to the creation operators and their Hermitian conjugated partners annihilation operators.

The Dirac vectors, presented in Appendix C, do not anticommute. Correspondingly, the anticommutativity of the creation operators must be postulated.

Let us introduce the momentum part of the creation and annihilation operators in the method that is briefly reviewed in Ref. [3], Section 3.3, and Appendix J.

$$
\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p>\,, \quad <\vec{p}\,| = <0_p\,|\,\hat{b}_{\vec{p}}\,, \\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}')\,, \quad \hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p}'} = \delta(\vec{p}-\vec{p}')\,,
\end{aligned}
\tag{A17}
$$

$$\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) \,=\, \sum_{m} c^{sm}{}_{f}(\vec{p})\, \hat{b}^{\dagger}_{\vec{p}} *_{T} \hat{b}^{m\dagger}_{f}\,. \tag{A18}$$

The vacuum state for fermions includes both spaces: internal $|\psi_{oc}>$ and momentum space $|0_{\vec{p}}>$. The coefficient $c^{sm}{}_{f}(\vec{p})$ is generally dependent on all powers of p^i.

When the kinematics of the right-handed weak chargeless u-quark of the color charge $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, with $\tau^4 = \frac{1}{6}$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$, is determined by the momenta in $d = (3+1)$, the Weyl equation applied on the u quark

$$\gamma^0 \gamma^a p_a \, \hat{\mathbf{u}}^{c1 s=1\dagger}_{Rf=1}(\vec{p})|\psi_{oc} > *_{T}|0_{\vec{p}} >= 0\,,$$

connects the creation operators with spin up and down, that is, the first two lines in Table A1, as follows

$$\begin{aligned}
&\text{Clifford odd creation operators in d} = (13+1) \\
p^0 &= |p^0|\,,\;\; c_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})\,,\;\; \Gamma^{(3+1)} = 1\,, \\
\Big(\hat{\mathbf{u}}^{c1 s=1\dagger}_{Rf=1}(\vec{p}) &= \beta \left(\stackrel{03}{(+i)}\stackrel{12}{[+]} + \frac{p^1 + i p^2}{p^0 + p^3} \stackrel{03}{[-i]}\stackrel{12}{(-)} \right) \cdot \stackrel{56}{[+]}\stackrel{78}{(+)} || \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]} \Big) \cdot \\
& \quad e^{-i(p^0 x^0 - \vec{p}\cdot\vec{x})}\,, \\
\Big(\hat{\mathbf{u}}^{c1 s=2\dagger}_{Rf=1}(\vec{p}) &= \beta^{*} \left(\stackrel{03}{[-i]}\stackrel{12}{(-)} - \frac{p^1 - i p^2}{p^0 + p^3} \stackrel{03}{(+i)}\stackrel{12}{[+]} \right) \cdot \stackrel{56}{[+]}\stackrel{78}{(+)} || \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]} \Big) \cdot \\
& \quad e^{-i(p^0 x^0 + \vec{p}\cdot\vec{x})}\,.
\end{aligned} \tag{A19}$$

The Hilbert space of the creation operators, as shown in Equation (A18), consists of any number of tensor products, $*_{T_H}$, for all possible creation operators

$$\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p'}) \cdots *_{T_H} \hat{\mathbf{b}}^{s''\dagger}_{f''}(\vec{p''}) \cdots *_{T_H} \cdots,$$

with a finite number of different (s, f) — $(2^{\frac{d}{2}-1})^2$—and continuous momentum $\vec{p}$ for each $\hat{b}^{s\dagger}_{f}$ ([3], Section 5).

In the case of the Clifford even "basis vectors", the creation operator for a free massless boson field, suggested in Ref. [15], can be written as

$${}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{s}\dagger}_{\mathbf{ff}}(\vec{p}) \,=\, \sum_{mf} \hat{b}^{\dagger}_{\vec{p}} *_{T} \mathcal{C}^{ms}{}_{f\alpha}(\vec{p})\; {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,. \tag{A20}$$

This suggestion needs further study.

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