symmetry

MDPI

Article

One-Dimensional Quaternion Fourier Transform with Application to Probability Theory

Wahyuni Ekasasmita [1,2], Mawardi Bahri [1,*], Nasrullah Bachtiar [1], Amran Rahim [1] and Muhammad Nur [1]

[1] Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia; wahyuni.ekasasmita@ith.ac.id (W.E.); nasrullahmipa.013@gmail.com (N.B); amran@science.unhas.ac.id (A.R.); muhammadnur@unhas.ac.id (M.N.)
[2] Department of Mathematics, Institut Teknologi Bacharuddin Jusuf Habibie, Parepare 91121, Indonesia
* Correspondence: mawardi.bahri@unhas.ac.id

Abstract: The Fourier transform occupies a central place in applied mathematics, statistics, computer sciences, and engineering. In this work, we introduce the one-dimensional quaternion Fourier transform, which is a generalization of the Fourier transform. We derive the conjugate symmetry of the one-dimensional quaternion Fourier transform for a real signal. We also collect other properties, such as the derivative and Parseval's formula. We finally study the application of this transformation in probability theory.

Keywords: quaternion Fourier transform; convolution; probability theory; quaternion characteristic function

check for updates

Citation: Ekasasmita, W.; Bahri, M.; Bachtiar, N; Rahim, A.; Nur, M. One-Dimensional Quaternion Fourier Transform with Application to Probability Theory. *Symmetry* **2023**, *15*, 815. https://doi.org/10.3390/sym15040815

Academic Editor: Charles F. Dunkl

Received: 28 February 2023
Revised: 17 March 2023
Accepted: 20 March 2023
Published: 28 March 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The classical Fourier transform is an indispensable tool in signal and image processing (see, e.g., [1,2]). On the one side, the quaternion Fourier transform (QFT) (for e.g., [3–9]) as a direct extension of the classical Fourier transform (FT) in the quaternion setting is also an indispensable tool for image processing for quaternion signals. A number of essential properties of the QFT have been demonstrated such as convolution, correlation, energy conservation, and inequalities. These properties are modifications of the corresponding properties of the FT. In [10], a general form of the QFT definition was discussed and the main properties and an application of the proposed transformation were also presented in detail. Moreover, many the mathematical problems are formulated in the language of quaternion algebras, such as the quaternion linear system [11] and linear quaternion differential equations [12].

It is well-nown that the FT plays crucial roles in probability theory. It is related to the characteristic function of any real-valued random variable used to compute the moment and the distribution function. Although the one-dimensional quaternion Fourier transform has been reported in [13–15] and the authors of [16] utilized it to construct the one-dimensional quaternion linear canonical transform, we have not yet come across its use in probability theory. Therefore, in this study, we first introduce the one-dimensional quaternion Fourier transform (1DQFT) and state some of its main properties. We develop its application in probability theory. In particular, we define the characteristic function and expected value in the quaternion setting. Distinct from the characteristic function and expected value in the real case, the quaternion characteristic function and expected value have four components. This fact causes several properties of the classical characteristic function to be modified in quaternion domains. We also study the relationship between the quaternion characteristic function and the 1DQFT. We apply this relation to obtain the moments and variance in the framework of a quaternion algebra. The results obtained play an important role in the development of probability theory in the context of a quaternion algebra [17,18].

The remainder of this paper is organized as follows. In Section 2, we briefly review the basic knowledge of quaternion used in the next section. In Section 3, we introduce some useful properties of the 1DQFT such as its convolution and correlation. An application of the properties of the 1DQFT is presented to obtain the inequality in that section. In Section 4, we present the application of the 1DQFT in probability theory. Some conclusions are drawn in Section 5.

2. Notations

Quaternions are a direct extension of complex numbers, which are associative but noncommutative over real numbers $\mathbb{R}$. Every element of a quaternion $\mathbb{H}$ can be written in the following form [19]:

$$\mathbb{H} = \{r = r^a + \mathbf{i}r^b + \mathbf{j}r^c + \mathbf{k}r^d \,:\, r^a, r^b, r^c, r^d \in \mathbb{R}\}, \tag{1}$$

which obeys the following algebraic rules:

$$\begin{aligned} &\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \tag{2}$$

For any quaternion $r = r^a + \mathbf{i}r^b + \mathbf{j}r^c + \mathbf{k}r^d \in \mathbb{H}$, we call r^a the scalar part of r. It is denoted by

$$S_c(r) = r^a, \tag{3}$$

and its vector part is

$$V(r) = \mathbf{i}r^b + \mathbf{j}r^c + \mathbf{k}r^d = \boldsymbol{r}.$$

From (2) we obtain the quaternion multiplications rz in the form

$$rz = r^a z^a - \boldsymbol{r} \cdot \boldsymbol{z} + r^a \boldsymbol{z} + z^a \boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{z}, \tag{4}$$

where

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{z} &= r^b z^b + r^c z^c + r^d z^d \\ \boldsymbol{r} \times \boldsymbol{z} &= \mathbf{i}(r^c z^d - r^d z^c) + \mathbf{j}(r^d z^b - r^b z^d) + \mathbf{k}(r^b z^c - r^c z^d). \end{aligned}$$

According to (4), one can verify that the scalar part (3) satisfies a cyclic multiplication symmetry, i.e.,

$$S_c(rpq) = S_c(prq) = S_c(qpr), \quad \forall r, p, q \in \mathbb{H}. \tag{5}$$

Analogous to the complex case, the quaternion conjugate of any quaternion r is defined by

$$\overline{r} = r^a - \mathbf{i}r^b - \mathbf{j}r^c - \mathbf{k}r^d, \tag{6}$$

which satisfies

$$\overline{rz} = \overline{z}\,\overline{r}, \quad \forall z, r \in \mathbb{H}. \tag{7}$$

It is easily seen from (7) that the quaternion conjugate changes the order of the multiplication. From (6), we obtain the norm or modulus of $r \in \mathbb{H}$ defined as

$$|r| = \sqrt{r\overline{r}} = \sqrt{(r^a)^2 + (r^b)^2 + (r^c)^2 + (r^d)^2}. \tag{8}$$

It is routine to check that

$$|r^2| = |r|^2, \quad |rz| = |r||z|, \quad \text{and} \quad |r + z| \le |r| + |z|, \quad \forall z, r \in \mathbb{H}. \tag{9}$$

From the quaternion conjugate (6) and the modulus of r, we obtain the inverse of a nonzero quaternion $r \in \mathbb{H}$ by

$$r^{-1} = \frac{\bar{r}}{|r|^2}. \tag{10}$$

This shows that $\mathbb{H}$ is a normed division algebra. For $|r| = 1, r$ is a unit quaternion and $r^a = 0, r$ is called a pure quaternion.

Similar to a complex number, we may define the inner product for two functions $f, g : \mathbb{R} \longrightarrow \mathbb{H}$ as

$$(f, g)_{L^2(\mathbb{R};\mathbb{H})} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx. \tag{11}$$

For $f = g$, we get the $L^2(\mathbb{R}; \mathbb{H})$-norm as

$$\|f\|_{L^2(\mathbb{R};\mathbb{H})} = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{12}$$

3. One-Dimensional Quaternion Fourier Transform with Properties

We start by introducing the definition of the one-dimensional quaternion Fourier transform (1DQFT). We recall its essential properties such as Parseval's formula and the convolution theorem. We propose an application of the properties which is the main result of this section. More details regarding the 1DQFT are discussed in [3].

Definition 1. *The one-dimensional quaternion Fourier transform is defined by*

$$\mathcal{F}_{\mathbf{i}}\{f\}(\xi) = \int_{\mathbb{R}} f(x)e^{\mathbf{i}\xi x}dx, \tag{13}$$

for $x, \xi \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; \mathbb{H}) \cap L^2(\mathbb{R}; \mathbb{H})$.

Below, we present the reconstruction formula for the 1DQFT.

Definition 2. *Let $f \in L^1(\mathbb{R}; \mathbb{H})$ and $\mathcal{F}_{\mathbf{i}}\{f\} \in L^1(\mathbb{R}; \mathbb{H})$. The inverse transform of the 1DQFT is computed by*

$$\mathcal{F}_{\mathbf{i}}^{-1}[\mathcal{F}_{\mathbf{i}}\{f\}](x) = f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}_{\mathbf{i}}\{f\}(\xi)e^{-\mathbf{i}\xi x}d\xi. \tag{14}$$

The following results related to the 1DQFT are useful.

Theorem 1. *Let $f \in L^2(\mathbb{R}; \mathbb{H})$, we have*

$$\int_{\mathbb{R}} |f(x)|^2 dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}_{\mathbf{i}}(\xi)|^2 d\xi, \tag{15}$$

which is known as Plancherel's formula for the 1DQFT.

Theorem 2. *Let $f \in L^1(\mathbb{R}; \mathbb{H}) \cap L^2(\mathbb{R}; \mathbb{H})$, one gets*

$$\overline{\mathcal{F}_{\mathbf{i}}\{f\}(\xi)} = \mathcal{F}_{\mathbf{i}}\{f^a\}(-\xi) - \mathbf{i}\mathcal{F}_{\mathbf{i}}\{f^b\}(-\xi) - \mathbf{j}\mathcal{F}_{\mathbf{i}}\{f^c\}(\xi) - \mathbf{k}\mathcal{F}_{\mathbf{i}}\{f^d\}(\xi). \tag{16}$$

In particular, when $f(x)$ is a real-valued function, (16) changes to

$$\overline{\mathcal{F}_{\mathbf{i}}\{f\}(\xi)} = \mathcal{F}_{\mathbf{i}}\{f\}(-\xi), \tag{17}$$

which is known as the conjugate symmetry for the 1DQFT.

Proof. It follows from (13) that

$$
\begin{aligned}
&\overline{\mathcal{F}_{\mathbf{i}}\{f\}(\xi)}\\
=&\int_{\mathbb{R}} e^{-\mathbf{i}\xi x}\overline{f(x)}dx\\
=&\int_{\mathbb{R}} e^{-\mathbf{i}\xi x}\left(f^a(x)-\mathbf{i}f^b(x)-\mathbf{j}f^c(x)-\mathbf{k}f^d(x)\right)dx\\
=&\int_{\mathbb{R}} f^a(x)e^{-\mathbf{i}\xi x}dx-\mathbf{i}\int_{\mathbb{R}} f^b(x)e^{-\mathbf{i}\xi x}dx-\mathbf{j}\int_{\mathbb{R}} f^c(x)e^{\mathbf{i}\xi x}dx-\mathbf{k}\int_{\mathbb{R}} f^d(x)e^{\mathbf{i}\xi x}dx\\
=&\mathcal{F}_{\mathbf{i}}\{f^a\}(-\xi)-\mathbf{i}\mathcal{F}_{\mathbf{i}}\{f^b\}(-\xi)-\mathbf{j}\mathcal{F}_{\mathbf{i}}\{f^c\}(\xi)-\mathbf{k}\mathcal{F}_{\mathbf{i}}\{f^d\}(\xi).
\end{aligned}
$$

This is the required result. □

It is known that one useful tool related to the 1DQFT is the convolution operator. Below, we briefly recall the convolution definition and its convolution theorem.

Definition 3. *The convolution of two quaternion function $f,g \in L^1(\mathbb{R};\mathbb{H})$ is defined by*

$$
(f*g)(x)=\int_{\mathbb{R}} f(y)g(x-y)dy. \tag{18}
$$

With Definition 3 above, we get the following [13].

Theorem 3. *Suppose that the quaternion functions $f \in L^1(\mathbb{R};\mathbb{H})$ and $g \in L^1(\mathbb{R};\mathbb{H})$. Then, the following holds:*

$$
\begin{aligned}
\mathcal{F}_{\mathbf{i}}\{f*g\}(\xi)=&\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^a\}(\xi)+\mathbf{i}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^b\}(\xi)\\
&+\mathbf{j}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^c\}(\xi)+\mathbf{k}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^d\}(\xi).
\end{aligned} \tag{19}
$$

Moreover,

$$
\begin{aligned}
(f*g)(x)=&\mathcal{F}_{\mu}^{-1}[\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^a\}(\xi)+\mathbf{i}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^b\}(\xi)\\
&+\mathbf{j}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^c\}(\xi)+\mathbf{k}\mathcal{F}_{\mathbf{i}}\{f\}(\xi)\mathcal{F}_{\mathbf{i}}\{g^d\}(\xi)](x).
\end{aligned}
$$

Definition 4. *The correlation for the 1DQFT of two quaternion functions $f,g \in L^1(\mathbb{R};\mathbb{H})$ is given by the integral*

$$
(f\circ g)(x)=\int_{\mathbb{R}} f(x+y)\overline{g(y)}dy. \tag{20}
$$

The next result will be very useful for deriving the main results of this work.

Theorem 4. *Suppose $f \in L^1(\mathbb{R};\mathbb{H})$ such that f(x) is continuous n-time differentiable; then, for* $\lim_{x\longrightarrow\pm\infty} f(x)=0$, *the following holds*

$$
\mathcal{F}_{\mathbf{i}}\left\{\frac{d^n}{dx^n}f\right\}(\xi)=\mathcal{F}_{\mathbf{i}}\{f\}\xi^n(-\mathbf{i})^n, n\in\mathbb{N}. \tag{21}
$$

Proof. Consider first the case $n = 1$. Direct computations yield

$$\begin{aligned}
\mathcal{F}_{\mathbf{i}}\left\{\frac{d}{dx}f\right\}(\xi) =& \int_{\mathbb{R}} \left(\frac{d}{dx}f(x)\right)e^{\mathbf{i}\xi x}\, dx \\
=& e^{\mathbf{i}\xi x}\, f(x)\mid_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\frac{d}{dx}e^{\mathbf{i}\xi x}\, dx \\
=& e^{\mathbf{i}\xi x}\, f(x)\mid_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\left(\mathbf{i}\xi\, e^{\mathbf{i}\xi x}\right)dx \\
=& -\int_{\mathbb{R}} f(x)e^{\mathbf{i}\xi x}dx\, \xi(\mathbf{i}) \\
=& \int_{\mathbb{R}} f(x)e^{\mathbf{i}\xi x}dx\, \xi(-\mathbf{i}) \\
=& \mathcal{F}_{\mathbf{i}}\{f\}\xi(-\mathbf{i}).
\end{aligned} \tag{22}$$

Using the mathematical induction, we can finish the proof. □

Let us now present the use of the properties of the 1DQFT to explore an inequality (compare to [20]), which is the main result in this section.

Theorem 5 (Szokefalvi-Nagy's inequality)**.** *Suppose that*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = E, \quad \text{and} \quad \int_{-\infty}^{\infty} \left|\frac{df}{dx}\right|^2 dx = E_1; \tag{23}$$

then,

$$|f(x)| \le \sqrt[4]{EE_1}. \tag{24}$$

The equality holds for $x = x_0$ if and only if

$$f(x) = \sqrt[4]{EE_1}e^{-\alpha|x-x_0|}, \quad \alpha = \sqrt{\frac{E_1}{E}}. \tag{25}$$

Proof. For every x_0 and $\alpha > 0$, we obtain from the inverse transform of the 1DQFT (14) that

$$|f(x_0)|^2 = \frac{1}{4\pi^2}\left|\int_{-\infty}^{\infty} \mathcal{F}_i(\xi)e^{-\mathbf{i}\xi x_0}d\xi\right|^2.$$

Applying the Cauchy-Schwartz inequality, we see that

$$\begin{aligned}
|f(x_0)|^2 &= \frac{1}{4\pi^2}\left|\int_{-\infty}^{\infty} \sqrt{\alpha^2+\xi^2}\mathcal{F}_{\mathbf{i}}(\xi)\frac{e^{-\mathbf{i}\xi x_0}}{\sqrt{\alpha^2+\xi^2}}d\xi\right|^2 \\
&\le \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left|\sqrt{\alpha^2+\xi^2}\mathcal{F}_{\mathbf{i}}(\xi)\frac{e^{-\mathbf{i}\xi x_0}}{\sqrt{\alpha^2+\xi^2}}\right|^2 d\xi \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left(\alpha^2+\xi^2\right)|\mathcal{F}_{\mathbf{i}}(\xi)|^2 d\xi \int_{-\infty}^{\infty}\frac{d\xi}{\alpha^2+\xi^2} \\
&= \frac{1}{4\pi^2}\left(\int_{-\infty}^{\infty}\alpha^2|\mathcal{F}_{\mathbf{i}}(\xi)|^2\, d\xi + \int_{-\infty}^{\infty}\xi^2|\mathcal{F}_{\mathbf{i}}(\xi)|^2 d\xi\right)\frac{\pi}{\alpha}.
\end{aligned}$$

By (15) and (21), we obtain

$$\begin{aligned}
|f(x_0)|^2 &\le \frac{1}{4\pi^2}\left(\alpha^2\int_{-\infty}^{\infty}|\mathcal{F}_{\mathbf{i}}(\xi)|^2 d\xi + \int_{-\infty}^{\infty}\xi^2|\mathcal{F}_{\mathbf{i}}(\xi)|^2 d\xi\right)\frac{\pi}{\alpha} \\
&= \frac{1}{4\pi^2}\left(\alpha^2 2\pi\int_{-\infty}^{\infty}|f(x)|^2 dx + 2\pi\int_{-\infty}^{\infty}\left|\frac{df}{dx}\right| dx\right)\frac{\pi}{\alpha}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
|f(x_0)|^2 &\leq \left(\alpha^2 E + E_1\right)\frac{\pi}{\alpha} \\
&= \frac{\alpha E}{2} + \frac{E_1}{2\alpha}.
\end{aligned} \tag{26}
$$

Since $\alpha = \sqrt{\frac{E_1}{E}}$, we get

$$|f(x_0)|^4 \leq EE_1.$$

The equality holds only if

$$\sqrt{\alpha^2+\xi^2}\mathcal{F}_{\mathbf{i}}(\xi) = \frac{ke^{-\mathbf{i}\xi x_0}}{\sqrt{\alpha^2+\xi^2}}. \tag{27}$$

Applying the inverse transform of the 1DQFT (14), relation (27) above leads to

$$f(x) = \frac{k}{\alpha}e^{-\alpha|x-x_0|}. \tag{28}$$

Substituting (28) into relation (23), we obtain

$$E = \frac{k^2}{4\alpha^3}, \quad E_1 = \frac{k^2}{4\alpha}. \tag{29}$$

From Equation (29), it is easily checked that

$$\alpha = \sqrt{\frac{E_1}{E}}. \tag{30}$$

Inserting Equation (30) into Equation (29) gives

$$k^2 = 4E\alpha^3 = 4\alpha^2 E\alpha = 4\alpha^2 E\sqrt{\frac{E_1}{E}}. \tag{31}$$

This means that

$$k = 2\alpha\sqrt[4]{EE_1}. \tag{32}$$

This ends the proof of the theorem. □

4. One-Dimensional Quaternion Fourier Transform in Probability Theory

In this part, we present the utility of the one-dimensional quaternion Fourier transform in probability theory. To begin with, we introduce the following definition:

Definition 5. *[21] Let X be a real random variable. A quaternion-valued function $f_{\mathbf{X}}(x) = f_X^a(x) + \mathbf{i}f_X^b(x) + \mathbf{j}f_X^c(x) + \mathbf{k}f_X^d(x)$ is called the quaternion probability density function of X if*

$$\int_{\mathbb{R}} f_X^i(x)dx = 1, \quad f_X^i(x) \geq 0, \quad \forall x \in \mathbb{R},\ i = a,b,c,d.$$

Here, f_X^i is a real probability density function. The quaternion cumulative distribution function is expressed as

$$f_X(x) = \frac{d}{dx}F_X(x), \tag{33}$$

where the probability P is related to F_X given by

$$F_X(x) = P(X \leq x). \tag{34}$$

Definition 6 (Expected value)**.** *Let X be a real random variable with the quaternion probability density function $f_X(x)$. The expected value $m = \mathrm{E}[X]$ is defined through*

$$\begin{aligned} m &= \mathrm{E}[X] \\ &= \int_{\mathbb{R}} x f_X(x) dx \\ &= \int_{\mathbb{R}} x(f_X^a(x) + \mathbf{i} f_X^b(x) + \mathbf{j} f_X^c(x) + \mathbf{k} f_X^d(x)) dx \\ &= \int_{\mathbb{R}} x f_X^a(x) dx + \mathbf{i} \int_{\mathbb{R}} x f_X^b(x) dx + \mathbf{j} \int_{\mathbb{R}} x f_X^c(x) dx + \mathbf{k} \int_{\mathbb{R}} x f_X^d(x) dx \\ &= \mathrm{E}[X_a] + \mathbf{i}\mathrm{E}[X_b] + \mathbf{j}\mathrm{E}[X_c] + \mathbf{k}\mathrm{E}[X_d], \end{aligned} \tag{35}$$

Here,

$$\mathrm{E}[X_i] = \int_{\mathbb{R}} x f_X^i(x) dx.$$

The expected value of the above definition is often called the mean in the quaternion setting. It is easily seen that

$$\overline{\mathrm{E}[X]} = \mathrm{E}[X_a] - \mathbf{i}\mathrm{E}[X_b] - \mathbf{j}\mathrm{E}[X_c] - \mathbf{k}\mathrm{E}[X_d], \tag{36}$$

and

$$|m|^2 = |\mathrm{E}[X]|^2 = \mathrm{E}[X]\overline{\mathrm{E}[X]} = \mathrm{E}^2[X_a] + \mathrm{E}^2[X_b] + \mathrm{E}^2[X_c] + \mathrm{E}^2[X_d]. \tag{37}$$

Definition 7. *Let X be a real random variable with the quaternion probability density function $f_X(x)$. The characteristic function of X, $\phi_X : \mathbb{R} \longrightarrow \mathbb{H}$, is defined by the formula*

$$\begin{aligned} \phi_X(t) &= \mathrm{E}[e^{\mathbf{i}tX}] \\ &= \int_{\mathbb{R}} f_X(x) e^{\mathbf{i}tx} dx. \end{aligned} \tag{38}$$

Relation (38) above may be expressed in the form

$$\begin{aligned} \phi_X(t) &= \int_{\mathbb{R}} (f_X^a(x) + \mathbf{i} f_X^b(x) + \mathbf{j} f_X^c(x) + \mathbf{k} f_X^d(x)) e^{\mathbf{i}tx} dx \\ &= \int_{\mathbb{R}} f_X^a(x) e^{\mathbf{i}tx} dx + \mathbf{i} \int_{\mathbb{R}} f_X^b(x) e^{\mathbf{i}tx} dx + \mathbf{j} \int_{\mathbb{R}} f_X^c(x) e^{\mathbf{i}tx} dx \\ &\quad + \mathbf{k} \int_{\mathbb{R}} f_X^d(x) e^{itx} dx \\ &= \phi_X^a(t) + \mathbf{i}\phi_X^b(t) + \mathbf{j}\phi_X^c(t) + \mathbf{k}\phi_X^d(t), \end{aligned}$$

where

$$\phi_X^i(t) = \int_{\mathbb{R}} x f_X^i(x) e^{\mathbf{i}tx} dx, \quad i = a, b, c, d. \tag{39}$$

Definition 8. *Let X be a real random variable with the quaternion probability density function $f_X(x)$. If $\int_{\mathbb{R}} |\phi_X(t)| dt < \infty$, the function $f_X(x)$ can be constructed using the formula*

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_X(t) e^{-\mathbf{i}tx} dt. \tag{40}$$

Some basic properties of the quaternion characteristic function are studied in the following results.

Lemma 1. *Let X be a real random variable. Then,*

$$\overline{\phi_X(t)} = \phi_X^a(-t) - \mathbf{i}\phi_X^b(-t) - \mathbf{j}\phi_X^c(t) - \mathbf{k}\phi_X^d(t). \tag{41}$$

Proof. In fact, we have

$$\begin{aligned}
\overline{\phi_X(t)} &= \int_{\mathbb{R}} \overline{f_X(x)e^{\mathbf{i}tx}}dx \\
&= \int_{\mathbb{R}} e^{-\mathbf{i}tx}\overline{f_X(x)}dx \\
&= \int_{\mathbb{R}} e^{-\mathbf{i}tx}\big(f_X^a(x) - \mathbf{i}f_X^b(x) - \mathbf{j}f_X^c(x) - \mathbf{k}f_X^d(x)\big)dx \\
&= \int_{\mathbb{R}} f_X^a(x)e^{-\mathbf{i}tx}dx - \mathbf{i}\int_{\mathbb{R}} f_X^b(x)e^{-\mathbf{i}tx}dx - \mathbf{j}\int_{\mathbb{R}} f_X^c(x)e^{\mathbf{i}tx}dx - \mathbf{k}\int_{\mathbb{R}} f_X^d(x)e^{\mathbf{i}tx}dx \\
&= \phi_X^a(-t) - \mathbf{i}\phi_X^b(-t) - \mathbf{j}\phi_X^c(t) - \mathbf{k}\phi_X^d(t).
\end{aligned}$$

This is the desired result. □

Lemma 2. *Let X be a real random variable. Then,*

$$\phi_{\overline{aX+b}}(t) = \phi_X(-at)e^{-\mathbf{i}tb},$$

where a and b are real constants.

Proof. Simple computations yield

$$\begin{aligned}
\phi_{\overline{aX+b}}(t) &= \int_{\mathbb{R}} f_X(x)\overline{e^{\mathbf{i}t(ax+b)}}dx \\
&= \int_{\mathbb{R}} f_X(x)e^{-\mathbf{i}tax}dx\; e^{-\mathbf{i}tb} \\
&= \phi_X(-at)\; e^{-\mathbf{i}tb},
\end{aligned}$$

and the proof is complete. □

It is easily seen that

$$|\phi_X(t)| = \left|\int_{\mathbb{R}} f_X(x)e^{itx}dx\right| \le \int_{\mathbb{R}} |f_X(x)|dx = \int_{\mathbb{R}} f_X(x)dx = 1. \tag{42}$$

Lemma 3. *For any real random variable X, one has*

$$\overline{\phi_{\overline{aX+b}}(t)} = \phi_X^a(at)\; e^{\mathbf{i}tb} - \mathbf{i}\phi_X^b(at)\; e^{\mathbf{i}tb} - \mathbf{j}\phi_X^c(-at)\; e^{-\mathbf{i}tb} - \mathbf{k}\phi_X^d(-at)\; e^{-\mathbf{i}tb}. \tag{43}$$

Proof. It follows from relation (38) that

$$\begin{aligned}
\overline{\phi_{\overline{aX+b}}(t)} &= \int_{\mathbb{R}} \overline{f_X(x)\overline{e^{\mathbf{i}t(ax+b)}}}dx \\
&= \int_{\mathbb{R}} e^{\mathbf{i}t(ax+b)}\overline{f_X(x)}dx \\
&= \int_{\mathbb{R}} \big(f_X^a(x) - \mathbf{i}f_X^b(x)\big)e^{\mathbf{i}tax}dxe^{\mathbf{i}tb} - \int_{\mathbb{R}} \big(\mathbf{j}f_X^c(x) - \mathbf{k}f_X^d(x)\big)e^{-\mathbf{i}tax}dxe^{-\mathbf{i}tb} \\
&= \big(\phi_X^a(at) - \mathbf{i}\phi_X^b(at)\big)\; e^{\mathbf{i}tb} - \big(\mathbf{j}\phi_X^c(-at) + \mathbf{k}\phi_X^d(-at)\big)\; e^{-\mathbf{i}tb}.
\end{aligned} \tag{44}$$

This is the desired result. □

With Definition 7, we obtain the following important result.

Theorem 6. *If the quaternion characteristic functions ϕ_X and ψ_X of the random variable X are defined by*

$$\phi_X(t) = \int_{\mathbb{R}} f_X(x) e^{\mathbf{i}tx} dx, \quad \psi_X(x) = \int_{\mathbb{R}} g_X(t) e^{\mathbf{i}tx} dt, \tag{45}$$

then the following holds:

$$\begin{aligned} \int_{\mathbb{R}} g_X(t)\phi_X(t)e^{-\mathbf{i}ty}dt &= \int_{\mathbb{R}} f_X(x)\psi_X^a(x-y)\,dx + \mathbf{i}\int_{\mathbb{R}} f_X(x)\psi_X^b(x-y)dx \\ &\quad + \mathbf{j}\int_{\mathbb{R}} f_X(x)\psi_X^c(y-x)dx + \mathbf{k}\int_{\mathbb{R}} f_X(x)\psi_X^d(y-x))dx. \end{aligned} \tag{46}$$

Proof. By virtue of the characteristic function (40), we obtain

$$\begin{aligned} \phi_X(t)e^{-\mathbf{i}ty} &= \int_{\mathbb{R}} f_X(x)e^{\mathbf{i}tx}dxe^{-\mathbf{i}ty} \\ &= \int_{\mathbb{R}} f_X(x)e^{\mathbf{i}t(x-y)}dx. \end{aligned} \tag{47}$$

Multiplying both sides of the above identity by $g_X(t)$ and then integrating with respect to dt, we see that

$$\begin{aligned} \int_{\mathbb{R}} g_X(t)\phi_X(t)e^{-\mathbf{i}ty}dt &= \int_{\mathbb{R}} g_X(t)\left(\int_{\mathbb{R}} f_X(x)e^{\mathbf{i}t(x-y)}dx\right)dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} (g_X^a(t) + \mathbf{i}g_X^b(t) + \mathbf{j}g_X^c(t) + \mathbf{k}g_X^d(t))f_X(x)e^{\mathbf{i}t(x-y)}dxdt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(g_X^a(t) + \mathbf{i}g_X^b(t) + \mathbf{j}\,g_X^c(t) + \mathbf{k}g_X^d(t)\right)f_X(x)e^{\mathbf{i}t(x-y)}dxdt. \end{aligned}$$

Fubini's theorem allows us to obtain

$$\begin{aligned} &\int_{\mathbb{R}} g_X(t)\phi_X(t)e^{-\mathbf{i}ty}dt \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f_X(x)g_X^a(t)e^{\mathbf{i}t(x-y)}dt + \mathbf{i}f_X(x)\int_{\mathbb{R}} g_X^b(t)\,e^{\mathbf{i}t(x-y)}dt\right)dx \\ &\quad + \int_{\mathbb{R}}\left(\mathbf{j}f_X(x)\int_{\mathbb{R}} g_X^c(t)\,e^{\mathbf{i}t(y-x)}dt + \mathbf{k}f_X(x)\int_{\mathbb{R}} g_X^d(t)\,e^{\mathbf{i}t(y-x)}dt\right)dx \\ &= \int_{\mathbb{R}}\left(f_X(x)\psi_X^a(x-y) + \mathbf{i}f_X(x)\psi_X^b(x-y)\right)dx \\ &\quad + \int_{\mathbb{R}}\left(\mathbf{j}f_X(x)\psi_X^c(y-x) + \mathbf{k}f_X(x)\psi_X^d(y-x)\right)dx \\ &= \int_{\mathbb{R}} f_X(x)\psi_X^a(x-y)dx + \mathbf{i}\int_{\mathbb{R}} f_X(x)\psi_X^b(x-y)dx \\ &\quad + \mathbf{j}\int_{\mathbb{R}} f_X(x)\,\psi_X^c(y-x)dx + \mathbf{k}\int_{\mathbb{R}} f_X(x)\psi_X^d(y-x))dx. \end{aligned}$$

This is the desired result. □

It follows from (33) that

$$\mathcal{F}_{\mathbf{i}}\left\{\frac{d}{dx}F_X\right\}(t) = \mathcal{F}_{\mathbf{i}}\{f_X\}(t) = \phi_X(t), \tag{48}$$

where $F_X(x)$ is the quaternion distribution function of random variable X. Furthermore, the application of (21) leads to

$$\phi_X(t) = \mathcal{F}_{\mathbf{i}}\{F_X\}(t)t(-\mathbf{i}), \tag{49}$$

and thus

$$\mathcal{F}_{\mathbf{i}}\{F_X\}(t) = \frac{\mathbf{i}}{t}\phi_X(t), \quad t \neq 0. \tag{50}$$

Based on (35), we define the nth moment of a real random variable X as

$$m_n = \mathrm{E}[\mathrm{X}^n] = \int_{\mathbb{R}} x^n f_X(x) dx, \quad n = 1, 2, 3, \cdots, \tag{51}$$

provided the integral exists. It is obvious that for $n = 1$ in (51), we obtain the first moment m_1, which is called the expectation of X.

Theorem 7. *If X is a real random variable, then there exists nth continuous derivatives for the quaternion characteristic function $\phi_X(t)$ which is given by the formula*

$$\frac{d^k}{dt^k}\phi_X(t) = \int_{\mathbb{R}} f_X(x)\, e^{\mathbf{i}tx} x^k dx\, \mathbf{i}^k. \tag{52}$$

Moreover,

$$m_k = \mathrm{E}[\mathrm{X}^k] = \frac{d^k}{dt^k}\phi_X(0)(-\mathbf{i})^k, \quad k = 1, 2, 3, ...n. \tag{53}$$

Proof. For $k = 1$, direct computations reveal that

$$\begin{aligned} \frac{d}{dt}\phi_X(t) &= \frac{d}{dt}\int_{\mathbb{R}} f_X(x) e^{\mathbf{i}tx} dx \\ &= \int_{\mathbb{R}} f_X(x) \left(\frac{d}{dt} e^{\mathbf{i}tx}\right) dx \\ &= \int_{\mathbb{R}} f_X(x)\, e^{\mathbf{i}tx}\, x dx\, \mathbf{i}. \end{aligned} \tag{54}$$

In view of relation (54), we further get

$$\begin{aligned} \frac{d^2}{dt^2}\phi_X(t) &= \frac{d}{dt}\left(\int_{\mathbb{R}} f_X(x)\, e^{\mathbf{i}tx}\, x dx \mathbf{i}\right) \\ &= \int_{\mathbb{R}} f_X(x)\, e^{\mathbf{i}tx}\, x^2 dx\, \mathbf{i}^2. \end{aligned} \tag{55}$$

This means that

$$\frac{d^k}{dt^k}\phi_X(t) = \int_{\mathbb{R}} f_X(x)\, e^{\mathbf{i}tx} x^k dx\, \mathbf{i}^k, \tag{56}$$

and

$$\frac{d^k}{dt^k}\phi_X(0) = \int_{\mathbb{R}} f_X(x)\, x^k dx\, \mathbf{i}^k. \tag{57}$$

Hence,

$$\frac{d^k}{dt^k}\phi_X(0)(\mathbf{i}^k)^{-1} = \int_{\mathbb{R}} f_X(x)\, x^k dx.$$

Or, equivalently,

$$\begin{aligned}\frac{d^k}{dt^k}\phi_X(0)(-\mathbf{i})^k &= \int_{\mathbb{R}} x^k f_X(x)dx \\ &= \mathrm{E}[X^k].\end{aligned}$$

The assertion is proved. □

Definition 9. *Let X be a any real random variable. The variance of X in the quaternion setting is defined by*

$$\begin{aligned}\sigma^2 &= \mathrm{E}\big[(X-\mathrm{E}[X])(\overline{X-\mathrm{E}[X]})\big] \\ &= \mathrm{E}\big[(X-\mathrm{E}[X])(X-\overline{\mathrm{E}[X]})\big] \\ &= \mathrm{E}\big[(X^2 - X\overline{\mathrm{E}[X]} - X\mathrm{E}[X] + |\mathrm{E}[X]|^2\big] \\ &= \mathrm{E}\big[X^2\big] - \mathrm{E}[X]\overline{\mathrm{E}[X]} - \mathrm{E}[X]\mathrm{E}[X] + |\mathrm{E}[X]|^2\big] \\ &= \mathrm{E}\big[X^2\big] - (\mathrm{E}[X])^2.\end{aligned} \tag{58}$$

The variance σ^2 of a real random variable in terms of the quaternion characteristic function can be obtained as

$$\sigma^2 = \frac{d^2}{dt^2}\phi_X(0)(-\mathbf{i})^2 - \left(\frac{d^2}{dt^2}\phi_X(0)(-\mathbf{i})\right)^2. \tag{59}$$

The following example illustrates the use of the results mentioned above.

Example 1. *The random variable X has the probability density function*

$$f(x) = \frac{1}{|\sigma|\sqrt{2\pi}} e^{-\frac{(x-|m|)^2}{2|\sigma|^2}}. \tag{60}$$

We find the first and second moments of X.

It follows from (40) that

$$\phi_X(t) = \int_{\mathbb{R}} \frac{1}{|\sigma|\sqrt{2\pi}} e^{-\frac{(x-|m|)^2}{2|\sigma|^2}} e^{\mathbf{i}tx}dx.$$

Performing a change of variable $x - |m| = y$, it is easily seen that

$$\begin{aligned}\phi_X(t) &= \int_{\mathbb{R}} \frac{1}{|\sigma|\sqrt{2\pi}} e^{-\frac{y^2}{2|\sigma|^2}} e^{\mathbf{i}t(|m|+y)}dy \\ &= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{2|\sigma|^2}+\mathbf{i}ty}dy.\end{aligned} \tag{61}$$

We further obtain

$$\begin{aligned}
\phi_X(t) &= \frac{e^{\mathbf{i}t|m|}}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma|^2}\left(y^2 - 2\sigma^2 \mathbf{i}ty\right)} dy \\
&= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma|^2}\left(\left(y-|\sigma|^2\mathbf{i}t\right)^2 - \left(|\sigma|^2\mathbf{i}t\right)^2\right)} dy \\
&= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{\left(|\sigma|^2\mathbf{i}t\right)^2}{2|\sigma|^2}} e^{-\frac{1}{2\sigma^2}\left(y-|\sigma|^2\mathbf{i}t\right)^2} dy \\
&= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} e^{\frac{\left(|\sigma|^2\mathbf{i}t\right)^2}{2|\sigma|^2}} \int_{\mathbb{R}} e^{-\frac{1}{2|\sigma|^2}\left(y-|\sigma|^2\mathbf{i}t\right)^2} dy \\
&= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} e^{\frac{|\sigma|^2\mathbf{i}^2t^2}{2}} \int_{\mathbb{R}} e^{-ax^2} dx \\
&= \frac{e^{\mathbf{i}t|m|}}{|\sigma|\sqrt{2\pi}} e^{\frac{-|\sigma|^2t^2}{2}} \sqrt{2\pi|\sigma|^2} \\
&= e^{\mathbf{i}t|m| - \frac{|\sigma|^2t^2}{2}}.
\end{aligned} \tag{62}$$

Therefore,

$$\begin{aligned}
\tfrac{d}{dt}\phi_X(t) &= \left(\mathbf{i}|m| - t|\sigma|^2\right) e^{\mathbf{i}t|m| - \frac{|\sigma|^2t^2}{2}} \\
\tfrac{d^2}{dt^2}\phi_X(t) &= \left(\mathbf{i}|m| - \left(t|\sigma|^2\right)^2 - |\sigma|^2\right) e^{\mathbf{i}t|m| - \frac{|\sigma|^2t^2}{2}} \\
\tfrac{d^3}{dt^3}\phi_X(t) &= \left(\mathbf{i}|m| - \left(t|\sigma|^2\right)^3 - 3|\sigma|^2\left(\mathbf{i}|m| - t|\sigma|^2\right)\right) e^{\mathbf{i}t|m| - \frac{|\sigma|^2t^2}{2}} \\
\tfrac{d^4}{dt^4}\phi_X(t) &= \left(\mathbf{i}|m| - \left(t|\sigma|^2\right)^4 - 6|\sigma|^2\left(\mathbf{i}|m| - t|\sigma|^2\right) + 3|\sigma|^4\right) e^{\mathbf{i}t|m| - \frac{|\sigma|^2t^2}{2}}.
\end{aligned} \tag{63}$$

Combining (53) and (63) yields

$$m_1 = \mathrm{E}[X] = \frac{d}{dt}\phi_X(0)(-\mathbf{i}) = (\mathbf{i}|m|)(-\mathbf{i}) = \left(-\mathbf{i}^2\right)|m| = |m|. \tag{64}$$

Similarly,

$$\begin{aligned}
m_2 &= \mathrm{E}[X^2] \\
&= \tfrac{d^2}{dt^2}\phi_X(0)(-\mathbf{i})^2 \\
&= \left(-\sigma^2 + \mathbf{i}|m|\right)(-\mathbf{i})^2 \\
&= \left(\sigma^2 - \mathbf{i}^2|m|\right).
\end{aligned} \tag{65}$$

5. Conclusions

In this paper, we introduced the one-dimensional quaternion Fourier transform (1DQFT) and utilized its properties for deriving the inequality related to this transformation. We demonstrated its use in probability theory. The characteristic function, expected value, and variance in the quaternion setting were studied in detail. These results play an important role in the development of probability theory in the context of quaternion algebra. In the future, uncertainty principles relating the quaternion probability density function and its characteristic function will be investigated.

Author Contributions: Conceptualization, M.B.; formal analysis, M.B.; funding acquisition, W.E. and M.N.; investigation, N.B. and A.R.; methodology, W.E. and M.N.; resources, A.R.; validation, N.B and M.N.; writing—original draft, W.E. and A.R.; writing review and editing, M.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bracewell, R. *The Fourier Transform and Its Applications*; McGraw Hill: Boston, MA, USA, 2000.
2. Mallat, S. *A Wavelet Tour of Signal Processing*; Academic Press: San Diego, CA, USA, 2001.
3. Bahri, M.; Amir, A.K.; Resnawati; Lande, C. The quaternion domain Fourier transform and its application in mathematical statistics. *IAENG Int. J. Appl. Math.* **2018**, *48*, 184–190.
4. Hitzer, E. The quaternion domain Fourier transform and its properties. *Adv. Appl. Clifford Algebras* **2016**, *26*, 969–984. [CrossRef]
5. Hitzer, E. Quaternion Fourier transform on quaternion fields and generalizations. *Adv. Appl. Clifford Algebras* **2007**, *17*, 497–517. [CrossRef]
6. Bahri, M.; Ashino, R.; Vaillancourt, R. Convolution theorems for quaternion Fourier transform: properties and applications. *Abstr. Appl. Anal.* **2013**, *2013*, 162769. [CrossRef]
7. Saima, S.; Li, B.Z.; Adnan, S.M. New sampling expansion related to derivatives in quaternion Fourier transform domain. *Mathematics* **2022**, *10*, 1217. [CrossRef]
8. Grigoryan, A.M.; Agaian, S. Tensor transform-based quaternion fourier transform algorithm. *Inf. Sci.* **2015**, *320*, 62–74. [CrossRef]
9. Hitzer, E. Quaternionic Wiener-Khinchine theorems and spectral representation of convolution with steerable two-sided quaternion Fourier transform. *Adv. Appl. Clifford Algebras* **2017**, *27*, 1313–1328. [CrossRef]
10. Bahri, M. Karim, S.A.A. A variation on inequality for quaternion Fourier transform, modified convolution and correlation theorems for general quaternion linear canonical transform. *Symmetry* **2022**, *14*, 1303. [CrossRef]
11. Fan, X.; Li, Y.; Liu, Z.; Zhao, J. Solving quaternion linear system based on semi-tensor product of quaternion matrices. *Symmetry* **2022**, *14*, 1359. [CrossRef]
12. Kou, K.I.; Xia, Y.H. Linear quaternion differential equations: basic theory and fundamental results. *Stud. Appl. Math.* **2018**, *141*, 3–45. [CrossRef]
13. Bahri, M.; Toaha, S.; Rahim, A.; Azis, M.I. On one-dimensional quaternion Fourier transform. *J. Phys. Conf. Ser.* **2019**, *1341*, 062004. [CrossRef]
14. Bihan, L.; Sangwine, S.J. Quaternionic spectral analysis of non-stationary improper complex signals. *Quat. Clifford Fourier Transform. Wavelets* **2013**, 41–56.
15. Eusebio, A.G.; Claudia, J.H.; Carlos, C. One-dimensional quaternion Fourier transform and some applications. *Bull. Comput. Appl. Math.* **2021**, *9*, 1–25.
16. Saima, S.; Li, B.Z. Quaternionic one-dimensional linear canonical transform. *Optik* **2021**, *244*, 166914. [CrossRef]
17. Took, C.C.; Mandic, D.P. Augmented second-order statistics of quaternion random signals. *Signal Process.* **2011**, *91*, 214–224. [CrossRef]
18. El-Melegy, M.T.; Kamal, A.T. Linear regression classification in the quaternion and reduced biquaternion domains. *IEEE Signal Process. Lett.* **2022**, *29*, 214–224. [CrossRef]
19. Morais, J.; Georgiev, S.; Sprosig, W. *Real Quaternion Calculus Handbook*; Springer: Basel, Switzerland, 2014.
20. Athanasios, P. *Signal Analysis*; McGraw-Hill, Inc.: New York, NY, USA, 1977.
21. Loots, M.T. The Development of the Quaternion Normal Distribution. Master's Theses, University of Pretoria, Pretoria, South Africa, 2010.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.